Projective Embeddings of Tori as Elliptic Curves

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Recall:

- Consider a lattice Λ. $T = \mathbb{C}/\Lambda$ is a compact Riemannian surface of genus 1.
- Additionally, it is an Abelian group $(\mathbb{C}, +)/(\Lambda, +)$.
- Every torus is biholomorphic equivalent to a torus with the normalized lattice $\Lambda_\tau := \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau,\ \tau \in \mathbb{H}$. (cf. Wehler 2019, sect 2.2)

- The **Weierstrass $\wp$-function** for Λ is the even elliptic function defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

  The poles of $\wp$ are the elements of Λ and are of order 2.
- $\wp$ satisfies the differential equation

$$\wp'^2 = 4 \cdot \wp^3 - g_2 \cdot \wp - g_3,$$

  with the **lattice constants**

$$g_2 := 60 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3 := 140 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

Very ampleness criterion

Consider a compact Riemann surface X. For an invertible sheaf $\mathcal{L}$ on X are equivalent:

- The sheaf $\mathcal{L}$ is very ample;
- For the point divisors $P, Q \in \mathrm{Div}(X)$ of two arbitrary, not necessarily distinct points $p, q \in X$ holds

$$\dim H^0\left(X, \mathcal{L}_{-(P+Q)}\right) = \dim H^0(X, \mathcal{L}) - 2.$$

We saw: If $g(X) = 1$, let $p \in X$, $P \in \mathrm{Div}(T)$ the corresponding point divisor. Then $\mathcal{O}_{3P}$ is very ample.

Projective embedding of a torus

Theorem

Consider a torus $T = \mathbb{C}/\Lambda$ and its Weierstrass $\wp$-function. Denote by $Z \in \mathrm{Div}(T)$ the point divisor corresponding to $0 \in T$. Set

$$D := 3Z \in \mathrm{Div}(T).$$

For the invertible sheaf $\mathcal{L} := \mathcal{O}_D$, the three sections

$$s_0 := 1, \quad s_1 := \wp, \quad s_2 := \wp'$$

are a basis of $H^0(T, \mathcal{L})$, defining the holomorphic embedding

$$\Phi_{\mathcal{L}} : T \to \mathbb{P}^2, \quad p \mapsto \begin{cases} \big(1 : \wp(p) : \wp'(p)\big) & p \neq 0 \\ (0 : 0 : 1) & p = 0 \end{cases}$$

Proof:

$\Phi_{\mathcal{L}}$ is holomorphic in a neighborhood of $0 \in T$:

Choose a chart z of T around 0 in a neighborhood U of 0. Then in $U \setminus \{0\}$:

$$\Phi_{\mathcal{L}} = \left(1 : \wp : \wp'\right) = \left(z^3 : z^3 \cdot \wp : z^3 \cdot \wp'\right)$$

which extends holomorphically to the singularity, setting

$$\Phi_{\mathcal{L}}(0) := (0 : 0 : 1) \in \mathbb{P}^2.$$

Recall:

$$\wp'^2 = 4 \cdot \wp^3 - g_2 \cdot \wp - g_3,$$

Weierstrass polynomial:

$$F(x, y) := y^2 - \left(4x^3 - g_2 \cdot x - g_3\right).$$

homogenization ($X = \frac{X_1}{X_0},\ Y = \frac{X_2}{X_0}$):

$$F_{\text{hom}}(X_0, X_1, X_2) = X_2^2 X_0 - \left(4X_1^3 - g_2 \cdot X_1 X_0^2 - g_3 \cdot X_0^3\right) \in \mathbb{C}[X_0, X_1, X_2]$$

$\rightsquigarrow \Phi_{\mathcal{L}}$ maps into the zero set of F_{hom} in $\mathbb{P}^2$

Proposition

The curve defined by the Weierstrass polynomial

$$F(x, y) := y^2 - \left(4x^3 - Ax - B\right) \in \mathbb{C}[x, y]$$

is non-singular iff

$$\Delta_F = A^3 - 27B^2 \neq 0$$

Proof: $(x_0, y_0) \in \mathbb{C}^2$ is a singular point of the curve iff

$$0 = y_0^2 - \left(4x_0^3 - Ax_0 - B\right), \frac{\partial F}{\partial y}(x_0, y_0) = 2y_0 = 0, \frac{\partial F}{\partial x}(x_0, y_0) = -12x_0^2 + A$$

introduce

$$f(x) := 4x^3 - Ax + B \in \mathbb{C}[x]$$

$$\rightsquigarrow f(x_0) = 0 \text{ and } f'(x_0) = 0$$

$\implies f$ has a double zero (at x_0). This is equivalent to *(cf. Knapp, Elliptic curves, ch. 3, Cor 3.4, Princeton University Press, 1992)*

$$A^3 - 27B^2 = 0$$

Proposition

The *discriminant form*

$$\Delta : \mathbb{H} \to \mathbb{C} \text{ with } \Delta(\tau) := g_2^3(\tau) - 27 g_3^2(\tau)$$

with

$$g_2(\tau) := 60 \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m\tau + n)^4},\ g_3(\tau) := 140 \cdot \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m\tau + n)^6}$$

has no zeros.

