Complex Analysis

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I prepared these notes for the students of my lecture. The present lecture took place during the summer term 2019 at the mathematical department of LMU (Ludwig-Maximilians-Universität) at Munich.

The first input are notes taken from former lectures by Otto Forster.

Compared to the oral lecture in class these written notes contain some additional material. In particular, at the end of several chapters I added an outlook to some more advanced topics from complex analysis of several variables and from the theory of complex spaces.

Please report any errors or typos to wehler@math.lmu.de

Release notes:

- Release 2.6: Chapter 8: Minor revisions.
- Release 2.5: Minor revisions.
- Release 2.4: Chapter 8 completed. List of results added.
- Release 2.3: Chapter 8, Section 8.2 added.
- Release 2.2: Definition 7.6 corrected.
- Release 2.1: Chapter 8, Section 8.1 added. References to problem sheet 12 removed
- Release 2.0: Chapter 7 completed. Chapter 5, references to problem sheet 11 removed.
- Release 1.9: Chapter 6 completed. References to problem sheet 12 to be removed later.
- Release 1.8: Minor revisions.
- Release 1.7: Chapter 6, Section 6.1 added.
- Release 1.6: Chapter 5 completed. References to problem sheet 11 to be removed later.
- Release 1.5: Chapter 5, Section 5.2 added.
- Release 1.4: Proposition 5.7: typo corrected.
- Release 1.3: Chapter 5, Section 5.1 added.
- Release 1.2: Chapter 4, Section 4.3 added.
- Release 1.1: Definition 3.10: typo corrected.
- Release 1.0: Remark 3.28 added, Chapter 4 completed.
- Release 0.9: Chapter 4, Section 4.1 created.
- Release 0.8: Section 2.2: Jacobi matrix replaced by transposed matrix, typo C-R differential equation corrected.
- Release 0.7: Chapter 3 completed.
- Release 0.6: Chapter 3, Section 3.2 created.
- Release 0.5: Chapter 3, Section 3.1 created.
- Release 0.4: Chapter 2 created, Section 1.4 updated.

- Release 0.3: Chapter 1 completed.
- Release 0.2: Section 1.2 created.
- Release 0.1: Chapter 1, Section 1.1 created.

Contents

Introduction

$$1 + e^{i\pi} = 0$$

Euler's identity is considered an example of beauty in mathematics. Some people even call it "the most famous formula in all mathematics".

Euler's identity relates the natural numbers 0 and 1 and the transcendental numbers e and π to the complex number i. The latter is the imaginary unit satisfying

$$i^2 = -1.$$

The formula above exemplifies the way mathematics reveals some unexpected structure, when one considers its concepts in the context of complex numbers.

Complex numbers form the context of complex analysis, the subject of the present lecture notes. Complex analysis investigates analytic functions. Locally, analytic functions are convergent power series. The well-known exponential series extends to the complex plane and evaluates at the non-real argument $i\pi$ from Euler's identity as

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} =$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} + i \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = cos\ x + i \cdot sin\ x.$$

The result

$$e^{i\pi} = cos\ \pi + i \cdot sin\ \pi$$

explains the key value

$$e^{i\pi} = -1,$$

hereby proving the Euler identiy.

Note. Not every formula has already revealed its deeper meaning. A famous riddle for contemporary physicists is the formula

$$L_{Planck} = \sqrt{\frac{\hbar G}{c^3}}$$

This formula comprises Planck's constant $\hbar$, Newton's constant of gravitation G, and the velocity of light c. The formula relates quantum theory, gravitation and the velocity of light to the Planck lenght L_{Planck}, the scale where spacetime becomes quantized.

Chapter 1
Analytic Functions

It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one.

Jacques Hadamard, 1945

The concept of holomorphy is accessible from different directions:

- Section 1.1 starts from *convergent power series*. They give reasons for the definition of *analytic functions* in Definition 1.8 (Access due to *Weierstrass*). All power series from calculus extend from intervals of the real line into the domain of complex numbers, see Section 1.3.

- Section 2.1 in Chapter 2 introduces a second path to holomorphy. It starts from *differentiability* in Definition 2.1 (Access due to *Cauchy*). Theorem 2.3 shows: Any analytic function in an open set $U \subset \mathbb{C}$ is differentiable. The derivative is again an analytic function.

- A third path starts from the *Cauchy-Riemann differential equations* for the partial derivatives in the real sense, see Theorem 2.6 (Access due to *Riemann*). The theorem also proves the equivalence of path two and path three.

Deeper is the fact that any differentiable function in an open set $U \subset \mathbb{C}$ is analytic. This result follows from a simple variant of Cauchy's integral formula. It will be demonstrated not until Chapter 3. The proof will be the final step in establishing the equivalence of the three paths to holomorphy.

Cauchy, Weierstrass and Riemann are the three protagonists of complex analysis in the 19th century.

1.1 Calculus of convergent power series

Analytic functions are those functions which expand locally into a convergent power series. Therefore the present section investigates the calculus of convergent power series.

Definition 1.1 (Compact convergence). A sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of functions

$$f_\nu : U \to \mathbb{C},\ \nu \in \mathbb{N},$$

on an open subset $U \subset \mathbb{C}$ is *compact convergent* to a function

$$f : U \to \mathbb{C}$$

if for every compact subset $K \subset U$ the sequence of restrictions $(f_\nu|K)_{\nu\in\mathbb{N}}$ is uniformly convergent to $f|K$.

Remark 1.2 (Convergence). Consider $f_n : U \to \mathbb{C},\ n \in \mathbb{N}$, and $f : U \to \mathbb{C}$. For

$$f = \lim_{n\to\infty} f_n$$

the convergence is

- *pointwise*:

$$\forall x \in U\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : |f(x) - f_n(x)| < \varepsilon$$

- *uniform*:

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in U\ \forall n \geq N : |f(x) - f_n(x)| < \varepsilon$$

- *compact*:

$$\forall K \subset U \text{ compact } \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in K\ \forall n \geq N : |f(x) - f_n(x)| < \varepsilon$$

Theorem 1.3 (Convergence of power series). *Consider a fixed point $a \in \mathbb{C}$ and a power series*

$$\sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

with center *a and complex coefficients*

$$c_n \in \mathbb{C},\ n \in \mathbb{N}.$$

1. *If the series is convergent for at least one*

$$z_0 \in \mathbb{C},\ z_0 \neq a,$$

then it is absolutely and compactly convergent in the whole open disk

$$D_r(a) := \{z \in \mathbb{C} :\ |z-a| < r\},\ r := |z_0 - a|.$$

The function

$$f : D_r(a) \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} c_n \cdot (z-a)^n,$$

is continuous.

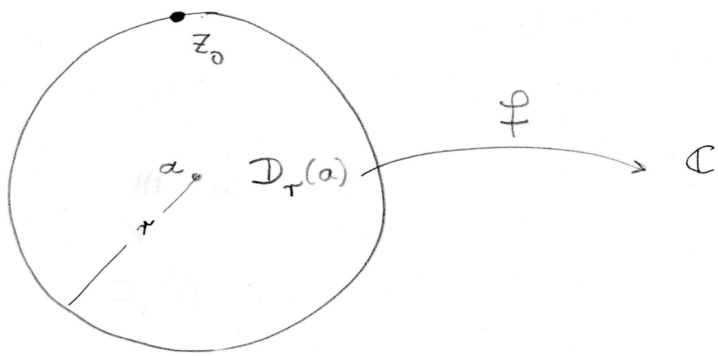


Fig. 1.1 Compact convergence in $D_r(a)$

2. *There exists a radius*

$$0 \leq R \leq \infty,$$

the radius of convergence*, which discriminates between the following two alternatives:*

- *The series is convergent for all $z \in \mathbb{C}$ with $|z-a| < R$ and*
- *divergent for all $z \in \mathbb{C}$ with $|z-a| > R$.*

The disk $D_R(a)$ is the named the disk of convergence*. If $R > 0$ then the power series is named a* convergent power series.

Proof. A statement about a power series with center a reduces by the linear substitution of variables

$$w := z - a$$

to a statement about a power series with center $= 0$. Hence, w.l.o.g. we assume $a = 0$.

1. Assume that the series is convergent for $z_0 \in \mathbb{C}$. Then

$$(c_n \cdot z_0^n)_{n \in \mathbb{N}}$$

is a null-sequence, in particular bounded: Exists $M > 0$ such that for all $n \in \mathbb{N}$

$$|c_n \cdot z_0^n| \leq M.$$

For any $z \in \mathbb{C}$ with $|z| < |z_0|$ we have

$$\sum_{n=0}^{\infty} |c_n \cdot z^n| = \sum_{n=0}^{\infty} |c_n \cdot z_0^n| \cdot \left|\frac{z}{z_0}\right|^n \leq M \cdot \sum_{n=0}^{\infty} \left|\frac{z}{z_0}\right|^n$$

and the dominating series is a convergent geometric series. Because the geometric series is compactly convergent for

$$\left|\frac{z}{z_0}\right| < 1 \text{ i.e. } |z| < |z_0|,$$

by dominated convergence the given power series is absolutely and compactly convergent for

$$|z| < |z_0|.$$

Continuity is a local property, i.e. a function is continuous iff its restriction to any open subset is continuous. For any $N \in \mathbb{N}$ the polynomial

$$\sum_{n=0}^{N} c_n \cdot (z-a)^n$$

is continuous. The uniform limit of continuous functions is continuous. Hence f is continuous.

2. Define

$$R := \sup\left\{|z| : \sum_{n=0}^{\infty} c_n \cdot z^n \text{ convergent}\right\}.$$

Due to part 1) the series is convergent for all $z \in D_R(0)$. By definition of R the series is divergent for all $z \in \mathbb{C}$ with

$$|z| > R, \; q.e.d.$$

Like in the proof of Theorem 1.3, we will often restrict to power series with center $a = 0$. If not stated otherwise, the general case then follows by a linear substitution of variables.

The following Theorem 1.4 uses the convention

$$1/0 = \infty \text{ and } 1/\infty = 0.$$

Theorem 1.4 (Radius of convergence). *Consider a power series*

$$\sum_{n=0}^{\infty} c_n \cdot z^n$$

and let R be its radius of convergence.

1. Root test*:*

i) Cauchy-Hadamard formula*:*

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}}$$

ii) If for suitable $r > 0$ and for all but finitely many indices $n \in \mathbb{N}$

$$\sqrt[n]{|c_n|} < \frac{1}{r}$$

then $R \geq r$.

iii) If for suitable $r > 0$ and for infinitely many indices $n \in \mathbb{N}$

$$\sqrt[n]{|c_n|} > \frac{1}{r}$$

then $R \leq r$.

2. Ratio test*:*

i) If for suitable $r > 0$ and for all but finitely many $n \in \mathbb{N}$

$$\left|\frac{c_{n+1}}{c_n}\right| < \frac{1}{r}$$

then $R \geq r$.

ii) If for suitable $r > 0$ and infinitely many $n \in \mathbb{N}$

$$\left|\frac{c_{n+1}}{c_n}\right| > \frac{1}{r}$$

then $R \leq r$.

iii) If the limit

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

Proof. 1. i) Define

$$r := \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}}.$$

- To prove convergence for $|z| < r$ we may assume $r > 0$. We choose a number r_1 with

  $$0 < |z| < r_1 < r.$$

  Then

  $$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{r} < \frac{1}{r_1}$$

  and for all but finitely many $n \in \mathbb{N}$

  $$|c_n| < \frac{1}{r_1^n} \ \text{ i.e. } \ |c_n| \cdot r_1^n < 1.$$

  Hence for suitable $M > 0$ and all $n \in \mathbb{N}$

  $$|c_n| \cdot r_1^n < M.$$

  Analogously to the proof of Theorem 1.3 we compute

  $$\sum_{n=0}^{\infty} |c_n \cdot z|^n \le \sum_{n=0}^{\infty} |c_n| \cdot \frac{|z|^n}{r_1^n} \cdot r_1^n \le M \cdot \sum_{n=0}^{\infty} \left| \frac{z}{r_1} \right|^n$$

  Domination by the convergent geometric series implies the absolute convergence of the series

  $$\sum_{n=0}^{\infty} c_n \cdot z^n.$$

  Hence $R \ge r$.

- To prove divergence for $|z| > r$ we may assume $r < \infty$. Then $|z| > r$ implies

  $$\frac{1}{|z|} < \frac{1}{r} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

  Hence infinitely many indices $n \in \mathbb{N}$ exist with

  $$\frac{1}{|z|} < \sqrt[n]{|c_n|} \ \text{ i.e. } \frac{1}{|z|^n} < |c_n| \ \text{ i.e. } \ 1 < |c_n \cdot z^n|.$$

  Therefore the sequence $(c_n \cdot z^n)_{n\in\mathbb{N}}$ cannot be a null-sequence, and the series

  $$\sum_{n=0}^{\infty} c_n \, z^n$$

cannot converge. Hence $R \leq r$.

