Harmonic Theory on Riemann Surfaces

Enya Hsiao

May 25, 2020

Harmonic Differential Forms

Definition

Let X be a compact Riemann surface. The **harmonic m-forms**

$$Harm^m(X) := ker[\Delta : H^0(X, \mathscr{E}^m) \to H^0(X, \mathscr{E}^m)], \quad 0 \leq m \leq 2$$

is the kernel of the Laplace operator $\Delta := d \circ \delta + \delta \circ d$, where

$$\delta : \mathscr{E}^{j+1} \to \mathscr{E}^j := (-1) \cdot (* \, d \, *)$$

is the formal adjoint of d with respect to the Hermitian scalar product

$$(-,-) := H^0(X, \mathscr{E}^m) \times H^0(X, \mathscr{E}^m) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge *\beta.$$

- We are interested in the case $m = 1$.

Main Results

We will prove the following results for compact Riemann surface X:

Theorem 1 (de Rham-Hodge)

$$H^1(X,\mathbb{C}) \cong Rh^1(X) \cong Harm^1(X).$$

Theorem 2 (de Rham-Hodge-Dolbeault decomposition)

$$H^1(X,\mathbb{C}) = \bigoplus_{p+q=1} H^q(X,\Omega^p).$$

As a immediate corollary to Theorem 2 by Serre duality,

$$b_1(X) = dim H^1(X,\mathbb{C}) = dim\, H^1(X,\mathscr{O}) + dim\, H^0(X,\Omega^1) = 2g(X).$$

The $*$-operator

Definition

Let X be a complex manifold with Hermitian metric h. For each $x \in X$, the **$*$-operator** is a $\mathbb{C}$-antilinear isomorphism

$$* : A^{p,q} \to A^{n-p,n-q}, \quad \beta \mapsto *\beta,$$

defined via the characteristic equation

$$\alpha \wedge *\beta = < \alpha, \beta >_h \mathrm{dvol}_g \qquad \forall \alpha \in A^{p,q},$$

where $g = \mathrm{Re}\, h$ is the induced Riemannian metric on the underlying real tangent bundle $T_{\mathbb{R}} X_{\text{smooth}}$.

- The $*$-operators on cotangent spaces glue to a $\mathbb{C}$-antilinear **$*$-operator of sheaves**

$$* : \mathscr{E}^{p,q} \to \mathscr{E}^{n-p,n-q}.$$

Computations: Types $(1,0)$ and $(0,1)$

Lemma

Let X a Riemann surface. For $\eta(x) = \eta_1(x) + \eta_2(x) \in A^{1,0} \oplus A^{0,1} = A^1$,

$$*(\eta_1(x) + \eta_2(x)) = i \cdot (\overline{\eta}_1(x) - \overline{\eta}_2(x)).$$

In particular, the $*$-operator on $A^{1,0}$ and $A^{0,1}$ is independent of the metric h.

Proof: For Riemann surfaces, the normalized volume form dvol_g has local expression

$$\mathrm{dvol}_g = \frac{dx \wedge dy}{||dx \wedge dy||_g} = i \cdot \frac{dz \wedge d\overline{z}}{||dz \wedge d\overline{z}||_h}, \quad ||\omega||_h = \sqrt{<\omega, \omega>_h}.$$

By the characteristic equation

$$dz \wedge *dz = < dz, dz >_h \cdot i \frac{dz \wedge d\overline{z}}{||dz \wedge d\overline{z}||_h} = i dz \wedge d\overline{z}.$$

Hence $* : A^{1,0} \to A^{0,1}, *dz = i \cdot d\,\overline{z}$. Similarly, $*d\,\overline{z} = -i \cdot dz$. □

$H^0(X, \mathscr{E}^{p,q})$: a unitary vector space

Theorem

On a compact Riemann surface X:

(i) The following conjugate linear map for $0 \leq p, q \leq 1$ is a well-defined Hermitian scalar product.

$$(-,-) : H^0(X, \mathscr{E}^{p,q}) \times H^0(X, \mathscr{E}^{p,q}) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge *\beta$$

(ii) $Harm^1(X) = H^0(X, \Omega^1) \oplus H^0(X, \overline{\Omega}^1)$.

