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Lie Groups

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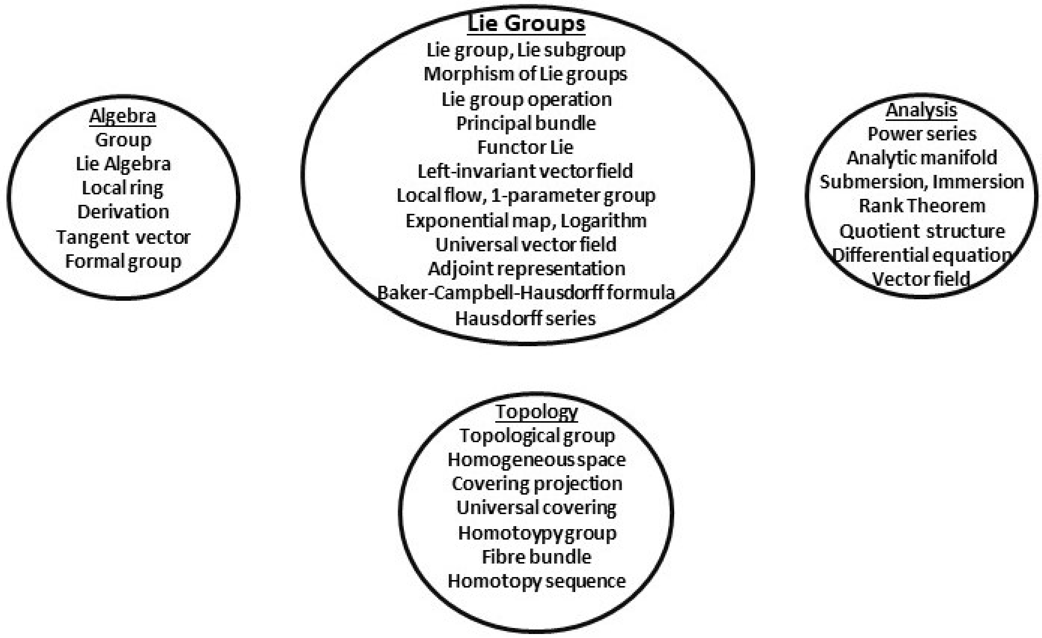
I have prepared these notes for the students of my lecture. The lecture took place during the summer semester 2017 at the mathematical department of LMU (Ludwig-Maximilians-Universität) at Munich.

Compared to the oral lecture in class these written notes contain some additional material.

Please report any errors or typos to wehler@math.lmu.de

Release notes:

- *Release 0.51*: Typos corrected.
- *Release 0.5*: Completed Chapter 3. Added Chapter 4, Chapter 5.
- *Release 0.4*: Shorter proof of Proposition 3.11. Added part of Chapter 3
- *Release 0.3*: Added Chapter 2.
- *Release 0.2*: Added first part of Chapter 2.
- *Release 0.1*: Document created, Chapter 1.


Lie Groups
Lie group, Lie subgroup
Morphism of Lie groups
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Principal bundle
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Left-invariant vector field
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Chapter 1
Topological groups

Before dealing with Lie groups, which are groups carrying an analytic structure, we investigate the more general case of topological groups.

1.1 Topology of topological groups

Definition 1.1 (Topological group). A *topological group* is a group (G,*) equipped with a topology such that both

- the group multiplication $m : G \times G \to G, (x,y) \mapsto m(x,y) := x * y$,
- and the inversion $\sigma : G \to G, x \mapsto x^{-1}$,

are continuous maps.

Note. Instead of $x * y$ we shall often write $x \cdot y$ or even xy.

In the following we will often require that the topology is a Hausdorff topology or even locally compact. Both properties are necessary requirements for a topological group to acquire the additional structure of a Lie group.

Lemma 1.2 (Canonical homeomophisms). *For a topological group G the following maps are homeomorphisms:*

- Left-translation*: For fixed $g \in G$*

$$L_g : G \to G, x \mapsto g \cdot x.$$

- Right-translation*: For fixed $g \in G$*

$$R_g : G \to G, x \mapsto x \cdot g.$$

- Inner automorphism*: For fixed* $g \in G$

$$\phi_g : G \to G, x \mapsto g \cdot x \cdot g^{-1}.$$

Proof. i) For fixed $g \in G$ the inclusion

$$j : G \to G \times G, x \mapsto (g,x)$$

is continuous by definition of the product topology. As a consequence the composition

$$L_g = [G \xrightarrow{j} G \times G \xrightarrow{m} G]$$

is continuous. The inverse map is $L_{g^{-1}}$.

iii) For fixed $g \in G$ the inner automorphism Φ_g is a composition of homeomorphisms:

$$\phi_g = L_g \circ R_{g^{-1}}.$$

Lemma 1.3 (Neighbourhood basis of a topological group). *Consider a topological group G and $\mathscr{U}$ a neighbourhood basis of the neutral element $e \in G$, i.e. $\mathscr{U}$ is a set of neighbourhoods of e such that any neighbourhood of e contains an element of $\mathscr{U}$.*

Then:

1. *For all $U \in \mathscr{U}$ exist $V \in \mathscr{U}$ such that $V \cdot V \subset U$.*
2. *For all $U \in \mathscr{U}$ exist $V \in \mathscr{U}$ such that $V^{-1} \subset U$.*
3. *For all $g \in G$ and for all $U \in \mathscr{U}$ exists $V \in \mathscr{U}$ such that $V \subset \phi_g(U)$.*

Proof. 1) Because the multiplication $m : G \times G \to G$ is continuous the pre-image $m^{-1}(U)$ is a neighbourhood of $(e,e) \in G \times G$ and contains a product neighbourhood $V \times V \subset m^{-1}(U)$ with $V \in \mathscr{U}$.

2) Because taking the inverse $\sigma : G \to G$ is continuous the inverse image $\sigma^{-1}(U)$ is a neighbourhood of $e \in G$, hence contains an element $V \in \mathscr{U}$. Then

$$V \subset \sigma^{-1}(U) \implies V^{-1} \subset U.$$

3) Because the inner automorphism ϕ_g is a homeomorphism the set $\phi_g(U)$ is a neighbourhood of $e \in G$, hence contains an element $V \in \mathscr{U}$, q.e.d.

In the opposite direction, i.e. to provide a group with the structure of a topological group, it is sufficient to consider the neutral element. To generate a topological group, a neighbourhood basis of e has to satisfy the following properties:

Lemma 1.4 (Defining a topological group by a neighbourhood basis). *Consider a group G and a non-empty set $\mathscr{U}$ of subsets of G with the following properties:*

- $U \in \mathscr{U} \implies e \in U$.
- $U_1, U_2 \in \mathscr{U} \implies (\exists\, U_3 \in \mathscr{U} : U_3 \subset U_1 \cap U_2)$.
- *$\mathscr{U}$ satisfies the three properties of Lemma 1.3.*

Then a unique topology $\mathscr{T}$ of G exists, such that $(G, \mathscr{T})$ is a topological group and $\mathscr{U}$ a neighbourhood basis of the neutral element $e \in G$.

Proof. For an arbitrary subset $U \in G$ we define

$$U \in \mathscr{T} \iff \forall\, x \in U\ \exists\, V \in \mathscr{U} : x \cdot V \subset U.$$

i) Apparently $\emptyset, G \in \mathscr{T}$.

ii) Consider two sets $U_1, U_2 \in \mathscr{T}$ and a point

$$x \in U := U_1 \cap U_2.$$

By assumption, for $j = 1, 2$ sets $V_j \in \mathscr{U}$ exist with $x \cdot V_j \subset U_j$. And by assumption a set $V \in \mathscr{U}$ exists with $V \subset V_1 \cap V_2$, i.e.

$$x \cdot V \subset U.$$

As a consequence $U \in \mathscr{T}$.

iii) Consider a family $(U_i)_{i \in I}$ of sets $U_i \in \mathscr{T}$ and a point

$$x \in U := \bigcup_{i \in I} U_i.$$

We choose an index $i_0 \in I$ and a set $V \in \mathscr{U}$ with $x \cdot V \subset U_{i_0}$. Then also $x \cdot V \subset U$ which proves $U \in \mathscr{T}$.

Part i) - iii) shows that $\mathscr{T}$ is a topology with $\mathscr{U}$ as neighbourhood basis of the neutral element and neighbourhood bases of all other elements determined by $\mathscr{U}$. In general, the topology of a topological space is uniquely determined by neighbourhood bases of all points of the space.

In a second step, we now prove that $(G, \mathscr{T})$ is a topological group.

iv) In order to prove the continuity of the multiplication $m : G \times G \to G$ we consider an element $g \in G$ and a neighbourhood U of g. Assume an arbitrary but fixed pair

$$(g_1, g_2) \in G \times G \text{ with } m(g_1, g_2) = g.$$

By definition of $\mathscr{T}$ there exist a set $V \in \mathscr{U}$ with $g \cdot V \subset U$ and a set $W \in \mathscr{U}$ with $W \cdot W \subset V$. Eventually, a set $B \in \mathscr{U}$ exists with

$$B \subset g_2 \cdot W \cdot g_2^{-1}.$$

As a consequence,

$$m(g_1 \cdot B, g_2 \cdot W) = g_1 \cdot B \cdot g_2 \cdot W \subset g_1 \cdot g_2 \cdot W \cdot W \subset g_1 \cdot g_2 \cdot V = g \cdot V \subset U$$

which implies

$$g_1 \cdot B \times g_2 \cdot W \subset m^{-1}(U)$$

with the neighbourhood

$$g_1 \cdot B \times g_2 \cdot W$$

of (g_1, g_2). As a consequence, any point

$$(g_1, g_2) \in m^{-1}(g)$$

has a neighbourhood contained in $m^{-1}(U)$. Therefore $m^{-1}(U)$ is open in the product topology of $G \times G$.

v) In order to prove the continuity of $\sigma : G \to G$ we consider an element $g \in G$ and an open neighbourhood U of g. There exist sets $V \in \mathscr{U}$ with $g \cdot V \subset U$ and $W \in \mathscr{U}$ with

$$W^{-1} \subset g \cdot V \cdot g^{-1}.$$

We obtain

$$\sigma(g^{-1} \cdot W) = W^{-1} \cdot g \subset g \cdot V \subset U.$$

As a consequence

$$g^{-1} \cdot W \subset \sigma^{-1}(U)$$

and $\sigma^{-1}(U)$ is a neighbourhood of the inverse image

$$\sigma^{-1}(g) = g^{-1} \in g^{-1} \cdot W.$$

q.e.d.

Proposition 1.5 (Subgroups and quotients). *Consider a topological group G and a subgroup $H \subset G$. Topologize H with the subspace topology and the quotient set G/H with the quotient topology. Then:*

1. *The subgroup $H \subset G$ is a topological group.*

2. *The canonical projection $\pi : G \to G/H$ is an open map.*

3. *The quotient space G/H is Hausdorff iff $H \subset G$ is closed.*

4. If $H \subset G$ is a normal subgroup, then G/H is a topological group.

Proof. The quotient topology is the finest topology on G/H such that $\pi : G \to G/H$ is continuous, i.e. a subset $V \subset G/H$ is open iff $\pi^{-1}(U) \subset G$ is open.

i) Consider the following commutative diagram with $j : H \hookrightarrow G$ the injection:

$$\begin{tikzcd} H \times H \arrow[r, "m_H"] \arrow[d, "j \times j"'] & H \arrow[d, "j"] \\ G \times G \arrow[r, "m_G"] & G \end{tikzcd}$$

Continuity of the map $m_G \circ (j \times j)$ implies the continuity of the map $j \circ m_H$. Therefore m_H is continuous due to the definition of the subspace topology on H. Analogously, the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\sigma_H"] \arrow[d, "j"'] & H \arrow[d, "j"] \\ G \arrow[r, "\sigma_G"] & G \end{tikzcd}$$

implies the continuity of the inversion σ_H.

ii) Consider an open set $U \subset G$. Then

$$\pi^{-1}(\pi(U)) = \{x \in G : \exists\, g \in U \text{ } with \text{ } xH = gH\} =$$

$$= \{x \in G : \exists\, g \in U \text{ } with \text{ } x \in gH\} = \bigcup_{h \in H} U \cdot h = \bigcup_{h \in H} R_h(U).$$

Each set $R_h(U)$ is open because R_g is a homeomorphism. As a union of open subsets the set $\pi^{-1}(\pi(U)) \subset G$ is open. By definition of the quotient topology the set $\pi(U) \subset G/H$ is open.

iii) Assume that G/H is a Hausdorff space. Then the singleton $\{\pi(e)\} \subset G/H$ is closed and as a consequence also the set $H = \pi^{-1}(\pi(e)) \subset G$.

For the opposite direction assume that $H \subset G$ is closed. The quotient G/H is a Hausdorff space iff the diagonal

$$\Delta \subset (G/H \times G/H)$$

is a closed subset. For $(x, y) \in G \times G$

$$(\pi(x), \pi(y)) \in \Delta \iff xH = yH \iff y^{-1}x \in H \iff \psi(x,y) \in H$$

with the continuous map

$$\psi : G \times G \to G, (x,y) \mapsto y^{-1}x.$$

Because $H \subset G$ is closed also $\psi^{-1}(H) \subset G \times G$ is closed. The representation

$$(\pi \times \pi)((G \times G) \setminus \psi^{-1}(H)) = (G/H \times G/H) \setminus \Delta$$

and the openness of $\pi \times \pi$ according to part ii) imply that the set

$$(G \times G) \setminus \psi^{-1}(H) \subset G \times G$$

is open. Therefore the diagonal $\Delta \subset (G/H \times G/H)$ is a closed subset.

iv) We consider the commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "\pi \times \pi"'] & G \arrow[d, "\pi"] \\ G/H \times G/H \arrow[r, "m_{G/H}"] & G/H \end{tikzcd}$$

From

$$\pi \circ m_G = m_{G/H} \circ (\pi \times \pi)$$

and from the quotient topology on $G/H \times G/H$ follows the continuity of $m_{G/H}$, q.e.d.

Corollary 1.6 (Hausdorff criterion). *A topological group G is a Hausdorff space iff the singleton $\{e\} \subset G$ is closed.*

Example 1.7 (Topological groups).

1. Any normed $\mathbb{K}$-vector space $(V,+)$ is a topological group.

2. The unit sphere
$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = U(1)$$
equipped with the subspace topology $S^1 \subset \mathbb{C}$ is a topological group with respect to multiplication. We have the homeomorphy
$$S^1 \simeq \{z \in \mathbb{R}^2 : \|z\| = 1\}.$$

3. The torus $T^n := (S^1)^n = S^1 \times ... \times S^1$, equipped with the product topology is a topological group.

4. Consider the subgroup $\mathbb{Z}^n \subset (\mathbb{R}^n, +)$ equipped with the subspace topology, i.e. the discrete topology. Then an isomorphism of topological groups exists

$$\mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\simeq} T^n.$$

Proof. The map

$$f : \mathbb{R}^n \to T^n, (x_1, ..., x_n) \mapsto (e^{2\pi i \cdot x_1}, ..., e^{2\pi i \cdot x_n}),$$

is a surjective morphism of topological groups. Due to the homomorphism theorem and the definition of the quotient topology f induces a bijective morphism of topological groups

$$\overline{f} : \mathbb{R}^n/\mathbb{Z}^n \to T^n$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] \arrow[d, "\pi"'] & T^n \\
\mathbb{R}^n/\mathbb{Z}^n \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

Compactness of $\mathbb{R}^n/\mathbb{Z}^n$ implies that $\overline{f}$ is a homeomorphism.

5. The multiplicative group

$$GL(n, \mathbb{K}) := \{A \in M(n \times n, \mathbb{K}) : det\ A \neq 0\}$$

equipped with the subspace topology of $M(n \times n, \mathbb{K}) \simeq \mathbb{K}^{n^2}$ is a topological group. In particular, $(\mathbb{K}^*, \cdot)$ is a topological group.

6. Provide $\mathbb{Z} \subset (\mathbb{C}, +)$ with the subspace topology. Then an isomorphism of topological groups exists

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*.$$

Proof. Consider the holomorphic map

$$f : \mathbb{C} \to \mathbb{C}^*, z \mapsto e^{2\pi i \cdot z}.$$

The map f is surjective. Moreover, f is continuous and as a holomorphic map also open. It induces the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C}^* \\
\mathbb{C}/\mathbb{Z} \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

The map $\overline{f}$ is continuous, hence a bijective morphism of topological groups. For any open subset $U \subset \mathbb{C}/\mathbb{Z}$ the image

$$\overline{f}(U) = f(\pi^{-1}(U)) \subset \mathbb{C}^*$$

is open because f is an open map.

Lemma 1.8 (Connectedness). *Consider a topological group G and a subgroup H. If H and G/H are connected, then also G is connected.*

Proof. Assume the existence of two non-empty open subsets $U, V \subset G$ with

$$G = U \cup V.$$

We claim $U \cap V \neq \emptyset$: If $\pi : G \to G/H$ denotes the canonical projection then

$$G/H = \pi(U) \cup \pi(V).$$

Because π is an open map according to Proposition 1.5 and because G/H is connected we have

$$\pi(U) \cap \pi(V) \neq \emptyset,$$

i.e. two elements $u \in U$ and $v \in V$ exist with $\pi(u) = \pi(v)$ or

$$u \cdot H = v \cdot H.$$

Then $v \in u \cdot H$ and

$$u \in u \cdot H \cap U \neq \emptyset \text{ and } v \in u \cdot H \cap V \neq \emptyset.$$

Connectedness of H and also of $u \cdot H$ and the representation

$$u \cdot H = (u \cdot H) \cap (U \cup V) = (u \cdot H \cap U) \cup (u \cdot H \cap V)$$

imply

$$(u \cdot H \cap U) \cap (u \cdot H \cap V) \neq \emptyset,$$

in particular $U \cap V \neq \emptyset$, q.e.d.

Proposition 1.9 (Component of the neutral element is normal subgroup). *For any topological group G the connected component $G^e \subset G$ of the neutral element $e \in G$ is a normal subgroup.*

Proof. The product of connected topological spaces is connected. Also the continuous image of a connected space is connected. As a consequence, the multiplication satisfies
$$m(G^e \times G^e) \subset G^e.$$
Similarly, the inversion satisfies
$$\sigma(G^e) \subset G^e.$$
As a consequence, $G^e \subset G$ is a subgroup.

According to Lemma 1.2: For each element $g \in G$ the group $\phi_g(G^e)$ is a connected subgroup of G. As a consequence
$$\phi_g(G^e) \subset G^e,$$
which proves normality of G^e, q.e.d.

Proposition 1.10 (Open subgroups are also closed). *Consider a topological group G. Any open subgroup $H \subset G$ is also closed.*

Proof. The group G decomposes in cosets with respect to H
$$G = H \dot\cup \bigcup_{g \notin H} gH.$$
For each element $g \in H$ the left-multiplicartion $L_g : G \to G$ is a homeomorphism according to Lemma 1.2. Openness of H implies that also $gH \subset G$ is open. Then the union of open sets
$$\bigcup_{g \notin H} gH \subset G$$
is open. Therefore its complement $H \subset G$ is closed, q.e.d.

Proposition 1.11 (Finite products of small elements). *Consider a connected topological group G and an arbitrary neighbourhood $V \subset G$ of the neutral element $e \in G$. Then for any element $x \in G$ finitely many elements $x_1, ..., x_n \in V$ exist with*
$$x = x_1 \cdot ... \cdot x_n.$$

Proof. Define the set

$$H := \{x \in G : \exists\, n \in \mathbb{N},\ \exists\, x_1, ..., n_n \in V \cap V^{-1}\ with\, x = x_1 \cdot ... \cdot x_n\}.$$

Apparently $H \subset G$ is a subgroup. It contains the neighbourhood $V \cap V^{-1}$ of $e \in G$. Therefore $H \subset G$ is open. According to Proposition 1.10 H is also closed. The decomposition $G = H \dot{\cup} (G \setminus H)$ and the connectedness of G imply $G \setminus H = \emptyset$, i.e. $G = H$, q.e.d.

The content of Proposition 1.11 can be stated as follows: For a topological group G any neighbourhood of $e \in G$ generates the component of the neutral element of G.

1.2 Continuous group operation

Continuous groups often appear as symmetry groups of a topological space. This issue is formalized by the concept of a *group operation*.

Definition 1.12 (Group operation and homogeneous space). Consider a topological group G with neutral element $e \in G$ and a topological space X.

1. A continous *left G-operation* on X is a continuous map

   $$\phi : G \times X \to X, (g,x) \mapsto g.x,$$

   which satisfies the following properties:

   - For all $x \in X$: $e.x = x$.
   - For all $g, h \in G$ and $x \in X$: $g.(h.x) = (g \cdot h).x$.

   The pair (G,X) is named a continous *left G-space*.

2. Consider a left G-space (G,X).

   - The *orbit map* of a point $x \in X$ is the continous map

     $$\phi_x : G \to X, g \mapsto g.x.$$

     Its image $\phi_x(G) \subset X$ is the *orbit* of x.

   - The *orbit space* of (G,X) is the quotient space X/G of X with respect to the equivalence relation

     $$x \sim y \iff y \in \phi_x(G).$$

- The group G operates *transitive* on X if the orbit space X/G is a singleton.
- The *isotropy group* of a point $x \in X$ is defined as

$$G_x := \{g \in G : g.x = x\}.$$

If $G_x = \{e\}$ for all $x \in X$ the group operation is *free*.

3. A continuous left G-space (G,X) is *homogeneous* iff both of the following conditions are satisfied:
   - The group operation is transitive.
   - A point $x \in X$ exists such that the canonical map

$$G/G_x \to X, g.G_x \mapsto g.x,$$

is a homeomorphism.

Concerning the notation in Definition 1.12 one should pay attention to the distinction between the dot *above* the line "$\cdot$", denoting the group multiplication, and the dot *on* the line ".", denoting the group action. We will omit the adjective "continuous" if the topological context of the concepts is clear.

If G operates on X then for each fixed $g \in G$ the left operation

$$X \to X, x \mapsto g.x$$

is the homeomorphism L_g with inverse the homeomorphism L_g^{-1}.

Analogously to a left G-operation one defines the concept of a continuous *right G-operation* on X

$$X \times G \to X, (x,g) \mapsto x.g.$$

For a transitive group action the whole space X is a single orbit. In this case $G/G_{x_0},\ x_0 \in G$ maps bijectively onto X. If this map is a homeomorphism then the G-space is homogenous and X is completely determined by the group operation. In the following we will derive a criterion, which assures that a G-space is homogeneous. Recall that a locally compact space is by definition a Hausdorff space.

Lemma 1.13 (Baire's theorem). *Consider a non-empty locally compact space X. If a sequence of closed subsets $A_\nu \subset X, \nu \in \mathbb{N}$, exist with*

$$X = \bigcup_{\nu \in \mathbb{N}} A_\nu$$

then the interior $A_{\nu_0}^\circ \neq \emptyset$ for at least one index $\nu_0 \in \mathbb{N}$, i.e. not all sets A_ν have empty interior.

For a proof see [8, Chap. XI, 10].

Definition 1.14 (σ-compactness). A locally compact topological space X is *σ-compact* if X is the countable union of compact subspaces.

Note that σ-compactnes is a global property.

Remark 1.15 (σ-compactness). The condition of σ-compactness is equivalent to the property that X has a countable exhaustion

$$X = \bigcup_{i \in \mathbb{N}} U_i$$

by relatively compact open subsets $U_i \subset\subset U_{i+1},\ i \in \mathbb{N}$, see [8, Chap. XI, 7].

Theorem 1.16 (Homogeneous space). *Any G-space (G,X) with a σ-compact topological group G and a locally compact space X is homogeneous.*

Proof. Consider a point $x \in X$. Its orbit map

$$\psi : G \to X, g \mapsto g.x,$$

induces a unique continuous and bijective map $\overline{\psi}$ in the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\pi"'] & X \\ G/G_x \arrow[ur, dashed, "\overline{\psi}"'] & \end{tikzcd}$$

We claim that the map $\overline{\psi}$ is also open. For the proof consider an open subset $S \subset G/G_x$. Then

$$\overline{\psi}(S) = \psi(\pi^{-1}(S)).$$

Therefore it suffices to show that the map ψ is open. Even more restrictive, it suffices to show: For any neighbourhood U of e in G the set

$$\psi(U) = U.x$$

is a neighbourhood of x in X.

The latter statement follows from Baire's category theorem: The theorem excludes that X is covered by a countable family of closed sets, all of them having empty interior. In the present context the closed sets will be even compact sets. They originate as translates of a fixed compact neighbourhood W of x. Due to local compacteness of G arbitrary small compact neighbourhoods W of e exist:

According to Lemma 1.3 a neighbourhood V of e in G exists with

$$V \cdot V \subset U.$$

Due to the local compactness of G a compact neighbourhood W of e exists with

$$W \subset V \cap V^{-1}.$$

It satisfies

$$W^{-1} \cdot W \subset U.$$

Each compact subset $K \subset G$ has a finite covering

$$K \subset \bigcup_{\nu=1}^{n} g_\nu \cdot W$$

with a finite index $n \in \mathbb{N}$ and elements $g_1, ..., g_n \in G$. By the assumption about σ-compactness the group G is the union of countably many compact subsets. Therefore a sequence $(s_\nu)_{\nu\in\mathbb{N}}$ of elements $s_\nu \in G$ exists with

$$G = \bigcup_{\nu\in\mathbb{N}} s_\nu \cdot W.$$

As a consequence

$$X = G.x \subset \bigcup_{\nu\in\mathbb{N}} (s_\nu \cdot W).x = \bigcup_{\nu\in\mathbb{N}} s_\nu.(W.x).$$

Compactness of W and continuity of the orbit map $\phi_x : G \to X$ imply the compactness of $W.x \subset X$. In addition, any element $s_\nu \in G, \nu \in \mathbb{N}$, operates as a homeomorphism on X. Therefore each set

$$s_\nu.(W.x) \subset X$$

is compact, in particular closed. According to Lemma 1.13 the local compactness of X implies the existence of at least one index $\nu_0 \in \mathbb{N}$ exists auch that

$$s_{\nu_0}.(W.x)$$

has non-empty interior. As a consequence, an element $w \in W$ exists with

$$s_{\nu_0}.(w.x) \in (s_{\nu_0}.(W.x))^\circ.$$

Therefore

$$w.x \in s_{v_0}^{-1}.(s_{v_0}.(W.x))^\circ = ((s_{v_0}^{-1} \cdot s_{v_0}).(W.x))^\circ = (W.x)^\circ$$

or

$$x \in w^{-1}.(W.x)^\circ = ((w^{-1} \cdot W).x)^\circ \subset (U.x)^\circ$$

which implies that $U.x$ is a neighbourhood of x, q.e.d.

Example 1.17 (Topological groups and group operations).

1. *Orthogonal group*: The orthogonal group

$$O(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : A \cdot A^\top = 1\}$$

is the zero set of continuous functions, and therefore a closed subgroup of $GL(n,\mathbb{R})$. The columns of an orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Therefore all columns are bounded and $O(n,\mathbb{R})$ is a compact topological group. Also its closed subgroup

$$SO(n,\mathbb{R}) := \{A \in O(n,\mathbb{R}) : det\ A = 1\}$$

is a compact topological group.

i) On the $(n-1)$-dimensional sphere

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\} \subset \mathbb{R}^n$$

there is a canonical left $O(n,\mathbb{R})$-operation:

$$O(n,\mathbb{R}) \times S^{n-1} \to S^{n-1}, (A,x) \mapsto Ax.$$

ii) The operation is transitive: Denote by $(e_j)_{j=1,\dots,n}$ the canonical basis of $\mathbb{R}^n$ and consider an arbitrary but fixed point $a \in S^{n-1}$. In order to determine a matrix $A \in O(n,\mathbb{R})$ with

$$Ae_1 = a$$

we extend the vector a to an orthonormal base

$$(a = a_1, a_2, ..., a_n)$$

of $\mathbb{R}^n$ - e.g., by using the Gram-Schmidt algorithm. Define

$$A := \begin{pmatrix} a_1 & \dots & a_n \\ | & \dots & | \\ | & \dots & | \end{pmatrix} \in O(n,\mathbb{R}).$$

For $n \geq 2$ also the induced operation

$$SO(n,\mathbb{R}) \times S^{n-1} \to S^{n-1}$$

is transitive: If $Ae_1 = a$ with $A \in O(n,\mathbb{R})$ then multiplying the last column of A by $(det\ A)^{-1}$ provides a matrix $A' \in SO(n,\mathbb{R})$ with $A'e_1 = a$.

iii) For the isotropy groups of the point $e_1 \in S^{n-1}$ we obtain

$$A \in O(n,\mathbb{R})_{e_1} \iff Ae_1 = e_1 \iff A := \begin{pmatrix} 1 & 0 \dots & 0 \\ 0 & & \\ \vdots & A' & \\ 0 & & \end{pmatrix}$$

with $A' \in O(n-1,\mathbb{R})$. Therefore

$$O(n,\mathbb{R})_{e_1} \simeq O(n-1,\mathbb{R}).$$

And analogously for $n \geq 2$

$$SO(n,\mathbb{R})_{e_1} \simeq SO(n-1,\mathbb{R}).$$

Applying Theorem 1.16, the sphere S^{n-1} can be described both as homogeneous $O(n,\mathbb{R})$-space and for $n \geq 2$ also as homogeneous $SO(n,\mathbb{R})$-space:

$$O(n,\mathbb{R})/O(n-1,\mathbb{R}) \xrightarrow{\simeq} S^{n-1}$$

and for $n \geq 2$

$$SO(n,\mathbb{R})/SO(n-1,\mathbb{R}) \xrightarrow{\simeq} S^{n-1}.$$

2. *Unitary group*: The unitary group

$$U(n) := \{A \in GL(n,\mathbb{C}) : A \cdot A^* = 1\},\ A^* := \overline{A}^{\top},$$

is the zero set of continuous functions, and therefore a closed subgroup of $GL(n,\mathbb{C})$. The columns of a unitary matrix are unit vectors. Therefore $U(n)$ is a compact topological group. Also its subgroup

$$SU(n) := \{A \in U(n) : det\ A = 1\}$$

is a compact topological group.

i) We identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ by the canonical map

$$\mathbb{C}^n \xrightarrow{\simeq} \mathbb{R}^{2n}, (z_1,...,z_n) = (x_1 + iy_1,...,x_n + iy_n) \mapsto (x_1,y_1,...,x_n,y_n).$$

Here $x_j := Re(z_j), y_j := Im(z_j), j = 1,...,n$. Then

$$S^{2n-1} = \{z \in \mathbb{C}^n : \|z\|^2 = \sum_{j=1}^{n} |z_j|^2 = 1\}.$$

Replacing the canonical Euclidean scalar product on $\mathbb{R}^{2n}$ by its Hermitian counterpart on $\mathbb{C}^n$ allows to mimic for the unitary groups the results just obtained for the orthogonal groups.

We have a canonical left $U(n)$-operation on S^{2n-1}

$$U(n) \times S^{2n-1} \to S^{2n-1}, (A,z) \mapsto Az,$$

and for $n \geq 2$ by restriction a canonical left $SU(n)$-operation on S^{2n-1}.

ii) Both operations are transitive. The proof is analogous to 1, part iii). If $Ae_1 = a$ with $A \in U(n)$ then multiplying the last column of A by $(det\ A)^{-1}$ provides a matrix $A' \in SU(n)$ with $A'e_1 = a$.

iii) The isotropy groups of e_1 are respectively

$$U(n)_{e_1} \simeq U(n-1) \text{ and } SU(n)_{e_1} \simeq SU(n-1).$$

As a consequence we obtain a description of the spheres as homogenous spaces

$$U(n)/U(n-1) \simeq S^{2n-1}$$

and for $n \geq 2$

$$SU(n)/SU(n-1) \simeq S^{2n-1}.$$

3. *Morphisms*: Consider a morphism $f : G \to G'$ of topological groups, i.e. a continous group homomorphism. Then (G,G') is a G-space with respect to the left G-operation

$$\phi : G \times G' \to G', (g,g') \mapsto f(g) \cdot g'.$$

- The orbit map of the point $e \in G'$ is

$$\phi_e = f : G \to G', e \mapsto g.e = f(g) \cdot e = f(g).$$

- The G-operation is transitive iff f is surjective.