The alternative from Theorem 1.3, and both parts together imply $r = R$.

ii) Assume for all but finitely many indices $n \in \mathbb{N}$

$$\sqrt[n]{|c_n|} < \frac{1}{r}$$

Then

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \frac{1}{r} \text{ i.e. } R \geq r.$$

iii) Assume an infinite set $I \subset \mathbb{N}$ such that for all $n \in I$

$$\sqrt[n]{|c_n|} > \frac{1}{r}$$

Then

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \geq \frac{1}{r} \text{ i.e. } R \leq r.$$

2. i) Assume for all but finitely many $n \in \mathbb{N}$

$$\left|\frac{c_{n+1}}{c_n}\right| < \frac{1}{r}.$$

We recall the ratio test for a series of complex numbers

$$\sum_{n=0}^{\infty} a_n :$$

If exists θ with

$$0 < \theta < 1$$

such that for all but finitely many $n \in \mathbb{N}$

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \theta,$$

then the series is absolutely convergent.

For all $z \in D_r(0),\ z \neq 0$, we apply the ratio test to the power series

$$\sum_{n=0}^{\infty} c_n \cdot z^n$$

and set

$$a_n := c_n \cdot z^n.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}}{c_n}\right| \cdot |z| < \frac{|z|}{r} =: \theta < 1.$$

Hence the power series converges absolutely for any $z \in D_r(0)$, which implies $R \geq r$.

ii) Assume an infinite set $I \subset \mathbb{N}$ such that for all $n \in I$

$$\left|\frac{c_{n+1}}{c_n}\right| > \frac{1}{r}$$

For all $z \in \mathbb{C}$ with $|z| > r$ and for all $n \in I$

$$|c_{n+1} \cdot z^{n+1}| > \frac{|z|}{r} \cdot |c_n \cdot z^n|$$

Hence, the sequence $(c_n \cdot z^n)_{n \in \mathbb{N}}$ cannot be a null-sequence, and the power series is divergent at z. We obtain $R \leq r$.

iii) Assume the existence of

$$\lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| = 1/r$$

Case $r > 0$: For small $\varepsilon > 0$ and all but finitely many indices $n \in \mathbb{N}$

$$\left|\frac{c_{n+1}}{c_n}\right| < \frac{1}{r - \varepsilon} \text{ and } \left|\frac{c_{n+1}}{c_n}\right| > \frac{1}{r + \varepsilon}.$$

Hence, for small $\varepsilon > 0$ according to the first two parts

$$R \geq r - \varepsilon \text{ and } R \leq r + \varepsilon,$$

i.e.

$$R = r.$$

Case $r = 0$: For any $\rho > 0$ and all but finitely many indices $n \in \mathbb{N}$

$$\left|\frac{c_{n+1}}{c_n}\right| > 1/\rho.$$

Part ii) implies $R \leq \rho$. Therefore

$$R = 0 = r, \ q.e.d.$$

We recall the following ordinary rearrangement theorem about linear orders of a series of complex numbers:

If a series converges absolutely with respect to a specific linear order of summation, then it converges absolutely with respect to each linear order of summation, and each of these orders produces the same limit of the series, see [8, §7, Satz 8].

More subtle is Theorem 1.5 about the rearrangement of double series.

Theorem 1.5 (Cauchy's rearrangement of double series). *Consider a double series*

$$\sum_{i,j\in\mathbb{N}} a_{ij}$$

of complex numbers.

1. *Assume that the double series is absolutely convergent for at least one linear order of summation. Then:*

   - *All row-sums*
     $$S_i := \sum_{j=0}^{\infty} a_{ij},\ i \in \mathbb{N},$$
     and all column-sums
     $$\tilde{S}_j := \sum_{i=0}^{\infty} a_{ij},\ j \in \mathbb{N},$$
     are absolutely convergent.

   - *The series of the row-sums and the series of the column-sums have the same value*
     $$\sum_{i=0}^{\infty} S_i = \sum_{j=0}^{\infty} \tilde{S}_j = \sum_{i,j\in\mathbb{N}} a_{ij}$$
     with respect to any linear order of the double sum on the right-hand side.

   - *The series of the row-sums*
     $$\sum_{i=0}^{\infty} S_i$$
     and the series of the column-sums
     $$\sum_{j=0}^{\infty} \tilde{S}_j$$
     are absolutely convergent.

2. Assume: All row-sums

$$S_i := \sum_{j=0}^{\infty} a_{ij},\ i \in \mathbb{N},$$

are absolutely convergent, and the series

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right)$$

is convergent. Then the double series

$$\sum_{i,j \in \mathbb{N}} a_{ij}$$

is absolutely convergent for at least one linear order of summation.

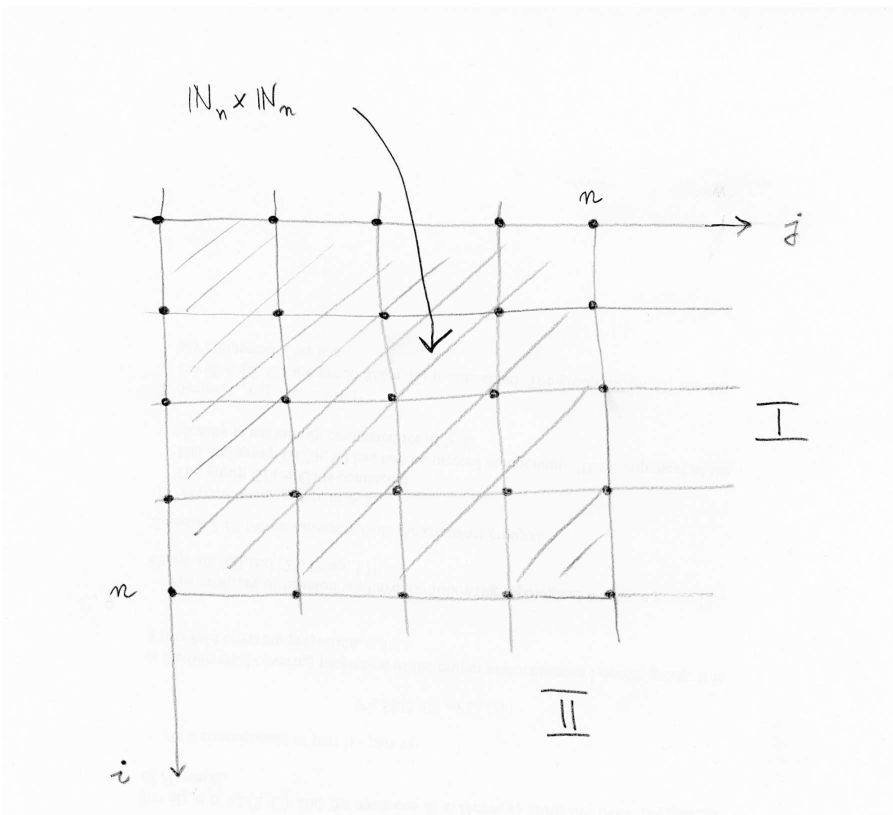


Fig. 1.2 Double indices

Note in Theorem 1.5 the difference between the order of summation of the two series on the left-hand and on the right-hand side of the equation

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{ij}\right) = \sum_{i,j\in\mathbb{N}} a_{ij}.$$

The ordinary rearrangement theorem deals with summing up all terms of the double series at the right-hand side in an arbitrary linear order. While the left-hand side considers two different types of summation:

$$\sum_{i=0}^{\infty} S_i$$

first computes all row-sums S_i, and afterwards computes the sum of all row-sums. While

$$\sum_{j=0}^{\infty} \tilde{S}_j$$

first computes all column-sums $\tilde{S}_j$, and afterwards computes the sum of all column-sums.

Proof. ad 1). Due to the ordinary rearrangement theorem the double series is absolutely convergent with respect to any linear order, and its limit does not depend on the order of summation. We choose the linear order

$$\lim_{n\to\infty} \sum_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} a_{ij} =: A$$

with

$$\mathbb{N}_n := \{k \in \mathbb{N} : \ k \le n\},$$

see Figure 1.2.

Consider an arbitrary $\varepsilon > 0$ which will be fixed in part i) and ii).

i) *Absolute convergence of each row-sum* $S_i = \sum_{i=0}^{\infty} a_{ij},\ i \in \mathbb{N}$:

There exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\sum_{\mathbb{N}\times\mathbb{N}\setminus(\mathbb{N}_n\times\mathbb{N}_n)} |a_{ij}| \le \varepsilon,$$

In particular, for any arbitrary but fixed $i \in \mathbb{N}$ and for all $n \ge n_0$

$$\sum_{j\ge n} |a_{ij}| < \varepsilon,$$

i.e. the series

$$S_i = \sum_{j=0}^{\infty} a_{ij}$$

is absolutely convergent.

ii) *Series of the row-sums*: Claim

$$\lim_{n\to\infty} \sum_{i=0}^{n} S_i = \lim_{n\to\infty} \sum_{i=0}^{n} \left(\sum_{j=0}^{\infty} a_{ij} \right) = A = \lim_{n\to\infty} \sum_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} a_{ij}.$$

There exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$\left| A - \sum_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} a_{ij} \right| < \varepsilon$$

The triangle inequality implies for $n \geq max\{n_0,\ n_1\}$

$$\left| \sum_{i=0}^{n} \left(\sum_{j=0}^{\infty} a_{ij} \right) - A \right| \leq \left| \sum_{i=0}^{n} \left(\sum_{j=0}^{\infty} a_{ij} \right) - \sum_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} a_{ij} \right| + \left| \sum_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} a_{ij} - A \right| \leq$$

$$\leq \left| \sum_{i=0}^{n} \left(\sum_{j=0}^{\infty} a_{ij} \right) - \sum_{i=0}^{n} \left(\sum_{j=0}^{n} a_{ij} \right) \right| + \varepsilon = \left| \sum_{i=0}^{n} \left(\sum_{j>n} a_{ij} \right) \right| + \varepsilon \leq \left| \sum_{i=0}^{n} \left(\sum_{j>n} |a_{ij}| \right) \right| + \varepsilon$$

$$\leq \left(\sum_{\mathbb{N}\times\mathbb{N}\setminus(\mathbb{N}_n\times\mathbb{N}_n)} |a_{ij}| \right) + \varepsilon \leq \varepsilon + \varepsilon = 2\varepsilon$$

This estimate finishes the proof of

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\sum_{j=0}^{\infty} a_{ij} \right) = A.$$

iii) *Absolute convergence of the series of the row-sums*: Part ii) applied to the series

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} |a_{ij}|$$

implies

$$\sum_{i=0}^{\infty} |S_i| = \sum_{i=0}^{\infty} \left(|\sum_{j=0}^{\infty} a_{ij}| \right) \leq \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right) = \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} |a_{ij}| < \infty$$

iv) *Interchanging the roles of i and j*: Apparently one may interchange the roles of i and j in part i) - iii).

ad 2) Assume for any $i \in \mathbb{N}$ the convergence of

$$\sum_{j=0}^{\infty} |a_{ij}|$$

and the convergence

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right) =: M < \infty.$$

The assumption implies for any index $n \in \mathbb{N}$ the estimate of the finite sum

$$\sum_{(i,j) \in \mathbb{N}_n \times \mathbb{N}_n} |a_{ij}| \leq M.$$

As a consequence the double sum

$$\sum_{i,j} a_{ij}$$

converges absolutely in a certain linear order of summation, q.e.d.

We shall apply Theorem 1.5 when proving

- the formula for the Cauchy product of power series (Corollary 1.6),
- the change of the centre of a convergent power series (Theorem 1.7), and
- the analyticity of the composition of two analytic functions (Proposition 1.19).

Corollary 1.6 (Cauchy product of power series). *Consider two power series*

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n \cdot z^n$$

with radius of convergence $\geq r$*. Then also the power series*

$$h(z) := \sum_{n=0}^{\infty} c_n \cdot z^n,$$

the Cauchy product of f and g*, with*

$$c_n := \sum_{k=0}^{n} a_{n-k} \cdot b_k = \sum_{k+m=n} a_k \cdot b_m, \; n \in \mathbb{N},$$

has radius of convergence $\geq r$*, and*

$$h(z) = f(z) \cdot g(z).$$

Proof. For $z \in D_r(0)$ we show: The double series

$$\sum_{k,m} A_{km}$$

with

$$A_{km} := a_k z^k \cdot b_m z^m = a_k b_m \cdot z^{k+m}$$

satisfies the assumption of Theorem 1.5, part 2, with $(i,j) = (k,m)$: First, we have

$$S_k := \sum_{m=0}^{\infty} A_{km}$$

and

$$\sum_{m=0}^{\infty} |A_{km}| = \sum_{m=0}^{\infty} |a_k \cdot z^k \cdot b_m \cdot z^m| = |a_k \cdot z^k| \cdot \sum_{m=0}^{\infty} |b_m \cdot z^m|.$$

The last series is convergent due to Theorem 1.3 applied to the power series $g(z)$.

