# Runge approximation theorem

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# Goals

**Theorem 1 (Vanishing theorem on open Riemann surfaces)**

*Let $X$ be an open Riemann surface. Then*

$$H^1(X, \mathcal{O}) = 0$$

**Theorem 2 (Runge approximation)**

*Let $X$ be an open Riemann surface and $Y \subset X$ a Runge domain. Then the restriction map*

$$\mathcal{O}(X) \to \mathcal{O}(Y)$$

*has a dense image in the uniform convergence on compact sets topology.*

# Plan

1. Put a topology in $\mathcal{E}(X)$ and $\mathcal{E}^{0,1}(X)$ & study their space of linear continuous functionals
2. Introduce Runge domains
3. Prove theorem 2 for a Runge domain
4. Extend the result to $X$
5. Use theorem 2 to prove theorem 1

**Definition 1 (Fréchet structure on $\mathcal{E}(X)$)**

*Let $X$ be an open Riemann surface, $(K_i)_{i\in\mathbb{N}}$ family of compact sets, each in a coordinate neighbourhood, such that $X = \bigcup_{i\in\mathbb{N}} \mathring{K}_i$.*

*Then we consider $\mathcal{E}(X)$ equipped with the topology induced by the family of semi-norms for $v = (v_1, v_2) \in \mathbb{N}^2$ given by*

$$p_{i,v}(f) := \sup\left\{|D^v f(x)| : x \in K_i\right\}$$

- Convergence implies uniform convergence of all derivatives.
- A neighborhood basis of zero is given by finite intersections of $\mathcal{U}\left(p_{j,v}, \varepsilon\right) := \{f \in \mathcal{E}(Y) : p_{jv}(f) < \varepsilon\}$

The same can be done for $\mathcal{E}^{0,1}$ since locally $\omega = f_i d\bar{z}_i$

## Lemma 1 (Functionals have compact support)

*Let $Y$ be an open subset of $X$. Then every continuous linear map $T : \mathcal{E}(Y) \to \mathbb{C}$ has compact support. Namely, there exists $K \subset Y$ compact such that $T[f] = 0$ for every $f \in \mathcal{E}(Y)$ with* $\mathrm{Supp}(f) \subset Y \setminus K$

## Proof (Sketch)

1. *Consider* $\mathcal{U} \subseteq \mathcal{E}(Y)$ *such that* $|T[f]| < 1$ *for all* $f \in \mathcal{U}$
2. $\mathcal{U}(p_{j_1,v_1}, \varepsilon) \cap \ldots \cap \mathcal{U}(p_{j_n,v_n}, \varepsilon) \subseteq \mathcal{U}$
3. *Set* $K = K_{j_1} \cup \ldots \cup K_{j_n}$ *and set* $f$ *with* $\mathrm{Supp}(f) \subset Y \setminus K$
4. *Scaling argument:* $\lambda f \in \mathcal{U} \implies |T[f]| < 1/\lambda$ *for* $\lambda > 0$

An analogous statement holds for $\mathcal{E}^{0,1}$

**Lemma 2**

*Let $Y$ be an open set on $X$. Consider a functional $S : \mathcal{E}^{0,1}(X) \to \mathbb{C}$ satisfying $S\,(d''g) = 0$ for all $g \in \mathcal{E}(X)$ with $\mathrm{Supp}(g) \subset\subset Y$. Then there exists $\sigma \in \Omega^1(X)$ such that*

$$S[\omega] = \iint_X \sigma \wedge \omega$$

*for all $\omega \in \mathcal{E}^{0,1}(X)$ such that $\mathrm{Supp}(\omega) \subset\subset Y$*

The proof relies strongly on the following theorem

**Theorem 3 (Weyl's Lemma)**

*Suppose $U$ is an open set in $\mathbb{C}$ and $T$ is a distribution on $U$ with $\Delta T = 0$. Then $T$ is can be realized as an integral whose kernel is a harmonic function.*