- The isotropy group of the neutral element $e \in G'$ is

$$G_e = \{g \in G : f(g) \cdot e = e\} = \{g \in G : f(g) = e\} = ker\ f.$$

The homomorphism theorem provides a canonical morphism of topological groups

$$G/ker\ f \to G'.$$

4. *Operation on cosets*: Consider a topological group G and a subgroup $H \subset G$. Then a transitive left G-operation on the topological space G/H of cosets exists

$$G \times (G/H) \to (G/H), (g_1, g_2H) \mapsto (g_1g_2)H.$$

The isotropy group at $H = eH$ is $G_{eH} = H$. The canonical map

$$G/G_{eH} \to G/H, gG_{eH} \mapsto gH,$$

is a homeomorphism. In particular, G/H is a homogenous G-space with respect to the G-left operation.

Lemma 1.18 (Connectedness of selected classical groups).

1. *For all* $n \geq 1$ *the following topological groups are connected:*

$$SO(n,\mathbb{R}), U(n), SU(n).$$

2. *For all* $n \geq 1$ *the following topological groups are not connected:*

$$O(n,\mathbb{R}), GL(n,\mathbb{R}).$$

3. *For all* $n \geq 1$ *the topological groups*

$$SL(n,\mathbb{R}),\ GL^+(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : det\ A > 0\},\ SL(n,\mathbb{C}),\ \textit{and}\ GL(n,\mathbb{C})$$

are connected.

Proof. 1) We prove the claim by induction on $n \in \mathbb{N}$ employing the representations from Example 1.17

$$SO(n,\mathbb{R})/SO(n-1,\mathbb{R}) \simeq S^{n-1}, n \geq 2$$

$$U(n)/U(n-1) \simeq SU(n)/SU(n-1) \simeq S^{2n-1}, n \geq 2.$$

For $n = 1$ we have the singletons $SO(1,\mathbb{R}) = SU(1) = \{id\}$ and $U(1) = S^1$. These sets are connected.

For the induction step $n-1 \mapsto n, n \geq 2$, the claim follows from the representation above and Lemma 1.8.

2) If $O(n,\mathbb{R})$ were connected then also its image under the continuous map

$$det : O(n,\mathbb{R}) \to \{\pm 1\}$$

were connected, a contradiction. Analogously follows the non-connectedness of $GL(n,\mathbb{R})$.

3) We prove the claim by induction on $n \in \mathbb{N}$. The case $n = 1$ is obvious.

For the induction step $n-1 \mapsto n, n \geq 2,$ we denote by G_n any of the groups in question. On the connected topological space $X := \mathbb{K}^n \setminus \{0\}$ we have the G_n-operation

$$G_n \times X \to X, (A,z) \mapsto Az.$$

The operation is transitive: Choose an arbitrary but fixed element $a \in X$ and extend it to a basis

$$(a = a_1, a_2, ..., a_n)$$

of $\mathbb{K}^n$. Define the matrix

$$A := \begin{pmatrix} a_1 & \dots & a_n \\ | & \dots & | \\ | & \dots & | \end{pmatrix} \in GL(n, \mathbb{K})$$

and $A' \in G_n$ as the matrix obtained by multiplying the last column of A by $(det\ A)^{-1}$. Then $A' \in G_n$ and $A'e_1 = a$.

For the isotropy group of e_1 holds

$$A \in (G_n)_{e_1} \iff Ae_1 = e_1 \iff A := \begin{pmatrix} 1 & \alpha_2 & \dots & \alpha_n \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix} \in G_n$$

and $det\ B = det\ A$, i.e. $B \in G_{n-1}$. As a consequence we have the homeomorphy

$$(G_n)_{e_1} \simeq G_{n-1} \times \mathbb{K}^{n-1}.$$

First, the induction assumption implies the connectedness of the isotropy group $(G_n)_{e_1}$. Secondly, because the group $G_n \subset \mathbb{K}^{n^2}$ is locally compact and σ-compact, Theorem 1.16 implies the homeomorphy

$$G_n/(G_n)_{e_1} \simeq X.$$

Eventually, Lemma 1.8 proves the connectedness of G_n, q.e.d.

1.3 Covering projections and homotopy groups

The present section recalls some results from algebraic topology. These results refer

- to covering spaces,
- to the fundamental group,
- and to higher homotopy groups.

General references are [36] and [19]. The results will be used in Chapter 2 to determine the fundamental groups of several classical groups. In particular we shall determine those groups which are simply connected.

Definition 1.19 (Covering). A *covering projection* is a continuous map

$$p : X \to B$$

between two topological spaces such that each point $b \in B$ has an open neighbourhood $V \subset B$ which is *evenly covered*, i.e. the inverse image splits into a set of disjoint open subsets $U_i \subset X$

$$p^{-1}(V) = \dot{\bigcup}_{i \in I} U_i,$$

and each restriction

$$p|U_i : U_i \to V, i \in I,$$

is a homeomorphism.

The space X is called the *covering space* and the space B the *base* of the covering projection.

Attached to each covering projection is a group of deck transformations.

Definition 1.20 (Deck transformation). Consider a covering projection $p : X \to B$.

1. A *deck transformation* of p is a homeomorphism

   $$f : X \to X$$

   such that the following diagram commutes

   $$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

   i.e. f permutes the points of each fibre.

2. With respect to composition the deck transformations of a covering projection p form a group, the deck transformation group $Deck(p)$.

The deck transformation group operates in a canonical way on the total space X:

$$Deck(p) \times X \to X, (f,x) \mapsto f(x).$$

Covering projections are important due to several reasons:

- Covering projections have the *homotopy lifting property*: Whether a map $f : Z \to B$ into the base of a covering projection

$$p : X \to B$$

lifts to a map into the covering space X only depends on the homotopy class of f.

- Covering projections facilitate the computation of the *fundamental group* of a topological space.

Proposition 1.21 (Homotop lifting property). *Consider a covering projection*

$$p : E \to B.$$

If a continous map $f : Z \to B$ into the base lifts to a map $\tilde{f} : Z \to E$ into the covering space then also any homotopy F of f lifts uniquely to a homotopy of $\tilde{f}$. Or expressing the homotopy lifting property *in a formal way:*

Assume the existence of

- *a continuous map $\tilde{f} : Z \to E$*
- *and a continuous map $F : Z \times I \to B$ with $F(-,0) = p \circ \tilde{f}$.*

Then a unique continuous map $\tilde{F} : Z \times I \to E$ exists such that the following diagram commutes:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & B \end{tikzcd}$$

The diagram from Proposition 1.21 has the following interpretation: The restriction

$$f := F(-,0) : Z \to B$$

is a continuous map with the lift $\tilde{f} : Z \to E$, i.e. $p \circ \tilde{f} = f$. The map

$$F : Z \times I \to B$$

is a homotopy of f. The homotopy lifting property ensures that the homotopy lifts to a continuous map

$$\tilde{F} : Z \times I \to E.$$

In particular, any map
$$(F-,t) : Z \to B, t \in I,$$
being homotopic to f, lifts to
$$\tilde{F}(-,t) : Z \to E.$$

The particular case of the singleton $Z = \{*\}$ shows that a covering projection has the *unique path lifting property*: Any path in B lifts to a unique path in E with fixed starting point. But in general, the lift $\tilde{\alpha}$ of a closed path α in B is no longer closed in E.

Moreover, the lifting criterion from Proposition 1.23 states: Whether a map
$$f : Z \to B$$
into the base B of a covering projection $p : X \to B$ lifts to a map into its covering space X only depends on the induced maps of the fundamental groups.

Definition 1.22 (Fundamental group). Consider a connected topological space X.

i) After choosing an arbitrary but fixed distinguished point $x_0 \in X$ the *fundamental group* $\pi_1(X, x_0)$ of X with respect to the basepoint x_0 is the set of homotopy classes of continuous maps
$$\alpha : [0,1] \to X \textit{ with } \alpha(0) = \alpha(1) = x_0$$
with the catenation
$$(\alpha_1 * \alpha_2)(t) := \begin{cases} \alpha_1(2t) & \text{if } 0 \le t \le 1/2 \\ \alpha_2(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$
as group multiplication.

ii) The topological space X is *simply-connected* if $\pi_1(X, x_0) = 0$.

Apparently the paths in question can also be considered as continuous maps
$$S^1 \to X.$$

One checks that the catenation defines a group structure on the set of homotopy classes. In addition, for path-connected X the fundamental group - as an abstract group - does not depend on the choice of the basepoint. In this case one often writes $\pi_1(X, *)$ or even $\pi_1(X)$.

A morphism
$$f : (X, x_0) \to (Y, y_0)$$

of pointed connected topological spaces, i.e. satisfying $f(x_0) = y_0$, induces a group homorphism of fundamental groups

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0), [\alpha] \mapsto [f \circ \alpha].$$

In case of a covering projection f the induced map $\pi_1(f)$ is injective. The fundamental group is a covariant functor from the homotopy category of pointed connected topological spaces to the category of groups.

We recall that for a locally path-connected topological space X the two properties *connectedness* and *path-connectedness* are equivalent.

Proposition 1.23 (Lifting criterion). *Consider a covering projection $p : E \to B$ and a continuous map $f : X \to B$ with X path-connected and locally path-connected. Then the following properties are equivalent:*

1. *The map f has a lift to E, i.e. a continuous map*

$$\tilde{f} : X \to E$$

exists such that the following diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

2. *The induced map of the fundamental groups*

$$\pi_1(f) : \pi_1(X, *) \to \pi_1(B, *)$$

satisfies

$$\pi_1(f)(\pi_1(X, *)) \subset \pi_1(p)(\pi_1(E, *)).$$

In particular, any continous map $f : X \to B$ from a simply-connected topological space X lifts to a continous map $\tilde{f}$ into the covering space E.

Definition 1.24 (Universal covering projection). Consider a topological space B. A *universal covering projection* of B is a covering projection $p : E \to B$ with the following universal property:

For any covering projection $f : X \to B$ a continuous map $\tilde{p} : E \to X$ exists such that the following diagram commutes

$$\begin{tikzcd}
 & E \arrow[dl, dashed, "\tilde{p}"'] \arrow[dd, "p"] \\
X \arrow[dr, "f"'] & \\
 & B
\end{tikzcd}$$

And after fixing base points the map $\tilde{p}$ is uniquely determined.

A universal covering exists if B satisfies certain properties with respect to its paths. A topological space X is *semilocally 1-connected* if any point $x \in X$ has a neighbourhood U in X such that any closed path in U is contractible in X to a point. This property as well as local path-connectedness is satisfied for any Lie group.

Proposition 1.25 (Simply connectedness and universal covering). *Consider a topological space B which is path-connected, locally path-connected and semilocally 1-connected. Then:*

1. *The space B has a unique universal covering projection*

$$p : E \to B.$$

2. *For a given covering projection $f : X \to B$ we have the equivalence:*
   - *f is the universal covering projection of B.*
   - $\pi_1(X, *) = 0$.

Proposition 1.26 (Deck transformation group of the universal covering projection). *Consider the universal covering projection*

$$p : E \to B$$

of the path-connected, locally path-connected and semilocally 1-connected topological space B. Choose a base point $b_0 \in B$ and a pre-image $\tilde{b} \in p^{-1}(b_0)$.

i) Then a group homomorphism exists

$$\Gamma : \pi_1(B, b_0) \to Deck(p), [\alpha] \mapsto \Gamma([\alpha]),$$

with $\Gamma([\alpha])$ the uniquely determined deck transformation which satisfies

$$\Gamma([\alpha])(\tilde{b}) = \tilde{\alpha}(1) \in p^{-1}(b_0)$$

with respect to the lift $\tilde{\alpha}$ of α.

ii) The group homomorphism Γ is an isomorphism:

$$\Gamma : \pi_1(B, b_0) \xrightarrow{\simeq} Deck(p)$$

Corollary 1.27 (Fundamental group of S^1). *The map*

$$p : \mathbb{R} \to S^1, t \mapsto e^{2\pi i \cdot t},$$

is a covering projection with deck transformation group the group of translations

$$Deck(p) = \{\mathbb{R} \to \mathbb{R}, t \mapsto t + n : n \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

In particular $\pi_1(S^1) \simeq \mathbb{Z}$.

The fundamental group $\pi_1(X, *)$ is the first in a series of homotopy groups of a connected topological space: For any $n \in \mathbb{N}^*$ the n-th *homotopy group* $\pi_n(X, *)$ of a connected topological space X is the set of homotopy classes of continuous maps

$$f : (S^n, *) \to (X, *)$$

equipped with a suitable group multiplication. In addition, one has for $n = 0$ the pointed set $\pi_0(X, *)$ of homotopy classes of continuous maps

$$f : S^0 = \{\pm 1\} \to X$$

with $f(1) = *$. Apparently, $\pi_0(X, *)$ is the set of path-components of X. Its distinguished element is the path-component of the base point $*$.

In order to compute the fundamental group of several classical groups we recall a result about the homotopy group of spheres.

Proposition 1.28 (Lower homotopy groups of spheres).

The homotopy groups of the spheres satisfy

$$\pi_i(S^n) = 0$$

for the lower indices $0 \leq i < n$.

For a simple proof presented as an excercise cf. [21, I.8, Ex. 3].

Definition 1.29 (Real projective space). The multiplicative topological group $\mathbb{R}^*$ defines a continuous right operation on the topological space $\mathbb{R}^{n+1} \setminus \{0\}$

$$\mathbb{R}^{n+1} \setminus \{0\} \times \mathbb{R}^* \to \mathbb{R}^{n+1} \setminus \{0\}, (x, \lambda) \mapsto x \cdot \lambda.$$

The orbit space, equipped with the quotient topology,

$$\mathbb{P}^n(\mathbb{R}) := (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$$

is the *real projective space* $\mathbb{P}^n(\mathbb{R})$.

The orbit space $\mathbb{P}^n(\mathbb{R})$ equals the set of lines in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. Apparently, $\mathbb{P}^n(\mathbb{R})$ is also the orbit space of the operation

$$S^n \times \mathbb{Z}_2 \to S^n, (x, \pm 1) \mapsto x \cdot (\pm 1).$$

The canoncial projection

$$p_n : S^n \to \mathbb{P}^n(\mathbb{R}) \simeq S^n/\mathbb{Z}_2$$

is a covering projection with deck transformation group $Deck(p_n) \simeq \mathbb{Z}_2$.

Lemma 1.30 (Fundamental group of the real projective spaces). *For $n \geq 2$ the real projective spaces are connected with fundamental group*

$$\pi_1(\mathbb{P}^n(\mathbb{R})) \simeq \mathbb{Z}_2.$$

Proof. Connectedness follows from the fact that $\mathbb{P}^n(\mathbb{R})$ is the continuous image of the connected space S^n. For $n \geq 2$ the sphere S^n is simply connected according to Proposition 1.28, and Proposition 1.26 implies

$$\pi_1(\mathbb{P}^n(\mathbb{R})) \simeq Deck(p_n) \simeq \mathbb{Z}_2, q.e.d.$$

Proposition 1.31 (Rotation group and real projective space). *The topological group $SO(3,\mathbb{R})$ and the real projective space $\mathbb{P}^3(\mathbb{R})$ are homeomorphic*

$$SO(3,\mathbb{R}) \simeq \mathbb{P}^3(\mathbb{R}).$$

Proof. i) $SO(3,\mathbb{R})$ *as a set of rotations*: Consider a unit vector $v \in \mathbb{R}^3$ and denote by $R_{v,\theta}$ the right-handed rotation of $\mathbb{R}^3$ around the axis v by the angle θ in the plane orthogonal to v. Choose an orthonormal basis (u_1, u_2) of the orthogonal plane such that the family (u_1, u_2, v) is a right-handed orthonormal system. By possibly changing v to $-v$ and simultaneously interchanging u_1 and u_2 one ensures $0 \leq \theta \leq \pi$. For any rotation by an angle θ distinct from 0 and from π the attached rotation matrix $R_{v,\theta} \in SO(3,\mathbb{R})$ with $0 < \theta < \pi$ is uniquely determined.

Conversely, we start with a matrix $A \in SO(3,\mathbb{R})$. To find the corresponding rotation axis we seek an eigenvalue $\lambda = 1$ of A: The characteristic polynomial $p_{char}(T) \in \mathbb{C}[T]$ of A has real coefficients. Therefore its eigenvalues are either real or appear in pairs of complex conjugates. The orthogonality of A implies $|\lambda| = 1$ for any eigenvalue. The product of all eigenvalues satisfies

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = det\ A = 1.$$

As a consequence, either all three eigenvalues are real, or one eigenvalue is real and the two other are complex conjugate. In both cases, at least one eigenvalue $\lambda = 1$ exists. We choose a unit vector v as corresponding eigenvector v. Due to the orthogonality of $A \in SO(3,\mathbb{R})$ the matrix A maps the orthogonal plane of v to itself. The restriction of A to this plane is a rotation $R_{v,\theta}$ by an angle $\theta \in [0,\pi]$ around the axis v.

ii) 3-*dimensional rotations are elements of* B^3: Denote by

$$B^3(\pi) := \{u \in \mathbb{R}^3 : \|u\| \leq \pi\}$$

the solid ball with radius π in $\mathbb{R}^3$. Due to the preliminary considerations we obtain a surjective map

$$\Phi : B^3(\pi) \to SO(3,\mathbb{R}), u \mapsto \begin{cases} R_{\hat{u},\|u\|} & u \neq 0, \hat{u} := u/\|u\| \\ \mathbb{1} & u = 0 \end{cases}$$

The restriction to the interior

$$\Phi|B^3(\pi)^\circ : B^3(\pi)^\circ \to SO(3,\mathbb{R})$$

is injective. On the boundary of $B^3(\pi)$ the map Φ identifies *antipodal* points: If $u \in \partial B^3(\pi)$ then

$$\Phi(u) = \Phi(-u).$$

As a consequence, the rotations from $SO(3,\mathbb{R})$ correspond bijectively to the union of all interior points of $B^3(\pi)$ with all pairs of antipodal points from the boundary $\partial B^3(\pi)$.

iii) *The ball* $(B^3/\sim)$ *as projective space* $\mathbb{P}^3(\mathbb{R})$: Here and in the rest of the proof the symbol $\sim$ denotes the equivalence relation which on the boundary identifies antipodal points. Consider the canonical projection

$$p_3 : S^3 \to \mathbb{P}^3(\mathbb{R}) \simeq S^3/\mathbb{Z}_2.$$

If we restrict the projection to the closed upper hemisphere

$$D^3 \subset S^3 \subset \mathbb{R}^4$$

and identify antipodal points on the boundary ∂D^3 then p_3 induces a homeomorphism

$$(D^3/\sim) \xrightarrow{\simeq} \mathbb{P}^3(\mathbb{R}).$$

Because

- D^3 is homeomorphic to the ball $B^3(1)$
- and the homeomorphism is compatible with the equivalence relation $\sim$

we obtain

$$SO(3,\mathbb{R}) \simeq (B^3(\pi)/\sim) \simeq (B^3(1)/\sim) \simeq (D^3/\sim) \simeq \mathbb{P}^3(\mathbb{R}), \ q.e.d.$$

Definition 1.32 (Continuous fibre bundle). Consider a topological space F. A continuous map $p : E \to B$ between two topological spaces is named continuous *fibre bundle* with typical fibre F, using the notation

$$F \hookrightarrow E \xrightarrow{p} B,$$

if each point $b \in B$ has an open neighbourhood U together with a homeomorphism

$$\phi : p^{-1}(U) \xrightarrow{\simeq} U \times F$$

such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

In particular, any covering projection $p : E \to B$ of a connectd space B is a fibre bundle with fibre $F = p^{-1}(b_0)$ for an arbitrary point $b_0 \in B$. We shall see more examples in Chapter 2.

The fundamental tool for the computation of homotopy groups is the long exact homotopy sequence of fibre bundles.

Theorem 1.33 (Homotopy sequence of fibre bundles). *Consider a continuous fibre bundle*

$$F \hookrightarrow E \xrightarrow{p} B.$$

Then a long exact sequence of groups $(n \geq 1)$ *and pointed sets* $(n = 0)$ *exists:*

$$\ldots \to \pi_2(B,*) \to \pi_1(F,*) \to \pi_1(E,*) \to \pi_1(B,*) \to \pi_0(F,*) \to \pi_0(E,*) \to \pi_0(B,*)$$

Chapter 2
Basic concepts from Lie group theory

In this chapter the base field is either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

2.1 Analytic manifolds

The following definition generalizes the concept of an analytic function of one variable defined on an open subset $U \subset \mathbb{K}$ of the base field.

Definition 2.1 (Analytic map of several variables). Consider an open subset $U \subset \mathbb{K}^n$.

1. A function
$$f : U \to \mathbb{K}$$
is $\mathbb{K}$-*analytic* if any point $z \in U$ has an open neighbourhood $V \subset U$ such that $f|V$ develops into a convergent power series around z, i.e. for all $w \in V$
$$f(w) = \sum_{\nu \in \mathbb{N}^n} c_\nu \cdot (w-z)^\nu = \sum_{\nu=(\nu_1,...,\nu_n) \in \mathbb{N}^n} c_\nu \cdot (w_1 - z_1)^{\nu_1} \cdot, ..., \cdot (w_n - z_n)^{\nu_n}, c_\nu \in \mathbb{K}.$$

2. A map
$$f = (f_1, ..., f_m) : U \to \mathbb{K}^m$$
is $\mathbb{K}$-*analytic* if each component function
$$f_i : U \to \mathbb{K}, i = 1, ..., m,$$
is $\mathbb{K}$-analytic.

A synonym for $\mathbb{C}$-analytic map is *holomorphic map*.

Definition 2.2 (Analytic structure and analytic manifold). Consider a topological space X.

1. A $\mathbb{K}$-*analytic atlas* for X is a family

$$\mathscr{A} = (\phi_i : U_i \to V_i)_{i \in I}$$

of homeomorphisms, named *charts*, onto open subsets $V_i \subset \mathbb{K}^{n_i}$, with $U_i \subset X$ open and $X = \bigcup_{i \in I} U_i$, such that all *chart transformations*

$$\phi_i \circ {\phi_j}^{-1} | \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \subset \mathbb{K}^{n_i}$$

are $\mathbb{K}$-analytic maps. The component functions of a chart $\phi = (\phi_1, ..., \phi_n)$ are named *coordinate functions*.

2. Two $\mathbb{K}$-analytic atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are *compatible*, if their union $\mathscr{A}_1 \cup \mathscr{A}_2$ is also a $\mathbb{K}$-analytic atlas. A $\mathbb{K}$-*analytic structure* on the topological space X is an equivalence class of $\mathbb{K}$-analytic atlases with respect to the compatibility relation.

3. A topological space X together with a $\mathbb{K}$-analytic structure is named a $\mathbb{K}$-*analytic manifold*. If $x \in X$ and $\phi : U \to V$ is a chart with $x \in U$ and $V \subset \mathbb{K}^n$ open, then the number $n \in \mathbb{N}$ is the *dimension* of X at x, denoted $dim_x X = n$.

   In case $\mathbb{K} = \mathbb{C}$ the manifold is named a *complex manifold*.

A chart *around a point* $x \in X$ is a chart

$$\phi : U \to V \subset \mathbb{K}^n$$

with $x \in U$ and $\phi(x) = 0 \in \mathbb{K}^n$.

We do no require that an analytic manifold is a Hausdorff space. But an analytic group structure, as it will be defined in Definition 2.16, implies that the underlying topological space is a Hausdorff space.

Definition 2.3 (Analytic functions and analytic maps between manifolds). Consider a $\mathbb{K}$-analytic manifold.

1. A map

$$f : V \to \mathbb{K}^m, V \subset X \; open,$$

is *analytic* if for any chart $\phi : U \to \mathbb{K}^n$ of X the composition

$$f \circ \phi^{-1} : \phi(V \cap U) \to \mathbb{K}^m$$

is an analytic map in the sense of Definition 2.1.

2. If Y is a second $\mathbb{K}$-analytic manifold then a continuous map
$$f : X \to Y$$
is *analytic* if for any chart
$$\psi : W \to \mathbb{K}^m$$
of Y the composition
$$\psi \circ f : f^{-1}(W) \to \mathbb{K}^m$$
is an analytic map.

Note: Requiring continuity of f in the second part of the definition is necessary to conclude that $f^{-1}(W) \subset X$ is open.

Again, if $\mathbb{K} = \mathbb{C}$ then analytic maps are named *holomorphic*.

Definition 2.4 (Germ of an analytic function). Consider a $\mathbb{K}$-analytic manifold X and a point $x \in X$. Two analytic functions
$$f_i : U_i \to \mathbb{K}, i = 1,2,$$
defined in open neighbourhoods $U_i \subset X$ of x are *equivalent* with respect to x if an open neighbourhood $V \subset U_1 \cap U_2$ of x exists with
$$f_1|V = f_2|V.$$
The set of equivalence classes equipped with the induced ring-structure is denoted by $\mathscr{O}_{X,x}$ and named the *ring of germs of analytic functions* at $x \in X$ or the *local ring* of X at x.

Remark 2.5 (Local ring). . In the case of the complex 1-dimensional analytic manifold $X = \mathbb{C}$ and the origin $0 \in \mathbb{C}$ we have
$$\mathscr{O}_{\mathbb{C},0} = \mathbb{C} < z >$$
the ring of convergent power series in one complex variable with expansion point the origin. Apparently, for any other expansion point $z_0 \in \mathbb{C}$ we have the isomorphy
$$\mathscr{O}_{\mathbb{C},z_0} = \mathbb{C} < z - z_0 > \simeq \mathbb{C} < z > .$$
More general: The local ring of any $\mathbb{K}$-analytic manifold X at a point $x \in X$ with $dim_x X = n$ is isomorphic to $\mathbb{K} < z_1, ..., z_n >$, the ring of convergent power series in n variables. Each of these power series has a positive convergence radius. But its value depends on the power series.

Definition 2.6 (Tangent space of derivations). Consider a $\mathbb{K}$-analytic manifold X and a point $x \in X$.

1. The *tangent space* of X at x is the $\mathbb{K}$-vector space

$$T_x X = Der(\mathscr{O}_{X,x}, \mathbb{K})$$

of $\mathbb{K}$-linear *derivations* of $\mathscr{O}_{X,x}$ into the base field $\mathbb{K}$, i.e. of $\mathbb{K}$-linear maps

$$D : \mathscr{O}_{X,x} \to \mathbb{K}$$

satisfying the product rule

$$D(f \cdot g) = D(f) \cdot g(x) + f(x) \cdot D(g), f, g \in \mathscr{O}_{X,x}.$$

2. Consider an analytic map $F : X \to Y$ with an analytic manifold Y. The *tangent map* of F at a point $x \in X$ with $y := F(x)$ is the induced $\mathbb{K}$-linear map

$$T_x F : T_x X \to T_y Y, D \mapsto T_x f(D),$$

with $(T_x(D))(f) := D(f \circ F), f \in \mathscr{O}_{Y,y}$.

Note: The product rule implies

$$D(1) = D(1 \cdot 1) = D(1) + D(1),$$

hence $D(1) = 0$ and by $\mathbb{K}$-linearity $D(a) = 0$ for all $a \in \mathbb{K}$.

A derivation $D \in T_{X,x}$ is defined on the germs of analytic functions. Because the definition uses representatives of the germ we will often write $D(f)$ instead of $D([f])$ for a germ $[f] \in \mathscr{O}_{X,x}$.

Proposition 2.7 (Algebraic characterisation of $T_x X$). *Consider a $\mathbb{K}$-analytic manifold and a point $x \in X$ and assume $dim_x X = n$. Denote by*

$$\mathfrak{m} := \{[f] \in \mathscr{O}_{X,x} : f(x) = 0\}$$

the maximal ideal in $\mathscr{O}_{X,x}$ of all germs vanishing at x.

1. Restriction defines a $\mathbb{K}$-linear map

$$T_x X = Der(\mathscr{O}_{X,x}, \mathbb{K}) \to Hom_{\mathbb{K}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{K})$$

which is an isomorphism of vector spaces.

2. *Each chart ϕ of X around x defines a $\mathbb{K}$-linear isomorphism*

$$\Phi : \mathscr{O}_{X,x} \xrightarrow{\simeq} \mathbb{K} < z_1, ..., z_n >, [f] \mapsto f \circ \phi^{-1},$$

onto the $\mathbb{K}$-algebra of convergent power series with expansion point $\phi(0) = 0$. The restriction of Φ on $\mathfrak{m}$ induces a $\mathbb{K}$-linear isomorphism of vector spaces

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\simeq} span_{\mathbb{K}} < z_1, ..., z_n > \simeq \mathbb{K}^n.$$

In particular dim $T_x X = n$.

3. *For $i = 1, ..., n$ define the i*-th derivational derivative *at x with respect to the chart ϕ around x as*

$$D_i := \frac{\partial}{\partial \phi_i} : \mathscr{O}_{X,x} \to \mathbb{K}, [f] \mapsto D_i([f]) := \frac{\partial (f \circ \phi^{-1})}{\partial z_i}(0),$$

with $z = (z_1, ..., z_n)$ the coordinates on $\Phi(U) \subset \mathbb{K}^n$.

The derivational derivatives $(D_i)_{i=1,...,n}$ are a basis of the tangent space $T_x X$.

Proof. The ideal $\mathfrak{m} \subset \mathscr{O}_{X,x}$ is maximal because the quotient

$$\mathscr{O}_{X,x}/\mathfrak{m} \xrightarrow{\simeq} \mathbb{K}, \overline{f} \mapsto f(0),$$

is a field. The ideal comprises all non-units, i.e. invertible elements of $\mathscr{O}_{X,x}$. Therefore $\mathfrak{m}$ is the unique maximal ideal of $\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,x}$ is a local ring.

ad 1) As a K-vector space the local ring decomposes as

$$\mathscr{O}_{X,x} \xrightarrow{\simeq} \mathbb{K} \oplus \mathfrak{m}, f \mapsto (f(0), f - f(0)).$$

- With respect to this decomposition any derivation $D \in Der(\mathscr{O}_{X,x}, \mathbb{K})$ satisfies $D|\mathbb{K} = 0$. Due to the product rule also
$$D|\mathfrak{m}^2 = 0,$$
i.e. D induces a linear map $\overline{D}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, "D|\mathfrak{m}"] \arrow[d, "\pi"'] & \mathbb{K} \\
\mathfrak{m}/\mathfrak{m}^2 \arrow[ur, dashed, "\overline{D}"'] &
\end{tikzcd}$$

Here π denotes the canonical projection.

- For the opposite direction consider a $\mathbb{K}$-linear map
$$\delta : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathbb{K}.$$
Extend the linear map
$$d := \delta \circ \pi : \mathfrak{m} \rightarrow \mathbb{K}$$
to a linear map
$$D : \mathscr{O}_{X,x} \simeq \mathbb{K} \oplus \mathfrak{m} \rightarrow \mathbb{K}, (a,m) \mapsto d(m).$$
We claim
$$D(f \cdot g) = D(f) \cdot g(0) + f(0) \cdot D(g)$$
for all $f,g \in \mathscr{O}_{X,x}$. The proof has to consider the following cases:
  - $f,g \in \mathbb{K}$ are constant functions: Then both sides of the equation vanish.
  - $f,g \in \mathfrak{m}$: Again both sides of the equation vanish.
  - $f \in \mathbb{K}$ and $g \in \mathfrak{m}$: Then the left-hand side
$$D(f \cdot g) = f \cdot D(g)$$
by linearity. While the right-hand side
$$D(f) \cdot g(0) + f(0) \cdot D(g) = 0 \cdot g(0) + f(0) \cdot D(g) = f \cdot D(g)$$
because $D|\mathbb{K} = 0$.

ad 3) The directional derivatives $D_i, i = 1,...,n$, are derivations. They are linearly independent: If $\phi = (\phi_1,...,\phi_n)$ then
$$D_i(\phi_j) = \delta_{ij}.$$
Hence $(D_i)_{i=1,...,n}$ is a basis of T_xX.