Secondly, the series

$$\sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} |A_{km}| \right) = \left(\sum_{k=0}^{\infty} |a_k \cdot z^k| \right) \cdot \sum_{m=0}^{\infty} |b_m \cdot z^m|$$

is convergent due to Theorem 1.3 applied to the power series $f(z)$.

Now Theorem 1.5, part 1) applies: On one hand, by summing the row-sums

$$\sum_{k=0} S_k = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} A_{km} \right) = \sum_{k=0}^{\infty} \left(a_k \cdot z^k \cdot \sum_{m=0}^{\infty} b_m \cdot z^m \right) =$$

$$= \sum_{k=0}^{\infty} (a_k \cdot z^k \cdot g(z)) = \left(\sum_{k=0}^{\infty} a_k \cdot z^k \right) \cdot g(z) = f(z) \cdot g(z).$$

On the other hand, by summing the double sum along the diagonals

$$\sum_{k,m} A_{km} = \sum_{n=0}^{\infty} \left(\sum_{k+m=n} A_{km} \right) = \sum_{n=0}^{\infty} c_n \cdot z^n = h(z), \ q.e.d.$$

Theorem 1.7 (Changing the centre of a convergent power series). *Consider a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

with center $= 0$ *and radius of convergence* $r > 0$. *Then for any point* $a \in D_r(0)$ *the power series*

$$\sum_{k=0}^{\infty} b_k \cdot (z-a)^k$$

with center $= a$ and coefficients

$$b_k := \sum_{n=k}^{\infty} \binom{n}{k} c_n \cdot a^{n-k},\ k \in \mathbb{N},$$

is convergent with radius of convergence at least

$$\rho := r - |a|,$$

and satisfies for all $z \in D_\rho(a)$

$$f(z) = \sum_{k=0}^{\infty} b_k \cdot (z-a)^k.$$

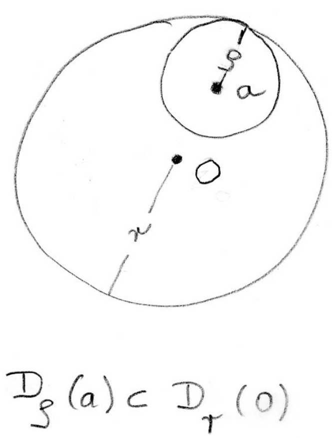


Fig. 1.3 Changing the center of a convergent power series

Proof. The binomial theorem implies

$$z^n = ((z-a)+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot (z-a)^k.$$

Hence

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n = \sum_{n=0}^{\infty} c_n \cdot \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot (z-a)^k \right).$$

For $z \in D_\rho(a)$ we introduce the coefficients

$$A_{nk} := \begin{cases} c_n \cdot \binom{n}{k} \cdot a^{n-k} \cdot (z-a)^k & k \leq n \\ 0 & otherwise \end{cases}$$

and the double series

$$\sum_{n,k} A_{nk}.$$

By definition, the series of the corresponding row-sums is $f(z)$. The double series satisfies the assumptions of Theorem 1.5, part 2:

- For each arbitrary but fixed $n \in \mathbb{N}$

  $$\sum_{k=0}^{\infty} |A_{nk}| = \sum_{k=0}^{n} |A_{nk}| < \infty$$

  because the series is finite.

- The series

  $$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} |A_{nk}| \right) = \sum_{n=0}^{\infty} \left(|c_n| \cdot \sum_{k=0}^{n} \binom{n}{k} \left| a^{n-k} \right| \cdot |z-a|^k \right) =$$

  $$= \sum_{n=0}^{\infty} |c_n| \cdot (|a| + |z-a|)^n = \sum_{n=0}^{\infty} |c_n| \cdot r_0^n < \infty$$

  with

  $$r_0 := |a| + |z-a| < |a| + \rho < |a| + r - |a| = r$$

  is convergent, because for any $w \in D_r(0)$ the series

  $$\sum_{n=0}^{\infty} c_n \cdot w^n$$

  is absolutely convergent due to Theorem 1.3 .

Theorem 1.5 implies that the series of row-sums equals the series of column-sums, which is

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} A_{nk} \right) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} c_n \cdot a^{n-k} \right) (z-a)^k = \sum_{k=0}^{\infty} b_k \cdot (z-a)^k, \ q.e.d.$$

If a power series $f(z)$ with center $= 0$ has radius of convergence at least r, then Theorem 1.7 states that the resulting power series $g(z)$ with center $= a$ has radius of convergence at least $\rho = r - |a|$. But the radius of convergence of $g(z)$ may be strictly larger than ρ. Then the power series $g(z)$ extends the function defined by the power series $f(z)$ to a larger domain of definition.

1.2 Fundamental properties of analytic functions

We are now ready to define Weierstrass' basic concept of complex analysis, the analytic function. Analytic functions are those functions which locally expand into a convergent power series.

Definition 1.8 (Analytic function). Consider an open subset $U \subset \mathbb{C}$. A function

$$f : U \to \mathbb{C}$$

is *analytic*, if for all points $a \in U$ a radius $r > 0$ with $D_r(a) \subset U$ exists such that f expands in the disk $D_r(a)$ into a convergent power series with center $= a$

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n.$$

Definition 1.8 shows that being analytic is a local property: A function f is analytic on an open set U iff the restriction of f to each open subset of U is analytic. Note that we avoid terms like "analytic at a point z_0". Being analytic always refers to an open set, e.g., an open neighbourhood of a point z_0. Moreover, analyticity does not require that any of the power series with center $= a$ is convergent for all $z \in U$.

Proposition 1.9 (Ring of analytic functions).

1. *Analytic functions are continuous.*

2. *If the functions $f, g : U \to \mathbb{C}$ are analytic and $\lambda \in \mathbb{C}$ then also the functions*

$$f + g,\ f \cdot g \text{ and } \lambda \cdot f$$

are analytic. Hence the set of analytic functions on an open set $U \subset \mathbb{C}$ is a ring with respect to addition and multiplication, and even a $\mathbb{C}$-algebra with respect to additional scalar multiplication. If f has no zeros in U then also $1/f$ is analytic in U.

3. *A power series*

$$\sum_{n=0}^{\infty} c_n \cdot (z-z_0)^n$$

which is convergent in $D_r(z_0),\ r > 0$, defines the analytic function

$$f : D_r(z_0) \to \mathbb{C},\ z \mapsto f(z) := \sum_{n=0}^{\infty} c_n \cdot (z-z_0)^n.$$

Proof. 1. Any analytic function is continuous, because a convergent power series is continuous.

2. The analyticity of the product $f \cdot g$ follows from the Cauchy product formula for power series, see Corollary 1.6.

   To prove the analyticity of $1/f$ we show: If a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

   satisfies $f(0) = c_0 \neq 0$, then for suitable $r > 0$ the function

$$1/f : D_r(0) \to \mathbb{C}$$

   expands into a convergent power series: W.l.o.g $c_0 = 1$. Set $d_0 := 1$ and define the power series

$$g(z) := \sum_{n=0}^{\infty} d_n \cdot z^n$$

   with recursively defined coefficients

$$d_1 := c_1 \text{ and } d_n := -c_1 \cdot d_{n-1} - c_2 \cdot d_{n-2} - \ldots - c_{n-1} \cdot d_1 - c_n,\ n \geq 2.$$

   Let R denote the radius of convergence of $f(z)$. We have $R \neq 0$. The formula of Cauchy-Hadamard from Theorem 1.4 implies the existence of a constant $M > 0$ such that for all indices $n \in \mathbb{N}$

$$|c_n| \leq M^n$$

   Hence, by induction, for all $n \geq 1$

$$|d_n| \leq |c_n| + \sum_{\nu=1}^{n-1} |c_\nu| \cdot |d_{n-\nu}| \leq M^n + (1/2) \cdot \sum_{\nu=1}^{n-1} M^\nu \cdot (2M)^{n-\nu} = (1/2) \cdot (2M)^n$$

   Hence g has radius of convergence at least

$$r := \frac{1}{2M} > 0,$$

   therefore both power series

$$f(z) \text{ and } g(z)$$

   are convergent. Their Cauchy product computes as

$$f(z) \cdot g(z) = 1 \text{ i.e. } g(z) = 1/f(z)$$

3. We have to show that f expands for any point $a \in D_r(z_0)$ into a convergent power series with center $= a$. The result follows from Theorem 1.7. We will give the details to exemplify how to reduce a statement about a power series with arbitrary

center to the analoguous statement for a power series with center $= 0$: First we make the linear change of the argument

$$w := z - z_0$$

to obtain a power series with center $= 0$. Then we apply Theorem 1.7. Eventually, we translate the result back to the original series with center $= a$.

After the substitution

$$w := z - z_0$$

the series

$$g(w) := \sum_{n=0}^{\infty} c_n \cdot w^n$$

converges for $w \in D_r(0)$. The substitution transforms the new center a to the point

$$a_0 := a - z_0.$$

If we define

$$\rho := r - |a_0| = r - |a - z_0| > 0$$

then Theorem 1.7 implies for $w \in D_\rho(a_0)$ the convergence of the series

$$f(z) = g(w) = \sum_{k=0}^{\infty} b_k \cdot (w - a_0)^k$$

with

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} c_n \cdot a_0^{n-k}.$$

Because

$$w - a_0 = (z - z_0) - (a - z_0) = z - a$$

we obtain for $z \in D_\rho(a)$

$$f(z) = \sum_{k=0}^{\infty} b_k \cdot (z - a)^k, \ q.e.d.$$

The importance of Proposition 1.9 is due to the explicit formulas to compute the coefficients of the product series and the series with changed centre, respectively. Proposition 1.9 will have an analogue for differentiable functions in Proposition 2.2.

Lemma 1.10 (Local behaviour of a power series). *Consider a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

1. If

$$c_0 \neq 0$$

then exists $r > 0$ *such that for all* $z \in D_r(0)$

$$f(z) \neq 0.$$

2. Otherwise, if for suitable $m \in \mathbb{N}$

$$c_0 = c_1 = \ldots = c_{m-1} = 0 \text{ but } c_m \neq 0$$

then exists $r > 0$ *such that for all* $z \in D_r(0) \setminus \{0\}$

$$f(z) \neq 0.$$

Proof. 1. The claim follows from the continuity of convergent power series, see Theorem 1.3.

2. We consider

$$f(z) = \sum_{n=m}^{\infty} c_n \cdot z^n = z^m \cdot \sum_{n=0}^{\infty} c_{n+m} \cdot z^n.$$

The convergent power series

$$f_1(z) := \sum_{n=0}^{\infty} c_{n+m} \cdot z^n$$

satisfies the assumption of part 1). Hence for suitable $r > 0$ and all $z \in D_r(0)$

$$f_1(z) \neq 0.$$

Also $z^m \neq 0$ for $z \in D_r(0) \setminus \{0\}$. Hence for $z \in D_r(0) \setminus \{0\}$

$$f(z) = z^m \cdot f_1(z) \neq 0, \ q.e.d.$$

Proposition 1.11 (Uniqueness of the power series expansion). *Consider two power series*

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n \cdot (z - z_0)^n$$

which converge in the disk $D_r(z_0),\ r > 0$. *If for all* $z \in D_r(z_0)$

$$f(z) = g(z)$$

then for all $n \in \mathbb{N}$

$$a_n = b_n.$$

Proof. For all $z \in D_r(z_0)$

$$h(z) := f(z) - g(z) = \sum_{n=0}^{\infty} (a_n - b_n) \cdot (z - z_0)^n = 0.$$

Hence the convergent power series $h(z)$ does not satisfy the conclusion from Lemma 1.10. As a consequence, $h(z)$ does not satisfy neither the assumption of part 1) nor the assumption of part 2) of the lemma, i.e. for all $n \in \mathbb{N}$

$$a_n - b_n = 0, \ q.e.d.$$

Combining Proposition 1.11 with the topological concept of *connectedness* will imply the identity theorem for analytic functions. We first recall the definition of connectedness and path-connectedness in the complex plane.

Definition 1.12 (Connectedness and path-connectedness). Consider an open set $U \subset \mathbb{C}$.

1. The set U is *disconnected* if two non-empty disjoint open subsets $U_1, U_2 \subset \mathbb{C}$ exist with
$$U = U_1 \cup U_2.$$
Otherwise the set U is *connected*.