**Definition 2**

*Let $Y$ be a subset of $X$ a Riemann surface. The* ***Runge hull*** *of $Y$ w.r.t $X$, denoted $h_X(Y)$ is the set of all relatively compact components of $X \setminus Y$ union with $Y$*

**Definition 3**

*A subset $Y \subset X$ is said to be a* ***Runge set*** *if it is own Runge hull i.e. $h_X(Y) = Y$.*

An essential property of Runge sets is the following:

**Proposition 1 (Existence of Runge exhaustions)**

*Let $X$ be an open Riemann surface $X$. Then $X$ admits an exhaustion $(Y_i)_{i \in \mathbb{N}}$ by relatively compact Runge domains.*

# Preliminary results

### Theorem 4 (Hahn-Banach)

*Let $E$ be a Fréchet space, $E_0 \subset E$ is a vector subspace and $\varphi_0 : E_0 \to \mathbb{C}$ is a continuous linear functional. Then there exists a continuous linear functional $\varphi : E \to \mathbb{C}$ such that $\varphi|_{E_0} = \varphi_0$*

### Corollary

*Consider the Fréchet spaces $A \subset B \subset E$. If every continuous linear functional $\varphi : E \to C$ such that $\varphi|_A = 0$ satisfies $\varphi|_B = 0$, then $A$ is dense in $B$*

### Proof (sketch)

*By contra positive, let $B' = \bar{A} \oplus b\mathbb{C} \subseteq B$ and $\varphi(a + \lambda b) = \lambda$ on $B'$*

# Preliminary results II

## Lemma 3

*Let* $Y \subset\subset Y'$ *open sets in* $X$ *an open Riemann surface. Then for any* $\omega \in H^0\left(Y', \mathcal{E}^{0,1}\right)$ *exists* $f \in H^0(Y, \mathcal{E})$ *with* $d''f = \omega|Y$
*Equivalently, we can say that* $\operatorname{Im}\left[H^1(\mathcal{O}, Y') \to H^1(\mathcal{O}, Y)\right] = 0$

The proof relies on the exactness of the Dolbeault sequence

$$0 \to \mathcal{O} \to \mathcal{E} \overset{d''}{\to} \mathcal{E}^{0,1} \to 0$$

and relies on the following result

## Lemma 4 (Finite dimension of the space of obstructions)

*For a relative compact pair* $Y \subset\subset Y'$ *the restriction map satisfies* $\dim im\left[H^1\left(Y', \mathcal{O}\right) \to H^1(Y, \mathcal{O})\right] < \infty$

## Theorem 5

*Let $Y \subset\subset X$ be an open Runge subset in an open Riemann manifold. Then for every open subset $Y \subset Y' \subset\subset X$ the image of the restriction map $\mathcal{O}(Y') \xrightarrow{r} \mathcal{O}(Y)$ is dense.*

## Proof

1. *By the Hahn-Banach theorem, we need to show that if $T : \mathcal{E}(Y) \to \mathbb{C}$ a linear functional with $T|_{r(\mathcal{O}(Y'))} = 0$ then $T|_{\mathcal{O}(Y)} = 0$*
2. *Consider the functional $S : \mathcal{E}^{0,1}(X) \to \mathbb{C}$ defined as follows*

$$S[\omega] := T[f|_Y]$$

   *with $d''f = \omega|_{Y'}$ in virtue of lemma 3.*
3. *$S$ is well-defined. If $g$ with $d''g = \omega|_{Y'}$, then $d''(f-g) = 0$ so $f - g \in \mathcal{O}(Y')$. Thus $T[f - g|_Y] = 0$ by assumption.*

## Proof

4. *Define* $V := \{(\omega, f) \in \mathcal{E}^{0,1}(X) \times \mathcal{E}(Y') : d''f = \omega | Y'\}$ *and consider the commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{r \circ pr_2} & \mathcal{E}(Y) \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{E}^{0,1}(X) & \overset{S}{\dashrightarrow} & \mathbb{C} \end{array}$$