The tangent map at the point x of an analytiv map F is the *Jacobi-matrix* of F with respect to charts around x and $F(x)$.

The local properties of an analytic map at a given point express themselves in corresponding properties of the tangent map.

- A local isomorphism maps a sufficiently small open neighbourhood isomorphically onto a neighbourhood of the image point,
- an immersion maps a sufficiently small neighbourhood isomorphically onto a coordinate slice of the image point,
- and a submersion splits a sufficiently small neighbourhood of a point in the domain of definition as a product and projects the product onto a neighbourhood of the image point.

Definition 2.8 introduces these concepts in a formal way, while Proposition 2.9 relates them to the rank of the tangent map.

Definition 2.8 (Local isomorphism, immersion, submersion). Consider an analytic map

$$f : X \to Y$$

between two $\mathbb{K}$-analytic manifolds and a point $x \in X$. The map f is named

- *local isomorphism* at x if open neighbourhoods

  $$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y$$

  exist such that the injections

  $$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y$$

  extend to the commutative diagram

  $$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "j_U"] \arrow[r, dashed, "\simeq", "f|U"'] & V \arrow[u, "j_V"'] \end{tikzcd}$$

- *immersion* at x if open neighbourhoods

  $$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m$$

  exist such that the injections

  $$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y,$$

  and the embedding onto a coordinate slice

  $$j = [U \xrightarrow{\simeq} U \times \{0\} \hookrightarrow U \times W]$$

  extend to a commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
U \arrow[u, "j_U"] \arrow[rr, "f|U"'] \arrow[dr, "j"'] & & V \arrow[u, "j_V"'] \\
& U \times W \arrow[ur, dashed, "\simeq"'] &
\end{tikzcd}
$$

- *submersion* at x if open neighbourhoods

$$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m$$

  exist such that the injections

$$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y,$$

  and the projection

$$pr_V : V \times W \to V$$

  extend to a commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
U \arrow[u, "j_U"] \arrow[rr, "f|U"'] \arrow[dr, dashed, "\simeq"'] & & V \arrow[u, "j_V"'] \\
& V \times W \arrow[ur, "pr_V"'] &
\end{tikzcd}
$$

- respectively *local isomorphism*, *immersion*, *submersion* if the corresponding local property holds for each point $x \in X$.

Proposition 2.9 (Local isomorphism, immersion, submersion). *Consider an analytic map*

$$f : X \to Y$$

between two $\mathbb{K}$-analytic manifolds, a point $x \in X$ with $y := f(x)$, and the tangent map

$$T_x f : T_x X \to T_y Y.$$

At x the map f is respectively

- *a local isomorphism iff the tangent map $T_x f$ is bijective.*
- *an immersion iff the tangent map $T_x f$ is injective.*
- *and a submersion iff the tangent map $T_x f$ is surjective.*

The proof uses the Inverse Mapping Theorem, see [33, Part II, Chap. III.9 and III.10].

The composition of two immersions is an immersion. The composition of two submersions is a submersion.

Proposition 2.10 (Criteria for analyticity of maps). *Consider analytic manifolds X, Y, Z and an analytic map $f : X \to Y$.*

1. Assume f to be an immersion. For a continous map $g : Z \to X$ then:

$$g \; analytic \iff f \circ g \; analytic.$$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f \circ g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

2. Assume f to be a submersion. For a continous map $g : Y \to Z$ then:

$$g \; analytic \iff g \circ f \; analytic.$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Proof. 1) Assume that the composition $f \circ g$ is analytic. The question is local in Z. We choose open sets

$$U \subset X, V \subset Y, W \subset \mathbb{K}^n$$

such that the restriction $f|V$ is an embedding. We set $U' := g^{-1}(U)$ and denote by $g' := g|U'$ the restriction. Due to the definition of an immersion we obtain a commutative diagram:

$$\begin{tikzcd}
Z \arrow[r, "g"] & X \arrow[rr, "f"] & & Y \\
U' := g^{-1}(U) \arrow[u] \arrow[r, "g'"] & U \arrow[u] \arrow[rr, "f|V"] \arrow[dr, "j"'] & & V \arrow[u] \\
 & & U \times W \arrow[ur, "h", "\simeq"'] &
\end{tikzcd}$$

By assumption the map $h \circ j \circ g'$ is analytic. As a consequence $j \circ g'$ is analytic, which implies that

$$g' = pr_U \circ j \circ g'$$

is analytic with $pr_U : U \times W \to U$.

2) Assume that the composition $g \circ f$ is analytic. The question is local in X. We choose open sets

$$U \subset X, V \subset Y, W \subset \mathbb{K}^n$$

such that the restriction $f|U$ is a projection. Due to the definition of a submersion the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[r, "g"] & Z \\
U \arrow[u] \arrow[rr, "f|U"] \arrow[dr, "h", "\simeq"'] & & V \arrow[u] \arrow[ur, "g' := g|V"'] & \\
 & V \times W \arrow[ur, "pr_V"'] & &
\end{tikzcd}$$

By assumption the map $g' \circ pr_V \circ h$ is analytic. As a consequence the map $g' \circ pr_V$ is analytic, which implies

$$g' = g' \circ pr_V \circ i$$

analytic with

$$i : V \to V \times W, i(x) = (x, *), q.e.d.$$

Corollary 2.11 (Uniqueness of analytic structure defined by immersions and submersions).

1. Consider an analytic manifold Y, a topological space X, and a continous map

$$j : X \to Y.$$

Then at most one analytic structure $\mathscr{A}$ exists on X such that

$$j : (X, \mathscr{A}) \to Y$$

is an immersion.

2. *Consider an analytic manifold X, a topological space Y, and a surjective continuous map*

$$f : X \to Y.$$

Then at most one analytic structure $\mathscr{A}$ exists on Y such that

$$f : X \to (Y, \mathscr{A})$$

is a submersion.

Proof. ad 1) Assume two analytic structures $(X, \mathscr{A}_1)$ and$(X, \mathscr{A}_2)$ which both satisfy the above condition. The composition

$$(X, \mathscr{A}_1) \xrightarrow{id} (X, \mathscr{A}_2) \xrightarrow{f} Y$$

is analytic because $(X, \mathscr{A}_1)$ satisfies the above condition. And because $(X, \mathscr{A}_2)$ satisfies the above condition Proposition 2.10 implies that the map

$$(X, \mathscr{A}_1) \xrightarrow{id} (X, \mathscr{A}_2)$$

is analytic. Analogously one proves that the map

$$(X, \mathscr{A}_2) \xrightarrow{id} (X, \mathscr{A}_1)$$

is analytic.

ad 2) Analogously, q.e.d.

Proposition 2.12 (Immersion and coordinate slice). *Consider an analytic manifold Y and a topological space X together with a continous map*

$$f : X \to Y.$$

Then the following properties are equivalent:

1. Immersion*: The topological space X has an analytic structure such that $f : X \to Y$ is an immersion.*

2. Zero set of cooordinate functions*: For each point* $x \in X$ *a neighbourhood* $U \subset X$ *of* x *and a chart of* Y

$$\phi = (\phi_1, ..., \phi_n) : V \to W \subset \mathbb{K}^n$$

in a neighbourhood $V \subset Y$ *of* $f(x)$ *exist, such that* $f(U) \subset V$ *and*

$$f|U : U \to f(U)$$

is a homeomorphism and

$$f(U) = \{y \in V : \phi_{m+1}(y) = ... = \phi_n(y) = 0\}$$

for a suitable integer $0 \le m \le n$.

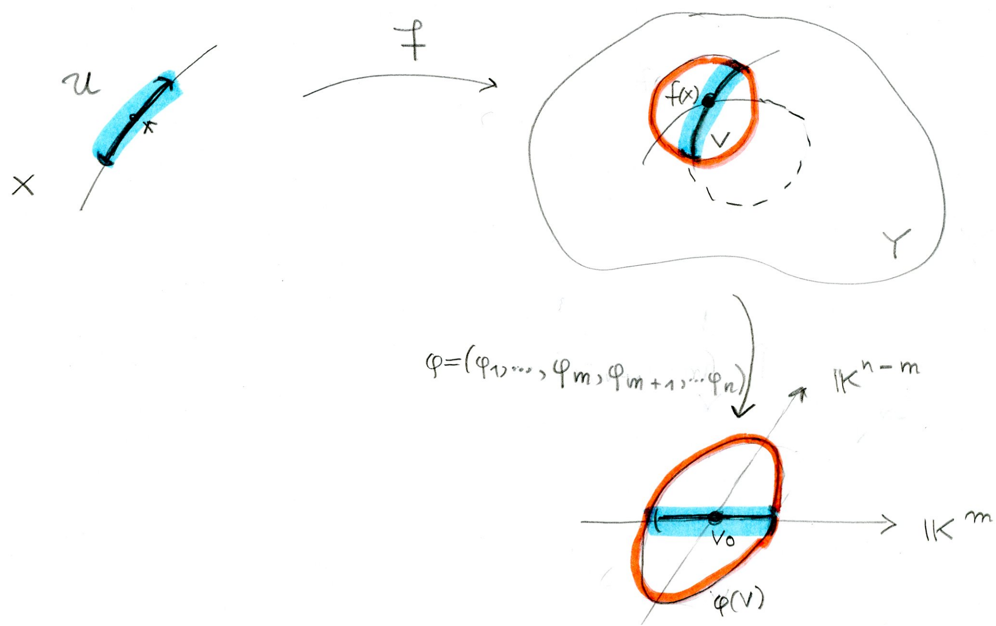

Fig. 2.1 Immersion

Proof. 1) $\implies$ 2) Assume that f is an immersion at a point $x \in X$. According to Proposion 2.9 a commutative diagram with the properties from Definition 2.8 exists:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
U \arrow[u, "j_U"] \arrow[rr, "f|U"] \arrow[dr, "j"'] & & V \arrow[u, "j_V"'] \\
& U \times W \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

with

$$j : U \to U \times W, x \mapsto (x, *).$$

Apparently

$$f(U) \simeq j(U) = U \times \{0\}$$

is the zero set of coordinate functions of a chart of Y defined on $V = U \times W$. The bijective continuous map

$$j : U \to j(U)$$

is open: For any open subset $U' \subset U$ holds

$$j(U') = U' \times \{0\} = (U' \times W) \cap j(U)$$

proving that $j(U')$ is an open subset of $j(U)$. Hence j is a homeomorphism onto its image and the same holds for $f|U$.

2) $\implies$ 1): Consider a point $x \in X$. We define a chart for X

$$\psi : U \to \mathbb{K}^m$$

around $x \in U$. By assumption a chart of Y exists

$$\phi = (\phi_1, ..., \phi_m, \phi_{m+1}, ..., \phi_n) : V \to \mathbb{K}^n$$

around $f(x) \in V$ with

$$f(U) = \{y \in V : \phi_{m+1}(y) = ... = \phi_n(y) = 0\}.$$

The set

$$V_0 := pr_{\mathbb{K}^m}(\phi(V) \cap (\mathbb{K}^m \times \{0\})) \subset \mathbb{K}^m$$

is open. The composition of the two homeomorphisms

$$\psi := [U \xrightarrow{f|U} f(U) \xrightarrow{(\phi_1, ..., \phi_m)} V_0]$$

is a homeomorphism. We provide $U \subset X$ with the analytic structure such that

$$\psi : U \to V_0$$

becomes an analytic isomorphism. The embedding

$$j : V_0 \simeq V_0 \times \{0\} \hookrightarrow V$$

is an immersion. Therefore also the composition

$$f|U = j \circ \psi : U \to V$$

has injective tangent map, i.e. is an immersion.

We obtain an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X and for each $i \in I$ an analytic structure $\mathscr{A}_i$ on U_i such that

$$f|U_i : (U_i, \mathscr{A}_i) \to f(U_i)$$

is an immersion. For each intersection $U_{ij} := U_i \cap U_j$ both compositions

$$(U_{ij}, \mathscr{A}_i | U_{ij}) \hookrightarrow (U_i, \mathscr{A}_i) \to Y$$

and

$$(U_{ij}, \mathscr{A}_j | U_{ij}) \hookrightarrow (U_j, \mathscr{A}_j) \to Y$$

are immersions. According to Corollary 2.11 both analytic structures on U_{ij} are equal. Therefore the local analytic structures $(\mathscr{A}_i)_{i \in I}$ induce a global analytic structure $(X, \mathscr{A})$ such that $f : (X, \mathscr{A}) \to Y$ is an immersion, q.e.d.

A submanifold of a manifold Y is locally the zero set of coordinate functions of Y.

Definition 2.13 (Submanifold). Consider an analytic manifold Y. A *submanifold* X of Y is a subset $X \subset Y$ equipped with the subspace topology such that the injection

$$j : X \hookrightarrow Y$$

satisfies condition 2) (*Zero set of cooordinate functions*) from Proposition 2.12, i.e. X is locally the zero set of coordinate functions of Y.

If $X \subset Y$ is closed then X is named a *closed submanifold* of Y.

A submanifold is locally closed, but in general not closed. Any submanifold of an analytic manifold is an analytic manifold itself. The specific case when taking the empty set of cooordinate functions show that any open subset of X is a submanifold.

A main theorem from the local theory of analytic manifolds is the Rank Theorem. It contains the Inverse Mapping Theorem as a particular case. We say in abbreviated form: An analytic map $f : X \to Y$ between two manifolds has *constant rank* $p \in \mathbb{N}$ iff each $x_0 \in X$ has a neighbourhood U in X with *rank* $T_x f = p$ for all $x \in U$.

Theorem 2.14 (Rank Theorem). *Consider an analytic map*

$$f : X \to Y$$

between two analytic manifolds. Assume a distinguished point $x_0 \in X$ such that the restriction

$$f|W : W \to Y$$

has constant rank p on a neighbourhood $W \subset X$ of x_0.

Then exist neighbourhoods $U \subset W$ of x_0, $V \subset Y$ of $f(x_0)$, and a p-dimensional submanifold B of V such that the restriction of f decomposes as

$$f|U = [U \xrightarrow{s} B \xhookrightarrow{i} V]$$

with

- *a submersion $s : U \twoheadrightarrow B$*
- *and an immersion $i : B \hookrightarrow V$.*

If f has constant rank on X then each fibre

$$X_y := f^{-1}(y) \subset X, y \in Y,$$

is a submanifold. At a point $x \in X_y$ its tangent space is the subspace of vertical tangent vectors

$$T_x X_y = ker\ [T_x f : T_x X \to T_y Y].$$

For the proof see [33, Part II, Chap. III.10].

The conclusion of the Rank Theorem is local in $x \in X$. Therefore V as well as the submanifold $B \subset V$ may depend on x, even if x varies in a fixed fibre.

The following theorem of Godement deals with the subtle task to provide the set of equivalence classes of an analytic equivalence relation with a suitable analytic structure.

Theorem 2.15 (Godement's theorem about analytic equivalence relations). *Consider an analytic manifold X, an equivalence relation $R \subset X \times X$, and the set of equivalence classes $Y := X/R$ with the canonical projection*

$$p : X \to Y.$$

Then are equivalent:

1. On Y exists an analytic structure such that $p : X \to Y$ becomes a submersion.

2. The equivalence relation $R \subset X \times X$ is a submanifold and $pr_2|R : R \to X$ is a submersion.

If these conditions are satisfied then:

$$R \subset X \times X \ closed \iff Y \ is \ a \ Hausdorff \ space.$$

In particular, the equivalence relation R is symmetric. Therefore the second projection $pr_2 : R \to X$ is a submersion iff the first projection $pr_1 : R \to X$ is a submersion.

For the proof see [33, Part II, Chap. III.12].

2.2 Lie groups

Definition 2.16 (Lie groups and morphisms).

1. A $\mathbb{K}$-*Lie group* G is a group which is also a $\mathbb{K}$-analytic manifold such that its group multiplication

$$m : G \times G \to G, (x,y) \mapsto x \cdot y := m(x,y),$$

   is an analytic map. The neutral group element is denoted by $e \in G$.

2. A *morphism* of Lie groups

$$f : G \to H$$

   is an analytic map which is also homomorphism of groups.

Note: We only require that the multiplication is analytic. Lemma 2.17 shows that the analyticity of the inversion is a consequence of the Inverse Mapping Theorem.

Lemma 2.17 (Analyticity). *Consider a Lie group G. Then each of the following maps is analytic:*

1. Inversion*:*

$$\sigma : G \to G, g \mapsto g^{-1},$$

For each arbitrary but fixed element $g \in G$

2. Left translation

$$L_g : G \to G, h \mapsto g \cdot h,$$

3. Right translation
$$R_g : G \to G, h \mapsto h \cdot g,$$

4. Inner automorphism
$$Ad\ g : G \to G, h \mapsto g \cdot h \cdot g^{-1},$$

Proof. The proof of part 2) - 4) is obvious. In order to prive part 1) we consider the analytic map
$$\theta : G \times G \to G \times G, (x,y) \mapsto (x, x \cdot y).$$
Its tangent map
$$T_{(x,y)}\theta : T_{(x,y)}(G \times G) \to T_{(x,y)}(G \times G)$$
at the point $(x,y) \in G \times G$ has the matrix - the columns corresponding to the components of θ -
$$T_{(x,y)}\theta = \begin{pmatrix} \mathbb{1} & T^1_{(x,y)}m \\ & \\ 0 & T^2_{(x,y)}m \end{pmatrix}$$
with
$$T^1_{(x,y)}m = T_x R_y \ for\ R_y : G \to G, x \mapsto x \cdot y$$
and
$$T^2_{(x,y)}m = T_y L_x \ for\ L_x : G \to G, y \mapsto x \cdot y.$$
The map θ is bijective, it's inverse is
$$\theta^{-1} : G \times G \to G \times G, (x,z) \mapsto (x, x^{-1} \cdot z).$$
Because the tangent map $T_{(x,y)}\theta$ is an isomorphism the map θ^{-1} is analytic due to Proposition 2.9. As a consequence, the composition
$$\sigma = [G \xrightarrow{j} G \times G \xrightarrow{\theta^{-1}} G \times G \xrightarrow{pr_2} G]$$
with
$$j(x) := (x,e)$$
is analytic.

Definition 2.18 (Lie subgroup). Consider a Lie group G. A subset $H \subset G$ is named a *Lie subgroup* if H is both,

- in the *algebraic* sense a subgroup of the group G and
- in the sense of *analysis* a submanifold of the manifold G

In these notes we use the term *Lie subgroup* in the same sense as Bourbaki, cf. [4, Chap. III, §1.3]. Some other textbooks, e.g. [9, Def. 1.10.1], use the term with a different meaning. We will not use the term *analytic subgroup* because this term is used with different meanings in the literature. Later on we will introduce a second type of subgroup, called *integral subgroup* by Bourbaki.

Having defined the concepts of a Lie group morphism and of a Lie subgroup we are confronted with the following questions:

1. Is a Lie subgroup always a Lie group?
2. Is the kernel of a morphism f of Lie groups a Lie subgroup of the domain of f?
3. Is the image of a morphism f of Lie groups a Lie subgroup of the codomain of f?
4. Is the quotient G/H of a Lie group G by a normal Lie subgroup H a Lie group? Is the corresponding projection

$$\pi : G \to G/H$$

a morphism of Lie groups?
5. Which properties for an analytic morphism f follow from the algebraic restriction to be a morphism of Lie groups?

We shall obtain the following answers:

1. The answer is affirmative: If a subset $H \subset G$ is a subobject with respect to all three mathematical structures Analysis, Topology, and Algebra, then these properties are compatible. They combine to the structure of a Lie group H such that the injection $H \hookrightarrow G$ is a morphism of Lie groups, see Lemma 2.19. Moreover, a Lie subgroup $H \subset G$ is automatically closed in G, see Proposition 2.24.

   On the contrary, even if a Lie group H is a subobject $H \subset G$ in the sense of algebra and in addition the inclusion $H \hookrightarrow G$ is a morphism of Lie groups, then H is not necessarily a subobject neither in the sense of analysis nor in the sense of topology. For a counter example see Example **??**.

2. The answer is affirmative: See Corollary 2.22.

3. The answer is negative: See Example **??**.

4. The answer is affirmative: See Proposition 2.24.

5. A surjective Lie group morphism f is an analytic fibre bundle with fibre the kernel *ker* f, see Theorem 2.32.

Lemma 2.19 (Lie subgroups are Lie groups). *Consider a Lie subgroup H of a Lie group G. Then H is a Lie group itself.*

Proof. If $j : H \to G$ denotes the injection, then the following diagram commutes

$$\begin{array}{ccc} H \times H & \xrightarrow{j \times j} & G \times G \\ {\scriptstyle m_H}\downarrow & & \downarrow{\scriptstyle m_G} \\ H & \xrightarrow{j} & G \end{array}$$

The map m_H is continous because for an open set $U \subset G$ the set

$$m_H^{-1}(H \cap U) = (H \times H) \cap m_G^{-1}(U)$$

is open in $H \times H$. From

$$j \circ m_H = m_G \circ (j \times j)$$

follows the analyticity of $j \circ m_H$. Proposition 2.10 implies that the map m_H is analytic because j is an immersion, q.e.d.

Lemma 2.20 (Locally compactness and σ-compactness in Lie groups). *Any Lie group G is a Hausdorff space and even locally compact. Any connected component of G is σ-compact.*

Proof. The singleton $\{e\} \subset G$ is closed because for any chart $\phi : U \to \mathbb{K}^n$ around e

$$e = \phi^{-1}(0).$$

Corollary 1.6 implies that G is a Hausdorff space. Any Hausdorff manifold is locally compact because a chart is a local homeomorphism onto an open subset of $\mathbb{K}^n$.

Consider the connected component G^e of $e \in G$. Proposition 1.11 and Proposition 1.9 show the existence of a compact neighbourhood K of e in G^e with

$$G^e = \bigcup_{n \in \mathbb{N}} K_n.$$

Here $K_n = K \cdot \ldots \cdot K \subset G^e$ denotes the k-fold product with respect to the group multiplication. Each set $K_n \subset G^e$ is compact because the canonical map from the compact Cartesian product K^n

$$K^n \to K_n$$

is continous and surjective. As a consequence G^e and any connected component of G is σ-compact, q.e.d.

Proposition 2.21 (Lie group operation). *Consider a Lie group G, an analytic manifold X, and an analytic left G-operation*

$$\phi : G \times X \to X, (g,x) \mapsto g.x.$$

Then for each point $x \in X$:

1. *The* orbit map
$$f := \phi_x : G \to X, g \mapsto g.x,$$
has constant rank.

2. *The isotropy group*
$$G_x := \{g \in G : g.x = x\}$$
is a Lie subgroup of G.

Proof. 1) For arbitrary but fixed $h \in G$ the commutative diagram of analytic maps

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\wr"', "L_h"] & X \arrow[d, "\wr"', "=: \psi"] \\ G \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r] \arrow[d] & g.x_0 \arrow[d] \\ h \cdot g \arrow[r] & (h \cdot g).x_0 \end{tikzcd}$$

induces the commutative diagram between tangent spaces with $g := e$

$$\begin{tikzcd} T_eG \arrow[r, "T_ef"] \arrow[d, "\wr"', "T_eL_h"] & T_{x_0}X \arrow[d, "\wr"', "T_x\psi"] \\ T_hG \arrow[r, "T_hf"] & T_{h.x_0}X \end{tikzcd}.$$

In particular,
$$rk\ T_ef = rk\ T_hf.$$

2) The commutative diagram

$$\begin{tikzcd} G_x \arrow[r, dashed] \arrow[d, dashed, "f_{\{x\}}"'] & G \arrow[d, "f"] \\ \{x\} \arrow[r] & X \end{tikzcd}.$$

represents the isotropy group as the fibre of an analytic map with constant rank. Therefore Theorem 2.14 proves the claim, q.e.d.

As a corollary we obtain a statement about the kernel of a Lie group morphism.

Corollary 2.22 (Lie group morphism). *Consider a Lie group morphism*

$$f : G \to H.$$

1. *The morphism f has constant rank.*

2. *The subgroup $ker\ f \subset G$ is a Lie subgroup of G.*

Proof. ad 1) One considers the analytic operation

$$\phi : G \times H \to H, (g,h) \mapsto f(g) \cdot h,$$

and applies Proposition 2.21 to the orbit map of the neutral element $e \in H$

$$G \to H, g \to f(g).$$

ad 2) $ker\ f = G_e$ for the analytic operation ϕ from part 1), q.e.d.

Example 2.23 (Examples of Lie groups).

1. The additive group $(\mathbb{K}^n, +)$ of a finite-dimensional $\mathbb{K}$-vector space is a $\mathbb{K}$-Lie group.

2. The *linear group* $(GL(n,\mathbb{K}), \cdot)$ is a Lie group: Being an open subset of the analytic manifold $\mathbb{K}^{n^2}$ it is an analytic manifold. The group multiplication and taking the inverse are analytic maps with respect to the coordinates from $\mathbb{K}^{n^2}$.

   Lie subgroups of $GL(n,\mathbb{K})$ are named *matrix groups*.

3. The *special linear group*

   $$SL(n,\mathbb{K}) := ker\ [GL(n,\mathbb{K}) \xrightarrow{det} \mathbb{K}^*]$$

   is a Lie subgroup according to Corollary 2.22, in particular a matrix group.

4. Consider the Lie group $GL(n,\mathbb{K})$ as a real Lie group and consider the $\mathbb{R}$-analytic operation

   $$GL(n,\mathbb{K}) \times GL(n,\mathbb{K}) \to GL(n,\mathbb{K}), (A,B) \mapsto A \cdot B \cdot A^*.$$

   Here $A^* := \overline{A}^\top$ denotes the Hermitian conjugate.

   According to Proposition 2.21 the following isotropy groups of the neutral element $e = \mathbb{1} \in GL(n,\mathbb{K})$ are Lie subgroups of the real Lie group $GL(n,\mathbb{K})$, in particular real matrix groups.

- $\mathbb{K}=\mathbb{R}$: $O(n,\mathbb{R}) = GL(n,\mathbb{R})_e$ (*Orthogonal group*)
- $\mathbb{K}=\mathbb{C}$: $U(n) = GL(n,\mathbb{C})_e$(*Unitary group*).

5. Consider a finite-dimensional associative $\mathbb{K}$-algebra A with product

$$m : A \times A \to A, (x,y) \mapsto x \cdot y.$$

Then its group of automorphisms

$$Aut(A) := \{\phi : A \to A | \mathbb{K} - linear\ automorphism\ with\ \phi(x \cdot y) = \phi(x) \cdot \phi(y)\}$$

is a $\mathbb{K}$-Lie group with respect to the composition as multiplication

$$Aut(A) \times Aut(A) \to Aut(A), (\phi_1, \phi_2) \mapsto \phi_1 \circ \phi_2.$$

In order to prove this claim consider the Lie group

$$G := \{\phi : A \to A | \mathbb{K} - linear\ automorphism\} \simeq GL(n,\mathbb{K})$$

with $n = dim_{\mathbb{K}} A$. For $\phi \in G$

$$\phi \in Aut(G) \iff m \circ (\phi \times \phi) = \phi \circ m \iff m = \phi \circ m \circ (\phi \times \phi)^{-1}$$

On the $\mathbb{K}$-vector space $Bil(A \times A, A)$ of $\mathbb{K}$-bilinear maps operates the group G according to

$$G \times Bil(A \times A, A) \to Bil(A \times A, A), (\phi, f) \mapsto \phi \cdot f := \phi \circ f \circ (\phi \times \phi)^{-1}.$$

The isotropy group $G_m = Aut(A)$ is a Lie subgroup of $G \simeq GL(n,\mathbb{K})$ according to Proposition 2.21. Due to Lemma 2.19 the group $Aut(A)$ is a matrix group, q.e.d.

The following Proposition deals with subgroups and quotients of a Lie group. The issue is more challenging than the other issues of this section. We will use Godement's theorem.

Proposition 2.24 (Lie subgroups and quotients). *Consider a Lie group G and a Lie subgroup $H \subset G$. Then:*

1. *There exists an analytic structure on G/H such that the canonical projection*

$$\pi : G \to G/H$$

is a submersion.

2. *The subgroup $H \subset G$ is closed and the quotient G/H is a Hausdorff space.*

3. If $H \subset G$ is a normal subgroup then the quotient G/H is a Lie group and π is a morphism of Lie groups.

Proof. 1. The map π induces the equivalence relation $R \subset G \times G$ defined as

$$R := \{(x,y) \in G \times G : \pi(x) = \pi(y)\} = \{(x,y) \in G \times G : y^{-1}x \in H\}.$$

We want to apply Godement's Theorem 2.15. We have to verify two preconditions:

- The analytic map
$$\psi : G \times G \to G, (x,y) \to y^{-1}x,$$
is a submersion: For an arbitrary but fixed element $y_0 \in G$ consider the injection
$$j : G \to G \times G, x \mapsto (x, y_0),$$
and the following commutative diagram

$$\begin{tikzcd} G \arrow[rr, "j"] \arrow[dr, "L_{y_0^{-1}}"', "\simeq"] & & G \times G \arrow[dl, "\psi"] \\ & G & \end{tikzcd}$$

The composition
$$\psi \circ j = L_{y_0^{-1}}$$
induces the composition of tangent maps
$$T_{(x,y_0)}\psi \circ T_x j = T_x L_{y_0^{-1}}.$$
The tangent map $T_x L_{y_0^{-1}}$ is an isomorphism. Therefore $T_{(x,y_0)}\psi$ is surjective and ψ is a submersion according to Proposition 2.9. Also the restriction to H according to the commutative diagram

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[d, dashed, "\psi_H"'] & G \times G \arrow[d, "\psi"] \\ H \arrow[r] & G \end{tikzcd}$$

defines a submersion ψ_H and provides
$$R = \psi^{-1}(H) \subset G \times G$$

with the structure of a submanifold.

- In addition, we have to check that the restriction

$$pr_2|R : R \to G$$

is a submersion: The following diagram

$$\begin{tikzcd} G \times H \arrow[d, "p"'] \arrow[dr, "pr_G"] & \\ R \arrow[r, "pr_2|R"] & G \end{tikzcd}$$

with the submersion pr_1 and the well-defined analytic map

$$p : G \times H \to R, (g,h) \mapsto (gh, g)$$

commutes. Considering the induced tangent maps shows that also

$$pr_2|R : R \to G$$

is a submersion.

Now Theorem 2.15 provides an analytic structure on

$$G/H \simeq G/R$$

such that the canonical map

$$\pi : G \to G/H$$

is a submersion.

2. Because G/H is a manifold the singleton $\{\pi(e)\} \subset G/H$ is closed. Continuity of π implies that

$$H = \pi^{-1}(\pi(e)) \subset G$$

is closed. Proposition 1.5 implies that G/H is a Hausdorff space.

3. Consider the commutative diagram with multiplication as horizontal maps

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "\pi \times \pi"'] & G \arrow[d, "\pi"] \\ G/H \times G/H \arrow[r, "m_{G/H}"] & G/H \end{tikzcd}$$

Because $H \subset G$ is a normal subgroup, the map $m_{G/H}$ is continuous according to Proposition 1.5. Because $\pi \times \pi$ is a submersion and

$$m_{G/H} \circ (\pi \times \pi) = \pi \circ m_G$$

is analytic, also $m_{G/H}$ is analytic due to Proposition 2.10, q.e.d.