2. The set U is *path-connected* if any two points $a, b \in U$ can be joined by a *path* in U, i.e. if a continuous map
$$\gamma : [0,1] \to U$$
exists satisfying
$$\gamma(0) = a \text{ and } \gamma(1) = b.$$

Proposition 1.13 (Connectedness and path-connectedness). *For an open set $U \subset \mathbb{C}$ holds the equivalence:*

$$U \textit{ connected} \iff U \textit{ path-connected}.$$

Proof. i) Assume U disconnected. Then

$$U = U_1 \,\dot{\cup}\, U_2$$

with two open, non-empty subsets $U_1,\ U_2 \subset \mathbb{C}$. To show that U is not path-connected we assume on the contrary that U is path-connected. Consider two arbitrary points $a \in U_1$ and $b \in U_2$ and a continuous map

$$\gamma : [0,1] \to U$$

with

$$\gamma(0) = a \text{ and } \gamma(1) = b.$$

Define

$$t_0 := \sup\{t \in [0,1] :\ \gamma(t) \in U_1\}$$

Then $t_0 \in [0,1]$, see Figure 1.4, on top. Hence

$$\gamma(t_0) \in U = U_1 \dot\cup U_2.$$

Continuity of γ and openess of U_1 and U_2 imply: Both subsets

$$\gamma^{-1}(U_i) \subset [0,1],\ i = 1,2,$$

are open in $[0,1]$, which excludes the boundaries $t_0 \in \{0,1\}$. Hence

$$t_0 \in \,]0,1[.$$

- If $\gamma(t_0) \in U_1$, then $\gamma^{-1}(U_1)$ is an open neighbourhood of t_0. In particular

  $$\sup\{t \in [0,1] :\ \gamma(t) \in U_1\} > t_0,$$

  a contradiction.

- Similarly, if $\gamma(t_0) \in U_2$, then $\gamma^{-1}(U_2) \subset I$ is an open neighbourhood of t_0. In particular

  $$\sup\{t \in [0,1] :\ \gamma(t) \in U_1\} < t_0,$$

  a contradiction.

The indirect proof shows that U is not path-connected.

ii) Assume U connected. Choose a fixed point $a \in U$ and decompose

$$U = U_1 \dot\cup U_2$$

with

$$U_1 := \{b \in U :\ b \text{ can be joined by a path to } a\}$$

$$U_2 := \{c \in U :\ c \text{ cannot be joined by a path to } a\}$$

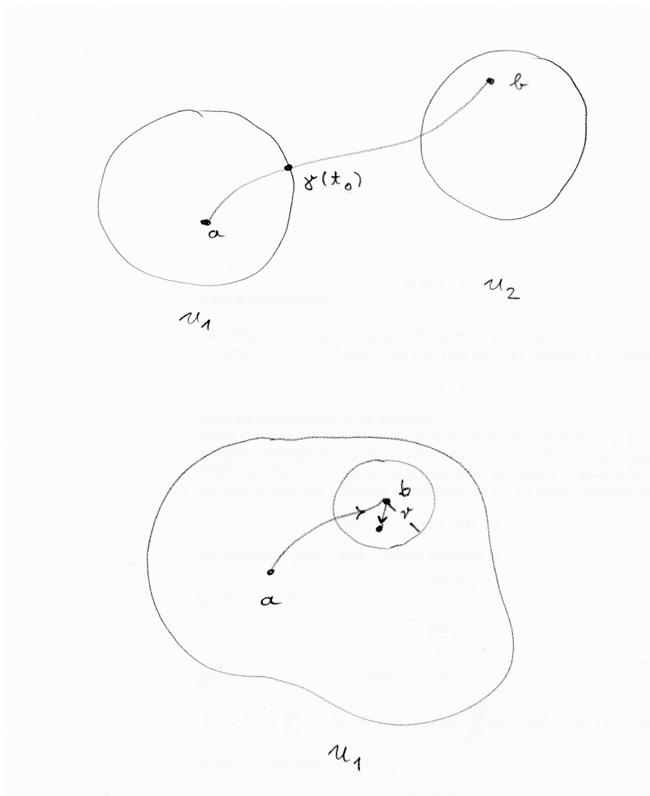


Fig. 1.4 Connectedness and path-connectedness

- We have $U_1 \neq \emptyset$ because $a \in U_1$.

- U_1 *is open*: We consider an arbitrary point $b \in U_1$. Openess of U implies the existence of $r > 0$ with $D_r(b) \subset U$. Path-connectedness of the disk $D_r(b)$ implies $D_r(b) \subset U_1$, see Figure 1.4, at bottom.

- U_2 *is open*: Any point $c \in U_2$ has a neighbourhood $D_r(c) \subset U$ with suitable $r > 0$. Path-connectedness of $D_r(c)$ implies $D_r(c) \subset U_2$. Hence $U_2 = \emptyset$, which finishes the proof, q.e.d.

Definition 1.14 (Domain). A *domain* G in $\mathbb{C}$ is a non-empty, connected open subset $G \subset \mathbb{C}$.

Definition 1.15 (Isolated point, accumulation point). Consider a subset $A \subset \mathbb{C}$.

- A point $a \in A$ is an *isolated point* if a has a neighbourhood $U \subset \mathbb{C}$ such that

$$U \cap A = \{a\}.$$

If A has only isolated points, then A is named a *discrete* set.

- A point $a \in \mathbb{C}$ is an *cluster point* or *accumulation point of* A if any neighbourhood $U \subset \mathbb{C}$ of a includes a point from $A \setminus \{a\}$, i.e. if

$$U \cap (A \setminus \{a\}) \neq \emptyset.$$

Denote by A' the cluster points of A. Then the isolated points in A form the complement

$$A \setminus A'.$$

Of course, A may have also cluster points belonging to $\mathbb{C} \setminus A$:

$$A' = \overline{A}.$$

All points of A are isolated iff A, equipped with the subspace topology from $\mathbb{C}$, is a *discrete* space, i.e. if each point set $\{a\},\ a \in A$, is open.

If $A \subset U$ for an open set $U \subset \mathbb{C}$, then A being discrete and closed in U means: The set A has

- no cluster point in A (discreteness) and
- no cluster point in U (discrete and closed in U).

Hence, if A has a cluster point $a_0 \in \mathbb{C}$ at all, then $a_0 \in \partial U$.

Note that *discreteness of* A is an intrinsic property of the topological space A equipped with its subspace topology. While A *closed in* U also refers to the boundary of A in the ambient space U. The set

$$A_1 := \{1/n : \ n \in \mathbb{N}^*\} \subset U := D_2(0)$$

is discrete, but not closed in U. The set A_1 has the cluster point

$$a_0 = 0 \in U \setminus A.$$

While the set

$$A_2 := \{2 - (1/n) : \ n \in \mathbb{N}^*\} \subset U$$

is discrete and closed in U. The set A_2 has no cluster point in U, because

$$a_0 = 2 \in \partial U.$$

Theorem 1.16 (Isolated zeros of an analytic function). *Consider a domain $G \subset \mathbb{C}$ and an analytic function f defined on G. Then:*

- *Either $f = 0$*

- *or each point of the zero set of f is isolated.*

Proof. We decompose G into the two sets

$U := \{a \in G :$ The power series expansion of f with center a vanishes identically$\}$

$V := \{a \in G :$ The power series expansion of f with center a does not vanish identically$\}$.

The uniqueness of the power series expansion due to Proposition 1.11 implies

$$G = U \,\dot{\cup}\, V.$$

i) U *is open*: If $a \in U$ then for suitable $r > 0$

$$f|D_r(a) = 0.$$

Proposition 1.11 implies that for any $b \in D_r(a)$ the derived power series expansion of f with center $= b$ vanishes. Hence $D_r(a) \subset U$.

ii) V *is open*: If $a \in V$ then the power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

is convergent in a neighbourhood $D_r(a)$ and by assumption

$$c_k \neq 0$$

for at least one $k \in \mathbb{N}$. Lemma 1.10 implies the existence of $\rho > 0$ such that for all $z \in D_\rho(a) \setminus \{a\}$

$$f(z) \neq 0.$$

As a consequence $D_\rho(a) \subset V$.

iii) *Connectedness of* G: The decomposition $G = U \,\dot{\cup}\, V$ and connectedness of G imply:

- Either $U = \emptyset$. Then $G = V$ and Lemma 1.10 implies that all zeros of f are isolated.

- Or $V = \emptyset$. Then $G = U$ and $f = 0$, q.e.d.

Theorem 1.17 is one of the fundamental results of complex analysis: An analytic function on a domain G is already determined by its values on a sequence of pairwise distinct points with a limit in G. In particular, the function is determined by its values on an arbitrary small, non-zero open subset of G.

Theorem 1.17 (Identity theorem). *Consider a domain $G \subset \mathbb{C}$ and two analytic functions*

$$f, g : G \to \mathbb{C}.$$

If exists a subset $A \subset G$ with an accumulation point $a \in G$ and

$$f|A = g|A,$$

then $f = g$.

Proof. Consider the difference

$$h := f - g,$$

which is analytic. Assume the existence of a sequence $(a_\nu)_{\nu \in \mathbb{N}}$ in $A \setminus \{a\}$ with

$$a := \lim_{\nu \to \infty} a_\nu \in A$$

such that for all $\nu \in \mathbb{N}$

$$h(a_\nu) = 0.$$

By continuity also $h(a) = 0$. The set

$$\{a_\nu : \nu \in \mathbb{N}\} \cup \{a\}$$

is a subset of the zero set of h. It contains the point a which is not an isolated point. Theorem 1.16 implies $h = 0$, i.e.

$$f = g, \ q.e.d.$$

Theorem 1.17 shows: An analytic function defined on a domain G is already determined by its power series expansion with center an arbitrary point $a \in G$. The radius of convergence of the power series is of no importance. The ring of all convergent power series with center $a \in \mathbb{C}$ is named the ring of *germs of analytic functions* in a neighbourhood of a.

Example 1.18 (Identity theorem). Consider

$$G := D_2(0) \text{ and } A := \{1/n : n \in \mathbb{N} \setminus \{0\}\}.$$

Then $0 \in G$ is an accumulation point of A. Theorem 1.17 implies: If two analytic functions $f,\ g$ on G satisfy for all $n \geq 1$

$$f(1/n) = g(1/n)$$

then

$$f = g.$$

Proposition 1.19 (Composition of analytic functions). *Consider two open sets $U,\ V \subset \mathbb{C}$ and two analytic functions*

$$f : U \to \mathbb{C} \text{ and } g : V \to \mathbb{C}$$

satisfying

$$f(U) \subset V.$$

Then the composition

$$g \circ f : U \to \mathbb{C}$$

is also analytic.

Proof. Consider an arbitrary but fixed point $a \in U$ and set

$$b := f(a) \in V.$$

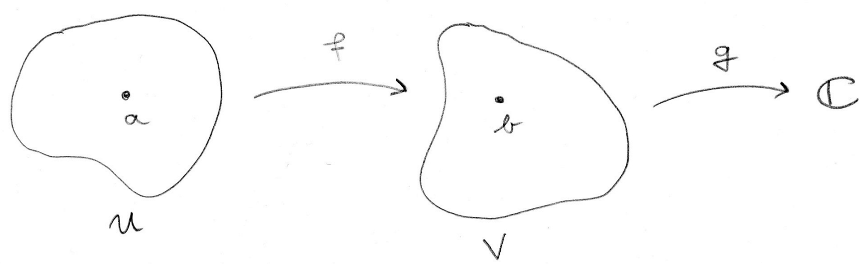


Fig. 1.5 Composition of analytic functions

i) *Power series expansion of f*: For suitable $r_1 > 0$ and all $z \in D_{r_1}(a)$ we have

$$f(z) = \sum_{n=0}^{\infty} b_n \cdot (z-a)^n = b + \sum_{n=1}^{\infty} b_n \cdot (z-a)^n.$$

The function

$$\phi(z) := f(z) - b = \sum_{n=1}^{\infty} b_n \cdot (z-a)^n$$

is analytic in $D_{r_1}(a)$ and satisfies

$$\phi(a) = 0.$$

By continuity of ϕ: For all $r > 0$ exists $s, 0 < s < r_1$, such that

$$\sum_{n=1}^{\infty} |b_n| \cdot s^n \leq r.$$

As a consequence for $z \in D_s(a)$

$$|\phi(z)| \leq r.$$

ii) *Power series expansion of g*: For suitable $r_2 > 0$ and all $z \in D_{r_2}(b)$

$$g(z) = \sum_{n=0}^{\infty} c_n \cdot (z-b)^n.$$

Theorem 1.3 implies for any r with $0 < r < r_2$

$$\sum_{n=0}^{\infty} |c_n| \cdot r^n < \infty.$$

As a consequence, we obtain

- for all $z \in D_s(a)$ the representation

$$(g \circ f)(z) = g(f(z)) = \sum_{n=0}^{\infty} c_n \cdot (f(z) - b)^n = \sum_{n=0}^{\infty} c_n \cdot \phi(z)^n$$

- and the estimate

$$\sum_{n=0}^{\infty} |c_n| \left(\sum_{k=1}^{\infty} |b_k| \cdot s^k \right)^n \leq \sum_{n=0}^{\infty} |c_n| \cdot r^n < \infty$$

iii) *Double series and rearrangement*: For any $n \in \mathbb{N}$ the power ϕ^n is analytic in $D_s(a)$ with convergent power series

$$\phi(z)^n = \sum_{k=n}^{\infty} d_{nk} \cdot (z-a)^k.$$

For $z \in D_s(a)$ we introduce the double series

$$\sum_{n,k} D_{nk}$$

with

$$D_{nk} := \begin{cases} c_n \cdot d_{nk} \cdot (z-a)^k & n \le k \\ 0 & n > k \end{cases}$$

The double series satisfies the assumptions of Theorem 1.5, part 2:

- For $z \in D_s(a)$ each row-sum

$$\sum_{k=0}^{\infty} D_{nk} = c_n \cdot \sum_{k=n}^{\infty} d_{nk} \cdot (z-a)^k$$

  is absolutely convergent due to part ii).