*so* $S$ *is continuous by the open mapping theorem*

5. *By lemma 1 we have compact sets* $K \subset Y$ *and* $L \subset X$ *such that*
   - $T[f] = 0$ *for every* $f \in \mathcal{E}(Y)$ *with* $\mathrm{Supp}(f) \subset Y \setminus K$
   - $S[\omega] = 0$ *for every* $\omega \in \mathcal{E}^{0,1}(X)$ *with* $\mathrm{Supp}(\omega) \subset X \setminus L$

6. *By lemma 2, we have that there exists* $\sigma \in \Omega(X \setminus K)$ *so that*

$$S[\omega] = \iint_{X \setminus K} \sigma \wedge \omega \quad \text{for } \omega \text{ with } \mathrm{Supp}(\omega) \subset X \setminus K$$

## Proof

7. *Combining these two results, we get* $\sigma|_{X\setminus(K\cup L)} = 0$
8. *The intersection* $(X \setminus (K \cup L)) \cap (X \setminus h(K))$ *is non-empty for each connected component of* $X \setminus h(K)$, *so* $\sigma|_{X\setminus h(K)} = 0$
9. *Since* $Y$ *is Runge,* $h(K) \subset (Y)$. *Thus, for any* $f \in \mathcal{O}(Y)$, *there is* $g \in \mathcal{E}(X)$ *with* $f - g = 0$ *on a neighbourhood of* $h(K)$ *and* $\mathrm{Supp}(g) \subset\subset Y$
10. *Therefore* $T[f] = T[g|_Y] = S[d''g]$
11. *Since* $f - g = 0$ *on a neighbourhood of* $h(K)$, *we get* $\mathrm{Supp}(d''g) \subset\subset X \setminus h(K)$, *so* $T[f] = S[d''g] = 0$

## Theorem 6 (Runge approximation)

*Let $X$ be an open Riemann surface and $Y \subset X$ a Runge domain. Then the restriction map*

$$\mathcal{O}(X) \to \mathcal{O}(Y)$$

*has a dense image in the uniform convergence on compact sets topology.*

## Proof

1. *Consider $(Y_i)_{i\in\mathbb{N}}$ a Runge exhaustion of $X$ with $Y = Y_0$*
2. *Use the previous theorem to build a sequence $f_i \in \mathcal{O}(Y_i)$ such that $\|f_n - f_{n-1}\|_K < \frac{1}{2^n}\varepsilon$*
3. *Hence, there exists $F \in \mathcal{O}(X)$ and $\|F - f\|_K < \varepsilon$*

## Theorem 7 (Vanishing theorem on open Riemann surfaces)

*Let $X$ be an open Riemann surface. Then*

$$H^1(X, \mathcal{O}) = 0$$

## Proof (Sketch for vanishing theorem on open Riemann surfaces)

1. *By Dolbeault's thm, we just need to prove that given a 1-form $\omega \in \mathcal{E}^{0,1}(X)$ there exists a function $f \in \mathcal{E}(X)$ with $d''f = \omega$*
2. *Set $(Y_i)_{i\in\mathbb{N}}$ a Runge exhaustion. By Lemma 3, we have $f_0 \in \mathcal{E}(Y_0))$ such that $d''f_o = \omega|_{Y_0}$*
3. *Inductively, let $g_{n+1}$ be given by Lemma 3. We get $g_{n+1} - f_n \in \mathcal{O}(Y_n)$ since $d''g_{n+1} = d''f_n$ on $Y_n$.*
4. *By Runge Approximation Theorem, we can modify $g_{n+1}$ by $h \in \mathcal{O}(Y_{n+1})$ so that $\|(g_{n+1} - f_n) - h\|_{Y_{n-1}} \leq 2^{-n}$*