(iii) $H^0(X, \mathscr{E}^1)$ decomposes as orthogonal direct sum

$$\textbf{(*)} \qquad H^0(X, \mathscr{E}^1) = Harm^1(X) \overset{\perp}{\oplus} dH^0(X, \mathscr{E}) \overset{\perp}{\oplus} \delta H^0(X, \mathscr{E}^2).$$

Proof of (ii): For any 1-form $\eta \in H^0(X, \mathscr{E}^1)$,

$$\Delta\, \eta = 0 \iff d\eta = 0,\ \delta\eta = 0 \iff d'\eta = d''\eta = 0 \iff (ii).$$

$Harm^1(X) = H^0(X, \Omega^1) \oplus H^0(X, \overline{\Omega}^1)$

First equivalence:

$$0 = (\Delta\, \eta, \eta) = ((\delta \circ d + d \circ \delta)\eta, \eta) = (d\eta, d\eta) + (\delta\eta, \delta\eta)$$

Second equivalence: Split η into its components

$$\eta = \eta_1 + \eta_2 \in H^0(X, \mathscr{E}^{1,0}) \oplus H^0(X, \mathscr{E}^{0,1}).$$

By our previous computation of the $*$-operator:

$$\delta\eta = 0 \iff d * \eta = d(i\overline{\eta}_1 - i\overline{\eta}_2) = i \cdot (d'\overline{\eta}_1 - d''\overline{\eta}_2) = 0.$$

Conjugation of the right hand side gives the equivalences

$$\begin{aligned} d\eta = 0,\, \delta\eta = 0 &\iff d''\eta_1 + d'\eta_2 = 0,\, d''\eta_1 - d'\eta_2 = 0 \\ &\iff d'\eta = d'\eta_1 = 0,\, d''\eta = d''\eta_2 = 0. \end{aligned}$$

□

Hodge decomposition of 1-forms

Proof of (iii): (i) *Claim:*

$$H^0(X, \mathscr{E}^{0,1}) = d'' H^0(X, \mathscr{E}) \oplus H^0(X, \overline{\Omega}^1).$$

By an application of Stokes' Theorem

$$d'' H^0(X, \mathscr{E}) \cap H^0(X, \overline{\Omega}^1) = 0.$$

Meanwhile Dolbeault's Theorem

$$H^0(X, \Omega^1) = \frac{H^0(X, \mathscr{E}^{0,1})}{im[H^0(X, \mathscr{E}) \xrightarrow{d''} H^0(X, \mathscr{E}^{0,1})]}$$

imply the dimension formula

$$dim\, H^0(X, \overline{\Omega}^1) = dim\, H^0(X, \Omega^1) = dim\, \frac{H^0(X, \mathscr{E}^{0,1})}{d''\, H^0(X, \mathscr{E})},$$

i.e.

$$dim\, H^0(X, \overline{\Omega}^1) + dim\, d'' H^0(X, \mathscr{E}) = dim\, H^0(X, \mathscr{E}^{0,1}),$$

which concludes the claim.

Hodge decomposition of 1-forms

(ii) Now taking the complex conjugate of part (i),

$$H^0(X, \mathscr{E}^{0,1}) = d'' H^0(X, \mathscr{E}) \oplus H^0(X, \overline{\Omega}^1),$$

and the decomposition of smooth 1-forms into its components,

$$\begin{aligned} H^0(X, \mathscr{E}^1) &= H^0(X, \mathscr{E}^{1,0}) \oplus H^0(X, \mathscr{E}^{0,1}) \\ &= d' H^0(X, \mathscr{E}) \oplus d'' H^0(X, \mathscr{E}) \oplus H^0(X, \Omega^1) \oplus H^0(X, \overline{\Omega}^1). \end{aligned}$$

The claim then follows from

$$Harm^1(X) = H^0(X, \Omega^1) \oplus H^0(X, \overline{\Omega}^1)$$

and the isomorphism

$$d\mathscr{E}(X) \oplus \delta \mathscr{E}^2(X) \cong d\mathscr{E}(X) \oplus *d\mathscr{E}(X) \to d'\mathscr{E}(X) \oplus d''\mathscr{E}(X)$$

defined by

$$(df, *dg) = (d'f + d''f, i(d''\overline{g} - d'\overline{g})) \mapsto (d'(f - i\overline{g}), d''(f + i\overline{g})).$$