2.3 Analytic bundles

First we introduce the analytic version of a fibre bundle. It is obtained from the continuous version in Definition 1.32 by requiring in addition that all maps are analytic.

Definition 2.25 (Analytic bundle).

1. Consider an analytic manifold H. An analytic map $p : X \to Y$ between two analytic manifolds is named *analytic fibre bundle with typical fibre H* or *analytic H-bundle*, using the notation
$$H \hookrightarrow X \xrightarrow{p} Y,$$
if each point $y \in Y$ has an open neighbourhood U together with a *bundle chart* or *local trivialization*, i.e. an analytic isomorphism
$$\phi : p^{-1}(U) \xrightarrow{\simeq} U \times H$$
such that the following diagram commutes
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times H \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

2. Let H be a Lie group. A *H-principal bundle*
$$p : X \to Y$$
is an analytic H-bundle
$$H \hookrightarrow X \xrightarrow{p} Y$$
with the additional property: Y has an open covering $\mathscr{U} = (U_i)_{i \in I}$ with bundle charts

$$\phi_i : p^{-1}(U_i) \xrightarrow{\simeq} U_i \times H, i \in I,$$

such that for each pair $i, j \in I$ the corresponding transition map derives from the group multiplication, i.e. a map

$$h_{ji} : U_i \cap U_j \to H$$

exists such that the transition map has the form

$$\phi_j \circ (\phi_i^{-1}|(U_i \cap U_j) \times H) : (U_i \cap U_j) \times H \to (U_i \cap U_j) \times H$$

$$(y,h) \mapsto (y, h_{ji}(y) \cdot h).$$

Note that the transition map h_{ij} operates on H by left-multplication with $h_{ij}(y)$.

In case of a principal bundle the maps h_{ij} are analytic, because they are the composition

$$h_{ji} = [U_i \cap U_j \xrightarrow{\alpha} U_i \cap U_j \times H \xrightarrow{\phi_j \circ \phi_i^{-1}} U_i \cap U_j \times H \xrightarrow{pr_H} H]$$

with $\alpha(y) = (y,e)$ for all $y \in U_i \cap U_j$.

Our next aim is to strengthen Proposition 2.24: We shall show that a surjective Lie group morphism

$$f : G \to H$$

is a principal bundle with fibre the kernel of the morphism, see Theorem 2.32. In order to derive this result we have to consider *proper* group operations.

Definition 2.26 (Proper map). A continuous map $f : X \to Y$ from a Hausdorff space X into a locally compact space Y is *proper* if $f^{-1}(K) \subset X$ is compact for any compact set $K \subset Y$.

Remark 2.27 (Proper map).

1. For a proper map $f : X \to Y$ also X is locally compact: Any $x \in X$ has the compact neighbourhood $f^{-1}(U)$ with $U \subset Y$ a compact neighbourhood of $f(x)$.

2. Any proper map $f : X \to Y$ is also closed, i.e. $f(A) \subset Y$ is closed for any closed $A \subset X$:

   Choose a covering $(U_i)_i \in I$ of Y by open sets with $\overline{U_i} \subset Y$ compact. Then $(f^{-1}(U_i))_{i \in I}$ is a covering of X by open sets with

   $$K_i := \overline{f^{-1}(U_i)} \subset f^{-1}(\overline{U}) \subset X$$

   compact, being a closed subset of a compact set. Each restriction

$$f_i := f|K_i : K_i \to \overline{U_i}$$

is a proper map.

Hence for the closed set $A \subset X$ and for each $i \in I$:

$$A \cap K_i \subset K_i$$

is compact, and

$$f_i(A \cap K_i) = f(A \cap K_i) = f(A) \cap \overline{U_i} \subset \overline{U_i}$$

is compact, in particular closed.

Eventually we prove the general result: Consider a family $(C_i)_{i \in I}$ of subsets $C_i \subset Y$ such that the family of interiors $(C_i^\circ)_{i \in I}$ is a covering of Y. Then a subset $B \subset Y$ is closed if for all $i \in I$ the subset

$$(B \cap C_i) \subset C_i$$

is closed.

For the proof consider the complements

$$\complement B \cap C_i = C_i \setminus (B \cap C_i), i \in I.$$

Therefore it suffices to consider the dual statement making the assumption that each

$$(B \cap C_i) \subset C_i, i \in I,$$

is open in C_i. Then each

$$B \cap C_i^\circ$$

is open in C_i° and therefore also open in Y. As a consequence

$$B = \bigcup_{i \in I} (B \cap C_i^\circ)$$

is open in Y.

Definition 2.28 (Proper group operation). Consider a topological group H and a locally compact space X. A right H-operation

$$X \times H \to X, (x,h) \mapsto x.h,$$

is *proper* iff the map

$$\theta : X \times H \to X \times X, (x,h) \mapsto (x.h, x),$$

is proper.

Intuitively speaking Definition 2.28 implies: If for a fixed point $x \in X$ the translates $h.x$ converge on the orbit then also a subsequence of the corresponding group elements h converges in H. Note that a H-operation with compact X and compact H is always proper.

Theorem 2.29 (Orbit space of a proper analytic operation). *Consider a Lie group H, an analytic manifold X, and an analytic right H-operation*

$$\Phi : X \times H \to X$$

which is free and proper. Denote by X/H its orbit space*, i.e. the set of equivalence classes of the equivalence relation $R \subset X \times X$ induced by Φ:*

$$x_1 \sim x_2 \iff \exists\, h : x_1.h = x_2.$$

Then an analytic structure exists on X/H such that the canonical map

$$p : X \to X/H$$

is a H-principal bundle.

Note: Because Definition 2.25 requires that the transition maps of a principal bundle operate from the *left*, we have to consider in Theorem 2.29 a *right* operation of H. Then the two multiplications do not interfere with each other.

Proof. i) *Injective immersion*: We show that the map

$$\theta := (\Phi, pr_X) : X \times H \to X \times X, (x,h) \mapsto (x.h, x),$$

is an injective immersion: First, the map is injective because the group operation Φ is free. Secondly, we consider an arbitrary but fixed point $x \in X \times H$. Denote by

$$f := \Phi_x : H \to X, h \mapsto x.h,$$

the orbit map of $x \in X$. The tangent map of θ at $(x,h) \in X \times H$ is the block matrix

$$T_{(x,h)}\theta = \begin{pmatrix} T^1_{(x,h)}\theta & T^2_{(x,h)}\theta \\ \mathbb{1} & 0 \end{pmatrix}$$

with the upper rows the tangent map of the first component of θ, which is the orbit map f. Therefore $T^2_{(x,h)}\theta = T_h f$. The lower rows of $T_{(x,h)}\theta$ are the tangent map of the second component of θ which is the projection $pr_X : X \times H \to X$.

Because the operation is free the map f is injective. It has constant rank according to Proposition 2.21. Therefore f is an immersion due to Theorem 2.14 and $T^2_{(x,h)}\theta$ is injective implying

$$rank\ T^2_{(x,h)} = dim\ H.$$

As a consequence, $T_{(x,h)}\theta$ has maximal rank:

$$rank\ T_{(x,h)}\theta = rank\ \mathbb{1} + rank\ T^2_{(x,h)} = dim\ X + dim\ H$$

which impllies that $T_{(x,h)}$ is injective. As a consequence θ is an immersion due to Proposition 2.9.

ii) *Applying Godement's Theorem*: Consider the commutative diagram

$$\begin{tikzcd}
X \times H \arrow[rr, "\theta"] \arrow[dr, "\tilde{\theta}"] \arrow[ddr, "pr_1"'] & & X \times X \arrow[ddl, "pr_2"] \\
& R \arrow[ur, "j_R"'] & \\
& X &
\end{tikzcd}$$

with the injection

$$j_R : R \hookrightarrow X \times X,$$

providing $R \subset X \times X$ with the subspace topology, and the restriction

$$\tilde{\theta} : X \times H \to R, \tilde{\theta}(x,h) := \theta(x,h) = (x.h,x).$$

Next we verify that both assumptions of Godements Theorem 2.15 are satisfied.

- First consider the upper triangle: The map θ is continuous and proper. Therefore the same holds for the restriction $\tilde{\theta}$ which is also bijective by definition. Hence $\tilde{\theta}$ is a homeomorphism according to Remark 2.27. We provide R with the analytic structure transferred from $X \times H$ via $\tilde{\theta}$. Then $\tilde{\theta}$ is an analytic isomorphism. The composition

  $$j_R = \theta \circ \tilde{\theta}^{-1} : R \to X \times X$$

  is an immersion due to part i), i.e.

  $$R \subset X \times X$$

  is a submanifold.

- Secondly, consider the lower triangle: The map

$$pr_2|R = pr_2 \circ j_R = pr_2 \circ \theta \circ \tilde{\theta}^{-1} = pr_1 \circ \tilde{\theta}^{-1} : R \to X$$

is a submersion.

According to Theorem 2.15 an analytic structure on $X/R = X/H$ exists such that

$$p : X \to X/R$$

is a submersion.

iii) *Bundle charts*: In order to show that p is an analytic H-bundle we construct an atlas of bundle charts for p: Consider an arbitrary but fixed point $y \in X/H$. Because p is a submersion an open subset $U \subset X/H$ and on U an analytic section against p exist, i.e. an analytic map

$$s : U \to p^{-1}(U)$$

with $p \circ s = id_U$.

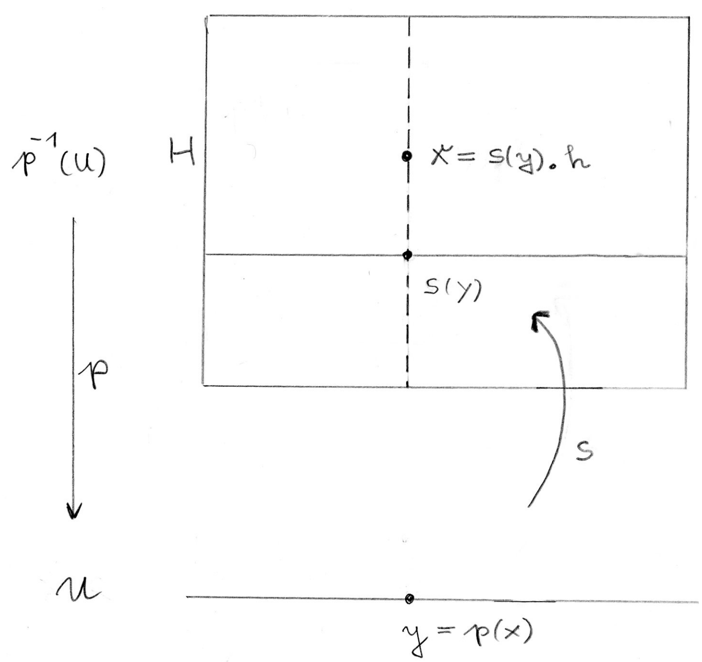

Fig. 2.2 Section against p

We claim that the analytic map

$$\psi : U \times H \to p^{-1}(U), (y,h) \mapsto s(y).h,$$

has an inverse ϕ which is a chart: Define the map

$$\phi : p^{-1}(U) \to U \times H, x \mapsto (p(x),h).$$

Here $h \in H$ is the unique element with

$$x = s(p(x)).h$$

The element h is well-defined because the group operation is free. Both maps ψ and ϕ are inverse to each other, because

$$(\phi \circ \psi)(y,h) = \phi(s(y).h) = (y,h)$$

$$(\psi \circ \phi)(x) = \psi(p(x),h) = s(p(x).h) = x.$$

In order to prove that ϕ is analytic, we consider the composition of the two maps

$$(s \times id) \circ \phi = [p^{-1}(U) \xrightarrow{\phi} U \times H \xrightarrow{s \times id} p^{-1}(U) \times H]$$

and

$$\theta|(p^{-1}(U) \times H) : p^{-1}(U) \times H \to p^{-1}(U) \times p^{-1}(U),$$

an injective immersion due to part i). In particular, $p^{-1}(U) \times H$ can be considered via the restriction of θ a submanifold of $p^{-1}(U) \times p^{-1}(U)$.

The composition of these maps

$$\theta|(p^{-1}(U) \times H) \circ (s \times id) \circ \phi$$

is analytic: It maps

$$x = s(p(x)).h \mapsto (p(x),h) \mapsto (s(p(x)),h) \mapsto (s(p(x)).h, s(p(x))) = (x, s(p(x))).$$

As a consequence the map

$$(s \times id) \circ \phi$$

is continuous and even analytic according to Poposition 2.10. Therefore also

$$\phi = (p \times id) \circ ((s \times id) \circ \phi)$$

is analytic.

iv) *Transition maps*: We finally show that the analytic H-bundle p is a H-principal bundle. We choose an open covering $(U_i)_{i \in I}$ of Y with analytic sections

$$s_i : U_i \to p^{-1}(U_i)$$

against p und define the charts ϕ_i alike to part iii). We have to check the form of the transition maps
$$\phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times H \to (U_i \cap U_j) \times H.$$
Consider an arbitrary, but fixed point $y \in U_i \cap U_j$. If for an element $h_i \in H$

$$(\phi_j \circ \phi_i^{-1})(y, h_i) = (\phi_j \circ \psi_i)(y, h_i) = \phi_j(s_i(y).h_i) = \phi_j(s_j(y).h_j) = (y, h_j)$$

then $h_j \in H$ is the unique element with

$$s_j(y).h_j = s_i(y).h_i.$$

If for a second element $h'_i \in H$

$$(\phi_j \circ \phi_i^{-1})(y, h'_i) = (y, h'_j)$$

then $h'_j \in H$ is the unique element with

$$s_j(y).h'_j = s_i(y).h'_i.$$

From
$$h'_i = h_i \cdot (h_i^{-1} \cdot h'_i)$$
follows

$$s_j(y).h'_j = s_i(y).h'_i = (s_i(y).h_i).(h_i^{-1} \cdot h'_i) = (s_j(y).h_j).(h_i^{-1} \cdot h'_i) = s_j(y).(h_j \cdot h_i^{-1} \cdot h'_i)$$

i.e.
$$h'_j = h_j \cdot h_i^{-1} \cdot h'_i \text{ or } h'_j \cdot h_i'^{-1} = h_j \cdot h_i^{-1}.$$
As a consequence the definition

$$h_{ji}(y) := h_j \cdot h_i^{-1} = h'_j \cdot h_i'^{-1}$$

does not depend on the choice of h_i or of h'_i. It satisfies

$$h_{ji}(y) \cdot h_i = h_j.$$

Varying $y \in U_i \cap U_j$ defines a map

$$h_{ji} : U_i \cap U_j \to H, y \mapsto h_{ji}(y),$$

with
$$(\phi_j \circ \phi_i^{-1})(y, h) = (y, h_{ji}(y) \cdot h), \ q.e.d.$$

Remark 2.30 (Principal bundles and group operations). Consider a Lie group H.

1. Due to Theorem 2.29 a proper and free right H-operation

$$\phi : X \times H \to X$$

induces a H-principal bundle

$$p : X \to X/H.$$

2. Conversely: A H-principal bundle

$$p : X \to Y$$

induces a right H-operation

$$\phi : X \times H \to X, (x,g) \mapsto x.g,$$

such that $Y \simeq X/H$ and $p : X \to Y$ is the canonical projection:

Consider an arbitrary but fixed point $x \in X$ and set $y := p(x) \in Y$. If

$$\phi_i : p^{-1}(U_i) \to U_i \times H$$

is a bundle chart of p and $x \in p^{-1}(U_i)$ with $\phi_i(x) = (y, h_i)$ then define

$$x.g := \phi_i^{-1}(y, h_i \cdot g).$$

The operation is well-defined: If also

$$\phi_j : p^{-1}(U_j) \to U_j \times H$$

is a bundle chart of p with $x \in p^{-1}(U_j)$ then a map

$$h_{ji} : U_i \cap U_j \to H$$

exists such that

$$\phi_j(x) = (y, h_j) = (y, h_{ji}(y) \cdot h_i).$$

Therefore

$$h_j \cdot g = (h_{ji}(y) \cdot h_i) \cdot g = h_{ji}(y) \cdot (h_i \cdot g)$$

which implies

$$\phi_j^{-1}(y, h_j \cdot g) = \phi_i^{-1}(y, h_i \cdot g).$$

Apparently, the group operation ϕ is free and Y equals the orbit space X/H of the group operation ϕ.

The group operation ϕ is also proper: The proof reduces to proving that for any bundle chart

$$p^{-1}(U) \xrightarrow{\sim} U \times H$$

the restricted operation

$$p^{-1}(U) \times H \to p^{-1}(U) \times p^{-1}(U), (x,g) \mapsto (x.g, x),$$

is proper. Therefore we may assume that the principal bundle p is a product, i.e.

$$X = Y \times H$$

and

$$\phi : (Y \times H) \times H \to Y \times H, ((y,h),g) \mapsto (y, h \cdot g)$$

with

$$\theta : (Y \times H) \times H \to (Y \times H) \times (Y \times H), ((y,h),g) \mapsto ((y, h \cdot g), (y,h)).$$

Up to permutation of two factors the latter map is the product

$$Y \times (H \times H) \xrightarrow{j \times \theta_H} (Y \times Y) \times (H \times H)$$

of the proper map

$$j : Y \to Y \times Y, y \mapsto (y,y),$$

and the map

$$\theta_H : H \times H \to H \times H, (h,g) \mapsto (h \cdot g, h).$$

The map θ_H is proper because for a compact set $K \subset H \times H$

$$(h,g) \in \theta_H^{-1}(K) \implies h \in pr_2(K), h \cdot g \in pr_1(K).$$

Using $g = h^{-1} \cdot (h \cdot g) \in H$ we obtain

$$\theta_H^{-1}(K) \subset pr_2(K) \times (pr_2(K)^{-1} \cdot pr_1(K)),$$

The latter set is the product of two compact subsets of H, hence itself a compact set, and its subset $\theta_H^{-1}(K)$ is closed, hence also compact.

Note: Here $pr_2(K)^{-1} \subset H$ denotes the inverse of the set $pr_2(K) \subset H$ with respect to the group structure of H.

As an application of Theorem 2.29 we prove that each orbit of a transitive group operation is a principal bundle with fibre the corresponding isotropy group.

Proposition 2.31 (Orbits as principal bundles). *Consider a connected Lie group G and a transitive analytic G-left operation on an analytic Hausdorff manifold X*

$$\phi : G \times X \to X.$$

Then for any point $x \in X$ its orbit map

$$f := \phi_x : G \to X, g \mapsto g.x,$$

is a G_x-principal bundle with G_x the isotropy group of $x \in X$.

Proof. The isotropy group $H := G_x$ is a Lie subgroup of G according to Proposition 2.21.

i) *Analytic homogenous space*: We consider the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & X \\ G/H \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

with the induced map $\overline{f}$.

First we show that $\overline{f} : G/H \to X$ is an analytic isomorphism: The connected Lie group G is σ-compact according to Lemma 2.20. Therefore, the map $\overline{f}$ is a homeomorphism according to Theorem 1.16.

We claim that $\overline{f}$ is even an analytic isomorphism: The map p is a submersion according to Proposition 2.24. Therefore $\overline{f}$ is analytic according to Proposition 2.10. The map p is open according to Proposition 1.5 which implies that also the analytic map

$$f = \overline{f} \circ p$$

is open. The map f has constant rank according to Corollary 2.22. Being an open map f is a submersion according to the Rank Theorem 2.14. Proposition 2.9 implies that also $\overline{f}$ is a submersion. Being a bijective map $\overline{f}$ is also an immersion, i.e. a local isomorphism and even an analytic isomorphism.

ii) *Principal bundle*: According to part i) we have to show that p is a H-principal bundle: The map p derives as the canonical projection of the right H-operation

$$G \times H \to G, (g,h) \mapsto g \cdot h.$$

We prove that the corresponding map

$$\theta : G \times H \to G \times G, (g,h) \mapsto (g \cdot h, g),$$

is proper. For any subset $K \subset G \times G$:

$$(g,h) \in \theta^{-1}(K) \iff \theta(g,h) = (g.h, g) \in K.$$

Using the representation $h = g^{-1} \cdot (g \cdot h)$ we obtain

$$\theta^{-1}(K) \subset pr_2(K) \times ((pr_2(K)^{-1} \cdot pr_1(K)) \cap H).$$

If K is compact then both factors of the latter product are also compact. Hence $\theta^{-1}(K)$ is compact, being a closed subset of a compact set. Theorem 2.29 proves that $\pi : G \to G/H$ is a H-principal bundle, q.e.d.

Specializing Proposition 2.31 gives an important result about morphisms between Lie groups.

Theorem 2.32 (Surjective Lie group morphisms are principal bundles). *Consider a connected Lie group G. Any surjective morphism of Lie groups*

$$f : G \to G'$$

is a H-principal bundle with $H := ker\ f$.

Proof. The map f is the orbit map ϕ_e of the neutral element $e \in G'$ of the left G-operation

$$\phi : G \times G' \to G', (g,g') \mapsto f(g) \cdot g'.$$

The operation is transitive because f is surjective. The isotropy group is $G_e = ker\ f$. Therefore Proposition 2.31 proves the claim, q.e.d.

As a further application of Theorem 2.29 we realize the projective spaces and more general the Grassmannians as the base manifolds of analytic principal bundles.

Example 2.33 (Complex projective space and Grassmannians).

1. *Projective space*: The n-dimensional projective space $\mathbb{P}^n(\mathbb{K})$ is the orbit space of the right operation of the Lie group $H := (\mathbb{K}^*, \cdot)$ on the analytic manifold $X := \mathbb{K}^{n+1} \setminus \{0\}$

$$\Phi : X \times H \to X, (x, \lambda) \mapsto x \cdot \lambda.$$

Apparently the operation is analytic and free. We prove that the operation is proper, i.e. we have to show properness of the map

$$\theta : X \times H \to X \times X, (x,h) \mapsto (x.h, x) :$$

For any two compact subsets $K_i \subset X, i = 1,2$, constants $K, m > 0$ exist such that

$$y \in K_1 \implies \|y\| \leq K \text{ and } x \in K_2 \implies m \leq \|x\|.$$

The second estimation uses the fact that $0 \notin X$. The set $\theta^{-1}(K_1 \times K_2) \subset X \times X$ is closed. It is also bounded because

$$(x, \lambda) \in \theta^{-1}(K_1 \times K_2) \implies x \in K_2, \lambda \cdot x \in K_1 \implies |\lambda| = \frac{\|\lambda \cdot x\|}{\|x\|} \leq \frac{K}{m}.$$

We obtain
$$\theta^{-1}(K_1 \times K_2) \subset K_2 \times B_{K/m}(0).$$

As a consequence, Theorem 2.29 implies that the projective space $\mathbb{P}^n(\mathbb{K})$ is the base manifold of a $\mathbb{K}^*$-principal bundle

$$\mathbb{K}^* \hookrightarrow \mathbb{K}^{n+1} \setminus \{0\} \xrightarrow{p} \mathbb{P}^n(\mathbb{K}).$$

One uses the notation

$$(z_0 : ... : z_n) := [(z_0, ..., z_n)] \ (\textit{homogeneous coordinate})$$

for the equivalence classes of $\mathbb{P}^n(\mathbb{K}) := (\mathbb{K}^{n+1} \setminus \{0\})/\mathbb{K}^*$. A point

$$(z_0 : ... : z_n) \in \mathbb{P}^n(\mathbb{K})$$

determines the values $z_0, ..., z_n$ only up to a common factor $\lambda \in \mathbb{K}^*$.

2. *Grassmann manifold*: Generalizing the concept of projective spaces we consider the Lie group
$$H := GL(k, \mathbb{K}),$$

the manifold

$$X := M_k(N \times k, \mathbb{K}) := \{A \in M(N \times k, \mathbb{K}) : \textit{rank } A = k\},$$

and the analytic right operation

$$\Phi : X \times H \to X, (A, B) \mapsto A \cdot B.$$

Note: The matrix product $A \cdot B$ is the reason for considering the operation of H as a right-operation.

i) The operation is free: If $A \cdot B = A$ then the linear maps corresponding to these matrices
$$\mathbb{K}^k \xrightarrow{f_B} \mathbb{K}^k \xrightarrow{f_A} \mathbb{K}^N$$

satisfy
$$f_A \circ f_B = f_A.$$

Hence for any $x \in \mathbb{K}^k$ we have $f_A(f_B(x)) = f_A(x)$. From the rank condition *rank* $A = k$ derives the injectivity of f_A. Hence $f_B(x) = x$. Therefore $f_B = id$ or

$$B = \mathbb{1} \in GL(k, \mathbb{K}).$$

Note: A different proof for the fact that the operation is free, can be obtained from the pseudo-inverse or *Moore-Penrose-inverse* of A: The rank condition *rank* $A = k$ implies that the matrix $A^* \cdot A$ is invertible and

$$A^+ := (A^* \cdot A)^{-1} \cdot A^*$$

is the pseudo-inverse of A, i.e. it satisfies

$$A^+ \cdot A = \mathbb{1}.$$

Therefore multiplying both sides of $A \cdot B = A$ by A^+ from the left implies

$$B = A^+ \cdot A \cdot B = A^+ \cdot A = \mathbb{1}.$$

ii) The operation

$$\theta : X \times H \to X \times X, (A,B) \mapsto (A \cdot B, A),$$

is proper: Consider a compact set $K = K_1 \times K_2 \subset X \times X$ and a sequence $(A_\nu, B_\nu)_{\nu \in \mathbb{N}}$ of elements $(A_\nu, B_\nu) \in \theta^{-1}(K)$ such that

$$\lim_{\nu\to\infty} A_\nu \cdot B_\nu = C \in K_1 \textit{ and } \lim_{\nu\to\infty} A_\nu = A \in K_2.$$

We have to show the existence of a matrix $B \in GL(k,\mathbb{K})$ such that

$$B = \lim_{\nu\to\infty} B_\nu.$$

The endomorphism f_A corresponding to the matrix $A \in M_k(N \times k, \mathbb{K})$ is injective. Therefore a matrix $A' \in M(k \times N, \mathbb{K})$ exists with $f_{A'} \circ f_A = id$, i.e.

$$A' \cdot A = \mathbb{1} \in GL(k,\mathbb{K}).$$

Because the determinant function is continous we may assume $det(A' \cdot A_\nu) \neq 0$, i.e.

$$A' \cdot A_\nu \in GL(k,\mathbb{K})$$

for all $\nu \in \mathbb{N}$. We obtain

$$\lim_{\nu\to\infty} B_\nu = \lim_{\nu\to\infty} [(A' \cdot A_\nu)^{-1} \cdot (A' \cdot A_\nu) \cdot B_\nu]$$

$$= \lim_{\nu\to\infty} [(A' \cdot A_\nu)^{-1}] \cdot A' \cdot \lim_{\nu\to\infty} [A_\nu \cdot B_\nu] = A' \cdot C =: B \in M(k \times k, \mathbb{K}).$$

From

$$A \cdot B = \lim_{\nu\to\infty} A_\nu \cdot \lim_{\nu\to\infty} B_\nu = \lim_{\nu\to\infty} (A_\nu \cdot B_\nu) = C \in M_k(N \times k, \mathbb{K})$$

follows $rank\ C = k = rank\ B$, i.e. $B \in GL(k,\mathbb{K})$.

Theorem 2.29 implies that the orbit space X/H is the basis of a $GL(k,\mathbb{K})$-principal bundle

$$GL(k,\mathbb{K}) \hookrightarrow M_k(N \times k, \mathbb{K}) \xrightarrow{p} X/H.$$

iii) From a geometric point of view the orbit space X/H classifies all k-dimensional subspaces of an N-dimensional vector space: The map

$$M_k(N \times k, \mathbb{K}) \to G(k,N)(\mathbb{K}), \begin{pmatrix} a_1 & \dots & a_k \\ | & \dots & | \\ | & \dots & | \end{pmatrix} \mapsto span_{\mathbb{K}} < a_1, ..., a_k >,$$

maps the column vectors to their span and induces a bijective map from the quotient manifold

$$X/H = M_k(N \times k, \mathbb{K})/GL(k, \mathbb{K})$$

to the *Grassmannian* $G(k,N)(\mathbb{K})$ of k-dimensional subspaces of $\mathbb{K}^N$. The particular case $k = 1$ is the projective space $\mathbb{P}^{N-1}(\mathbb{K})$.

As a further application of Proposition 2.31 we compute the fundamental group of the classical groups introduced in Example 1.17.

Corollary 2.34 (Fundamental group of selected classical groups). *All of the following groups are connected. Their fundamental groups are:*

1. $SL(n,\mathbb{R}),\ SO(n,\mathbb{R})$:

$$\pi_1(SL(n,\mathbb{R}) \simeq \pi_1(SO(n,\mathbb{R})) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}_2, & n \geq 3 \end{cases}$$

2. $U(n)$: For all $n \geq 1$

$$\pi_1(U(n)) = \mathbb{Z}.$$

3. $SL(n,\mathbb{C}),\ SU(n)$: For all $n \geq 1$

$$\pi_1(SL(n,\mathbb{C})) \simeq \pi_1(SU(n)) = \{0\}.$$

Proof. Connectedness has been shown in Lemma 1.18.

ad 1) i) First, the isomorphism of topological groups

$$SO(2,\mathbb{R}) \xrightarrow{\simeq} S^1, \begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix} \mapsto e^{i\theta} = cos\ \theta + i \sin\ \theta,$$

induces the isomorphy

$$\pi_1(SO(2,\mathbb{R})) \simeq \pi_1(S^1) \simeq \mathbb{Z}$$

according to Corollary 1.27.

ii) According to Example 1.17, part 1, and Proposition 2.31 we have an anaytic, in particular continuous fibre bundle for $n \geq 2$

$$SO(n-1,\mathbb{R}) \hookrightarrow SO(n,\mathbb{R}) \to S^{n-1}.$$

From its long exact homotopy sequence according to Theorem 1.33 we consider the section

$$\pi_2(S^{n-1}) \to \pi_1(SO(n-1,\mathbb{R})) \to \pi_1(SO(n,\mathbb{R})) \to \pi_1(S^{n-1})$$

For $n \geq 4$ we have $\pi_2(S^{n-1}) = \pi_1(S^{n-1}) = 0$ due to Proposition 1.28 and as a consequence

$$\pi_1(SO(n-1,\mathbb{R})) \simeq \pi_1(SO(n,\mathbb{R})).$$

iii) We are left to show

$$\pi_1(SO(3,\mathbb{R})) = \mathbb{Z}_2.$$

This claim follows from the homeomorphy

$$SO(3,\mathbb{R}) \simeq \mathbb{P}^3(\mathbb{R}),$$

see to Proposition 1.31, and the isomorphy

$$\pi_1(\mathbb{P}^3(\mathbb{R})) \simeq \mathbb{Z}_2$$

according to Lemma 1.30.

iv) Consider a matrix $A \in SL(n,\mathbb{R})$. In order to orthogonalize A one considers the rows $a_1,...,a_n$ of A as a basis of $\mathbb{R}^n$ and applies the Gram-Schmidt algorithm: Defining $v_1 := a_1$ and

$$v_i := -\left(\sum_{k=1}^{i-1} < a_i, v_k > \cdot \frac{v_k}{\|v_k\|^2}\right) + a_i,\ i = 2,...,n,$$

one obtains a matrix

$$\delta := \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \in GL(n,\mathbb{R})$$

such that the rows $v_1,...,v_n$ of the product

$$v := \delta \cdot A$$

form an orthogonal basis. Multiplying from the left with a diagonal matrix with entries the length of the basis vectors gives a matrix

$$D \in \mathscr{D} := \{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix} \in SL(n,\mathbb{R}) : \lambda_i > 0, i = 1,...,n\}$$

such that

$$D \cdot A \in SO(n).$$

One obtains the homeomorphy

$$SL(n,\mathbb{R}) \simeq \mathbb{R}^{(n^2-n)/2} \times (\mathbb{R}^*)^{n-1} \times SO(n) \simeq \mathbb{R}^{((n^2+n)/2)-1} \times SO(n).$$

Because the first factor is contractible we get

$$\pi_1(SL(n,\mathbb{R})) \simeq \pi_1(SO(n)).$$

ad 2) According to Proposition 2.31 and Example 1.17, part 2, we have for $n \geq 2$ the analytic, in particular fibre bundle

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}.$$

Its long exact homotopy sequence contains the section

$$\pi_2(S^{2n-1}) \to \pi_1(U(n-1)) \to \pi_1(U(n)) \to \pi_1(S^{2n-1}).$$

From Proposition 1.28 we obtain $\pi_2(S^{2n-1}) = \pi_1(S^{2n-1}) = \{0\}$ for $n \geq 2$. Therefore

$$\pi_1(U(n-1)) \simeq \pi_1(U(n)).$$

And $U(1) \simeq S^1$ implies for all $n \geq 1$

$$\pi_1(U(n)) \simeq \pi_1(U(1)) \simeq \mathbb{Z}.$$

ad 3) i) For $n \geq 2$ we use the fibre bundle representation from Example 1.17, part 2,

$$SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$$

and the section of its long exact homotopy sequence

$$\pi_2(S^{2n-1}) \to \pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(S^{2n-1})$$

with $\pi_2(S^{2n-1}) = \pi_1(S^{2n-1}) = \{0\}$ due to Proposition 1.28. We obtain for $n \geq 2$

$$\pi_1(SU(n-1)) \simeq \pi_1(SU(n)).$$

Because $SU(1) = \{*\}$ we obtain for $n \geq 1$

$$\pi_1(SU(n)) \simeq \pi_1(SU(1)) = \{0\}.$$

ii) Analogously to part 1, iv) one obtains a homeomorphism

$$SL(n,\mathbb{C}) \simeq \{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix} \in SL(n,\mathbb{C}) : \lambda_i \in \mathbb{R}^*_+, 0, i = 1, ..., n\} \times SU(n).$$

As a consequence $SL(n,\mathbb{C}) \simeq \mathbb{R}^{n^2-1} \times SU(n)$. We obtain

$$\pi_1(SL(n,\mathbb{C})) \simeq \pi_1(SU(n)),\ q.e.d.$$

Any analytic manifold, in particular any Lie group is locally path-connected and semilocal 1-connected. Therefore the results from Section 1.3 concerning covering spaces apply to Lie groups.