- To estimate the double series

$$\sum_{n,k=0}^{\infty} |D_{nk}|$$

  note: The coefficients $(d_{nk})_{n \le k}$ of the power series of $\phi^n(z)$ are polynomials in the coefficients $(b_k)_{k \in \mathbb{N}}$ of the power series of $\phi(z)$. Hence for each fixed $n \in \mathbb{N}$ they satisfy the estimate

$$\sum_{k=n}^{\infty} |d_{nk}| \cdot s^k \le \left(\sum_{k=1}^{\infty} |b_k| \cdot s^k \right)^n$$

  With the estimate from part ii) we get

$$\sum_{n=0}^{\infty} |c_n| \left(\sum_{k=n}^{\infty} |d_{nk}| \cdot s^k \right) \le \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} |b_k| \cdot s^k \right)^n < \infty$$

  i.e.

$$\sum_{n,k=0}^{\infty} |D_{nk}| = \sum_{n=0}^{\infty} |c_n| \cdot \sum_{k=n}^{\infty} |d_{nk}| \cdot |(z-a)^k| \le \sum_{n=0}^{\infty} |c_n| \cdot \left(\sum_{k=n}^{\infty} |d_{nk}| \cdot s^k \right) < \infty$$

Now Theorem 1.5, part 1, applies: We obtain the rearrangement

$$g(f(z)) = \sum_{n=0}^{\infty} c_n \cdot \left(\sum_{k=n}^{\infty} d_{nk} \cdot (z-a)^k \right) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} c_n \cdot d_{nk} \right) \cdot (z-a)^k$$

The series on the right-hand side is the power series expansion with center $= a$ of the composition

$$g \circ f$$

for $z \in D_s(a)$, q.e.d.

The proof of Proposition 1.19 allows to derive an explicit formula for the coefficents of the convergent power series of the composition. An analogue for differentiable functions will be obtained in Proposition 2.2.

1.3 Exponential map and related analytic functions

In the present section we study functions which relate to the exponential function and its power series expansion. For a real argument the exponential function

$$exp : \mathbb{R} \to \mathbb{R}^*_+ := \{x \in \mathbb{R} : x > 0\}$$

expands into the exponential series which is convergent for all arguments. The function is bijective with derivative $(e^x)' = e^x$. Therefore the inverse, the logarithm,

$$ln : \mathbb{R}^*_+ \to \mathbb{R}$$

is well-defined and locally expands into a convergent power series. The exponential function satisfies the functional equation

$$exp(x_1 + x_2) = exp(x_1) \cdot exp(x_2).$$

We will investigate how the exponential function and the logarithm extend to complex arguments. In particular, we will study the unexpected behaviour of the complex logarithm.

Theorem 1.20 (The exponential function). *The exponential series*

$$exp(z) := e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is convergent for all $z \in \mathbb{C}$. Its functional equation is the addition theorem

$$exp(z_1 + z_2) = exp(z_1) \cdot exp(z_2).$$

In particular, for all $z \in \mathbb{C}$

$$exp(z) \neq 0 \text{ and } (exp\ z)^{-1} = exp(-z).$$

Proof. The radius of convergence $R = \infty$ follows from the ratio test according to Proposition 1.4

$$\frac{1}{R} = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

The functional equation follows from taking the Cauchy product according to Corollary 1.6:

$$exp(z_1) \cdot exp(z_2) = \sum_{n=0}^{\infty} c_n \cdot z^n$$

with

$$c_n = \sum_{k=0}^{n} \frac{z_1^k}{k!} \cdot \frac{z_2^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{1}{n!} \cdot \binom{n}{k} \cdot z_1^k \cdot z_2^{n-k} = \frac{1}{n!} \cdot (z_1 + z_2)^n$$

Hence

$$exp(z_1) \cdot exp(z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (z_1 + z_2)^n = exp(z_1 + z_2)$$

In particular

$$1 = exp(0) = exp(z + (-z)) = exp(z) \cdot exp(-z), \ q.e.d.$$

If one presupposes the functional equation for real arguments x_1, x_2, then the functional equation for complex arguments z_1, z_2 follows from the Identity Theorem 1.17: Because the left-hand side and the right-hand side are analytic and coincide on the subset $\mathbb{R}$. But the argument is misleading because the proof in the real case also uses the Cauchy product.

For any complex argument $z \in \mathbb{C}$ the exponental series splits according to even and odd indices as

$$exp(iz) = \sum_{n=0}^{\infty} \frac{i^n \cdot z^n}{n!} = \sum_{n=0}^{\infty} i^{2n} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \frac{z^{2n+1}}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \cdot \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

The last two series extend the well-known real power series for

$$cos(x) \text{ and } \sin(x), \ x \in \mathbb{R},$$

to complex arguments. They define the complex *cos*-function and the complex *sin*-function.

Definition 1.21 (Complex *cos*- and *sin*-function). For complex $z \in \mathbb{C}$ one defines the analytic functions

$$cos : \mathbb{C} \to \mathbb{C}, \ cos(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

$$sin : \mathbb{C} \to \mathbb{C}, \ sin(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Remark 1.22 (Euler formula).

1. Due to Definition 1.21 for any complex $z \in \mathbb{C}$ the values $cos(z)$ and $sin(z)$ of the trigonometric functions relate to the value of the exponential function as

$$exp(iz) = cos(z) + i \cdot sin(z) \text{ (Euler formula).}$$

We used this formula for real arguments $x = z \in \mathbb{R}$ in the introduction chapter. Definition 1.21 shows that *cos* is an *even* function, i.e.

$$cos(z) = cos(-z),$$

while *sin* is an odd function, i.e.

$$sin(-z) = -(sinz).$$

Therefore respectively, adding and subtracting the two equations

$$exp(iz) = cos(z) + i \cdot sin(z) \text{ and } exp(-iz) = cos(z) - i \cdot sin(z)$$

implies for all $z \in \mathbb{C}$

$$cos\ z := \frac{1}{2}(e^{iz} + e^{-iz}) \text{ and } sin\ z := \frac{1}{2i}(e^{iz} - e^{-iz})$$

2. Similar formulas hold for the *hyperbolic* trigonometric functions: For $z \in \mathbb{C}$

$$cosh\ z = \frac{1}{2}(e^{z} + e^{-z}) \text{ and } sinh\ z = \frac{1}{2}(e^{z} - e^{-z}),$$

when defining

$$cosh\ z := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

and

$$sinh\ z := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

A remarkable consequence of the Euler formula allows to compute the value of the exponential function directly from the real- and imaginary part of its argument. It shows that the complex exponential is a mixture of the real exponential and the real trigonometric functions. The complex exponential inherits from the real exponential the exponential growth along the real axis, and it inherits from the trigonometric functions the periodicity along the imaginary axis.

Proposition 1.23 (Basic properties of the exponential function).

The exponential of a complex number

$$z = x + iy \in \mathbb{C}$$

with real part x and imaginary part y has the form

$$e^z = e^x \cdot (cos\ y + i \cdot sin\ y),$$

see Figure 1.6. In particular, the modulus is determined by the real part alone

$$|e^z| = e^x,$$

and

$$exp(z + 2\pi i) = exp(z) \text{ (Periodicity } 2\pi i\text{).}$$

Proof. Theorem 1.20 and the Euler formula for the argument iy, see Remark 1.22, imply

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot (cos\ y + i \cdot sin\ y).$$

Hence

$$|e^z| = |e^x| \cdot |e^{iy}| = e^x \cdot |cos(y) + i \cdot sin(y)| = e^x$$

because

$$|cos\ y + i \cdot sin\ y|^2 = (cos\ y + i \cdot sin\ y) \cdot \overline{(cos\ y + i \cdot sin\ y)} =$$

$$= (cos\ y + i \cdot sin\ y) \cdot (cos\ y - i \cdot sin\ y) = cos^2\ y + sin^2\ y = 1.$$

Moreover

$$exp(z + 2\pi i) = exp(z) \cdot exp(2\pi i) =$$

$$= exp(z) \cdot (cos\ 2\pi + i \cdot sin\ 2\pi) = exp(z),\ q.e.d.$$

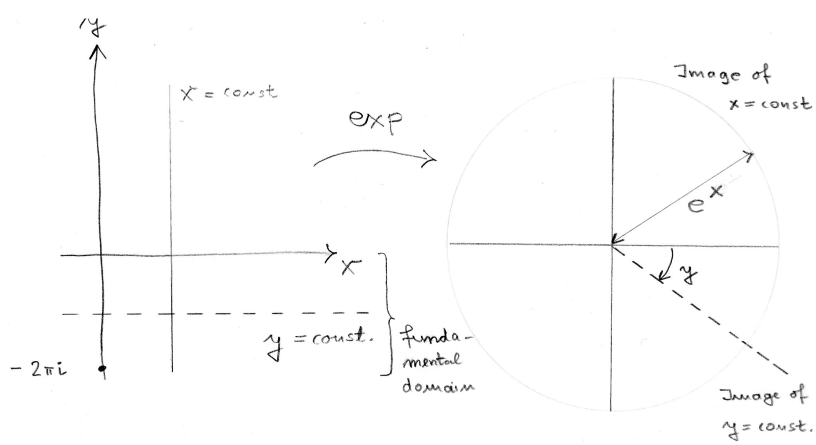


Fig. 1.6 Exponential map

Different from the real exponential the complex exponential function

$$exp : \mathbb{C} \to \mathbb{C}^*$$

is no longer bijective. The map is surjective. We will study the set of all arguments which are mapped to the same value. The locally defined inverse maps are the branches of the complex logarithm. The logarithm also serves to define the complex roots of complex numbers.

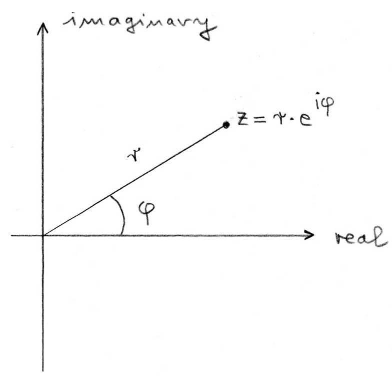


Fig. 1.7 Polar coordinates

Proposition 1.24 (Logarithm, polar coordinates, and roots of unity).

1. *The complex solutions $w \in \mathbb{C}$ of the equation*

$$e^w = 1$$

are named the logarithms of 1. *They form the set of imaginary numbers*

$$\{k \cdot 2\pi i : \; k \in \mathbb{Z}\}.$$

2. *The polar coordinates (r, ϕ) of a complex number*

$$z \in \mathbb{C} \simeq \mathbb{R}^2$$

determine the representation

$$z = r \cdot e^{i\phi},$$

see Figure 1.7. For $z \neq 0$ the representation determines the angle ϕ up to an integer multiple of 2π.

3. *For any $z \in \mathbb{C}$, $z \neq 0$, the equation*

$$e^w = z$$

has infinitely many solutions $w \in \mathbb{C}$: If w_0 denotes one specific solution then any solution has the form

$$w = w_0 + k \cdot 2\pi i, \ k \in \mathbb{Z}.$$

4. *For $n \in \mathbb{N}, \ n \geq 1$, the complex solutions w of the equation*

$$w^n = 1$$

are named the n-th roots of unity. *They all are located on the unit circle and form the set*

$$\{e^{(k/n)\cdot 2\pi i}, \ k = 0, ..., n-1\},$$

see Figure 1.8.