□

De Rham-Hodge Theorem

Theorem 1

On a compact Riemann surface X,

$$H^1(X,\mathbb{C}) \cong Rh^1(X) \cong Harm^1(X).$$

Proof: (i) We first show that

$$ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)] = dH^0(X,\mathscr{E}) \oplus Harm^1(X).$$

- One inclusion is clear, from $d \circ d = 0$ and $ker\,\Delta \subset ker\,d$.
- For the opposite inclusion, we show

$$ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)] \perp \delta H^0(X,\mathscr{E}^2).$$

Consider $\eta \in H^0(X,\mathscr{E}^1)$ with $d\eta = 0$. For any $\xi \in H^0(X,\mathscr{E}^2)$,

$$(\eta, \delta\xi) = (d\eta, \xi) = 0.$$

De Rham-Hodge Theorem

Then by the Hodge decomposition

$$H^0(X, \mathscr{E}^1) \overset{*}{=} Harm^1(X) \overset{\perp}{\oplus} dH^0(X, \mathscr{E}) \overset{\perp}{\oplus} \delta H^0(X, \mathscr{E}^2),$$

we have the result

$$ker[H^0(X, \mathscr{E}^1) \xrightarrow{d} H^0(X, \mathscr{E}^2)] = dH^0(X, \mathscr{E}) \oplus Harm^1(X).$$

(ii) Now the de Rham Theorem states

$$H^1(X, \mathbb{C}) \cong Rh^1(X) = \frac{ker[H^0(X, \mathscr{E}^1) \xrightarrow{d} H^0(X, \mathscr{E}^2)]}{im[H^0(X, \mathscr{E}) \xrightarrow{d} H^0(X, \mathscr{E}^1)]}.$$

By part (i), the right hand side is

$$\frac{dH^0(X, \mathscr{E}) \oplus Harm^1(X)}{im[H^0(X, \mathscr{E}) \xrightarrow{d} H^0(X, \mathscr{E}^1)]} = Harm^1(X).$$

□

$(p,q)-$Harmonic forms

Definition

The **harmonic** p,q**-forms**

$$Harm^{p,q}(X) := ker[\Box : H^0(X, \mathscr{E}^{p,q}) \to H^0(X, \mathscr{E}^{p,q})], \quad 0 \le p.q \le 1$$

is the kernel of the Laplace-Beltrami operator $\Box := d'' \circ \delta'' + \delta'' \circ d''$, where

$$\delta'' : \mathscr{E}^{p,q+1} \to \mathscr{E}^{p,q} := (-1) \cdot (* \, d'' \, *)$$

is the formal adjoint of d'' with respect to $(-,-)$ on $H^0(X, \mathscr{E}^{p,q})$.

Theorem (Proportionality of Δ and $\Box$)

On a compact Riemann surface X, the Laplace and Laplace-Beltrami operators satisfy: For $\eta = \eta_1 + \eta_2 \in \mathscr{E}^{1,0}(X) \oplus \mathscr{E}^{0,1}(X)$,

$$\Delta \, \eta = 2 \cdot \Box \, \eta_1 + 2 \cdot \Box \, \eta_2.$$

Relating Δ and $\square$ on Riemann Surfaces

Proof: Direct computation.

$$
\begin{array}{ccccc}
 & & H^0(X,\mathscr{E}^1) & \xrightarrow{\ d\ } & H^0(X,\mathscr{E}^2) \\
 & \swarrow^{\delta} & \downarrow {\color{red}\Delta} & \swarrow_{\delta} & \\
H^0(X,\mathscr{E}) & \xrightarrow{\ d\ } & H^0(X,\mathscr{E}^1) & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & H^0(X,\mathscr{E}^{0,1}) & & H^0(X,\mathscr{E}^{1,0}) & \xrightarrow{\ d''\ } & H^0(X,\mathscr{E}^2) \\
 & \swarrow^{\delta''} & \downarrow {\color{blue}\square} & & {\color{blue}\square} \downarrow & \swarrow_{\delta''} & \\
H^0(X,\mathscr{E}) & \xrightarrow{\ d''\ } & H^0(X,\mathscr{E}^{0,1}) & & H^0(X,\mathscr{E}^{1,0}) & &
\end{array}
$$