Proposition 2.35 (Characterization of the universal covering of Lie groups). *Consider a surjective morphims of Lie groups*

$$p : \tilde{G} \to G$$

with $\tilde{G}$ connected and simply connected. Assume that the Lie subgroup

$$H := ker\ f \subset \tilde{G}$$

is discrete, i.e. it carries the discrete topology.

Then p is the universal covering projection of G and $\pi_1(G,e) \simeq H$.

Proof. According to Theorem 2.32 the map p is a H-principal bundle, in particular a covering projection. According to Proposition 1.25 the map p is the universal covering projection of G. According to Theorem 1.33 the long exact homotopy sequence of p contains the exact sequence

$$\{e\} = \pi_1(\tilde{G},e) \to \pi_1(G,e) \to \pi_0(H) \simeq H \to \pi_0(\tilde{G}) = \{*\}$$

Theorem 2.36 (The universal covering of a Lie group is a Lie group). *For any connected Lie group G the universal covering projection exists as a morphism of Lie groups*

$$p : \tilde{G} \to G$$

with $\tilde{G}$ a unique connected and simply connected Lie group G. It satisfies

$$ker\ p \simeq \pi_1(G,e),$$

and $ker\ p \subset \tilde{G}$ is a discrete subgroup.

Proof. According to Proposition 1.25 in the topological context a unique universal covering projection
$$p : \tilde{G} \to G$$
exists as a continuous map. We recall the relevant properties of p, see also [12, Kap. I, § 5].

i) As a set
$$\tilde{G} = \{(x,[\alpha]) : x \in G, \alpha : I \to G \; continous, \alpha(0) = e, \alpha(1) = x\}.$$
Here $I = [0,1] \subset \mathbb{R}$ and $[\alpha]$ is the homotopy class of the path α. One has
$$p : \tilde{G} \to G, (x,[\alpha]) \mapsto x.$$
The group multiplication in $\tilde{G}$ derives from the group multiplication in G:
$$m_{\tilde{G}} : \tilde{G} \times \tilde{G} \to \tilde{G}, m_{\tilde{G}}((x,[\alpha]),(y,[\beta])) := (x \cdot y, [\alpha \cdot \beta]).$$
The map is independent from the choice of repesentatives. The group mutiplication is associative. The neutral element is the pair $(e,[\alpha_e])$ with $\alpha_e(t) \equiv e$. The inverse element is $(x,[\alpha])^{-1} = (x,[\alpha^{-1}])$.

ii) Concerning the topology of $\tilde{G}$: For $(x,[\alpha]) \in \tilde{G}$ and U an open, connected, and simply connected neighbourhood of x in G define the set
$$[U,[\alpha]] := \{(y,[\beta]) \in \tilde{G} : y \in U, \beta = \alpha * \gamma \; with \; a \; path \; \gamma : I \to U, \alpha(0) = x, \alpha(1) = y\}$$

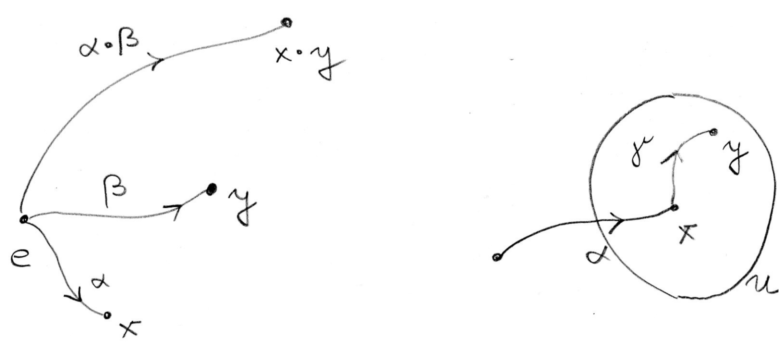

Fig. 2.3 Universal covering space

The set of these sets $[U,[\alpha]]$ is a neighbourhood basis of $(x,[\alpha])$ in $\tilde{G}$. Then $\tilde{G}$ is simply connected.

iii) Analytic structure: Because $p : \tilde{G} \to G$ is a local homeomorphism a unique analytique structure $\tilde{\mathscr{A}}$ exists on $\tilde{G}$ such that

$$p : (\tilde{G}, \tilde{\mathscr{A}}) \to G$$

is a local analytic isomorphism. In addition

$$p(m_{\tilde{G}}(g_1, g_2)) = m_G(p(g_1), p(g_2)).$$

The commutativity of the diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{m_{\tilde{G}}} & \tilde{G} \\ {\scriptstyle p \times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{m_G} & G \end{array}$$

proves that the multiplication $m_{\tilde{G}}$ is analytic.

iv) Kernel of p: We have

$$ker\ p = p^{-1}(e) = \{(e, [\alpha]) : [\alpha] \in \pi_1(G, e))\} \simeq \pi_1(G, e).$$

Because p is a covering projection the fibre $p^{-1}(e)$ is a discrete topological space, q.e.d.

Definition 2.37 (Spin group). The universal covering spaces of the special orthogonal groups $SO(n, \mathbb{R})$ are named *spin groups* $Spin(n), n \in \mathbb{N}^*$.

Remark 2.38 (Spin groups). For $n \geq 3$ the universal covering projections of $SO(n, \mathbb{R})$ are real-analytic principal $\mathbb{Z}_2$-bundles

$$\mathbb{Z}_2 \hookrightarrow Spin(n) \to SO(n, \mathbb{R})$$

according to Proposition 2.34, Theorem 2.36, and Theorem 2.32. The low-dimensional spin groups are:

$$Spin(1) = \{e\}, Spin(2) = (\mathbb{R}, +), Spin(3) = SU(2),$$

$$Spin(4) = SU(2) \times SU(2), Spin(5) = U(2, \mathbb{H})$$

with $\mathbb{H}$ the division ring of quaternions.

The representation $Spin(3) = SU(2)$ has been studied in detail in Example **??**.

Chapter 3
The functor Lie from Lie groups to Lie algebras

The law of multiplication of a Lie group G is already determined in a neigbourhood of the neutral element $e \in G$. Therefore one considers a chart ϕ of G around e and translates the multiplication m_G in a neighbourhood V of e to a formal group structure: An analytic function $F(X,Y)$ defined on the open subset $\phi(V) \times \phi(V)$ of a number space. From the second order terms of F derives a Lie bracket. It measures the second order derivation of m_G from being Abelian. The bracket provides the tangent space with the structure of a Lie algebra $(T_eG, [-,-]_F)$.

One checks that the construction does not depend on the choice of the chart ϕ. Therefore one has attached to G a Lie algebra *Lie G*. This attachment is compatible with morphisms of Lie groups. It defines a covariant functor *Lie* from the category of Lie groups to the category of Lie algebras.

3.1 The Lie algebra of a Lie group

In the present section

$$\mathbb{K} < Z > = \mathbb{K} < Z_1, ..., Z_m >$$

denotes the $\mathbb{K}$-algebra of convergent power series in m variables with expansion point $0 \in \mathbb{K}^m$

$$f(X) = \sum_{\nu \in \mathbb{N}^m} c_\nu \cdot Z^\nu = \sum_{\nu = (\nu_1, ..., \nu_m) \in \mathbb{N}^m} c_\nu \cdot Z_1^{\nu_1} \cdot ... \cdot Z_m^{\nu_m}, c_\nu \in \mathbb{K}.$$

Moreover,

$$\mathfrak{m} < Z > := \{ f \in \mathbb{K} < Z > : \ f(0) = 0 \}$$

denotes the maximal ideal of all convergent power series vanishing at $0 \in \mathbb{K}^m$.

For an n-dimensional $\mathbb{K}$-Lie group G we want to mirror the group multiplication of G by a corresponding multiplication on the set of n-tuples of convergent power series from

$$\mathfrak{m} < X >^{\oplus n},$$

the direct sum of n copies of $\mathfrak{m} < X >$. This task can be achieved by choosing a chart around the neutral element $e \in G$. We express the multiplication using this chart, but show that the result does not depend on the choice of the chart.

Lemma 3.1 (Group multiplication in coordinates). *Consider an n-dimensional $\mathbb{K}$-Lie group G with neutral element $e \in G$ and choose a chart of G around e*

$$\phi : U \to \mathbb{K}^n,$$

and an open neighbourhood V of e with

$$V \cdot V \subset U.$$

Then the group multiplication $m : V \times V \to U$ defines an element

$$F = (F_1, ..., F_n)^\top \in \mathbb{K} < X, Y >^{\oplus n}$$

with respect to the variables

$$X = (X_1, ..., X_n), Y = (Y_1, ..., Y_n),$$

such that each power series $F_j \in \mathbb{K} < X, Y >$, $j = 1, ..., n$, converges on $\phi(V) \times \phi(V)$ and the following diagram commutes:

$$\begin{tikzcd}
V \times V \arrow[r, "m"] \arrow[d, "\phi \times \phi"', "\simeq"] & U \arrow[d, "\phi"', "\simeq"] \\
\phi(V) \times \phi(V) \arrow[r, dashed, "F"] & \phi(U)
\end{tikzcd}$$

The element $F \in \mathbb{K} < X, Y >^{\oplus n}$ has the following properties:

1. Zero order*: $F(0,0) = 0$, i.e. $F \in \mathfrak{m} < X, Y >^{\oplus n}$.*

2. First order*: $F(X,0) = X \in \mathfrak{m} < X >^{\oplus n}$, $F(0,Y) = Y \in \mathfrak{m} < Y >^{\oplus n}$.*

3. Associativity*: $F(F(X,Y),Z) = F(X, F(Y,Z)) \in \mathfrak{m} < X, Y, Z >^{\oplus n}$.*

Proof. 1. Because $e \in G$ is the neutral element we have $m(e,e) = g$ which implies $F(0,0) = 0$:

$$F \in \mathfrak{m} < X, Y >^{\oplus n}.$$

2. More general, for all $g \in V$

$$m(g,e) = m(e,g) = g$$

implies $F(X,0) = X$ *and* $F(0,Y) = Y$.

3. Due to part 1), for any two elements $U,V \in \mathbb{K} < X >^{\oplus n}$ the composition
$$F(U(X),V(X))$$
is well defined. As a consequence, both sides of the claim are well-defined. The law of associativity follows from the corresponding property of the group multiplication
$$m(m(g_1,g_2),g_3) = m(g_1,m(g_2,g_3))$$
for elements $g_i, i = 1,2,3$, in a suitable open neighbourhood of e in G, q.e.d.

In the following we consider the tuple
$$F = (F_1,...,F_n) \in \mathfrak{m} < X,Y >^{\oplus n}$$
of convergent power series, which derives from the Lie group multiplication, as a separate object, a *formal group structure*, and investigate its properties.

Definition 3.2 (Formal group). Using the notation $X = (X_1,...,X_n), Y = (Y_1,...,Y_n)$ a *formal group structure* on $\mathfrak{m} < X >^{\oplus n}$ is an element
$$F = (F_1,...,F_n)^\top \in \mathfrak{m} < X,Y >^{\oplus n}$$
which satisfies:

- $F(X,0) = X \in \mathfrak{m} < X >^{\oplus n}$ *and* $F(0,Y) = Y \in \mathfrak{m} < Y >^{\oplus n}$
- $F(X,F(Y,Z)) = F(F(X,Y),Z) \in \mathfrak{m} < X,Y,Z >^{\oplus n}$.

The map defines a multiplication on $\mathfrak{m} < X >^{\oplus n}$
$$* : \mathfrak{m} < X >^{\oplus n} \times \mathfrak{m} < X >^{\oplus n} \rightarrow \mathfrak{m} < X >^{\oplus n}, U * V := F(U,V).$$

Lemma 3.3 (Inversion in a formal group). *Consider a formal group structure $F \in \mathfrak{m} < X,Y >^{\oplus n}$.*

Any element $U \in \mathfrak{m} < X >^{\oplus n}$ has a unique element $U^- \in \mathfrak{m} < Y >^{\oplus n}$ satisfying
$$U * U^- = U^- * U = 0 \in \mathfrak{m} < X >^{\oplus n} .$$
The element U^{-1} is named the inverse *of U.*

Proof. The partial derivatives at $(0,0) \in \mathbb{K}^n \times \mathbb{K}^n$ are
$$D^1F(0,0) = D^2F(0,0) = id.$$

According to the Implicit Function Theorem unique elements $\Phi, \Psi \in \mathfrak{m} < X >^{\oplus n}$ exist with

$$0 = F(\Phi(X), U(X)) = F(U(X), \Psi(X)).$$

Accordingly the element $\Phi \in \mathfrak{m} < X >^{\oplus n}$ is a left-inverse of U and the element $\Psi \in \mathfrak{m} < X >^{\oplus n}$ is a right-inverse of U.

Invoking in addition the two required properties of F we obtain

$$\Phi(X) = F(\Phi(X), 0) = F(\Phi(X), F(U(X), \Psi(X))) =$$

$$= F(F(\Phi(X), U(X)), \Psi(X)) = F(0, \Psi(X)) = \Psi(X),$$

i.e. the left-inverse equals the right-inverse, q.e.d.

Proposition 3.4 (Jacobi identity of a formal group structure). *A formal group structure*

$$F \in \mathfrak{m} < X, Y >^{\oplus n}$$

has the power series expansion

$$F(X,Y) = X + Y + B(X,Y) + \sum_{\alpha,\beta \in \mathbb{N}^n, |\alpha|+|\beta| \geq 3, |\alpha|,|\beta| \geq 1} c_{\alpha\beta} \cdot X^{\alpha} \cdot Y^{\beta},\ c_{\alpha\beta} \in \mathbb{K}^n.$$

Referring to the quadratic terms of F we define the alternating $\mathbb{K}$-bilinear map

$$[-,-]_F := B(X,Y) - B(Y,X) : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n.$$

Here we consider $B(X,Y)$ and $B(Y,X)$ with respect to the variables X and Y as maps on $\mathbb{K}^n \times \mathbb{K}^n$. Then:

1. Inverse: $X^{-} = -X + B(X,X) + O(3) \in \mathfrak{m} < X >^{\oplus n}$

2. Conjugation: $X^{Y} := Y^{-} * X * Y = X + [X,Y]_F + O(3) \in \mathfrak{m} < X, Y >^{\oplus n}$

3. $(X,Y) := X^{-} * Y^{-} * X * Y = [X,Y]_F + O(3) \in \mathfrak{m} < X, Y >^{\oplus n}$

4. Hall identity: $(X^{Y}, (Y,Z)) * (Y^{Z}, (Z,X)) * (Z^{X}, (X,Y)) = 0 \in \mathfrak{m} < X, Y, Z >^{\oplus n}$

5. Jacobi identity: $[X,[Y,Z]_F]_F + [Y,[Z,X]_F]_F + [Z,[X,Y]_F]_F = 0 \in \mathfrak{m} < X, Y, Z >^{\oplus n}$

Note: In order to indicate that the map

$$[-,-]_F : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$$

depends on the two sets of variables $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ the map is also denoted $[X,Y]_F$. Its value at the point $(x,y) \in \mathbb{K}^n \times \mathbb{K}^n$ is

$$[x,y]_F := [X,Y]_F(x,y) := B(X,Y)(x,y) - B(Y,X)(x,y) = B(x,y) - B(y,x) \in \mathbb{K}^n.$$

Proof. Due to the first property of a formal group structure all summands of F

$$c_{\alpha\beta} \cdot X^{\alpha} Y^{\beta}$$

of order ≥ 2 satisfy

$$|\alpha|, |\beta| \geq 1.$$

To simplify the notation we omit the subscript F and set $[-,-] := [-,-]_F$.

1. The ansatz
$$X^- = \Phi_1(X) + \Phi_2(X) + O(3), deg\ \Phi_i = i,$$
implies
$$0 = F(X, X^-) = X + X^- + B(X, X^-) + O(3) =$$
$$= X + \Phi_1(X) + \Phi_2(X) + B(X, \Phi_1(X)) + O(3).$$
As a consequence
$$\Phi_1(X) = -X, \Phi_2(X) = -B(X, \Phi_1(X)) = B(X, X).$$
2. By definition and using part 1)
$$X^Y := F(Y^-, F(X,Y)) = Y^- + F(X,Y) + B(Y^-, F(X,Y)) + O(3) =$$
$$(-Y + B(Y,Y)) + (X + Y + B(X,Y)) + B(-Y, X+Y) + O(3) =$$
$$-Y + B(Y,Y) + X + Y + B(X,Y) - B(Y,X) - B(Y,Y) + O(3) =$$
$$X + [X,Y] + O(3)$$
3. By definition and using the previous parts
$$(X,Y) := X^- * Y^- * X * Y = X^- * (Y^- * X * Y) = X^- * X^Y = F(X^-, X^Y) =$$
$$= X^- + X^Y + B(X^-, X^Y) + O(3) = (-X + B(X,X)) + (X + [X,Y]) + B(-X,X) + O(3) =$$
$$= [X,Y] + O(3)$$
4. We compute
$$(X^Y, (Y,Z)) := (X^Y)^- * (Y,Z)^- * X^Y * (Y,Z) =$$
$$(Y^- * X^- * Y) * (Z^- * Y^- * Z * Y) * (Y^- * X * Y) * (Y^- * Z^- * Y * Z) =$$
$$= (Y^- * X^- * Y * Z^- * Y^-) * (Z * X * Z^- * Y * Z)$$
Using cyclic permutation set
   - $U := Z * X * Z^- * Y * Z$,
   - $V := X * Y * X^- * Z * X$,
   - $W := Y * Z * Y^- * X * Y$.

Then

- $(X^Y,(Y,Z)) = W^- * U$,
- $(Y^Z,(Z,X)) = U^- * V$,
- $(Z^X,(X,Y)) = V^- * W$.

As a consequence

$$(X^Y,(Y,Z)) * (Y^Z,(Z,X)) * (Z^X,(X,Y)) = 0.$$

5. First, we apply part 2) and part 3):

$$(X^Y,(Y,Z)) = (X + O(2), [Y,Z] + O(3))$$

Then we apply part 3) a second time and use the bilinearity of [-,-]:

$$(X + O(2), [Y,Z] + O(3)) = [X + O(2), [Y,Z] + O(3)] = [X,[Y,Z]] + O(4).$$

We obtain

$$(X^Y,(Y,Z)) = [X,[Y,Z]] + O(4)$$

and analogous terms by cyclic permutation.

Eventually, we apply part 4):

$$0 = (X^Y,(Y,Z)) * (Y^Z,(Z,X)) * (Z^X,(X,Y)) =$$

$$= ([X,[Y,Z]] + O(4)) + ([Y,[Z,X]] + O(4)) + ([Z,[X,Y]] + O(4)) =$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] + O(4).$$

Here the first three terms have order $= 3$. As a consequence we obtain the Jacobi identity, q.e.d.

We recall some basic definitions from $\mathbb{K}$-Lie algebra theory:

1. A *Lie algebra* over the field $\mathbb{K}$ is a $\mathbb{K}$-vector space L together with a $\mathbb{K}$-bilinear map

$$[-,-] : L \times L \to L \ (\textit{Lie bracket})$$

such that

- $[x,x] = 0$ for all $x \in L$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$ (*Jacobi identity*).

2. A *morphism of Lie algebras* is a $\mathbb{K}$-linear map $f : L_1 \to L_2$ between two $\mathbb{K}$-Lie algebras with

$$f([x_1,x_2]) = [(f(x_1), f(x_2))], x_1, x_2 \in L_1.$$

3. A *Lie subalgebra* of a Lie algebra L is a subspace $M \subset L$ which is closed with respect to the Lie bracket of L, i.e. for all $m_1, m_2 \in M$ also
$$[m_1, m_2] \in M.$$

Theorem 3.5 (The Lie algebra of a formal group structure).

1. *For a formal group structure*
$$F(X,Y) \in \mathfrak{m} < X,Y >^{\oplus n}.$$
the $\mathbb{K}$*-bilinear map from Proposition 3.4*
$$[-,-]_F : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$$
defines a $\mathbb{K}$*-Lie algebra* $(\mathbb{K}^n, [-,-]_F)$.

2. *Consider an n-dimensional Lie group G, choose a chart*
$$\phi = (\phi_1, ..., \phi_n) : U \to \mathbb{K}^n$$
around $e \in G$*, denote by F the corresponding formal group structure (Lemma 3.1), and by*
$$\psi : \mathbb{K}^n \xrightarrow{\sim} T_eG, (x_1, ..., x_n) \mapsto \sum_{i=1}^{n} x_i \cdot \frac{\partial}{\partial \phi_i}$$
the isomorphism onto the tangent space at $e \in G$ *induced by* ϕ *(Proposition 2.7).*

Then the bracket
$$[-,-] : T_eG \times T_eG \to T_eG, (u,v) \mapsto [u,v] := \psi([\psi^{-1}(u), \psi^{-1}(v)]_F),$$
provides the tangent space of G at e with a $\mathbb{K}$*-Lie algebra structure* $(T_eG, [-,-])$.

Proof. For the Jacobi identity see Proposition 3.4, part 5).

Definition 3.6 (Lie algebra of a Lie group). The *Lie algebra of a Lie group G* with neutral element $e \in G$ is the Lie algebra from Theorem 3.5
$$Lie\ G := (T_eG, [-,-]).$$

Remark 3.7 (Lie algebra).

1. We will show in Lemma 3.9 that the Lie algebra structure on T_eG does not depend on the choice of the chart ϕ and its corresponding formal group structure F.

2. Speaking in a descriptive way: The Lie bracket measures the second order deviation of the group multiplication from commutativity.

Lemma 3.8 (Low order approximations). *A formal group structure*

$$F \in \mathfrak{m} < X, Y >^{\oplus n}$$

satisfies the following approximations:

1. If $U, V, U', V' \in \mathfrak{m} < X >^{\oplus n}$ and $U' = U + O(2), V' = V + O(2)$ then

$$B(U', V') - B(V', U') = B(U, V) - B(V, U) + O(3)$$

2. If $U, V \in \mathfrak{m} < X >^{\oplus n}$ then

$$F(U, V) - F(V, U) = B(U, V) - B(V, U) + O(3)$$

3. If $U, U' \in \mathfrak{m} < X >^{\oplus n}$ and $U' = U + O(2)$ then

$$U'(F(X, Y)) - U'(F(Y, X)) = U(F(X, Y)) - U(F(Y, X)) + O(3).$$

Proof. 1. The relation between respectively U and U' and between V and V' implies

$$B(U', V') - B(V', U') = B(U + O(2), V + O(2)) - B(V + O(2), U + O(2)) =$$

$$= B(U, V) - B(V, U) + O(3).$$

2. Set

$$U = U_1 + U_2 + O(3), V = V_1 + V_2 + O(3), deg\ U_i = deg\ V_i = i.$$

Then

$$F(U, V) = U_1 + V_1 + U_2 + V_2 + B(U_1, V_1) + O(3)$$

and

$$F(V, U) = V_1 + U_1 + V_2 + U_2 + B(V_1, U_1) + O(3).$$

Therefore using part i):

$$F(U, V) - F(V, U) = B(U_1, V_1) - B(V_1, U_1) + O(3) = B(U, V) - B(V, U) + O(3).$$

3. The expansion

$$U' - U = \sum_{|\alpha| \geq 2} c_\alpha X^\alpha$$

implies

$$U'(F(X, Y)) - U'(F(Y, X)) - U(F(X, Y)) + U(F(Y, X)) =$$

$$= U'(F(X,Y)) - U(F(X,Y)) - (U'(F(Y,X)) - U(F(Y,X)))$$
$$= \sum_{|\alpha| \geq 2} c_\alpha \cdot (F(X,Y)^\alpha - F(Y,X)^\alpha)$$

We expand the summands using the binomial formula

$$F(X,Y)^\alpha - F(Y,X)^\alpha = [F(Y,X) + (F(X,Y) - F(Y,X))]^\alpha - F(Y,X)^\alpha =$$

$$= (\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} F(Y,X)^\beta \cdot (F(X,Y) - F(Y,X))^{\alpha - \beta}) - F(Y,X)^\alpha =$$

$$= \sum_{\beta < \alpha} \binom{\alpha}{\beta} F(Y,X)^\beta \cdot (F(X,Y) - F(Y,X))^{\alpha - \beta}).$$

Each summand is at least $O(3)$ because

$$\binom{\alpha}{\beta} F(Y,X)^\beta \cdot (F(X,Y) - F(Y,X))^{\alpha - \beta})$$

has the order - with the factor 2 due to part 2) -

$$|\beta| + 2(|\alpha| - |\beta|) =$$

$$= |\beta| + 2(|\alpha| - |\beta|) = 2|\alpha| - |\beta| = |\alpha| + (|\alpha| - |\beta|) > |\alpha| \geq 2.$$

Here we have used the notation: If

$$\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$$

then

$$|\alpha| := \sum_{i=1}^n \alpha_i \textit{ and } \alpha! := \prod_{i=1}^n \alpha_i! \textit{ and } \binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha - \beta)! \cdot \beta!}, \; q.e.d.$$

Lemma 3.9 (Independence of the Lie algebra structure). *Consider two Lie groups G and G' and a morphism $f : G \to G'$ of Lie groups.*

Consider two charts $\phi : U \to \mathbb{K}^n$ and $\phi' : U' \to \mathbb{K}^{n'}$ around their neutral elements with $f^{-1}(U') \subset U$. Denote by

$$F \in \mathfrak{m} < X, Y >^{\oplus n}, F' \in \mathfrak{m} < X', Y' >^{\oplus n'}$$

the induced formal group structures and by

$$[-,-]_{(F)} \textit{ and } [-,-]_{(F')}$$

the induced Lie brackets on respectively T_eG and $T_{e'}G'$.

Then the tangent map

$$T_e f : (T_e G, [-,-]_{(F)}) \to (T_{e'} G', [-,-]_{(F')})$$

is a morphism of Lie algebras.

In particular, choosing $G' = G$ and $f = id$ shows: The Lie algebra structure Lie G on the tangent space $T_e G$ is independent from the choice of the chart ϕ.

Proof. The claim is local in a neighbourhood of $e \in G$. Assume two charts of respectively G and G'

$$\phi : U \to \mathbb{K}^n \text{ and } \phi' : U' \to \mathbb{K}^{n'}$$

which induce the analytic map

$$g := \phi' \circ f \circ \phi^{-1} \in \mathfrak{m} < X >^{\oplus n'}$$

in the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"', "\simeq"] & U' \arrow[d, "\phi'"', "\simeq"] \\ \phi(U) \arrow[r, dashed, "g"] & \phi'(U') \end{tikzcd}$$

i) First we consider the linear part g_1 of g. We obtain

$$\begin{aligned}
[g_1(X), g_1(Y)]_{F'} :=& \\
B'(g_1(X), g_1(Y)) - B'(g_1(Y), g_1(X)) =& \qquad \text{(Lemma 3.8, part 1)} \\
B'(g(X), g(Y)) - B'(g(Y), g(X)) + O(3) =& \qquad \text{(Lemma 3.8, part 2)} \\
F'(g(X), g(Y)) - F'(g(Y), g(X)) + O(3) =& \qquad (f \text{ is a homomorphism}) \\
g(F(X,Y) - g(F(Y,X)) + O(3) =& \qquad \text{(Lemma 3.8, part 3)} \\
g_1(F(X,Y)) - g_1(F(Y,X)) + O(3) =& \qquad (g_1 \text{ is linear}) \\
g_1(F(X,Y) - F(Y,X)) + O(3) =& \qquad \text{(Lemma 3.8, part 2)} \\
g_1(B(X,Y) - B(Y,X)) + O(3) =:& \\
g_1([X,Y]_F) + O(3)&
\end{aligned}$$

Both functions

$$B'(g_1(X), g_1(Y)) - B'(g_1(Y), g_1(X)) \text{ and } g_1(B(X,Y) - B(Y,X))$$

are quadratic. Therefore they are equal:

$$[g_1(X), g_1(Y)]_{F'} = B'(g_1(X), g_1(Y)) - B'(g_1(Y), g_1(X)) =$$

$$= g_1(B(X,Y) - B(Y,X)) = g_1([X,Y]_F).$$

ii) Secondly, we transfer the result about the linear approximation of g_1 from part i) to the corresponding result about the tangent map $T_e f$, using the commutative diagram

$$\begin{tikzcd}
\mathbb{K}^n \arrow[r, "g_1"] \arrow[d, "\psi"', "\simeq"] & \mathbb{K}^{n'} \arrow[d, "\psi'"', "\simeq"] \\
T_e G \arrow[r, "T_e f"] & T_e G'
\end{tikzcd}$$

For $u, v \in T_G$ we employ the definition of the Lie bracket on the tangent spaces according to Theorem 3.5 and the result from part i). We obtain

$$T_e f([u,v]_{(F)}) = T_e f \circ \psi([\psi^{-1}(u), \psi^{-1}(v)]_F) =$$

$$= \psi' \circ g_1([\psi^{-1}(u), \psi^{-1}(v)]_F) =$$
$$= \psi'([g_1(\psi^{-1}(u)), g_1(\psi^{-1}(v))]_{F'}) =$$
$$= \psi'([\psi'^{-1}(T_e f(u)), \psi'^{-1}(T_e f(v))]_{F'}) =$$
$$= [T_e f(u), T_e f(v)]_{(F')}, \ q.e.d.$$

Definition 3.10 (The functor Lie).