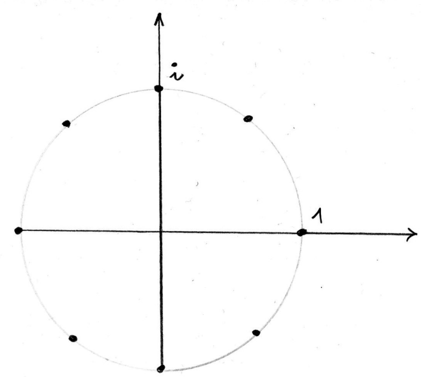


Fig. 1.8 The 8-th roots of unity

Proof. 1. Proposition 1.23 implies for $w = x + i \cdot y$

$$1 = |e^w| = e^x \implies x = 0 \text{ and } cos(y) + i \cdot sin(y) = 1.$$

Hence

$$y = k \cdot 2\pi.$$

2. For $z \neq 0$ set $r := |z|$ and consider

$$\zeta := \frac{z}{r}$$

with $|\zeta| = 1$. For suitable $\phi \in \mathbb{R}$

$$\zeta = cos(\phi) + i \cdot sin(\phi),$$

hence

$$z = r \cdot e^{i\phi}.$$

According to part 1)

$$r \cdot e^{i\phi} = r \cdot e^{i\psi} \iff e^{i(\phi-\psi)} = 1 \iff \phi - \psi = k \cdot 2\pi,\ k \in \mathbb{Z}.$$

3. The right-hand side in polar coordinates (r, ϕ)

$$z = r \cdot e^{i\phi},$$

and the ansatz

$$w_0 = u + i \cdot \mathrm{v}$$

require

$$e^u \cdot e^{i\mathrm{v}} = r \cdot e^{i\phi}$$

Setting

$$u := ln(r)$$

with ln the logarithm of positive real numbers, and

$$\mathrm{v} := \phi$$

provides the specific solution

$$w_0 = ln(r) + i\phi.$$

For the general solution part 1) implies

$$e^w = e^{w_0} \iff e^{w-w_0} = 1 \iff w - w_0 = k \cdot 2\pi i,\ k \in \mathbb{Z}.$$

4. If $w = r \cdot e^{i\phi}$ is an n-th root of unity, then $r = 1$ because $|w| = 1$. And

$$w^n = e^{i \cdot n\phi} = 1$$

implies due to part 1) for suitable $k \in \mathbb{Z}$

$$n \cdot \phi = k \cdot 2\pi, \text{ i.e. } \phi = (k/n) \cdot 2\pi.$$

The opposite claim

$$\left(e^{(k/n) \cdot 2\pi i}\right)^n = 1$$

is obvious, q.e.d.

We now consider the different solutions w of the equation $e^w = z$ for a given complex number $z \neq 0$: Do the solutions extend to analytic functions in a neighbourhood of each w when varying z? We first require continuous dependency.

Definition 1.25 (Principal value and branches of the logarithm function).

1. Any solution w of the equation
$$e^w = z$$
for a given
$$z = r \cdot e^{i\phi} \in \mathbb{C}^*$$
is named a *logarithm*
$$w = log(z)$$
of z. The uniquely determined solution
$$w = ln(r) + i \cdot \phi,\ \phi \in]-\pi,\ \pi]$$
is named
$$w = Log(z),$$
the *principal value of the logarithm* of z.

2. Consider an open subset $U \subset \mathbb{C}^*$. A continuous function
$$f : U \to \mathbb{C}$$
is named a *branch of the logarithm function* if for all $z \in U$
$$e^{f(z)} = z.$$

Lemma 1.26 (Comparing two branches of the logarithm function). *Consider a domain $G \subset \mathbb{C}^*$ and two branches of the logarithm function*
$$f,\ g : G \to \mathbb{C}.$$
If
$$f(z_0) = g(z_0)$$
for at least one $z_0 \in G$, then
$$f = g.$$

Proof. By assumption
$$e^{f-g} = 1.$$
Because both functions f and g are branches of the logarithm function, Proposition 1.24 provides for each $z \in G$ an integer
$$k = k(z) \in \mathbb{Z}$$
satisfying
$$g(z) = f(z) + k(z) \cdot 2\pi i.$$
Continuity of f and g and connectedness of G imply that $k(z)$ does not depend on z. Because

$$f(z_0) = g(z_0)$$

we have $k = 0$, q.e.d.

Example 1.27 proves the existence of a logarithm function on a certain domain in $\mathbb{C}^*$. Because the exponential function has no zeros the logarithm does not extend to the argument $z = 0$. But it remains open why the example has to exclude also the negative real axis from the domain of definition. We will return to this question later, see Remark 7.2.

Example 1.27 (Logarithm on the sliced plane). Consider the sliced complex plane

$$G := \mathbb{C}^- := \mathbb{C} \setminus]-\infty, 0]$$

i.e. the complex plane minus the negative real axis and minus the origin, see Figure 1.9. Any $z \in G$ has a unique representation by polar coordinates

$$z = r \cdot e^i \phi, \ \phi \in]-\pi, \ \pi[.$$

The principal branch of the logarithm function in G is the function

$$Log : G \to \mathbb{C}, \ Log(r \cdot e^i \phi) := ln(r) + i\phi, \ \phi \in]-\pi, \ \pi[.$$

For each branch of the logarithm function in G a unique $k \in \mathbb{Z}$ exists satisfying for all $z \in G$

$$log(z) = Log(z) + k \cdot 2\pi i.$$

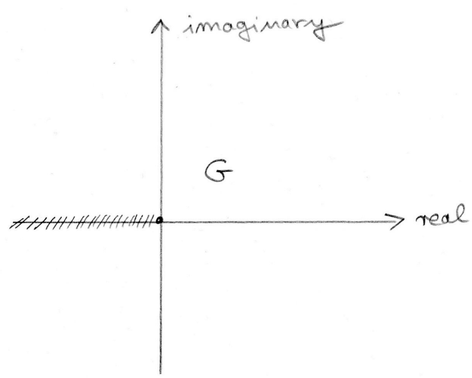


Fig. 1.9 Branch of logarithm on the sliced plane

The branches of the complex logarithm do not necessarily satisfy the functional equation known from real arguments. In general

$$log(z_1 \cdot z_2) \neq log(z_1) + log(z_2).$$

Instead the equation

$$log(z_1 \cdot z_2) = log(z_1) + log(z_2) + k \cdot 2\pi i,\ k \in \{-1, 0, 1\}$$

relates two branches of the complex logarithm function. The relation indicates that the inverse function of the exponential map has to consider all branches together: The canonical domain of definition for the complex logarithm function is not the sliced plane from Example 1.27 but a covering of $\mathbb{C}^*$ with infinitely many leaves: The canonical domain of the complex logarithm function is a *Riemann surface*.

Theorem 1.28 (Analytic local branch of the logarithm function). *Any point $z_0 \in \mathbb{C}^*$ has an open neighbourhood $U \subset \mathbb{C}^*$ with an analytic branch of the logarithm function*

$$f : U \to \mathbb{C}.$$

Proof. i) *Real logarithm series*: We recall the power series with center $a = 0$ of the real logarithm function

$$ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$$

which is convergent for

$$x \in \mathbb{R},\ |x| < 1.$$

The substitution

$$y := 1 + x$$

defines for $|y-1| < 1$ the power series

$$ln(y) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(y-1)^n}{n}$$

ii) *Extension to complex arguments*: For $z_0 \in \mathbb{C}^*$ we choose a logarithm $w_0 \in \mathbb{C}$ according to Proposition 1.24, i.e. satisfying

$$e^{w_0} = z_0.$$

For $z \in \mathbb{C}$ we have

$$\left| \frac{z - z_0}{z_0} \right| = \left| \frac{z}{z_0} - 1 \right| < 1 \iff |z - z_0| < |z_0|.$$

Hence we set $r := |z_0|$ and define the domain

$$U := D_r(z_0) = \{z \in \mathbb{C}^* : |z - z_0| < |z_0|\}.$$

For $z \in U$ the series

$$f(z) := w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{z - z_0}{z_0}\right)^n = w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot z_0^n} \cdot (z - z_0)^n$$

is convergent due to the ratio test. Hence the function f is analytic in U.

iii) *Analytic branch of the logarithm function*: Claim: For all $z \in U$

$$e^{f(z)} = z.$$

For real numbers $\alpha \in \mathbb{R}$ with distance $|\alpha - 1| < 1$ we consider the following function of the argument α

$$f(\alpha z_0) = w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{\alpha z_0 - z_0}{z_0}\right)^n = w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (\alpha - 1)^n =$$

$$= w_0 + ln(\alpha),$$

see Figure 1.10. As a consequence

$$e^{f(\alpha z_0)} = e^{w_0 + ln(\alpha)} = e^{w_0} \cdot e^{ln(\alpha)} = \alpha z_0.$$

Therefore the two analytic functions

$$e^f,\ id : U \to \mathbb{C}$$

coincide for all arguments

$$z = \alpha z_0 \in U$$

with α in an open real interval. By Theorem 1.17 both functions agree, i.e. for all $z \in U$

$$e^{f(z)} = z.$$

Hence f is an analytic branch of the logarithm function in U, q.e.d.

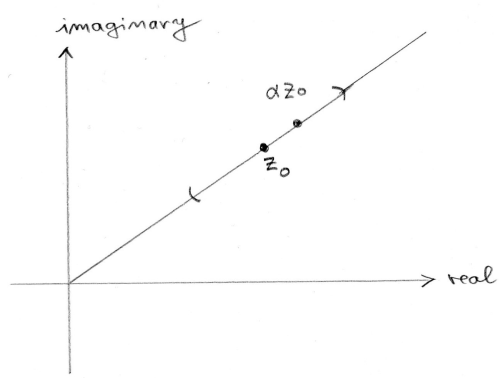


Fig. 1.10 Being equal on a subset parametrized by a real interval

Theorem 1.28 and Lemma 1.26 imply: The principal value of the logarithm from Example 1.27

$$Log : \mathbb{C}^- \to \mathbb{C}$$

is analytic.

Corollary 1.29 (Analytic branches of the root function). *Consider a number $n \in \mathbb{N}^*$. Any point $z_0 \in \mathbb{C}^*$ has an open neighbourhood $U \subset \mathbb{C}^*$ with an analytic branch*

$$F : U \to \mathbb{C}$$

of the n-th root, i.e.

$$f^n = id_U.$$

If U is a domain then two analytic branches differ by an n-th root of unity.

Proof. Theorem 1.28 implies the existence of an analytic branch of the logarithm

$$log : U \to \mathbb{C}^*.$$

We define

$$f : U \to \mathbb{C}^*,\ f(z) := e^{(1/n)\cdot log(z)}.$$

Then for all $z \in U$

$$f(z)^n = \left(e^{(1/n)\cdot log(z)}\right)^n = e^{n\cdot(1/n)\cdot log(z)} = e^{log(z)} = z.$$

The analytic branch is determined up to an n-th root of unity due to Proposition 1.24, q.e.d.

1.4 Outlook

Making the first step into the field of several complex variables is quite similar to the case of one complex variable in the present chapter. Possibly the main difficulty for the beginner, is handling the multi-indices and the multi-products of exponents used for power of several variables. To obtain an impression of the similarities concerning the basic concepts of convergent power series, it is worthwile to browse the first pages of a textbook like [14, Chap. I, Sect. A] or of the lecture [5, §1].

How do the results of the present chapter generalize to the case of several complex variables?

The algebraic properties of the ring of analytic functions are the subject of [14, Chap. II, Sect. A]. For each point $w \in \mathbb{C}^n$ one obtains a local ring $\mathscr{O}_w$ by identifying analytic functions which coincide in an arbitrary neighbourhood of w. All these rings are isomorphic to $\mathbb{C}\{z_1, ..., z_n\}$.

The ring of convergent power series $\mathbb{C}\{z_1, ..., z_n\}$ in several variables is a local ring with residue field $\mathbb{C}$. The non-units form the maximal ideal. From the algebraic point of view one studies the ideals and notably the prime ideals as well as the quotient rings with respect to these ideals. [20, Abschn. 4.4] deals with the 1-dimensional case, while [14] treats the case of several complex variables.

The whole book [10], an advanced text, is devoted to the study of the quotient rings, named *analytic algebras*. Analytic algebras characterize the singularities of complex spaces, a generalization of complex manifolds. Accordingly, the category of analytic algebras is the means to study the local structure of complex spaces.

A different kind of generalizing power series and analytic functions

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open},$$

is to replace the image field $\mathbb{C}$ by a complex Banach algebra A. A typical example from functional analysis is the Banach algebra A of all continuous linear endomorphisms of a complex Banach space. For an introduction and the application to spectral theory see [2, Chap. V, XI] and [18, Kap. XIII, Num. 95-99].

Chapter 2
Differentiable Functions and Cauchy-Riemann Differential Equations

The present chapter investigates Cauchy's access to complex analysis by differentiability. And also Riemann's access by real differentiability and $\mathbb{C}$-linearity of the functional matrix. In addition, we start the proof of the equivalence of all three approaches.

2.1 Differentiability

We start with Cauchy's approach.

Definition 2.1 (Differentiability). Consider a function

$$f : U \to \mathbb{C}$$

defined on an open subset $U \subset \mathbb{C}$.

1. The function f is *differentiable at a point* $z_0 \in U$ if the differential quotient

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

   exists. In this case the limit is written $f'(z_0)$ and named the *derivative* of f at z_0.