Decomposition of Harmonic forms

Corollary

On a compact Riemann surface X,

$$Harm^m(X) = \bigoplus_{p+q=m} Harm^{p,q}(X), \quad m = 0, 1, 2.$$

Proof: The canonical map by projection into (p, q)-components

$$Harm^m(X) \to \bigoplus_{p+q=m} Harm^{p,q}(X),\ \eta \mapsto \sum_{p+q=m} \eta^{p,q}$$

is well defined, since $\Delta\,\eta = 0$ implies $\Box\,\eta = 0$.

Injectivity is clear, while surjectivity follows since $\eta^{p,q}$ is an m-form and

$$\Box\,\eta^{p,q} \Rightarrow \Delta\,\eta^{p,q} = 0.$$

□

De Rham-Dolbeault-Hodge Decomposition

Theorem 2

On a compact Riemann surface X,

$$H^1(X, \mathbb{C}) = \bigoplus_{p+q=1} H^q(X, \Omega^p).$$

Proof: From Theorem 1 and the decomposition of harmonic forms

$$H^1(X, \mathbb{C}) = Harm^1(X) = Harm^{1,0}(X) \oplus Harm^{0,1}(X).$$

Recall the decomposition of 1-forms

$$H^0(X, \mathscr{E}^1) = Harm^1(X) \oplus d'' H^0(X, \mathscr{E}) \oplus \delta'' H^0(X, \mathscr{E}^2).$$

where we have employed the isomorphism

$$d' H^0(X, \mathscr{E}) \xrightarrow{\sim} \delta'' H^0(X, \mathscr{E}^2)$$
$$i \cdot d' f \mapsto \delta''(f \cdot \mathrm{dvol}_g)$$

De Rham-Dolbeault-Hodge Decomposition

Separating the $(1,0)$ and $(0,1)$ forms

$$H^0(X,\mathscr{E}^{1,0}) = Harm^{1,0}(X) \oplus \delta'' H^0(X,\mathscr{E})$$
$$H^0(X,\mathscr{E}^{0,1}) = Harm^{0,1}(X) \oplus \delta'' H^0(X,\mathscr{E}^2).$$

By Dolbeault's Theorem

$$H^1(X,\mathscr{O}) \cong \frac{H^0(X,\mathscr{E}^{0,1})}{im[H^0(X,\mathscr{E}) \xrightarrow{d''} H^0(X,\mathscr{E}^{0,1})]} = Harm^{0,1}(X).$$

On the other hand, Dolbeault's Theorem show

$$\begin{aligned} H^0(X,\Omega^1) &\cong ker[H^0(X,\mathscr{E}^{1,0}) \xrightarrow{d''} H^0(X,\mathscr{E}^2)] \\ &= ker[H^0(X,\mathscr{E}^{1,0}) \xrightarrow{\Box} H^0(X,\mathscr{E}^2)] = Harm^{1,0}(X). \end{aligned}$$

We therefore have our claim

$$H^1(X,\mathbb{C}) = Harm^{1,0}(X) \oplus Harm^{0,1}(X) = H^0(X,\Omega^1) \oplus H^1(X,\mathscr{O}).$$

Generalization to higher dimension

- *Remark 1*: The de Rham-Hodge Theorem generalizes to higher dimension for all smooth manifolds.
- *Remark 2*: In contrast, the de Rham-Hodge-Dolbeault decomposition generalizes to higher dimensions only for **Kähler manifolds**. (Crux: Kähler identities.)
- On compact Kähler manifolds X the **Betti numbers** and **Hodge numbers** are finite:

$$b^m := dim\, H^m(X, \mathbb{C})\, , h^{p,q} := dim\, H^q(X, \Omega^p) < \infty$$

$$h^{0,0} = b^0 = 1$$

$$h^{0,1} = g(X) \qquad + \qquad h^{1,0} = g(X) \qquad (= b^1)$$

$$h^{1,1} = b^2 = 1$$