1. Consider a morphism
$$f : G \to H$$
between two Lie groups. The induced morphism between their Lie algebras (see Lemma 3.9) is denoted by
$$Lie\ f := T_e f : Lie\ G \to Lie\ H.$$

2. Denote by $LieGrp_{\mathbb{K}}$ the category of $\mathbb{K}$-Lie groups and $\mathbb{K}$-Lie group morphisms and by $LieAlg_{\mathbb{K}}$ the category of $\mathbb{K}$-Lie algebras and $\mathbb{K}$-Lie algebra morphisms. Attaching
   - to each Lie Group G its Lie algebra $Lie\ G$ and
   - to each morphism $f : G \to H$ of Lie groups its tangent map $Lie\ f : Lie\ G \to Lie\ H$

   defines the covariant functor
$$Lie : LieGrp_{\mathbb{K}} \to LieAlg_{\mathbb{K}}$$

Proposition 3.11 (The functor Lie is faithful). *On the subcategory of connected Lie groups the functor Lie is faithful, in particular two morphisms*

$$f_1, f_2 : G \to H$$

between Lie groups with connected G agree, if

$$Lie\ f_1 = Lie\ f_2 : Lie\ G \to Lie\ H.$$

Proof. The formal group law $F(X,Y) = X + Y + O(2)$ of H implies for the tangent map of the multiplication

$$T_{(e,e)} m_H : T_e H \times T_e H \to T_e H, (x,y) \mapsto x + y.$$

As a consequence, $m_H(x, \sigma(x)) = e$ implies

$$T_e id + T_e \sigma = id + T_e \sigma = 0,$$

i.e. taking the inverse

$$\sigma_H : H \to H, \sigma(x) := x^{-1},$$

has the tangent map

$$(T_e \sigma)(x) = -x.$$

Consider the morphism of Lie groups

$$f := f_1 \circ f_2^{-1} : G \to H.$$

By assumption, it satisfies

$$Lie\ f = Lie\ f_1 - Lie\ f_2 = 0.$$

Apparently, f has rank $= 0$ at $e \in G$. According to Corollary 2.22 the map f has constant rank, i.e.

$$rank\ T_g f = 0$$

for all $g \in G$. Theorem 2.14 and the connectedness of G and imply that f is constant, i.e. $f_1(g) = f_2(g)$ for all $g \in G$, q.e.d.

3.2 Vector fields and local flows

Definition 3.12 (Derivations and vector fields). Consider a $\mathbb{K}$-analytic manifold X.

1. For any open subset $U \subset X$ denote by

$$\mathscr{O}_X(U) := \{f : U \to \mathbb{K} : \ f\ analytic\}$$

the $\mathbb{K}$-*algebra of analytic functions on* U. For an open subset $V \subset U$ the restriction

$$\rho_V^U : \mathscr{O}_X(U) \to \mathscr{O}_X(V), f \mapsto f|V,$$

is well-defined.

For any $x \in V \subset U$ the following diagram with the induced canonical maps to the local ring $\mathscr{O}_{X,x}$ commutes

$$\begin{tikzcd}
\mathscr{O}_X(U) \arrow[rr, "\rho_V^U"] \arrow[dr, dashed, "\pi_x^U"'] & & \mathscr{O}_X(V) \arrow[dl, dashed, "\pi_x^V"] \\
& \mathscr{O}_{X,x} &
\end{tikzcd}$$

2. A *derivation* on $\mathscr{O}_X(U)$ is a $\mathbb{K}$-linear map

$$D_U : \mathscr{O}_X(U) \to \mathscr{O}_X(U)$$

satisfying the product rule

$$D_U(f \cdot g) = D_U(f) \cdot g + f \cdot D_U(g), f, g \in \mathscr{O}_X(U).$$

3. An *analytic vector field* on X is a family $D = (D_U)_{U \subset X \ open}$ of derivations

$$D_U : \mathscr{O}_X(U) \to \mathscr{O}_X(U)$$

such that for each pair of open sets $V \subset U$ the following diagram commutes

$$\begin{tikzcd}
\mathscr{O}_X(U) \arrow[r, "D_U"] \arrow[d, "\rho_V^U"'] & \mathscr{O}_X(U) \arrow[d, "\rho_V^U"] \\
\mathscr{O}_X(V) \arrow[r, "D_V"] & \mathscr{O}_X(V)
\end{tikzcd}$$

4. We denote by $\Theta(X)$ the $\mathbb{K}$-*vector space of analytic vector fields* on X.

Note:

1. For an analytic vector field

$$D = (D_U)_{U \subset X \ open}$$

and two open subsets $U, V \subset X$ both derivations D_U and D_V induce on the intersection $U \cap V$ the same derivation $D_{U\cap V}$ according to the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(U) \arrow[r, "D_U"] \arrow[d, "\rho^U_{U\cap V}"'] & \mathscr{O}_X(U) \arrow[d, "\rho^U_{U\cap V}"] \\
\mathscr{O}_X(U\cap V) \arrow[r, "D_{U\cap V}"] & \mathscr{O}_X(U\cap V) \\
\mathscr{O}_X(V) \arrow[r, "D_V"] \arrow[u, "\rho^V_{U\cap V}"] & \mathscr{O}_X(V) \arrow[u, "\rho^V_{U\cap V}"']
\end{tikzcd}$$

2. An analytic verctor field $D \in \Theta(X)$ is a whole family $D = (D_U)_{U\subset X \ open}$, not just the single derivation

$$D_X : \mathscr{O}_X(X) \to \mathscr{O}_X(X).$$

Lemma 3.13 (Vector fields and tangent vectors).

1. Consider a $\mathbb{K}$-analytic manifold X. For any point $x \in X$ two $\mathbb{K}$-linear maps exist

$$\varepsilon_x : \Theta(X) \to Der_{\mathbb{K}}(\mathscr{O}_{X,x}, \mathscr{O}_{X,x}), D \mapsto D_x,$$

and

$$\varepsilon(x) : \Theta(X) \to T_xX = Der_{\mathbb{K}}(\mathscr{O}_{X,x}, \mathbb{K}), D \mapsto D(x),$$

satisfying for any open neighbourhood U of x the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(U) \arrow[r, "D_U"] \arrow[d, "\pi^U_x"'] & \mathscr{O}_X(U) \arrow[d, "\pi^U_x"] \\
\mathscr{O}_{X,x} \arrow[r, dashed, "D_x"] \arrow[dr, dashed, "D(x)"'] & \mathscr{O}_{X,x} \arrow[d, "\pi(x)"] \\
 & \mathscr{O}_{X,x}/\mathfrak{m}_x
\end{tikzcd}$$

Here

$$\pi(x) : \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathfrak{m}_x \simeq \mathbb{K}, [f] \mapsto f(x),$$

denotes the evaluation of germs of analytic functions at the point $x \in X$.

2. *Consider an analytic analytic vector field $D \in \Theta(X)$, an analytic function $f \in \mathcal{O}(U)$ on an open subset $U \subset X$, and a point $x \in X$. Denote by $f_x \in \mathcal{O}_{X,x}$ the germ of f at x. Then*
$$D_x(f_x) = D_U(f)_x \in \mathcal{O}_{X,x}$$
and
$$D(x)(f_x) = (D_U(f))(x) \in \mathbb{K}.$$
In particular:
$$D = 0 \iff D_U = 0 \; for\ all\ open\ U \subset X \iff D_x = 0 \; for\ all\ x \in X \iff$$
$$D(x) = 0 \; for\ all\ x \in X.$$

Proof. 1. The main task is to show that the induced map D_x does not depend on the choice of the neighbourhood of x. Therefore, for two neighbourhoods U, V of x one uses the compatibility of D_U and D_V on $\mathcal{O}_X(U \cap V)$ according to Definition 3.12, part 3.

2. The proof follows from part 1).

Lemma 3.14 (Lie algebra of vector fields). *Consider a $\mathbb{K}$-analytic manifold X.*

1. *If $A, B \in \Theta(X)$ then also*
$$[A, B] \in \Theta(X).$$
Here $[A, B]$ denotes the family
$$[A, B] := ([A_U, B_U])_{U \subset X\ open}$$
with the commutator
$$[A_U, B_U] := A_U \circ B_U - B_U \circ A_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U),\ U \subset X\ open.$$

2. *The pair $(\Theta(X), [-,-])$ is a $\mathbb{K}$-Lie algebra, the* Lie algebra of analytic vector fields on X.

Proof. ad 1) The commutator of two vector fields is a vector field again: For $U \subset X$ open, $\alpha := A_U, \beta := B_U$, and $f, g \in \mathcal{O}(U)$:

$$[\alpha, \beta](f \cdot g) = \alpha(\beta(f \cdot g)) - \beta(\alpha(f \cdot g)) = \alpha((\beta f) \cdot g + f \cdot \beta(g)) - \beta((\alpha f) \cdot g - f \cdot \alpha(g)) =$$
$$= (\alpha(\beta f)) \cdot g + \beta f \cdot \alpha g + \alpha f \cdot \beta g + f \cdot (\alpha(\beta g)) - (\beta(\alpha f)) \cdot g - \alpha f \cdot \beta g - \beta f \cdot \alpha g - f \cdot (\beta(\alpha g)) =$$
$$(\alpha(\beta f) - \beta(\alpha f)) \cdot g + f \cdot (\alpha(\beta g) - \beta(\alpha g)) = ([\alpha, \beta](f)) \cdot g + f \cdot ([\alpha, \beta](g))$$

ad 2) The Jacobi identity for vector fields follows by the same computation like the proof of the Jacobi identity for the commutator of matrices, q.e.d.

Note: In the following we will always consider $\Theta(X)$ as the Lie algebra with the Lie bracket from Lemma 3.14.

Lemma 3.15 (Group operation on vector fields). *Any Lie group G operates on the Lie algebra $\Theta(G)$ of its analytic vector fields according to*

$$G \times \Theta(G) \to \Theta(G), (g, D) \mapsto g.D.$$

Here the analytic vector field

$$g.D := ((g.D)_U)_{U \subset X \ open} \in \Theta(G)$$

is defined on $U \subset X$ by the derivation

$$(g.D)_U : \mathscr{O}_G(U) \to \mathscr{O}_G(U)$$

induced from the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_G(U) \arrow[r, "\tau_g", "\simeq"'] \arrow[d, dashed, "(g.D)_U"'] & \mathscr{O}_G(g^{-1} \cdot U) \arrow[d, "D_{g^{-1} \cdot U}"] \\
\mathscr{O}_G(U) & \mathscr{O}_G(g^{-1} \cdot U) \arrow[l, "\tau_g^{-1}"', "\simeq"]
\end{tikzcd}$$

with

$$\tau_g : \mathscr{O}_G(U) \to \mathscr{O}_G(g^{-1} \cdot U), f \mapsto f \circ L_g,$$

and analogously $\tau_g^{-1} = \tau_{g^{-1}}$.

The operation has the following properties:

1. *For all $g \in G$*
$$g : \Theta(G) \to \Theta(G), D \mapsto g.D,$$
is an isomorphism of Lie algebras.

2. *For all $g, x \in G$ and $D \in \Theta(G)$ the evaluation to tangent vectors satisfies*
$$(g.D)(x) = (T_{g^{-1}x} L_g)(D(g^{-1}x)) \in T_x G$$
i.e. the following diagram commutes

$$\begin{tikzcd} \Theta(G) \arrow[r] \arrow[d, "\varepsilon(g^{-1}\cdot x)"'] & \Theta(G) \arrow[d, "\varepsilon(x)"] \\ T_{g^{-1}x}G \arrow[r, "T_{g^{-1}x}L_g"] & T_xG \end{tikzcd}$$

Here the horizontal map

$$\Theta(G) \to \Theta(G), D \mapsto g.D,$$

results from the group operation.

Paraphrasing Lemma 3.15 in a descriptive way: The transformed vector field $g.D$, when applied to a function $f \in \mathscr{O}(G)$, evaluates at the point $x \in G$ to the same value like the original vector field D applied to the left-translate of f evaluates at the point $g^{-1}(x)$.

Definition 3.16 (Left-invariant vector fields). Consider a Lie group G. The vector space

$$\Theta(G)^G := \{A \in \Theta(G) : g.A = A \ for \ all \ g \in G\}$$

is named the vector space of *left-invariant* analytic vector fields on G.

Theorem 3.17 (Lie algebra of left-invariant vector fields). *Consider a Lie group G and denote by*

$$L(G) := \Theta(G)^G$$

the vector space of left-invariant vector fields on G.

1. *Then*

$$L(G) = \{A \in \Theta(G) : A(g) = (T_eL_g)(A(e)) \in T_gG \ for \ all \ g \in G\}$$

and $L(G) \subset \Theta(G)$ is a Lie subalgebra.

2. *The evaluation of left-invariant vector fields to tangent vectors at $e \in G$*

$$\varepsilon = \varepsilon(e) : L(G) \to Lie\ G = (T_eG, [-,-]), A \mapsto A(e),$$

is an isomorphism of Lie algebras.

Proof. During the proof we will apply several times the characterization of the group operation as the equality of tangent vectors

$$(\gamma.A)(x) = (T_{\gamma^{-1}x}L_\gamma)(A(\gamma^{-1}x)) \in T_xG, A \in \Theta(G), \gamma, x \in G,$$

from Lemma 3.15.

1) i) *Lie subalgebra*: Consider an arbitrary but fixed $\gamma \in G$. In order to prove that

$$\Theta(G)^\gamma := \{A \in \Theta(G) : \gamma.A = A\} \subset \Theta(G)$$

is a Lie subalgebra, we consider two vector fields $A, B \in \Theta(G)$ with $\gamma.A = A, \gamma.B = B$. From the definition by the commutative diagram in Lemma 3.15 follows

$$\gamma.[A,B] = [\gamma.A, \gamma.B] :$$

- The derivation $(\gamma.[A,B])_U$ originates from $[A,B]_{g^{-1}.U}$.
- The derivation $[\gamma.A, \gamma.B]_U$ derives from $[A_{g^{-1}.U}, B_{g^{-1}.U}]$.

Because the Lie bracket of vector fields is defined by the Lie bracket of the corresponding derivations, we have

$$[A,B]_{g^{-1}.U} = [A_{g^{-1}.U}, B_{g^{-1}.U}].$$

We obtain

$$\gamma.[A,B] = [\gamma.A, \gamma.B] = [A,B],$$

which proves that $\Theta(G)^\gamma \subset \Theta(G)$ is a Lie subalgebra.

Therefore, the intersection of Lie subalgebras

$$L(G) = \bigcap_{\gamma \in G} \Theta(G)^\gamma \subset \Theta(G)$$

is a Lie subalgebra.

ii) *Condition on tangent vectors*: A vector field $A \in L(G)$ satisfies $\gamma.A = A$ for all $\gamma \in G$. Therefore the formula for the tangent vectors specializes as

$$(T_eL_\gamma)(A(e)) = A(\gamma).$$

Concerning the opposite direction consider a vector field $A \in \Theta(G)$ satisfying for all $\gamma \in G$

$$(T_eL_\gamma)(A(e)) = A(\gamma).$$

Applying again Lemma 3.15, part 2) gives for all $x \in X$

$$(\gamma.A)(x) = (T_{\gamma^{-1}}L_\gamma)(A(\gamma^{-1}x)) = (T_{\gamma^{-1}}L_\gamma)(T_eL_{\gamma^{-1}x})(A(e)) = (T_eL_x)(A(e)) = A(x)$$

or

$$(\gamma.A - A)(x) = 0.$$

As a consequence $\gamma.A - A = 0$ or $\gamma.A = A$ due to Lemma 3.13, i.e. $A \in L(G)$.

2) i) *Injectivity*: The evaluation $\varepsilon : L(G) \to Lie(G)$ is injective: If $A \in L(G)$ and $A(e) = 0$ then also

$$A(g) = (T_e L_g)(A(e)) = 0$$

for all $g \in G$. Therefore $A = 0$ due to Lemma 3.13. In particular: $L(G)$ is a finite-dimensional Lie algebra.

ii) *Transformation to coordinates*: We choose a chart of G around $e \in G$

$$\phi = (\phi_1, ..., \phi_n) : U \to \mathbb{K}^n$$

and a neighbourhood V of e with $V \cdot V \subset U$. We set $W := \phi(U) \subset \mathbb{K}^n$. The chart induces

- the isomorphism of derivations

  $$Der(\mathscr{O}_G(U), \mathscr{O}_G(U)) \to Der(\mathscr{O}_{\mathbb{K}^n}(W), \mathscr{O}_{\mathbb{K}^n}(W)), A_U \mapsto \tilde{A} := \phi \circ A_U \circ \phi^{-1},$$

  according to the commutative diagram

  $$\begin{tikzcd} \mathscr{O}_G(U) \arrow[r, "A_U"] \arrow[d, "\simeq"'] & \mathscr{O}_G(U) \arrow[d, "\simeq"] \\ \mathscr{O}_{\mathbb{K}^n}(W) \arrow[r, dashed, "\tilde{A}"] & \mathscr{O}_{\mathbb{K}^n}(W) \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r] \arrow[d] & A_U(f) \arrow[d] \\ f \circ \phi^{-1} \arrow[r] & A_U(f) \circ \phi^{-1} \end{tikzcd}$$

- the isomorphism of vector spaces

  $$\psi : \mathbb{K}^n \xrightarrow{\sim} T_e G, (a_1, ..., a_n) \mapsto \sum_{i=1}^{n} a_i \cdot \left. \frac{\partial}{\partial \phi_i} \right|_{z=0}.$$

- and the formal group structure F according to the commutative diagram

  $$\begin{tikzcd} V \times V \arrow[r, "m_G"] \arrow[d, "\phi \times \phi"'] & U \arrow[d, "\phi"] \\ \phi(V) \times \phi(V) \arrow[r, dashed, "F"] & W \end{tikzcd}$$

  satisfying

$$F(X,Y) = X + Y + B_F(X,Y) + \sum_{|\alpha|+|\beta|\geq 3, |\alpha|,|\beta|\geq 1} c_{\alpha\beta} \cdot X^{\alpha} Y^{\beta},\ deg\ B_F = 2.$$

iii) *Morphism of Lie algebras*: Consider two invariant vector fields $A, B \in L(G)$. We have to show

$$[A,B](e) = [A(e), B(e)]_F \in T_e G,$$

or equivalently - using the definition $[-,-]_F$ from Proposition 3.4 -

$$\psi^{-1}([A,B](e)) = \psi^{-1}([A(e),B(e)]_F) := [\psi^{-1}(A(e)), \psi^{-1}(B(e))]_F :=$$

$$:= B_F(\psi^{-1}(A(e)), \psi^{-1}(B(e)) - B_F(\psi^{-1}(B(e)), \psi^{-1}(A(e))) \in \mathbb{K}^n.$$

First we introduce on both sides of the latter equation the corresponding derivations on $\mathscr{O}_{\mathbb{K}^n(W)}$:

The derivation A_U induces

$$\tilde{A} = \sum_{\nu=1}^{n} g_\nu \cdot \frac{\partial}{\partial z_\nu} \in \Theta(W)$$

with row-vector

$$g = (g_1, ..., g_n) \in \mathscr{O}_{\mathbb{K}^n}(W)^{\oplus n}\ and\ g(0) = \psi^{-1}(A(e)) \in \mathbb{K}^n.$$

The derivation $B_U : \mathscr{O}_G(U) \to \mathscr{O}_G(U)$ induces

$$\tilde{B} = \sum_{\nu=1}^{n} h_\nu \cdot \frac{\partial}{\partial z_\nu} \in \Theta(W)$$

with row-vector

$$h = (h_1, ..., h_n) \in \mathscr{O}_{\mathbb{K}^n}(W)^{\oplus n}\ and\ h(0) = \psi^{-1}(B(e)) \in \mathbb{K}^n.$$

- *Right-hand side*: We obtain

$$B_F(\psi^{-1}(A(e)), \psi^{-1}(B(e)) - B_F(\psi^{-1}(B(e)), \psi^{-1}(A(e))) =$$

$$= B_F(g(0), h(0)) - B_F((h(0), g(0)) \in \mathbb{K}^n.$$

- *Left-hand side*: Because second order terms cancel we obtain

$$[\tilde{A}, \tilde{B}] = \left[\sum_{\mu=1}^{n} g_\mu \frac{\partial}{\partial x_\mu}, \sum_{\nu=1}^{n} h_\nu \frac{\partial}{\partial x_\nu}\right] = \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{n} g_\mu \frac{\partial h_\nu}{\partial x_\mu} - h_\mu \frac{\partial g_\nu}{\partial x_\mu}\right) \frac{\partial}{\partial x_\mu} =$$

$$= (g \cdot \nabla h - h \cdot \nabla g) \cdot \nabla$$

Here the Nabla operator ∇ is considered a column vector and the dots denote matrix multiplication. As a consequence

$$\psi^{-1}([A,B](e)) = (g\cdot\nabla h - h\cdot\nabla g)(0) \in \mathbb{K}^n.$$

In order to compute

$$(g\cdot\nabla h - h\cdot\nabla g)(0)$$

we analyze the assumption that both vector fields A and B are left-invariant. The corresponding condition

$$A(\gamma) = (T_eL_\gamma)(A(e)) \; for\ any\ \gamma \in V$$

implies: For all $f \in \mathscr{O}_{\mathbb{K}^n}(W)$ and all $a := \phi(\gamma) \in W$

$$(\tilde{A}(f))(a) = \tilde{A}(f\circ\tilde{L}_a)(0)$$

with $\tilde{L}_a(x) := F(a,x)$. These equations impose a condition on the coefficients of $\tilde{A} = g\cdot\nabla$:

$$(\tilde{A}(f))(a) = \sum_{\nu=1}^{n} g_\nu(0)\cdot\left.\frac{\partial f(F(a,x))}{\partial x_\nu}\right|_{x=0} \in \mathbb{K}^n,$$

or after replacing (a,x) by (x,y)

$$(\tilde{A}(f))(x) = \sum_{\nu=1}^{n} g_\nu(0)\cdot\left.\frac{\partial f(F(x,y))}{\partial y_\nu}\right|_{y=0} \in \mathbb{K}^n.$$

We specialize the last result to the particular case of the projections

$$f = pr_j|\phi(U) : \phi(U) \to \mathbb{K}, (z_1,...,z_n) \mapsto z_j, j = 1,...,n,$$

and obtain

$$g_j(x) = (\tilde{A}(pr_j))(x) = \sum_{\nu=1}^{n} g_\nu(0)\cdot\left.\frac{\partial F_j(x,y)}{\partial y_\nu}\right|_{y=0} \in \mathbb{K}^n.$$

Each component function of $F = (F_1,...,F_n)$ expands as

$$F_j(x,y) = x_j + y_j + B_{F,j}(x,y) + O(3),$$

hence

$$\left.\frac{\partial F_j(x,y)}{\partial y_\nu}\right|_{y=0} = \delta_{j\nu} + B_{F,j}(x,e_\nu) + O(2)$$

with $e_\nu \in \mathbb{K}^n$ the ν-th row vector of the canonical basis. We obtain

$$g_j(x) = \sum_{\nu=1}^{n} g_\nu(0) \cdot (\delta_{j\nu} + B_{F,j}(x, e_\nu)) + O(2) =$$

$$= g_j(0) + B_{F,j}(x, g(0)) + O(2).$$

We conclude

$$\frac{\partial g_j}{\partial x_\mu}(0) = B_{F,j}(e_\mu, g(0)).$$

Analogously

$$\frac{\partial h_j}{\partial x_\mu}(0) = B_{F,j}(e_\mu, h(0)).$$

We obtain

$$(g \cdot \nabla h - h \cdot \nabla g)(0) = g(0) \cdot (\nabla h)(0) - h(0) \cdot (\nabla g)(0) =$$

$$= B_F(g(0), h(0)) - B_F(h(0), g(0))$$

Hence the left-hand side equals the right-hand side which proves the claim: ε is a morphism of Lie algebras.iv) *Surjectivity*: We have to extend a given tangent vector

$w \in T_eG$ at the point e to a left-invariant vector field $A \in \Theta(G)$: Using the chart ϕ we choose a vector field

$$\alpha \in \Theta(U) \; with \; \alpha(e) = w$$

and extend it to a vector field

$$\alpha' := (0, \alpha) \in \Theta(G \times U).$$

We embed G as a retract into the product $G \times U$, taking the multiplication m as retraction: When defining

$$j : G \to G \times U, x \mapsto (x, e),$$

then

$$id = [G \xrightarrow{j} G \times U \xrightarrow{m} G].$$

Note: The retraction projects the point $(g,u) \in G \times U$ to $m(g,u) \in G$, but not necessarily to its first component $g \in G$.

The retraction m splits the tangent space of Z at each point

$$z = (g, u) \in Z := G \times U$$

as the direct sum of a horizontal and a vertical subspace according to

$$T_zZ = im[T_gj : T_gG \xrightarrow{\simeq} T_{(g,e)}Z \subset T_zZ] \oplus ker[T_zm : T_zZ \to T_{m(z)}G].$$

Therefore the pointwise projection of the vector field $\alpha' \in \Theta(Z)$ onto its first summand is a vector field $A \in \Theta(G)$. The equation

$$m \circ j = id_G$$

implies for any point $g \in G$ for the tangent maps

$$id_{T_gG} = [T_gG \xrightarrow{T_gj} T_{(g,e)} \xrightarrow{T_{(g,e)}m} T_gG,\ A(g) \mapsto pr_1(\alpha'(g,e)) \mapsto A(g)].$$

At any point $g \in G$ the tangent vector is

$$A(g) = (T_{(g,e)}m)(pr_1(\alpha'(g,e))) = (T_{(g,e)}m)(\alpha'(g,e)) \in T_gG.$$

We obtain

$$A(g) = (T_{(g,e)}m)(\alpha'(g,e)) = (T_{(g,e)}m)(\alpha(e)) = (T^2_{(g,e)}m)(\alpha(e))$$

$$= (T^2_{(g,e)}m)(w) = (T_eL_g)(w),$$

in particular

$$A(e) = w.$$

Here T^2 denotes derivation with respect to the second component of $(g,u) \in G \times U$.

As a consequence

$$A \in \Theta(G)^G = L(G) \text{ and } \varepsilon(A) = w,\ q.e.d.$$

Thanks to Theorem 3.17 we have two equivalent ways to obtain the Lie algebra of a Lie group G. The first employs a local method: The quadratic terms of the formal group law provide the tangent space of G at the neutral element with the structure of a Lie algebra *Lie G*. The second way starts with the global object of analytic vector fields on G. The Lie algebra is then obtained as the subalgebra of left-invariant vector fields L(G) with respect to the commutator of vector fields.

In the following we will identify the two Lie algebras *Lie G* and $L(G)$, using for both the notation *Lie G*.

Definition 3.18 (Integral curves and local flows). Consider a $\mathbb{K}$-analytic manifold X and an analytic vector field $A \in \Theta(X)$.

1. An *integral curve* of A passing through the point $x \in X$ is an analytic map

$$\alpha : B \to X$$

with $B \subset \mathbb{K}$ a connected open neighbourhood of $0 \in \mathbb{K}$, such that

- $\alpha(0) = x$ (*Initial condition*)

- and for all $t \in B$

$$\dot{\alpha}(t) = A(\alpha(t)) \in T_{\alpha(t)}X. (\textit{Differential equation}).$$

The last equation on tangent vectors means

$$A(\alpha(t)) = (T_t\alpha)(\frac{d}{d\tau}|_{\tau=t}) \in T_{\alpha(t)}X.$$

2. A *local flow* of A is an analytic map

$$\Phi : V \to X$$

with an open subset $V \subset \mathbb{K} \times X$ with $\{0\} \times X \subset V$, such that for each $x \in X$

$$\Phi(-,x) : \{t \in \mathbb{K} : (t,x) \in V\} \to X$$

is an integral curve passing through x.

Speaking in a descriptive way: A local flow of A is a family of integral curves of A such that for each point $x \in X$ one member of the family passes through x.

A short notation to memorize the flow condition is

$$\dot{\Phi} = A(\Phi) \; and \Phi(0,-) = id_X.$$

Remark 3.19 (Local flows and ordinary differential equations). Consider a $\mathbb{K}$-analytic manifold and an analytic vector field $A \in \Theta(X)$.

1. If $\phi = (\phi_1, ..., \phi_n) : U \to \mathbb{K}^n$ is a chart of X then $A \in \Theta(X)$ has on U the representation
$$A_U = \sum_{i=1}^{n} A_i \cdot \frac{\partial}{\partial \phi_i}, \; A_i \in \mathscr{O}_X(U), \; i = 1, ..., n.$$
For an analytic map $\alpha : B \to U$ defined on an open neighbourhood $B \subset \mathbb{K}$ of $0 \in \mathbb{K}$ we have the equivalence:
   - α is an integral curve of A on X
   - The component functions

$$\phi \circ \alpha = (\phi_1 \circ \alpha, ..., \phi_n \circ \alpha) : B \to \mathbb{K}^n$$

solve for all $t \in B$ the system of first order ordinary differential equations

$$\frac{d}{dt}(\phi_i \circ \alpha)(t) = A_i(\alpha(t)) = (A_i \circ \phi^{-1})((\phi \circ \alpha)(t)).$$

2. According to the theorem on the existence and uniqueness of solutions the Picard-Lindelöf algorithm constructs for each $x \in X$ a constant $r(x) > 0$ and an unique integral curve of A

$$\alpha_x : B(x) := \{t \in \mathbb{K} : |t| < r(x)\} \to X$$

passing through x. Because the vector field is analytic, the constructed solution α_x is analytic too. Even more: The integral curve α_x depends analytically on the initial condition $x \in X$, i.e. on an open subset $V \subset \mathbb{K} \times X$ with

$$\{0\} \times X \subset V$$

exists a local flow of A

$$\Phi : V \to X, \Phi(t,x) := \alpha_x(t).$$

Lemma 3.20 (Additivity of local flows). *Consider a $\mathbb{K}$-analytic manifold X and an analytic vector field $A \in \Theta(X)$ with a local flow*

$$\Phi : V \to X,\ V \subset \mathbb{K} \times X \ \mathit{open}.$$

Then for all points $x \in X$ and for all parameters $s,t \in \mathbb{K}$ with $(s,x), \Phi(t,\Phi(s,x)), (t+s,x) \in V$

$$\Phi(t,\Phi(s,x)) = \Phi(t+s,x).$$

Proof. Choose an arbitrary but fixed point $(s,x) \in V$ and define on suitable neigbourhoods of $0 \in \mathbb{K}$ the two functions

$$\alpha : V_1 \to X, \alpha(t) := \phi(t,\phi(s,x)) \ \mathit{and} \ \beta : V_2 \to X, \beta(t) := \phi(t+s,x).$$

Both functions satisfy the same initial condition

$$\alpha(0) = \phi(s,x) = \beta(0)$$

and the same differential equation

$$A(\alpha(t)) = A(\phi(t,\phi(s,x)) = \dot{\phi}(t,\phi(s,x)) = \dot{\alpha}(t)$$

and

$$A(\beta(t)) = A(\phi(t+s,x)) = \dot{\phi}(t+s,x) = \dot{\beta}(t).$$

The theorem on the uniqueness of the solution of a system of differential equations with analytic coefficients implies $\alpha = \beta$, q.e.d.

3.3 One-parameter subgroups and the exponential map

Definition 3.21 (One-parameter subgroup). A 1-*parameter subgroup* of a $\mathbb{K}$-Lie group G is a morphism of Lie groups

$$f : (\mathbb{K}, +) \to G.$$

In general, the integral curves of vector fields are defined only locally, i.e. in a small neighbourhood of zero. We show: Under the additional assumption, that the vector field is a left-invariant vector field on a Lie group, the integral curves are even defined globally.