2. The function f is *differentiable* if the derivative $f'(z)$ exists for all $z \in U$.

For functions $f : U \to \mathbb{C}$ depending on one complex variable the term "differentiable" refers to differentiability in the complex sense, i.e. the differential quotient considers all complex arguments $z \neq z_0$ with limit z_0. To emphasize this property some textbooks use the term "complex differentiability". If not stated otherwise, these lecture notes will use the term “differentiable” always in the sense of Definition 2.1.

The Cauchy-Riemann differential equations, see Theorem 2.6 will relate differentiablility of f as a function of one *complex* variable to partial differentiability of f, which refers to two *real* variables. These variables are the real- and the imaginary part of the complex variable.

Proposition 2.2 (Ring of differentiable functions). *Consider an open set $U \subset \mathbb{C}$.*

1. *The set*
$$\mathscr{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ differentiable}\}$$
is a ring with respect to addition and multiplication of functions.

2. *For two elements $f, g \in \mathscr{O}(U)$ the derivative satisfies*
$$(f \pm g)' = f' \pm g' \text{ (Linearity)}, \ (f \cdot g)' = f' \cdot g + f \cdot g' \text{ (Product rule)}.$$
If g has no zeros then
$$(f/g)' = \frac{f' \cdot g - f \cdot g'}{g^2} \text{ (Quotient rule)}$$

3. *Consider two differentiable functions*
$$f : U \to \mathbb{C} \text{ and } g : V \to \mathbb{C},\ V \subset \mathbb{C} \text{ open},$$
satisfying
$$f(U) \subset V.$$
Then the composition
$$g \circ f : U \to \mathbb{C}$$
is differentiable and the derivative satisfies
$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \text{ (Chain rule)}.$$

Proof. In all cases the proof is literally the same as the proof for differentiable functions of one real variable, [6, Kap. 15], q.e.d.

Any power series
$$\sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$
has a formal derivative. It is obtained by differentiating every summand separately. The result is the power series

$$\sum_{n=1}^{\infty} n \cdot c_n \cdot (z-a)^{n-1}.$$

If the original series is a convergent power series then also the formal derivative is convergent with the same radius of convergence: One applies the formula of Cauchy-Hadamard, Theorem 1.4, and uses

$$\sqrt[n]{|c_{n+1}| \cdot (n+1)} = \sqrt[n]{|c_{n+1}|} \cdot \sqrt[n]{n+1}$$

and

$$\lim_{n\to\infty} \sqrt[n]{n+1} = 1.$$

This result prompts the questions:

- Is a convergent power series differentiable in the sense of Definition 2.1?
- If yes, is the derivative equal to the formal derivative.

The answer to both questions is "yes", see Theorem 2.3.

Theorem 2.3 (Differentiability of analytic functions).

1. *Consider a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

with radius of convergence $= R$*. Then the function*

$$f : D_R(a) \to \mathbb{C}$$

is differentiable. Its derivative f' *is the power series obtained by formal derivation of* f*. The derivative has radius of convergence* $= R$*.*

2. *Any analytic function*

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \textit{ open},$$

is differentiable.

Iterated application of Theorem 2.3 shows: Any analytic function has derivatives of arbitrary order, and each of these derivatives is again analytic.

Proof. W.l.o.g. we assume $a = 0$. The function

$$g : D_R(0) \to \mathbb{C},\ g(z) := \sum_{n=1}^{\infty} n \cdot c_n \cdot z^{n-1},$$

is well-defined. To prove the existence of $f'(z_0)$ for arbitrary $z_0 \in D_R(0)$, we compute for any index $n \in \mathbb{N}^*$

$$\frac{z^n - z_0^n}{z - z_0} =: q_n(z)$$

with

$$q_n(z) = z^{n-1} + z^{n-2} z_0 + \dots + z^{n-j} z_0^{j-1} + \dots + z_0^{n-1}.$$

The proof of the last equation employs the classical division algorithm. Note

$$q_n(z_0) = n \cdot z_0^{n-1}.$$

As a consequence

$$f(z) - f(z_0) = \sum_{n=0}^{\infty} c_n \cdot z^n - \sum_{n=0}^{\infty} c_n \cdot z_0^n = \sum_{n=1}^{\infty} c_n \cdot (z^n - z_0^n) =$$

$$= \sum_{n=1}^{\infty} c_n \cdot (z - z_0) \cdot q_n(z) = (z - z_0) \cdot \sum_{n=1}^{\infty} c_n \cdot q_n(z),$$

or

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \cdot q_n(z) =: f_1(z).$$

We have

$$f_1(z_0) = \sum_{n=1}^{\infty} n \cdot c_n \cdot z_0^{n-1} = g(z_0).$$

It remains to show that f_1 is continuous at z_0, i.e. to show

$$\lim_{z \to z_0} f_1(z) = f_1(z_0):$$

We choose a radius r with

$$|z_0| < r < R.$$

Then for $z \in D_r(0)$

$$\sum_{n=1}^{\infty} |c_n \cdot q_n(z)| \le \sum_{n=1}^{\infty} |c_n| \cdot n \cdot r^{n-1}.$$

The last series is convergent because the formal derivative has also radius of convergence $= R$. As a consequence, the series $f_1(z)$ is uniformly convergent for $z \in D_r(0)$, and the resulting function f_1 is continuous, q.e.d.

2.2 Cauchy-Riemann differential equations

We now go on to Riemann's approach to complex analysis.

Open subsets of $\mathbb{C}$ can be identified with open subsets of $\mathbb{R}^2$. Hereby, complex points

$$z = x + i \cdot y \in \mathbb{C}$$

become identified with real pairs

$$(x,y) \in \mathbb{R}^2.$$

As a consequence, the value of a function f defined on U will be written either $f(z)$ in the complex context or $f(x,y)$ in the real context.

Proposition 2.4 (Differentiability and partial derivatives). *Consider an open subset*

$$U \subset \mathbb{C} \simeq \mathbb{R}^2$$

and a differentiable function

$$f : U \to \mathbb{C}.$$

Then the partial derivatives of f, considered as a function of two real variables, exist at all points

$$z = x + i \cdot y \in U, (x,y) \in \mathbb{R}^2,$$

and satisfy

$$\frac{\partial f}{\partial x}(x,y) = f'(z) \text{ and } \frac{\partial f}{\partial y}(x,y) = i \cdot f'(z).$$

Proof. We compute

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^*}} \frac{f(z+h) - f(z)}{h} = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}^*}} \frac{f(z+h) - f(z)}{h} = f'(z).$$

Similarly

$$\frac{\partial f}{\partial y}(x,y) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^*}} \frac{f(z + i \cdot h) - f(z)}{h} =$$

$$= i \cdot \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^*}} \frac{f(z + i \cdot h) - f(z)}{i \cdot h} = i \cdot \lim_{\substack{h \to 0 \\ h \in \mathbb{C}^*}} \frac{f(z + i \cdot h) - f(z)}{i \cdot h} = i \cdot f'(z), \ q.e.d.$$

Corollary 2.5 (Partial derivatives with respect to polar coordinates). *Consider an open subset*

$$U \subset \mathbb{C} \simeq \mathbb{R}^2$$

and a differentiable function

$$f : U \to \mathbb{C}.$$

Then with respect to polar coordinates, see Proposition 1.24,

$$z = x + iy = r \cdot e^{i\phi}$$

holds

$$\frac{\partial f}{\partial \phi}(r, \phi) = ir \cdot \frac{\partial f}{\partial r}(r, \phi)$$

Proof. We have

$$x = r \cdot \cos(\phi) \text{ and } y = r \cdot \sin(\phi).$$

The chain rule implies

$$\frac{\partial f(x(r,\phi), y(r,\phi)))}{\partial r} = \frac{\partial f(x,y)}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f(x,y)}{\partial y} \cdot \frac{\partial y}{\partial r}$$

or shortened as equality of differential operators

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \cdot \frac{\partial}{\partial y} \text{ and similarly } \frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \cdot \frac{\partial}{\partial y}$$

As a consequence

$$\frac{\partial f}{\partial r} = \cos(\phi) \cdot \frac{\partial f}{\partial x} + \sin(\phi) \cdot \frac{\partial f}{\partial y} \text{ and } \frac{\partial f}{\partial \phi} = -r\sin(\phi) \cdot \frac{\partial f}{\partial x} + r\cos(\phi) \cdot \frac{\partial f}{\partial y}$$

Proposition 2.4 implies

$$\frac{\partial f}{\partial r} = \cos(\phi) \cdot f' + \sin(\phi) \cdot if' = e^{i\phi} f'$$

and

$$\frac{\partial f}{\partial \phi} = -r\sin(\phi) f' + ir\cos(\phi) f' = ire^{i\phi} f', \ q.e.d.$$

Theorem 2.6 (Cauchy-Riemann differential equations and differentiability). *Consider an open subset*

$$U \subset \mathbb{C} \simeq \mathbb{R}^2$$

and a function

$$f : U \to \mathbb{C}.$$

Denote by

$$f = u + i \cdot v$$

its decomposition into real part

$$u := Re\ f : U \to \mathbb{R},\ u(z) := Re\ f(z),$$

and imaginary part

$$v := Im\ f : U \to \mathbb{R},\ v(z) := Im\ f(z).$$

1. *If f is differentiable then the partial derivatives of u and v exist, and satisfy the Cauchy-Riemann differential equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

2. *If the partial derivatives of u and v exist as continuous functions, and satisfy the Cauchy-Riemann differential equations, then f is differentiable.*

Proof. 1) $\implies$ 2): On one hand, Proposition 2.4 implies

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \cdot \frac{\partial \mathrm{v}}{\partial x}$$

or

$$i \cdot f' = i \cdot \frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial x}.$$

On the other hand,

$$i \cdot f' = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \cdot \frac{\partial \mathrm{v}}{\partial y}.$$

Equating real- and imaginary part in the last two equations finishes the proof.

2) $\implies$ 1): Consider an arbitrary but fixed point

$$z = x + i \cdot y \in U$$

and decompose

$$f = u + i \cdot \mathrm{v}.$$

The existence of the continuous partial derivatives of u implies

$$u((x,y)+(h_1,h_2)) - u(x,y) = u_x(x,y) \cdot h_1 + u_y(x,y) \cdot h_2 + o(h), \text{ i.e.}$$
$$u(z+h) - u(z) = u_x(z) \cdot h_1 + u_y(z) \cdot h_2 + o(h)$$

with arbitrary

$$h := h_1 + i \cdot h_2, \ h_1, h_2 \in \mathbb{R},$$

and using $o(h)$ as shorthand for any function $\phi(h)$ satisfying

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{\phi(h)}{||h||} = 0.$$

Note: The continuity of the partial derivatives assures that the rest term is $o(h)$, see [7, §6, Satz 2]. Analogously

$$\mathrm{v}(z+h) - \mathrm{v}(z) = \mathrm{v}_x(z) \cdot h_1 + \mathrm{v}_y(z) \cdot h_2 + o(h).$$

Hence

$$f(z+h) - f(z) = u(z+h) - u(z) + i \cdot (\mathrm{v}(z+h) - \mathrm{v}(z)) =$$
$$= u_x(z) \cdot h_1 + u_y(z) \cdot h_2 + i \cdot (\mathrm{v}_x(z) \cdot h_1 + v_y(z) \cdot h_2) + o(h)$$

Now the Cauchy-Riemann differential equations replace the partial derivatives with respect to y by partial derivatives with respect to x

$$f(z+h) - f(z) = u_x(z) \cdot h_1 - \mathrm{v}_x(z) \cdot h_2 + i \cdot (\mathrm{v}_x(z) \cdot h_1 + u_x(z) \cdot h_2) + o(h) =$$
$$= u_x(z) \cdot (h_1 + i \cdot h_2) + i \cdot \mathrm{v}_x(z) \cdot (h_1 + i \cdot h_2) + o(h) =$$
$$= u_x(z) \cdot h + i \cdot \mathrm{v}_x(z) \cdot h + o(h)$$

Hence

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = u_x(z) + i \cdot \mathrm{v}_x(z) = f_x(z), \ q.e.d.$$

The theory of differentiable functions in open subsets $U \subset \mathbb{R}^n, \ n \geq 2$, distinguishes between the following types of differentiability:

- Existence of continuous partial derivatives,
- existence of the total derivative as a linear map, and
- existence of partial derivatives.

It is well known: Each of the three conditions is strictly stronger than its follower. Therefore Theorem 2.6, part 2) has to presuppose that the partial derivatives are continuous to obtain the existence of the derivative of f. But that's no restriction because we will later prove a remarkable property of the complex context: The partial derivatives of any differentiable function are continuous.