Proof: see Wehler 2019, sect. 4.1

Lemma

The point $O := (0 : 0 : 1)$ in the projective curve E defined by the homogenization of

$$F(x, y) := y^2 - \left(4x^3 - Ax - B\right) \in \mathbb{C}[x, y]$$

is non-singular.

Proof: Consider standard coordinates of $\mathbb{P}^2$ around O:

$$\phi_2 : U_2 \to \mathbb{C}^2, \quad (z_0 : z_1 : z_2) \mapsto (u, v) := \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right)$$

introduce

$$f(u, v) := u - (4 \cdot v^3 - A \cdot u^2 v - B \cdot u^3)$$

Then

$$\phi_2(E \cap U_2) = \{(u, v) \in \mathbb{C}^2 | f(u, v) = 0\}$$

check explicitly: $\nabla f(0, 0) = (1, 0) \neq 0$

Corollary

For any $\tau \in \mathbb{H}$ the projective curve defined by the homogenization of the Weierstrass polynomial

$$F(x, y) = y^2 - (4x^3 - g_2(\tau) \cdot x - g_3(\tau))$$

is non-singular.

Definition

An **elliptic curve** is a non-singular curve $X \subset \mathbb{P}^n$ of genus $g(X) = 1$.

Corollary

Consider a torus $T = \mathbb{C}/\Lambda$ with normalized lattice $\Lambda = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$, $\tau \in \mathbb{H}$, and lattice constants g_2, $g_3 \in \mathbb{C}$. The image of the embedding $\Phi_{\mathcal{L}} : T \to \mathbb{P}^2$ is the elliptic curve $E \subset \mathbb{P}^2$ with Weierstrass polynomial

$$F(x, y) = y^2 - (4x^3 - g_2 \cdot x - g_3).$$

Proof: $\Phi_{\mathcal{L}} : T \to E$ is surjective:
Let $(1 : x : y) \in E$. Since $\wp : T \to \mathbb{P}^1$ is non-constant, holomorphic
$\implies \exists z \in T$ s.t. $\wp(z) = x \implies \wp(-z) = x$
From

$$y^2 = 4x^3 - g_2 \cdot x - g_3 = 4\wp(z)^3 - g_2 \cdot \wp(z) - g_3 = \wp'(z)^2$$

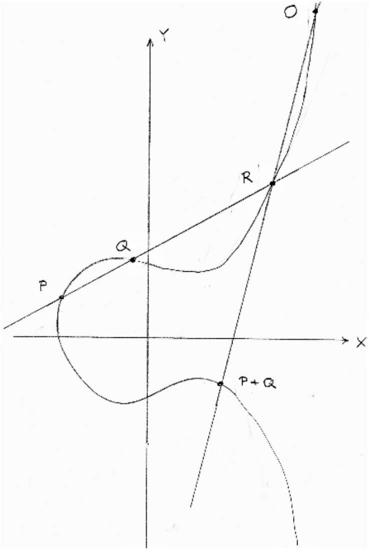
$\implies$ either:

$$y = \wp'(z) \implies \phi_{\mathcal{L}}(z) = (1 : x : y)$$

or:

$$y = -\wp'(z) = \wp'(-z) \implies \phi_{\mathcal{L}}(-z) = (1 : x : y)$$

The group structure of elliptic curves



- **Bezout's theorem:** number of intersection points of two algebraic curves in $\mathbb{P}^2$ counted by multiplicity equals product of degrees of the curves
  (cf. Hartshorne 1977, ch.1, sect.7)
- $\Phi_{\mathcal{L}}$ is a group homomorphism
  (cf. Wehler 2019, sect. 3)

Further comments and outlook

- Converse statement: *(cf. Wehler 2019, sect. 4.2)*
  Every elliptic curve is biholomorphic equivalent to a torus
- More refined definition of elliptic curves: *(cf. Hartshorne 1977, ch.4, sect.4)*
  - projective curve E is **defined** over a subfield $k \subset \mathbb{C}$ (write E/k) if coefficients of defining polynomials are in k
  - for $k \subset K \subset \mathbb{C}$ the K-**valued points** are $(z_0 : ... : z_n) \in E$ s.t. $z_j \in K \ \forall j$
  - an **elliptic curve** is a pair $(E/k, O)$, where E/k is a non-singular projective curve of genus 1, $O \in E$ a k-valued point