Theorem 3.22 (Integration of left-invariant vector fields to 1-parameter subgroups). *Consider a $\mathbb{K}$-Lie group G and a left-invariant analytic vector field*

$$A \in Lie\ G = \Theta(G)^G.$$

1. *There exists a unique* 1*-parameter subgroup of G*

   $$f_A : (\mathbb{K}, +) \to G$$

   with $\dot{f}_A(0) = A(e)$.

2. *The* 1*-parameter subgroup f_A satisfies for all $t \in \mathbb{K}$*

   $$\dot{f}_A(t) = A(f_A(t)),$$

   i.e. f_A is the global integral curve of A passing through $e \in G$.

Proof. 1. Due to Remark 3.19 the vector field $A \in L(G)$ has a local flow

$$\phi : V \to G, \{0\} \times G \subset V \text{ with } V \subset \mathbb{K} \times G \text{ open}.$$

i) *Left invariance of the flow*: If $(t, g) \in V$ and $(t, e) \in V$ then

$$\phi(t, g) = L_g(\phi(t, e)) :$$

For each fixed $g \in G$ exists an $\varepsilon > 0$ such that

$$|t| < \varepsilon \implies (t, e) \in V \text{ and } (t, g) \in V.$$

Set

$$B(\varepsilon, 0) := \{t \in \mathbb{K} : |t| < \varepsilon\}$$

and consider

$$\alpha : B(\varepsilon, 0) \to X, \alpha(t) := \phi(t, g)$$

and
$$\beta : B(\varepsilon,0) \to X, \beta(t) := L_g(\phi(t,e)).$$
Both functions satisfy the same intial condition
$$\alpha(0) = g = \beta(0)$$
and the same differential equation:
$$\dot{\alpha}(t) = A(\alpha(t))$$
and
$$\dot{\beta}(t) = (T_{\phi(t,e)}L_g) \circ (T_t\phi(-,e))(\frac{d}{d\tau}) = (T_{\phi(t,e)}L_g)(A(\phi(t,e)) = A(L_g(\phi(t,e)) = A(\beta(t))$$
Here we have applied Lemma 3.15 to the left invariant vector field A to obtain the penultimate equality. The theorem on the uniqueness of the solution of a system of differential equations with analytic coefficients implies $\alpha = \beta$.

ii) *Germ of the* 1*-parameter group*: Choose $\varepsilon > 0$ such that
$$s,t \in B(\varepsilon,0) \implies \phi(t+s,e), (t, \phi(s,e), (s, \phi(t,e) \in V.$$
Then according to Lemma 3.20 for all $(s,t) \in B(\varepsilon,0)$
$$\phi(t+s,e) = \phi(t, \phi(s,e)) \textit{ and } \phi(s+t,e) = \phi(s, \phi(t,e)).$$
Part i) with the choice $g := \Phi(s,e)$ implies
$$\phi(t+s,e) = \phi(s,e) \cdot \phi(t,e) = \phi(t,e) \cdot \phi(s,e).$$
iii) *Extension to a group morphism*: Consider a fixed but arbitrary $t \in \mathbb{K}$ and choose $\varepsilon > 0$ alike to part ii). For a suitable $n \in \mathbb{N}$ holds $|t|/n < \varepsilon$. Define
$$f_A(t) := \Phi(t/n,e)^n \in G.$$
The value is well-defined: If also $m \in \mathbb{N}$ with $|t|/m < \varepsilon$ then part ii) implies
$$\phi(t/n,e)^n = (\phi(t/(n \cdot m) + ... + t/(n \cdot m), e))^n = (\phi(t/(n \cdot m), e)^m)^n =$$
$$= (\phi(t/(n \cdot m), e)^n)^m = \phi(t/m,e)^m.$$
The map
$$f_A : (\mathbb{K}, +) \to G$$
is a group homomorphism: Assume $s,t \in \mathbb{K}$. For a suitable $n \in \mathbb{N}$ part ii) implies
$$f_A(s+t) = \phi((s+t)/n,e)^n = (\phi(s/n,e) \cdot \phi(t/n,e))^n =$$

$$= \phi(s/n,e)^n \cdot \phi(t/n,e)^n = f_A(s) \cdot f_A(t).$$

The map f_A is analytic in a neighbourhood of $0 \in \mathbb{K}$. Being a group homorphism it is analytic in a neighbourhoof of any point.

iv) *Uniqueness*: By construction $\dot{f}_A(0) = A(e)$. This condition determines the morphism f_A according to Proposition 3.11 because $\dot{f}_A(0) = Lie\ f_A$.

2. The following diagram commutes because f_A is a group homomorphism according to part iii)

$$\begin{tikzcd} \mathbb{K} \arrow[r, "L_t"] \arrow[d, "f_A"'] & \mathbb{K} \arrow[d, "f_A"] \\ G \arrow[r, "L_{f_A(t)}"] & G \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r] \arrow[d] & t+u \arrow[d] \\ f_A(u) \arrow[r] & f_A(t) \cdot f_A(u) \end{tikzcd}$$

Therefore Theorem 3.17, part 1) applied to the left-invariant vector field A, the condition $\dot{\phi}(0) = A(e)$, and the differential equation of the flow imply

$$A(f_A(t)) = (T_e L_{f_A(t)})(A(e)) = (T_e L_{f_A(t)} \circ T_0 f_A)(\frac{d}{du}|_{u=0}) =$$

$$= (T_t f_A \circ T_e L_t)(\frac{d}{du}|_{u=0}) = (T_t f_A)(\frac{d}{du}|_{u=t}) = \dot{f}_A(t),\ q.e.d.$$

Definition 3.23 (The exponential map of a Lie group). Consider a $\mathbb{K}$-Lie group with neutral element $e \in G$. The *exponential map* of G is the map

$$exp : Lie\ G \to G, A \mapsto exp(A) := f_A(1).$$

Here f_A denotes the uniquely determined 1-parameter group with

$$\dot{f}_A(0) = A(e)$$

from Theorem 3.22.

Lemma 3.24 (The exponential map as integration of left-invariant vector fields). *The exponential*

$$exp : L(G) \to G$$

satisfies for all $A \in Lie\ G$ and for all parameter values $t \in \mathbb{K}$

$$exp(t \cdot A) = f_A(t),$$

i.e. for any left-invariant vector field A the map

$$\mathbb{K} \to G, t \mapsto exp(t \cdot A),$$

is the integral curve of A passing through $e \in G$.

Proof. For arbitrary but fixed $t \in \mathbb{K}$ the composition

$$g := [\mathbb{K} \xrightarrow{j} \mathbb{K} \xrightarrow{f_A} G], \ j(s) := t \cdot s$$

is a 1-parameter group on G satisfying

$$g(0) = e$$

and

$$\dot{g}(0) = (T_0 g)(\frac{d}{ds}|_{s=0}) = (T_0 f_A)(t\frac{d}{ds}|_{s=0}) = t \cdot (T_0 f_A)(\frac{d}{ds}|_{s=0}) = t \cdot A(e) = (t \cdot A)(e).$$

As a consequence

$$g = f_{t \cdot A}.$$

Moreover, on one hand, $g(1) = exp(t \cdot A)$. On the other hand $g(1) = f_A(t)$ by definition. Therefore

$$exp(t \cdot A) = f_A(t), \ q.e.d.$$

Theorem 3.25 (Analyticity of exp and tangent map). *The exponential map of a Lie group G*

$$exp : Lie\ G \to G$$

is an analytic map. Its tangent map at $0 \in Lie\ G$ is

$$T_0 exp = id \in End(Lie\ G).$$

Proof. We have to show that the family of 1-parameter subgroups $(f_A)_{A \in Lie\ G}$ depends analytically on the parameter $A \in Lie\ G$.

i) *Universal vector field*: The proof introduces a relative version of Theorem 3.22 on the existence of the flow of an invariant vector field.

We introduce the analytic manifold

$$X := Lie\ G \times G,$$

the Cartesian product of the parameter space *Lie G* and the base manifold *G*. The first projection

$$pr_1 : X \to Lie\ G$$

considers X as a family of copies of G, parametrized by *Lie G*. On X the *universal vector field*

$$U \in \Theta(Lie\ G \times G)$$

is defined by the family of tangent vectors

$$U(A,g) := (0, A(g)) \in Lie\ G \oplus T_g G = T_{(A,g)}(Lie\ G \times G)$$

for all $x = (A,g) \in X$. Consider the local flow of U on an open subset $V \subset \mathbb{K} \times X$ with $\{0\} \times X \subset V$

$$\Phi_U = (\Psi, \Phi) : V \to X = Lie\ G \times G.$$

By definition, Φ_U has for all $x \in X$ the initial value

$$\Phi_U(0,x) = x \in X$$

and satisfies for all $x \in X$ the differential equation

$$(\dot{\Psi}(t,x), \dot{\Phi}(t,x)) = \dot{\Phi}_U(t,x) = U(\Phi_U(t,x)) =$$

$$= U((\Psi(t,x), \Phi(t,x)) = (0, \Psi(t,x)(\Phi(t,x))).$$

Here the last term $\Psi(t,x)(\Phi(t,x)))$ evaluates the vector field $\Psi(t,x) \in Lie\ G$ at the point $\Phi(t,x) \in G$. The differential equation splits into the two differential equations

- $\dot{\Psi}(t,x) = 0$, which implies the constant value

  $$\Psi(t,x) = \Psi(0,x)$$

  for all $t \in \mathbb{K}$ with $(t,x) \in V$,

- and as a consequence

  $$\dot{\Phi}(t,x) = \Psi(t,x)(\Phi(t,x))) = \Psi(0,x)(\Phi(t,x))),$$

  i.e. with $x = (A,g) \in Lie\ G \times G$ therefore

  $$\dot{\Phi}(t,A,g) = A(\Phi(x,A,g)),$$

  because $\Phi_U(0,x) = x = (A,g)$ implies for the first component $\Psi(0,x) = A$.

For arbitrary but fixed $A \in Lie\ G$ the integral curve of A passing through $e \in G$ is the map

$$\mathbb{K} \to G,\ t \mapsto \Phi(t,A,e).$$

According to Theorem 3.22 the integral curve extends to the 1-parameter group

$$\Phi(-,A,e) : \mathbb{K} \to G, \Phi(t,A,e) := f_A(t).$$

ii) *Analyticity*: We claim that the map

$$\mathbb{K} \times Lie\ G \to G,\ (t,A) \mapsto \Phi(t,A,e)$$

is analytic. Because $\Phi_U = (\Psi, \Phi)$ is a local flow of the universal vector field U, the second component

$$\Phi : V \to G$$

is defined and analytic in a suitable neighbourhood of any arbitrary, but fixed

$$(0,x) = (0,A,e) \in \mathbb{K} \times Lie\ G \times G,$$

in particular $\Phi(-,-,e)$ is analytic in a neighbourhood of any arbitrary, but fixed $(0,A) \in \mathbb{K} \times Lie\ G$.

Consider a fixed, but arbitrary point $(t_0,A_0) \in \mathbb{K} \times Lie\ G$. There exist relatively compact open neighbourhoods

$$W_1 \subset\subset \mathbb{K}$$

of t_0 and

$$W_2 \subset\subset Lie\ G$$

of A_0 and a number $n_0 \in \mathbb{N}$ such that: For all $(t,A) \in W_1 \times W_2$ and all $n \geq n_0$

$$(t/n,A,e) \in V.$$

For $(t,A) \in W_1 \times W_2$ and $n \geq n_0$

$$f_A(t) = f_A(t/n)^n = \Phi(t/n,A,e)^n$$

which proves analyticity with respect to (t,A) in a neighbourhood of (t_0,A_0). In particular, the restriction

$$exp : Lie\ G \to G, A \mapsto f_A(1,A,e)$$

is analytic.

iii) *Tangent map of the exponential*: For arbitrary but fixed vector field $A \in Lie\ G$ consider the composition

$$\alpha := [\mathbb{K} \xrightarrow{j} Lie\ G \xrightarrow{exp} G]\ with\ j(t) := t \cdot A.$$

Then $\alpha(t) = exp(t \cdot A)$. According to Lemma 3.24 $\alpha(t) = f_A(t)$. Therefore

$$A = \dot{\alpha}(0)(\frac{d}{dt}) = ((T_0\ exp) \circ (T_0 j))(\frac{d}{dt}) = (T_0\ exp)(A)$$

which proves $T_0\ exp = id \in End(Lie\ G)$, q.e.d

Corollary 3.26 (The exponential map is a local isomorphism). *The exponential map of a Lie group G*

$$exp : Lie\ G \to G$$

is a local isomorphism.

Proof. The proof follows from Theorem 3.25 by applying Proposition 2.9.

Corollary 3.27 (The image of the exponential map generates the component of the neutral element). *The connected component $G^e \subset G$ of $e \in G$ is the group generated by the subset $exp(Lie\ G) \subset G$.*

Proof. Because exp is a local isomorphism at $0 \in Lie\ G$, the set $exp(Lie\ G)$ is a neighbourhood of $e \in G$. It generates G^e according to Proposition 1.11, q.e.d.

A topological group G has *no small subgroups* if a neighbourhood U of the neutral element $e \in G$ exists such that $H := \{e\}$ is the only subgroup $H \subset G$ with $H \subset U$.

Corollary 3.28 (Lie groups have no small subgroups). *Any Lie group G has no small subgroups.*

Proof. Denote by $L := Lie\ G$ the n-dimensional Lie algebra of G. According to Corollary 3.26 a ball $B := B(r,0) \subset \mathbb{K}^n$ of radius $r > 0$ around $0 \in \mathbb{K}^n$ and a neighbourhood $U \subset G$ of $e \in G$ exist such that the restriction

$$exp|B : B \xrightarrow{\simeq} U$$

is an analytic isomorphism. Set $V := B(r/2,0)$. We claim that any subgroup $H \subset G$ with

$$H \subset exp(V)$$

reduces to $H = \{e\}$:

Consider an element $h = exp(v_1) \in H$ for a suitable element $v_1 \in V$. Because $h^2 \in H$ an element $v_2 \in V$ exists with

$$h^2 = exp(v_2).$$

We obtain

$$exp(v_2) = exp(v_1)^2 = exp(2v_1).$$

We conclude

$$v_2 = 2v_1 \in V.$$

It is important to conlude that the element v_2 also belongs to V.

Iterating the argument with $(h^2)^n = h^{2n},\ n = 2,3,4,$ provides a sequence of elements

$$v_n = 2^n v_1 \in V, n \in \mathbb{N}^*.$$

Therefore $v_1 = 0$ and $h = e$, q.e.d.

Note: According to a theorem of Gleason and Montgomery-Zippin any locally compact topological group without small subgroups can be provided with the structure of a real Lie group, see [29].

We recall some results about matrices from our lecture on Lie algebras.

Example 3.29 (Exponential of matrices). The map

$$e: \; M(n \times n, \mathbb{K}) \to GL(n, \mathbb{K}), A \mapsto e^A := \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!}$$

is well-defined, because the complex power series

$$e^z := \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} \in \mathbb{C} < z >$$

has radius of convergence $= \infty$. The exponential map satisfies the following properties:

1. If $A \in M(n \times n, \mathbb{K})$ then
$$e^{-A} = (e^A)^{-1}.$$
2. If $A, \; B \in M(n \times n, \mathbb{K})$ and $[A, B] = 0$ then
$$e^{A+B} = e^A \cdot e^B \in GL(n, \mathbb{K}),$$
3. If $A \in M(n \times n, \mathbb{K})$ and $S \in GL(n, \mathbb{K})$ then
$$S \cdot e^A \cdot S^{-1} = e^{S \cdot A \cdot S^{-1}}$$
4. If $\lambda \in \mathbb{C}$ eigenvalue of A, then $e^\lambda \in \mathbb{C}$ eigenvalue of e^A.
5. For all $A \in M(n \times n, \mathbb{K})$ holds $det \; e^A = e^{tr \; A}$.
6. If $A \in M(n \times n, \mathbb{K})$ and $t \in \mathbb{K}$ then
$$\frac{d}{dt}(e^{t \cdot A})|_{t=0} = A.$$
7. The Lie group $GL(n, \mathbb{K})$ has the Lie algebra
$$Lie \; GL(n, \mathbb{K}) = gl(n, \mathbb{K}) := (M(n \times n, \mathbb{K}), [-, -]).$$
8. The e-function is the exponential map
$$exp : gl(n, \mathbb{K}) \to GL(n, \mathbb{K})$$
of the Lie group $GL(n, \mathbb{K})$.

Proof. For the proof of part 1) - 5) see Proposition **??**.

7) On a suitable open neighbourhood U of $\mathbb{1} \in GL(n,\mathbb{K})$ the map

$$\phi : U \to \mathbb{K}^{n^2} \simeq M(n \times n, \mathbb{K}), A \mapsto A - \mathbb{1},$$

is a chart around the neutral element $\mathbb{1} \in GL(n,\mathbb{K})$.

There exists a neighbourhood $V \subset U$ of $\mathbb{1} \in GL(n,\mathbb{K})$ such that $V \cdot V \subset U$. The corresponding formal group structure

$$F \in \mathfrak{m} < X,Y >^{\oplus n^2}$$

with $X = (X_i)_{1 \le i \le n^2}$ and $Y = (Y_i)_{1 \le i \le n^2}$ is defined by the following commutative diagram

$$\begin{tikzcd}
V \times V \arrow[r, "m"] \arrow[d, "\phi \times \phi"', "\simeq"] & V \arrow[d, "\phi"', "\simeq"] \\
\phi(V) \times \phi(V) \arrow[r, "F"] & \phi(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,B) \arrow[r] \arrow[d] & A \cdot B \arrow[d] \\
(A - \mathbb{1}, B - \mathbb{1}) \arrow[r] & A \cdot B - \mathbb{1}
\end{tikzcd}$$

Because

$$A \cdot B - \mathbb{1} = (A - \mathbb{1}) + (B - \mathbb{1}) + (A - \mathbb{1}) \cdot (B - \mathbb{1})$$

we obtain

$$F(X,Y) = X + Y + B(X,Y),\ B(X,Y) := X \cdot Y,$$

and

$$[-,-]_F := B(X,Y) - B(Y,X) = X \cdot Y - Y \cdot X.$$

8) For each $A \in M(n \times n, \mathbb{K})$ the map

$$f : \mathbb{K} \to GL(n,\mathbb{K}), t \mapsto e^{t \cdot A},$$

is a 1-parameter subgroup of $GL(n,\mathbb{K})$ with

$$\frac{d}{dt}(e^{t \cdot A})|_{t=0} = A$$

and

$$f(0) = \mathbb{1} = e \in GL(n,\mathbb{K}).$$

According to Theorem 3.25 the 1-parameter group is uniquely determined by these properties, q.e.d.

Proposition 3.30 (Exponential and tangent map commute). *Consider a morphism $\phi : G \to H$ of Lie groups. Then the following diagram commutes:*

$$
\begin{tikzcd}
G \arrow[r, "\phi"] & H \\
Lie\ G \arrow[u, "exp_G"] \arrow[r, "Lie\ \phi"] & Lie\ H \arrow[u, "exp_H"']
\end{tikzcd}
$$

Proof. Consider a vector field $A \in Lie\ G$ and set $B := Lie\ \phi(A) \in Lie\ H$. Denote by

$$f_A : \mathbb{K} \to G \ and\ f_B : \mathbb{K} \to H$$

the corresponding one-parameter groups. Both 1-parameter groups $\Phi \circ f_A$ and f_B habe the same derivation:

$$\frac{d}{dt}(\Phi \circ f_A)(0) = (Lie\ \Phi)(\dot{f}_A(0)) = (Lie\ \Phi)(A) = B = \dot{f}_B(0).$$

As a consequence, they are equal

$$\Phi \circ f_A = f_B,$$

which implies

$$exp_H(Lie\ (\Phi(A)) = \exp_H(B) = f_B(1) = \Phi(f_A(1)) = \Phi(exp_G\, A),\ q.e.d.$$

Corollary 3.31 (The Lie algebra of a subgroup). *Consider an injective morphism*

$$j : H \to G$$

of $\mathbb{K}$-Lie groups. Then:

$$Lie\ H = \{A \in Lie\ G : exp(t \cdot A) \in H\ for\ all\ t \in \mathbb{K}\}.$$

The corollary states: If the whole integral curve of a tangent vector A from G belongs to the subgroup H then A itself is tangent to the subgroup H.

Proof. According to Proposition 3.11 the following diagram commutes

$$\begin{tikzcd}
H \arrow[r, "j"] & G \\
Lie\ H \arrow[u, "exp"] \arrow[r, "Lie\ j"] & Lie\ G \arrow[u, "exp"']
\end{tikzcd}$$

According to Corollary 2.22 the map j has constant rank. According to the rank theorem, Theorem 2.14, the injectivity of j implies the injectivity of the tangent map $Lie\ j$.

i) Apparently $Lie\ H \subset \{A \in Lie\ G : exp(t \cdot A) \in H\ for\ all\ t \in \mathbb{K}\}$.

ii) In order to prove the opposite inclusion, we apply Corollary 3.26: There exists an open neighbourhood $V \subset Lie\ G$ of $0 \in Lie\ G$ and an open neighbourhood $W \subset G$ of $e \in G$ such that

$$exp|V : V \xrightarrow{\sim} W$$

is an isomorphism. The following diagram commutes and the vertical maps are isomorphism

$$\begin{tikzcd}
exp(V \cap Lie\ H) \arrow[r, "j"] & W \\
V \cap Lie\ H \arrow[u, "exp"] \arrow[r, "Lie\ j"] & V \arrow[u, "exp"']
\end{tikzcd}$$

In particulat, the set $exp(V \cap Lie\ H)$ is a neighbourhood of e in H.

Consider an arbitrary $A \in Lie\ G$ with $exp(s \cdot A) \in H$ for all $s \in \mathbb{K}$. Then a parameter value $t \in \mathbb{K}, t \neq 0$, exists such that

$$t \cdot A \in exp(V \cap Lie\ H),$$

i.e. an element $B \in Lie\ H$ exists with $exp(B) = exp(t \cdot A)$. We obtain

$$B = t \cdot A$$

or

$$A = (1/t) \cdot B \in Lie\ H,\ q.e.d.$$

The subsequent Lemma 3.32 generalizes Corollary 2.22 including also the corresponding Lie algebras.

Lemma 3.32 (The pre-image of Lie subgroups). *Consider a morphism*

$$f : G \to H$$

of Lie groups and a Lie-subgroup $H' \subset H$. Then the pre-image

$$G' := f^{-1}(H')$$

is a Lie subgroup of G with Lie algebra

$$Lie\ G' = (Lie\ f)^{-1}(Lie\ H').$$

Proof. According to Proposition 2.24 the quotient H/H' is an analytic Hausdorff manifold and the canonical projection

$$\pi : H \to H/H'$$

is a submersion. According to Theorem 2.14 the fibre $H' = \pi^{-1}(eH')$ has the tangent space at $e \in H'$

$$Lie\ H' = T_e H' = ker\ T_e\pi.$$

The analytic left-operation

$$G \times (H/H') \to H/H', (g, hH') \mapsto (f(g) \cdot h)H'$$

has the isotropy group at $eH' \in H/H'$

$$G_{eH'} = \{g \in G : f(g) \in H'\} = f^{-1}(H') = G'.$$

Being an isotropy group $G' \subset G$ is a Lie subgroup according to Proposition 2.21.

The composition

$$\pi \circ f : G \to H/H'$$

has constant rank. According to Theorem 2.14 the fibre

$$G' = (\pi \circ f)^{-1}(eH')$$

has the tangent space at $e \in G'$

$$T_e G' = ker\ (T_e(\pi \circ f)) = (T_e f)^{-1}(ker\ T_e\pi)) = (T_e f)^{-1}(Lie\ H') = (Lie\ f)^{-1}(Lie\ H'),\ q.e.d.$$

Part II
Advanced Lie Group Theory

Chapter 4
The functional equation of the exponential map

4.1 The Baker-Campbell-Hausdorff formula

Any Lie algebra has a distinguished representation, namely a canonical morphism to the Lie algebra of endomorphisms of L. We recall from Lie algebra theory:

The *adjoint representation* of a finite dimensional Lie algebra $(L, [-,-]_L)$ is the Lie algebra morphism

$$ad : L \to gl(L) := (End(L), [-,-]), x \mapsto ad\ x,$$

with

$$ad\ x : L \to L, y \mapsto [x,y]_L.$$

In close relation to the adjoint representation of Lie algebras any Lie group G has a distinguished representation, namely a canonical morphism to the Lie group of automorphisms of *Lie G*:

Definition 4.1 (Adjoint representation of a Lie group). The *adjoint representation* of a Lie group G is the map

$$Ad : G \to GL(Lie\ G), g \mapsto Ad\ g := T_e\Phi_g.$$

Here

$$T_e\Phi_g : Lie\ G \to Lie\ G$$

is the tangent map at the neutral element $e \in G$ of the inner automorphism

$$\Phi_g : G \to G, h \mapsto g \cdot h \cdot g^{-1},$$

which is an isomorphism of Lie groups.

Proposition 4.2 (Tangent map of the adjoint representation). *Consider a Lie group G.*

1. *The adjoint representation of G*

$$Ad : G \to GL(Lie\ G)$$

is a morphism of Lie groups.

2. *The tangent map of the adjoint representation of G at the neutral element $e \in G$ is the adjoint representation of Lie G, i.e.*

$$Lie(Ad) = ad : Lie\ G \to Lie\ G.$$

Proof. i) *Group homomorphism*: The map

$$Ad : G \to GL(Lie\ G)$$

is a group homomorphism: For $g, h \in G$

$$Ad(g_1 \cdot g_2) = T_e\phi_{g_1 \cdot g_2} = T_e(\phi_{g_1} \circ \phi_{g_2}) = T_e\phi_{g_1} \circ T_e\phi_{g_2} = Ad\ g_1 \circ Ad\ g_2.$$

ii) *Analyticity*: We identify

$$GL(Lie\ G) \simeq GL(n, \mathbb{K}),\ n := dim_{\mathbb{K}} G.$$

Consider a chart $\tau : U \to \mathbb{K}^n$ around $e \in G$ and denote by F its formal group law. The commutative diagram

$$\begin{tikzcd} U \arrow[r, "Ad"] \arrow[d, "\tau"', "\sim"] & GL(n,\mathbb{K}) \\ \phi(U) \arrow[ur, dashed, "Ad_\tau"'] & \end{tikzcd}$$

defines the coordinate representation Ad_τ of Ad. We have

$$Ad_\tau : \tau(U) \to GL(n, \mathbb{K}),\ x \mapsto\ linear\ part\ of\ \phi_g,\ g := \tau^{-1}(x).$$

With respect to the chart τ the inner automorphism ϕ_g has the approximation from Proposition 3.4, part ii)

$$XYX^{-} = Y + [X,Y]_F + \sum_{|\alpha|+|\beta|\geq 3} c_{\alpha,\beta} x^\alpha y^\beta.$$

It shows that $Ad_\tau(x)$ is the linear part of the map

$$y \mapsto y + [x,y]_F + \sum_{|\alpha|+|\beta|\geq 3, |\beta|=1} c_{\alpha,\beta} x^\alpha y^\beta,$$

with fixed x and variable y. The power series representation shows that this map and therefore also its linear part depend analytically on x. Therefore Ad is analytic in a neighbourhood of $e \in G$. Being a group homomorphism, Ad is analytic.

iii) *Tangent map*: The tangent map $T_e Ad$ is the linear part of the map

$$Ad_\tau : \tau(U) \to GL(n, \mathbb{K}).$$

Here Ad_τ is a map defined for the argument x. According to part ii) the linear part of Ad_τ is the linear part of the map

$$\tau(U) \to End(\mathbb{K}^n), x \mapsto Ad_\tau(x) - id + O(2),$$

i.e. the linear part of Ad_τ is the map

$$ad : \mathbb{K}^n \to End(\mathbb{K}^n)$$

with

$$ad\ x : \mathbb{K}^n \mapsto \mathbb{K}^n,\ y \mapsto [x,y]_F,\ q.e.d.$$

Lemma 4.3 (Adjoint representation and exponential map). *Consider a Lie group G. Then:*

1. The following diagram commutes

$$\begin{tikzcd} G \arrow[r, "Ad"] & GL(Lie\ G) \\ Lie\ G \arrow[u, "exp"] \arrow[r, "ad"] & gl(Lie\ G) \arrow[u, "e"'] \end{tikzcd},$$

i.e. for all $x \in Lie\ G$

$$Ad(exp\ x) = e^{ad\ x} \in GL(Lie\ G).$$

2. For all $x, y \in Lie\ G$:

$$exp\ x \cdot exp\ y \cdot (exp\ x)^{-1} = exp(e^{ad\ x}(y)) \in G.$$

Proof. 1. The commutativity of the diagram follows from Proposition 4.2, Proposition 3.30, and Example 3.29.

2. For any $a \in G$ the following diagram commutes by definition of $Ad\ a$

$$\begin{tikzcd} G \arrow[r, "\phi_a"] & G \\ Lie\ G \arrow[u, "exp"] \arrow[r, "Ad\ a"] & Lie\ G \arrow[u, "exp"'] \end{tikzcd}$$

Therefore for all $y \in Lie\ G$

$$\phi_a(exp\ y) = exp((Ad\ a)y).$$

Notably for $a := exp\ x,\ x \in Lie\ G$, and according to the formula from part 1)

$$(exp\ x)\ (exp\ y)\ (exp\ x)^{-1} = exp(Ad(exp\ x)(y)) = exp(e^{ad\ x}(y)),\ q.e.d.$$

Proposition 4.4 (The functional equation in the Abelian case). *Consider a Lie group G and two elements $A, B \in Lie\ G$ with $[A,B] = 0$. Then*

$$exp(A+B) = exp(A) \cdot exp(B).$$

Proof. We consider the corresponding one-parameter groups

$$f_A : \mathbb{K} \to G, t \mapsto exp(t \cdot A) \text{ and } f_B : \mathbb{K} \to G, t \mapsto exp(t \cdot B).$$

i) *Commutation of f_A and f_B*: The assumption implies $(ad\ A)(B) = 0$. Hence

$$e^{ad(sA)}(tB) = \sum_{\nu=0}^{\infty} \frac{(ad\ sA)^{\nu}(tB)}{\nu!} = tB.$$

Lemma 4.3 shows

$$f_A(s) \cdot f_B(t) = exp(sA) \cdot exp(tB) = exp(e^{ad(sA)}(tB)) \cdot exp(sA) = exp(tB) \cdot exp(sA) = f_B(t) \cdot f_A(s).$$

ii) 1-*parameter group $f_A \cdot f_B$*: Due to part i)

$$(f_A \cdot f_B)(t+s) := f_A(t+s) \cdot f_B(t+s) = f_A(t) \cdot (f_A(s) \cdot f_B(t)) \cdot f_B(s) =$$

$$= f_A(t) \cdot (f_B(t) \cdot f_A(s)) \cdot f_B(s) =: (f_A \cdot f_B)(t) \cdot (f_A \cdot f_B)(s)$$

Therefore also the product

$$f_A \cdot f_B : \mathbb{K} \to G$$

is a one-parameter group. The tangent map of the composition

$$f_A \cdot f_B = [\mathbb{K} \xrightarrow{f_A \cdot f_B} G \times G \xrightarrow{m} G]$$

is

$$T_0(m \circ (f_A, f_B)) = T_0 f_A + T_0 f_B = A + B.$$

Hence $f_A \cdot f_B$ is the uniquely determined one-parameter group passing through $e \in G$ with tangent vector $A + B$, i.e.

$$f_A \cdot f_B = f_{A+B}.$$

In particular

$$exp(A) \cdot exp(B) = (f_A \cdot f_B)(1) = f_{A+B}(1) = exp(A+B),\ q.e.d.$$

To obtain the general functional equation of the exponential map, which also holds for non-commuting vector fields, one needs some refined formulas for calculations with binomial coefficients:

Lemma 4.5 (Formulas with binomial coefficients).

1. Binomial coefficients*:*

$$(-1)^l \binom{n}{l} = \sum_{k=0}^{l} (-1)^k \binom{n+1}{k},\ 1 \le l \le n \in \mathbb{N}.$$

2. Derivation of power series*: For a convergent power series $g \in \mathbb{K} < X, Y >$ holds*

$$\left(\frac{d}{dt}\right)^n g(-t,t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot g_{k,n-k}(-t,t)$$

with

$$g_{k,n-k} := \frac{\partial^n g}{\partial x^k \partial y^{n-k}}.$$

3. Powers of ad*: Consider a Lie group G and two invariant vector fields $X, Y \in Lie\ G = \Theta(G)^G$. Then for all $n \in \mathbb{N}$:*

$$(ad\ X)^n(Y) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, X^{n-k}\, Y\, X^k \in Lie\ G.$$

Proof. i) Induction on n.

ii) and iii) Induction on n with induction step $n-1 \mapsto n$ using

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Lemma 4.6 (Taylor development along one integral curve). *Consider an analytic manifold Z, an analytic vector field $X \in \Theta(Z)$, and an integral curve of X*

$$\alpha : B \to Z$$

defined on a connected neighourhood B of $0 \in \mathbb{K}$ and passing through a point $z \in Z$.

Then for any open neighbourhood U of $z \in Z$, and any analytic function $f \in \mathscr{O}_Z(U)$ holds the Taylor development in one variable t:

$$f(\alpha(t)) = \sum_{n=0}^{\infty} < X^n, f > (z) \cdot \frac{t^n}{n!} \in \mathbb{K} < t > .$$

Here the notation

$$< X^n, f > := X^n(f)$$

denotes the n-fold application of the vector field X - or more precisely its derivation X_U - to f.

Proof. We compute by induction on $n \in \mathbb{N}$ the coefficients of the Taylor series as

$$\left(\frac{d}{dt}\right)^n (f \circ \alpha)(t) = < X^n, f > (\alpha(t)) :$$

The induction start $n = 0$ is obvious.

Induction step $n-1 \mapsto n$: First, the differential equation

$$X(\alpha(t)) = \dot{\alpha}(t)$$

implies

$$X(f)(\alpha(t)) = \frac{d}{dt}(f \circ \alpha)(t),$$

i.e. the directional derivation of f along the vector field X equals the time derivative of f on the integral curve α of X.

Secondly, the induction assumption applied to the analytic function

$$X(f) \in \mathscr{O}_Z(U)$$

shows:

$$\left(\frac{d}{dt}\right)^n (f \circ \alpha)(t) = \left(\frac{d}{dt}\right)^{n-1} \left(\frac{d}{dt}(f \circ \alpha)\right)(t) = < X^{n-1}, X(f) > (\alpha(t)) =$$

$$= < X^n, f > (\alpha(t)), \ q.e.d.$$

Remark 4.7 (Exponentiation of a sum of vector fields). Consider an analytic manifold Y, a number $s \in \mathbb{N}$, and a set

$$V_s := \{X_j \in \Theta(Y) : 1 \leq j \leq s\}$$

of s vector fields. In general, these vector fields do no commute. Therefore the binomial theorem does not apply for the computation of powers like

$$(t_1X_1 + ... + t_sX_s)^k,\ (t_1, ..., t_s) \in \mathbb{K}^s,\ k \in \mathbb{N}.$$

We have to replace the binomial theorem by a more general formula:

$$(t_1X_1 + ... + t_sX_s)^k = \sum_{\underline{n}=(n_1,...,n_s),\ |\underline{n}|=k} X(\underline{n}) \cdot {t_1}^{n_1} \cdot ... \cdot {t_s}^{n_s},\ |\underline{n}| := n_1 + ... + n_s.$$

For each tuple of exponents

$$\underline{n} = (n_1, ..., n_s) \in \mathbb{N}^s \text{ with } |\underline{n}| = k$$

the vector field $X(\underline{n}) \in \Theta(Y)$ is the sum of all those products of k vector fields from V_s, where the product contains each vector field $X_j,\ 1 \leq j \leq s$, as a factor n_j-times.

The following proposition 4.8 generalizes Lemma 4.6 to the case of several vector fields.

Proposition 4.8 (Taylor formula along integral curves of a Lie group). *Consider a Lie group G, a finite set of left invariant vector fields $X_1, ..., X_s \in Lie\ G$, and a point $z \in G$.*

Then: Then for any open neighbourhood U of $z \in Z$ and any analytic function $f \in \mathscr{O}_Z(U)$ holds the Taylor development

$$f(z \cdot exp(t_1X_1 + ... + t_sX_s)) = \sum_{\underline{n}=(n_1,...,n_s)\in\mathbb{N}^s} < X(\underline{n}), f > (z) \cdot \frac{{t_1}^{n_1} \cdot ... \cdot {t_s}^{n_s}}{|\underline{n}|!} \in \mathbb{K} < t_1, ..., t_s >,$$

using the notation $X(\underline{n})$ from Remark 4.7.

Proof. Consider a point $t = (t_1, ..., t_s) \in \mathbb{K}^s$, sufficiently small such that

$$F(t_1, ..., t_s) := f(z \cdot exp(t_1X_1 + ... + t_sX_s))$$

is well-defined. On the line passing through t and $0 \in \mathbb{K}^s$ define the convergent power series $u(w) \in \mathbb{K} < w >$ as

$$u(w) := F(w \cdot (t_1, ..., t_n)).$$

On one hand, $u(w)$ has the power series expansion which derives from the Taylor series of F in s variables $(t_1,...,t_s)$

$$u(w) = F(w\cdot t) = \sum_{k\in\mathbb{N}} \frac{1}{k!} \sum_{\underline{n}=(n_1,...,n_s),\ |\underline{n}|=k} ({D_1}^{n_1}\cdot ...\cdot {D_s}^{n_s}F)(0)\cdot (w\cdot t)^{\underline{n}} =$$

$$= \sum_{k\in\mathbb{N}} \frac{w^k}{k!} \sum_{\underline{n}=(n_1,...,n_s),\ |\underline{n}|=k} ({D_1}^{n_1}\cdot ...\cdot {D_s}^{n_s}F)(0)\cdot t^{\underline{n}}$$

On the other hand, the map

$$w \mapsto z\cdot exp(w\cdot(t_1X_1+...+t_sX_s))$$

is the integral curve of the vector field

$$t_1X_1+...+t_sX_s \in Lie\ (G)$$

passing through the point z. Lemma 4.6 implies for the Taylor series in one variable w

$$u(w) = f(\alpha(w)) = \sum_{k=0}^{\infty} < (t_1X_1+...+t_sX_s)^k, f > (z)\cdot \frac{w^k}{k!}.$$

For each arbitrary, but fixed $k\in\mathbb{N}$ we equate the terms of order k with respect to w from both representations of u(w), using the representation of $X(\underline{n})\in Lie\ G$ from Remark 4.7:

$$\sum_{\underline{n}\in\mathbb{N}^s,|\underline{n}|=k} ({D_1}^{n_1}\cdot ...\cdot {D_s}^{n_s}F)(0)\cdot t^{\underline{n}} =< (t_1X_1+...+t_sX_s)^k, f > (z) =$$

$$= \sum_{\underline{n}\in\mathbb{N}^s,|\underline{n}|=k} < X(\underline{n}), f > (z)\cdot t^{\underline{n}}.$$

Evaluating $u(w)$ at the argument $w=1$ gives

$$F(t) = u(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\underline{n}\in\mathbb{N}^s,|\underline{n}|=k} ({D_1}^{n_1}\cdot ...\cdot {D_s}^{n_s}F)(0)\cdot t^{\underline{n}} =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\underline{n}\in\mathbb{N}^s,|\underline{n}|=k} < X(\underline{n}), f > (z)\cdot t^{\underline{n}},\ q.e.d.$$

Lemma 4.9 (Some distinguished analytic functions).

1. The function

$$g:\mathbb{C}\to\mathbb{C},\ g(z) := \frac{1-e^{-z}}{z},$$

is analytic and expands into the convergent power series

$$g(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu+1)!} \cdot z^\nu$$

with radius of convergence $= \infty$. *In particular* $g(0) = 1$.

2. *Denote by* $\Delta(\pi, 0) \subset \mathbb{C}$ *the complex disc with radius* $= \pi$. *The inverse function* $1/g$ *is defined and analytic on* $\Delta(\pi, 0)$, *and the analytic function*

$$f_{Bern} : \Delta(\pi, 0) \to \mathbb{C}, z \mapsto f(z) := \frac{1}{g(z)} - \frac{z}{2}$$

is an even function satisfying $f_{Bern}(0) = 1$.

Proof. The only zero of the function $1 - exp(-z)$ in the domain $\Delta(\pi, 0)$ is the point $z = 0$. The power series expansion of $exp(-z)$ shows that $z = 0$ is a removable singularity of the function

$$\frac{z}{1 - exp(-z)}$$

and $f_{Bern}(0) = 1$. Therefore f_{Bern} is a holomorphic function. One checks by explicit computation

$$f_{Bern}(z) = f_{Bern}(-z),$$

i.e. f_{Bern} is an even function, q.e.d.

Remark 4.10 (Generator of the Bernoulli numbers). It is well known, that the function $f_{Bern}(z)$ from Lemma 4.9 generates the *Bernoulli numbers* B_n, named in honour of Jacob Bernoulli, see [24, Chap. 15, §1] and [31, Kap. 3. Euler]. These numbers B_n vanish for all odd indices $n \geq 3$ and

$$f_{Bern}(z) = 1 + \sum_{\nu=1}^{\infty} \frac{B_{2\nu}}{(2\nu)!} \cdot z^{2\nu}$$

with radius of convergence $= \pi$. The first Bernoulli numbers are

$$B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ B_8 = -1/30,\ B_{10} = 5/66.$$

The following Theorem 4.11 generalizes Theorem 3.25 which considers only the constant term $= 1$ of the power series g from Lemma 4.9.

Theorem 4.11 (Tangent of the exponential map). *Consider a Lie group G. The tangent map of the exponential map*

$$exp : Lie\ G \to G$$

at a point $X \in Lie\ G$ is

$$T_X exp = T_e L_{exp\ X} \circ g(ad\ X) =: Lie\ G \to T_{exp\ X} G.$$

Here we employ the power series $g \in \mathbb{C} < t >$ from Lemma 4.9.

Note: Theorem 4.11 claims the commutativity of the following diagram

$$\begin{tikzcd}
Lie\ G = T_X Lie\ G \arrow[r, "T_X exp"] \arrow[d, "g(ad\ X)"'] & T_{expX} G \\
T_e G = T_X Lie\ G \arrow[ur, "T_e L_{expX}"'] &
\end{tikzcd}$$

under the usual identification

$$T_X Lie\ G = Lie\ G = T_e G.$$

Proof.

Definition 4.12 (Logarithm on a Lie group). Consider a Lie group G and a connected open neighbourhood V of $0 \in Lie\ G$, such that an open neighbourhood U of $e \in G$ exists with

$$exp|V : V \xrightarrow{\sim} U$$

an analytic isomorphism, cf. Corollary 3.26. The inverse map

$$log := (exp|V)^{-1} : U \xrightarrow{\sim} V$$

is analytic, it is named the *logarithm* of G on U.

One of the main steps to obtain the Baker-Campbell-Hausdorff formula is the differential equation satisfied by the logarithm. For the rest of the section we follow [37, Chap. 2.15].

Theorem 4.13 (Differential equation of logarithm). *Consider a Lie group G and two elements $X, Y \in Lie\ G$. Then for suitable $\varepsilon = \varepsilon(X,Y) > 0$ the map, depending on one variable,*

$$F : B(\varepsilon, 0) \to Lie\ G, t \mapsto log(exp(tX) \cdot exp(tY)),$$

solves the system of ordinary first order differential equations

$$\dot{F}(t) = < f_{Bern}(ad\ F(t)), X + Y > +(1/2) < ad\ F(t), Y - X >$$

with the initial condition $F(0) = 0 \in Lie\ G$.

Here the notation

$$< f_{Bern}(ad\ F(t)), X + Y >$$

means the result from applying the endomorphism

$$f_{Bern}(ad\ F(t)) \in End(Lie\ G)$$

to the argument $X + Y \in Lie\ G$. Because the complex power series f_{Bern} has the radius of convergence π, the endomorphism $f_{Bern}(ad\ F(t))$ is well-defined for small $t \in \mathbb{K}$.

We will use the differential equation of F to derive a recursion relation for the coefficents in the Taylor series of F, cf. Lemma 4.14.

Proof. For suitable $\varepsilon > 0$ consider the analytic map

$$Z : B(\varepsilon, 0) \times B(\varepsilon, 0) \to Lie\ G, (u, v) \mapsto log(exp(uX) \cdot exp(vY)).$$

It satisfies for $(u, v) \in B(\varepsilon, 0) \times B(\varepsilon, 0)$ the equation

$$exp\ Z(u, v) = exp(uX) \cdot exp(vY).$$

i) *Partial differentiation with respect to* v: For any fixed $u \in B(\varepsilon, 0)$ the right-hand side of the latter equation is the composition

$$B(\varepsilon, 0) \xrightarrow{j} Lie\ G \xrightarrow{exp} G \xrightarrow{L_{exp\ uX}} G,\ j(v) := v \cdot Y.$$

It has the partial differentiation with respect to v:

$$\frac{\partial}{\partial v}(exp(uX) \cdot exp(vY)) = (T_{exp\ vY} L_{exp\ uX} \circ T_{vY} exp)(Y).$$

Theorem 4.11 implies

$$\frac{\partial}{\partial v}(exp(uX) \cdot exp(vY)) = (T_{exp\ vY} L_{exp\ uX} \circ T_e L_{exp\ vY} \circ g(ad(vY)))(Y).$$

And similar for the partial derivation of the left-hand side of the equation above:

$$\frac{\partial}{\partial v} exp(Z(u, v) = (T_e L_{exp\ Z(u,v)} \circ g(ad(Z(u, v))))(\frac{\partial Z}{\partial v}(u, v)).$$

Equating the right-hand side with the left-hand side gives

$$(T_{exp\ vY}L_{exp\ uX} \circ T_eL_{exp\ vY} \circ g(ad(vY)))(Y) = (T_eL_{exp\ Z(u,v)} \circ g(ad(Z(u,v))))(\frac{\partial Z}{\partial v}(u,v)).$$

The map

$$T_{exp\ vY}L_{exp\ uX} \circ T_eL_{exp\ vY} = T_eL_{exp(uX)\cdot exp(vY)}$$

is an isomorphism. Therefore

$$g(ad(vY))(Y) = g(ad(Z(u,v))(\frac{\partial Z}{\partial v}(u,v)).$$

Using on the left-hand side

$$g(ad(vY)(Y) = Y$$

implies

$$Y =< g(ad\ Z), \frac{\partial Z}{\partial v} >: B(\varepsilon,0) \times B(\varepsilon,0) \to Lie\ G.$$

In order to solve for $\frac{\partial Z}{\partial v}$ we use the relation

$$g^{-1}(z) = f_{Bern}(z) + z/2$$

from Lemma 4.9. It gives

$$g(ad\ Z)^{-1} = f_{Bern}(ad\ Z) + (1/2)\cdot ad\ Z$$

i.e.

$$\frac{\partial Z}{\partial v} =< f_{Bern}(ad\ Z) + (1/2)\cdot ad\ T, Y > .$$

ii) *Partial differentiation with respect to u*: Replacing (u,v) by $(-u,-v)$ in the equation above gives

$$exp(-Z(u,v)) = exp(-vY)\cdot exp(-uX).$$

For any fixed $v \in B(\varepsilon,0)$ partial differentiation with respect to u of the latter equation gives for its left-hand side in an analogous fashion

$$-X =< g(ad(-Z)), -\frac{\partial Z}{\partial u} >$$

$$-\frac{\partial Z}{\partial u} =< f_{Bern}(ad(-Z)), -X > +(1/2) < ad\ Z, X >$$

$$\frac{\partial Z}{\partial u} =< f_{Bern}(ad(-Z)) - (1/2) < ad\ Z, X > .$$

iii) *Differential equation*: From $F(t) = Z(t,t)$ and $f(ad(-Z)) = f_{Bern}(ad\ Z)$ we obtain - using that the function f_{Bern} is even -

$$\dot{F}(t) = \frac{\partial Z}{\partial u}(t,t) + \frac{\partial Z}{\partial v}(t,t) = < f_{Bern}(ad\ F(t), X+Y) > +(1/2) < ad\ F(t), Y-X >,\ q.e.d.$$

Lemma 4.14 (Recursion relation). *Consider a Lie group G and its logarithm*

$$log : U \to V$$

on a suitable connected open neighbourhood U of $e \in G$, see Definition 4.12. Then the coefficients

$$c_n = c_n(X,Y) \in Lie\ G, n \in \mathbb{N}^*$$

of the power series

$$F(t) = log(exp(tX) \cdot exp(tY)) = \sum_{n=1}^{\infty} c_n \cdot t^n \in Lie\ G < t >$$

satisfy the recursion

$$c_1 = X + Y$$

$$(n+1) \cdot c_{n+1} =$$

$$= (1/2)[X-Y, c_n] + \sum_{1 \le p \le \lfloor n/2 \rfloor} \frac{B_{2p}}{(2p)!} \sum_{\nu=(\nu_1,...,\nu_{2p}),|\nu|=n} [c_{\nu_1}, [...[c_{\nu_{2p}}, X+Y]...],\ n \ge 1, c_0 := 0.$$

The coefficients B_{2p} are the Bernoulli numbers from Remark 4.10.

Proof. Due to Theorem 4.13 the map F satisfies the differential equation

$$\dot{F}(t) = < f_{Bern}(ad\ F(t)), X+Y > +(1/2) \cdot [F(t), Y-X]$$

with the initial condition $F(0) = 0$. Here f_{Bern} is the generator of the Bernoulli numbers from Lemma 4.9. We compare on both sides of the differential equation the coefficients of the power series up to finite order: For arbitrary $n \in \mathbb{N}^*$

- Left-hand side:

$$\dot{F}(t) = \sum_{\nu=1}^{n+1} \nu \cdot t^{\nu-1} \cdot c_\nu + O(n+1) \in End(Lie\ G)$$

- Right-hand side:

$$ad\ F = \sum_{\nu=1}^{n} t^\nu \cdot ad\ c_\nu + O(n+1)$$

$$(ad\ F)^{2p} = \sum_{2p \le s \le n} t^s \cdot \sum_{\nu=(\nu_1,...,\nu_{2p}),|\nu|=s} (ad\ c_{\nu_1} \cdot ... \cdot ad\ c_{\nu_{2p}}) + O(n+1)$$

$$f_{Bern}(ad\ F) = id + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!}(ad\ F)^{2p} =$$

$$= id + \sum_{1 \le p \le \lfloor n/2 \rfloor} \frac{B_{2p}}{(2p)!} \sum_{2p \le s \le n} t^s \cdot \sum_{\nu=(\nu_1,\ldots,\nu_{2p}),|\nu|=s} (ad\ c_{\nu_1} \cdot \ldots \cdot ad\ c_{\nu_{2p}}) + O(n+1)$$

Comparing on both sides the terms of order n gives

- $n = 0$:
$$c_1 = X + Y$$
- $n \ge 1$:
$$(n+1) \cdot c_{n+1} =$$
$$= \frac{1}{2}[X-Y, c_n] + \sum_{1 \le p \le \lfloor n/2 \rfloor} \frac{B_{2p}}{(2p)!} \sum_{\nu=(\nu_1,\ldots,\nu_{2p}),|\nu|=n} [c_{\nu_1}, [c_{\nu_2}, [\ldots, , [c_{\nu_{2p}}, X+Y]\ldots], \ q.e.d.$$

Each coefficient $c_n(X,Y) \in Lie\ G, n \in \mathbb{N}^*$ from Lemma 4.14 comprises terms with n Lie brackets. The functions

$$(X,Y) \mapsto c_n(X,Y),\ n \in \mathbb{N},$$

are well-defined for any finite-dimensional Lie algebra L, independently from any Lie group. They do not refer to a fixed Lie algebra.

Definition 4.15 (Hausdorff polynomials). Consider a Lie algebra L. For each $n \in \mathbb{N}^*$ the map

$$H_n : L \times L \to L, H_n(X,Y) := c_n(X,Y),$$

with the element $c_n(X,Y) \in L$ from Lemma 4.14 is named the n-th *Hausdorff polynomial*.

Remark 4.16 (Hausdorff polynomials).

In low order the Hausdorff polynomials are

$$H_1(X,Y) = X + Y$$
$$H_2(X,Y) = (1/2)[X,Y]$$
$$H_3(X,Y) = (1/12)[[X,Y],Y] - [[X,Y],X])$$
$$H_4(X,Y) = -(1/24)[Y,[X,[X,Y]]]$$

Theorem 4.17 (Baker-Campbell-Hausdorff formula). *Consider a Lie group G. Then: For any norm $\|-\|$ on the vector space Lie G a constant $\varepsilon > 0$ exists such that the* Hausdorff series

$$H : B(\varepsilon,0) \times B(\varepsilon,0) \to Lie\ G, H(X,Y) := \sum_{n=1}^{\infty} H_n(X,Y),$$

is absolute and compact convergent. In particular, the exponential map of G satisfies the functional equation

$$exp(X) \cdot exp(Y) = exp\ (H(X,Y))$$

for all $X,Y \in B(\varepsilon,0)$.

Proof. i) *Majorisation and coefficient relations*: The power series $f_{Bern}(z)$ is absolute convergent. Therefore also

$$\tilde{f}(z) := 1 + \sum_{n=1}^{\infty} \frac{|B_{2n}|}{(2n)!} \cdot z^n$$

is a convergent power series. Therefore exists $\delta > 0$ such that the differential equation

$$\dot{y} = \tilde{f}(y) + (1/2) \cdot y$$

with initial condition $y(0) = 0$ has a unique solution

$$y : B(\delta,0) \to \mathbb{K}.$$

Note that δ does not depend on the Lie group G. Inserting the solution

$$y(z) = \sum_{n=1}^{\infty} \gamma_n \cdot z^n$$

into the differential equation provides the recursion formula for its coefficients:

- Index $= 1$:
$$\gamma_1 = 1$$
- Index ≥ 2:
$$(n+1) \cdot \gamma_{n+1} = (1/2) \cdot \gamma_n + \sum_{1 \leq p \leq \lfloor n/2 \rfloor} \frac{|B_{2p}|}{(2p)!} \sum_{\nu=(\nu_1,\ldots,\nu_{2p}),|\nu|=n} \gamma_{\nu_1} \cdot \ldots \cdot \gamma_{\nu_{2p}}, \gamma_0 := 0.$$
  Here $\lfloor n/2 \rfloor$ denotes the largest integer $\leq n/2$. In particular, $\gamma_n \geq 0$ for all $n \in \mathbb{N}$.

ii) *Estimation of Hausdorff polynomials*: Set

$$M := \max\{\|[A,B]\| \in \mathbb{R} :\ A,B \in Lie\ G \cap B(1,0)\}.$$

Then for all $X,Y \in Lie\ G$

$$\| \, [X,Y] \, \| \leq M \cdot \|X\| \cdot \|Y\|.$$

Consider two arbitrary, but fixed $X,Y \in Lie\ G$ and set $\alpha := \max\{\|X\|, \|Y\|\}$. Then the values of the Hausdorff polynomials

$$c_n := H_n(X,Y)$$

satisfy for all $n \in \mathbb{N}^*$ the estimation:

$$\|c_n\| \leq M^{n-1} \cdot (2\alpha)^n \cdot \gamma_n :$$

The estimation holds for $n = 1$ because

$$\|c_1\| = \|X+Y\| \leq 2\alpha.$$

Induction step $n \mapsto n+1$: The recursion formula from Lemma 4.14 for the values

$$c_n = H_n(X,Y), \ n \in \mathbb{N},$$

of the Hausdorff polynomials implies

$$(n+1) \cdot \|c_{n+1}\| \ \leq$$

$$\leq (1/2) \cdot 2\alpha \cdot M \cdot \|c_n\| + 2\alpha \cdot \sum_{1 \leq p \leq \lfloor n/2 \rfloor} \frac{|B_{2p}|}{(2p)!} \sum_{\nu=(\nu_1,\ldots,\nu_{2p}),|\nu|=n} M^{2p} \cdot M^{n-2p} \cdot (2\alpha)^n \cdot \gamma_{\nu_1} \cdot \ldots \cdot \gamma_{\nu_{2p}}$$

i.e.

$$(n+1) \cdot \|c_{n+1}\| \leq$$

$$M^n \cdot (2\alpha)^{n+1} \cdot ((1/2) \cdot \gamma_n + \sum_{1 \leq p \leq \lfloor n/2 \rfloor} \frac{|B_{2p}|}{(2p)!} \cdot \gamma_{\nu_1} \cdot \ldots \cdot \gamma_{\nu_{2p}}) =$$

$$= M^n \cdot (2\alpha)^{n+1} \cdot (n+1) \cdot \gamma_{n+1}$$

with the last equation due to part i). Eventually

$$\|c_{n+1}\| \leq M^n \cdot (2\alpha)^{n+1} \cdot \gamma_{n+1}.$$

iii) *Domain of convergence*: Set

$$\varepsilon := \delta/(2M) > 0$$

with the universal constant $\delta > 0$ from part i).

Then for all $X,Y \in Lie\ G \cap B(\varepsilon, 0)$ the Hausdorff series

$$\sum_{n=1}^{\infty} H_n(X,Y)$$

converges absolute and uniform: For the proof note $\alpha < \varepsilon$ and apply part ii) to obtain

$$\|H_n(X,Y)\| = \|c_n\| < M^{n-1} \cdot (2\varepsilon)^n \cdot \gamma_n \leq (1/M) \cdot \gamma_n \cdot \delta^n.$$

The convergence of

$$\sum_{n=1}^{\infty} \gamma_n \cdot z^n$$

for $|z| < \delta$, see part i), implies by majorisation the absolute and uniform convergence of the Hausdorff series for $(X,Y) \in B(\varepsilon,0) \times B(\varepsilon,0)$.

iv) *Functional equation*: By definition of the Hausdorff polynomials for all $X,Y \in Lie\ G$ holds the equality

$$F(t) := log(exp\ tX \cdot exp\ tY) = \sum_{n=1}^{\infty} H_n(tX,tY) \in (Lie\ G) < t >$$

as equality of two convergent power series with coefficients from the normed vector space *Lie G*. We have just shown that the right-hand side converges for $t = 1$ and $X,Y \in B(\varepsilon,0)$. Therefore the functional equation

$$exp\ X \cdot exp\ Y = exp(\sum_{n=1}^{\infty} H_n(X,Y))$$

holds for all $X,Y \in B(\varepsilon,0) \subset Lie\ G$, q.e.d.

Theorem 4.17 shows that the multiplication in a Lie group is determined by the group-specific exponential map and the group-independent Hausdorff series.

Chapter 5
Outlook

5.1 Lie group - Lie algebra

Definition (Integral subgroup) An *integral* subgroup of a Lie group G is a connected Lie group H, which is a subgroup (in the algebraic sense) of G, such that the injection

$$H \hookrightarrow G$$

is an immersion. See [4, III,§6,2].

Note: An integral subgroup $H \subset G$ is not necessary a Lie subgroup. A counterexample is a line H with irrational slope on the real torus

$$G := T^2 = \mathbb{R}^2/\mathbb{Z}^2.$$

There is a close relationship between the algebraic properties of a Lie group G or at least the connected component $G^e \subset G$ of the neutral element and the algebraic propertie of its Lie algebra *Lie G*. We list a series of examples:

Theorem (Dictionary Lie group - Lie algebra). Consider a Lie group G.

1. *Adjoint representation*: $Lie(Ad) = ad$, see Proposition 4.2.

2. *Normal subgroup versus ideal*: If G is connected and $H \subset G$ an integral subgroup, then are equivalent:
   - $H \subset G$ is a normal subgroup.
   - $Lie\ H \subset Lie\ G$ is an ideal.

3. *Center*: The center of G is defined as

$$Z(G) := \{g \in G : g \cdot h = h \cdot g \ for\ all\ h \in G\}.$$

Then:

- The center $Z(G) \subset G$ is a Lie subgroup.

- For connected G the Lie algebra of the center is the center of *Lie G*, i.e.

$$Lie\ Z(G) = Z(Lie\ G) := \{x \in Lie\ G : [x, Lie\ G] = 0\}.$$

4. *Commutativity*: If G is connected, then are equivalent:

- G is Abelian

- *Lie G* is Abelian.

Note. In general, properties of the Lie algebra *Lie G* can be transferred only to the connected component $G^e \subset G$ of G. In [9, Chapter 1.9] a non-Abelian example

$$G = N(T) = T \dot{\cup} T' \subset SU(2)$$

is constructed with $G^e = T$ is Abelian. As a consequence

$$Lie\ G = Lie\ T$$

is Abelian but G is not.

According to Ado's theorem any Lie algebra embedds into a Lie algebra of matrices. The analogue does not hold for Lie groups:

Proposition (Coverings of $SL(n, \mathbb{R})$)

The total space of the universal covering projection of the matrix group $SL(n, \mathbb{R})$

$$p : G \to SL(n, \mathbb{R}), n \geq 2,$$

is a Lie group G which is not a matrix group, i.e. G is not a closed subgroup of $GL(N, \mathbb{R})$ for any $N \in \mathbb{N}$.

5.2 Two of Lie's Theorems

The following Theorem 5.2 is also named Lie's First Theorem .

Theorem (Lifting of a Lie algebra morphism)

Consider two Lie groups G and H and a morphism between their Lie algebras

$$\phi : Lie\ G \to Lie\ H.$$

If G is connected and simply connected then a unique morphism of Lie groups

$$\Phi : G \to H$$

exists with $Lie\ \Phi = \phi$, i.e. the map

$$Lie : Hom(G,H) \to Hom(Lie\ G, Lie\ H), \Phi \mapsto Lie\ \Phi,$$

is bijective for G connected and simply connected.

Theorem (Lie's third theorem)

Any $\mathbb{K}$-Lie algebra is the Lie algebra of a uniquely determined connected and simply connected $\mathbb{K}$-analytic Lie group.

5.3 Analytic versus smooth, differentiable, and continuous

Theorem (Cartan's theorem on closed subgroups)

Consider a real Lie group G. Any closed subgroup $H \subset G$ is a real Lie subgroup.

In the real context a smooth or C^k-Lie group is respectively a smooth or C^k-manifold with a group multiplication of the corresponding class. If $k \geq 2$ one can develop the whole Lie group theory including the Baker-Campbell-Hausdorff formula in the smooth or in the C^k-category. But the Lie algebra of a Lie group from this category has to be obtained as the Lie algebra of left-invariant global vector fields, it cannot be derived from the power series of a formal group law.

Using as a next step the Baker-Campbell-Hausdorff formula, one shows that the smooth or C^k-structure is already real analytic: Any Lie group from the smooth or C^k-category is a Lie group in the real-analytic category, see [9, Theor. 1.6.3].

The final step to integrate also the continous category had been listed as Hilbert's fifth problem in 1900. Its solution is due to Gleason, Montgomery, and Zippin in 1952:

Theorem (Solution of the Fifth Hilbert Problem) Any locally compact topological group with no small subgroups has a unique structure of a real-analytic Lie group.

References

The main references for these notes are

- Lie algebra: Hall [18], Humphreys [23] and Serre [32]
- Lie group: Serre [33].

In addition,

- References with focus on mathematics:
  [2], [3], [5], [20], [21], [22], [23], [37], [38]
- References with focus on physics:
  [1], [13], [16], [17],[28]
- References with focus on both mathematics and physics:
  [14], [18], [34]

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