Remark 2.7 ($\mathbb{C}$-linearity of the Jacobi matrix). When identifying the field $\mathbb{C}$ with the real vector space $\mathbb{R}^2$ one identifies

$$1 \in \mathbb{C} \text{ with } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \text{ and } i \in \mathbb{C} \text{ with } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$$

1. The multiplication

$$(x + i \cdot y) \cdot (u + i \cdot \mathrm{v}) \in \mathbb{C}$$

translates to

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ \mathrm{v} \end{pmatrix} = \begin{pmatrix} xu - y\mathrm{v} \\ x\mathrm{v} + yu \end{pmatrix}$$

A matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2 \times 2, \mathbb{R})$$

defines a $\mathbb{C}$-linear map

$$T:\mathbb{C}\to\mathbb{C}$$

iff

$$T(i)=i\cdot T(1).$$

The latter equation states

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}*\begin{pmatrix} a \\ c \end{pmatrix}$$

i.e.

$$\begin{pmatrix} b \\ d \end{pmatrix}=\begin{pmatrix} -c \\ a \end{pmatrix},\ \text{ i.e. } a=d \text{ and } b=-c$$

2. As a consequence: If a function

$$f=u+i\cdot \mathrm{v}:U\to\mathbb{C},\ U\subset\mathbb{C} \text{ open},$$

has partial derivatives under the identification $\mathbb{C}\simeq\mathbb{R}^2$, then its Jacobi matrix

$$Jac(f)=\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial \mathrm{v}}{\partial x} & \dfrac{\partial \mathrm{v}}{\partial y} \end{pmatrix}$$

defines a $\mathbb{C}$-linear map iff the partial derivatives of the real and imaginary part of f satisfy the Cauchy-Riemann differential equations.

Corollary 2.8 (Relation between real and imaginary part). *Consider a domain $G\subset\mathbb{C}$ and two differentiable functions*

$$f,g:G\to\mathbb{C}$$

with $Re(f)=Re(g)$. Then

$$Im(f)-Im(g)=c$$

for a suitable constant $c\in\mathbb{R}$.

Proof. Theorem 2.6 implies

$$\frac{\partial(Im(f)-Im(g))}{\partial x}=\frac{\partial(Im(f)-Im(g))}{\partial y}=0$$

which finishes the proof, q.e.d.

A deep theorem proves that a differentiable function is analytic, hence has derivations of arbitrary order. In particular, the derivative of a differentiable function is continuous. The proof will be given in Chapter 3 when showing the equivalence of all three approaches to complex analysis.

Chapter 3
Cauchy's Integral Theorem for disk and annulus

The two main methods in analysis are differentiation and integration. Chapter 2 dealt with differentiability. We now add to Complex Analysis a series of results obtained by integration.

3.1 Cauchy kernel and Cauchy integration

The section investigates the integration of complex differentiable functions defined in disks and annuli. We prove a simple variant of Cauchy's integral theorem. It is a consequence of the Cauchy-Riemann differential equations expressed in polar coordinates. The integral theorem implies Cauchy's integral formula which represents the values of a complex differentiable function by an integral. As an application Theorem 3.8 collects several equivalent conditions for a function to be holomorphic.

Remark 3.1 (Path integral). We integrate continuous functions

$$f : U \to \mathbb{C}$$

defined on subsets $U \subset \mathbb{C}$ along paths contained in U.

1. First, we consider a continuously differentiable path

$$\gamma : [a,b] \to U$$

depending on a real parameter from a compact interval $[a,b] \subset \mathbb{R}$. One defines

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt :=$$

$$:= \int_a^b Re\left(f(\gamma(t))\cdot\gamma'(t)\right)\,dt + i\cdot\int_a^b Im\left(f(\gamma(t))\cdot\gamma'(t)\right)\,dt.$$

One checks that the integral does not depend on the choice of the parametrization γ, i.e. the integral is invariant with respect to parameter transformations.

Secondly, a *piecewise continuously differentiable path* γ is a continuous path composed of finitely many successive continuously differentiable paths

$$\gamma_j,\ j = 1,...,n.$$

One defines

$$\int_\gamma f(z)\,dz := \sum_{k=1}^{n}\int_{\gamma_k} f(z)\,dz.$$

2. For an annulus

$$A := \{z \in \mathbb{C}:\ r_1 \le |z| \le r_2\} \subset U,\ 0 < r_1 < r_2,$$

one chooses the orientation of the two continuously differentiable component paths $\gamma_j\ j = 1,2,$ of the boundary

$$\partial A = D_{r_1}(0)\ \cup\ D_{r_2}(0)$$

as follows, see Figure 3.1: When moving along $\gamma_j,\ j = 1,2,$ the interior of A is on the left-hand side. Hence

$$\gamma_2 : [0,2\pi] \to \mathbb{C},\ \gamma_2(t) = r_2\cdot e^{it},$$

and

$$\gamma_1 : [0,2\pi] \to \mathbb{C},\ \gamma_1(t) = r_1\cdot e^{-it}.$$

3. A similar rule holds for integrating along the boundary of a disk $D_r(0)$, the limit of an annulus with $r_2 = r$ and $r_1 = 0$.

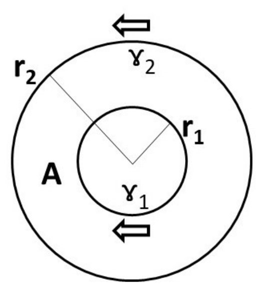


Fig. 3.1 Orientation of boundary paths of an annulus

Lemma 3.2 prepares the proof of Cauchy's integral theorem, Theorem 3.3. The main step in the proof is to employ the relation between the partial derivatives with respect to polar coordinates of a differentiable function.

Lemma 3.2 (Integration of differentiable functions defined in disk or annulus). *Consider two radii*

$$0 \leq r_1 < r_2 < \infty$$

and the open annulus

$$A := \{z \in \mathbb{C} : \; r_1 < |z| < r_2\}.$$

For a differentiable function

$$f : A \to \mathbb{C}.$$

the integral

$$I(r) := \int_{|z|=r} f(z)\; dz$$

does not depend on the parameter

$$r \in]r_1, r_2[.$$

Proof. The function

$$F : A \to \mathbb{C}, \; F(z) := z \cdot f(z)$$

is also differentiable. Then

$$I(r) = \int_{|z|=r} F(z) \cdot \frac{dz}{z}$$

Using the standard parametrization of the circle with radius r by polar coordinates

$$z = re^{i\phi} \text{ and } \phi \in [0, 2\pi].$$

we get

$$dz = ire^{i\phi} d\phi \text{ and } \frac{dz}{z} = i \cdot d\phi.$$

Hence

$$I(r) = i \int_0^{2\pi} F(re^{i\phi}) d\phi.$$

It suffices to show $\dfrac{dI}{dr} = 0$. Using the formula from Corollary 2.5 we compute

$$\frac{dI}{dr}(r) = i \cdot \int_0^{2\pi} \frac{\partial F}{\partial r}(re^{i\phi}) d\phi = \frac{1}{r} \cdot \int_0^{2\pi} \frac{\partial F}{\partial \phi}(re^{i\phi}) d\phi =$$

$$= \frac{1}{r} \cdot \left[F(re^{i\phi}) \right]_{\phi=0}^{\phi=2\pi} = 0, \; q.e.d.$$

Theorem 3.3 (Cauchy's integral theorem for disk and annulus). *Consider two radii*

$$0 \le r_1 < r_2 < \infty$$

and the closed annulus or disk respectively

$$A := \{z \in \mathbb{C} : \ r_1 \le |z| \le r_2\}.$$

Any differentiable function

$$f : U \to \mathbb{C}$$

defined on an open neighbourhood

$$A \subset U \subset \mathbb{C}$$

satisfies

$$\int_{\partial A} f(z)\ dz = 0.$$

Proof. i) Case $0 < r_1$: Lemma 3.2 implies

$$\int_{\partial A} f(z)\ dz = \int_{|z|=r_2} f(z)\ dz - \int_{|z|=r_1} f(z)\ dz = 0.$$

Here we used the convention from Remark 3.1 concerning the orientation of the paths in ∂A.

ii) Case $0 = r_1$: We have to show

$$\lim_{r_1 \to 0} \int_{|z|=r_1} f(z)\ dz = 0.$$

Set

$$M := max\{|f(z)| : |z| \le r_1\} < \infty$$

and estimate integrand and curve length as

$$\left| \int_{|z|=r_1} f(z)\ dz \right| \le M \cdot \left| \int_{|z|=r_1} dz \right| \le M \cdot 2\pi r_1.$$

Hence

$$\lim_{r_1 \to 0} \int_{|z|=r_1} f(z)\ dz = 0.$$

The claim follows from part i), q.e.d.

The most important integrand of path integration in complex analysis is the *Cauchy kernel*

$$\frac{1}{z - \zeta}$$

In addition to Lemma 3.2, a second step to obtain Cauchy's integral formula is to represent the winding number of a path as a path integral of the Cauchy kernel, see Lemma 3.4. The proof expands the Cauchy kernel into a convergent geometric series.

Lemma 3.4 (Winding number). *Consider a point $a \in \mathbb{C}$ and a radius*

$$r > 0,\ r \neq |a|.$$

Then

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{dz}{z-a} = \begin{cases} 0 & \text{if } |a| > r \\ 1 & \text{if } |a| < r \end{cases}$$

Hence the integral is considered the *winding number* $n(\gamma;\ a)$ of the path

$$\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) := r \cdot e^{it},$$

with respect to the point $a \in \mathbb{C}$.

Proof. Case $|a| > r$: In a suitable neighbourhood

$$U \supset \{z \in \mathbb{C} : |z| > r\}$$

the function

$$U \to \mathbb{C},\ z \mapsto \frac{1}{z-a},$$

is differentiable. Hence the claim of the lemma follows from Theorem 3.3.

Case $|a| < r$, see Figure 3.2:

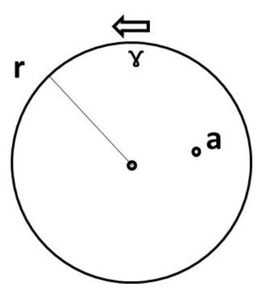


Fig. 3.2 Winding number

The Cauchy kernel expands into the convergent geometric series

$$\frac{1}{z-a} = (1/z) \cdot \frac{1}{1-(a/z)} = (1/z) \cdot \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} a^n \cdot z^{-n-1}.$$

The last series is uniformly convergent for $|z| = r$. Using the standard parametrization of the circle with radius r we obtain

$$\int_{|z|=r} \frac{dz}{z-a} = \sum_{n=0}^{\infty} a^n \cdot \int_{|z|=r} z^{-n-1} dz = \sum_{n=0}^{\infty} a^n \cdot \int_0^{2\pi} r^{-n-1} e^{(-n-1)i\phi} ir \cdot e^{i\phi}\, d\phi =$$

$$= i \cdot \sum_{n=0}^{\infty} a^n \cdot r^{-n} \int_0^{2\pi} e^{-ni\phi}\, d\phi$$

Because

$$\int_0^{2\pi} e^{-ni\phi}\, d\phi = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

we obtain

$$\int_{|z|=r} \frac{dz}{z-a} = 2\pi i, \ q.e.d.$$

Theorem 3.5 (Cauchy's integral formula for disk and annulus). *Consider two radii*

$$0 \leq r_1 < r_2 < \infty$$

and the closed annulus or disk respectively

$$A := \{z \in \mathbb{C} : \ r_1 \leq |z| \leq r_2\}.$$

Moreover, consider an open neighbourhood $A \subset U$ and a differentiable function

$$f : U \to \mathbb{C}.$$

Then for any $a \in \mathring{A}$

$$f(a) = \frac{1}{2\pi i} \int_{\partial A} \frac{f(z)}{z-a}\, dz$$

Theorem 3.5 is a first example how to obtain the values of a differentiable function at a point a by a path integral around a. Evidently, Theorem 3.5 generalizes Lemma 3.4 about the winding number.

Proof. For a given point $a \in \mathring{A}$ we split the Cauchy integral as

$$\frac{1}{2\pi i} \int_{\partial A} \frac{f(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_{\partial A} \frac{f(z) - f(a)}{z-a}\, dz + \frac{1}{2\pi i} \int_{\partial A} \frac{f(a)}{z-a}\, dz$$

After applying Lemma 3.4 to the second integral we obtain

$$\frac{1}{2\pi i}\int_{\partial A}\frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_{\partial A}\frac{f(z)-f(a)}{z-a}\,dz + f(a).$$

We claim

$$\frac{1}{2\pi i}\int_{\partial A}\frac{f(z)-f(a)}{z-a}\,dz = 0.$$

For $r \in [r_1, r_2]$ set

$$I(r) := \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)-f(a)}{z-a}\,dz.$$

We consider the integrand as a function of z which extends to U:

$$U \to \mathbb{C},\ z \mapsto \begin{cases} \dfrac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\[2ex] f'(a) & \text{if } z = a \end{cases}$$

The function is differentiable at all points $z \in U \setminus \{a\}$. At $z = a$ the integrand is continuous because

$$\lim_{\substack{z\to a\\ z\neq a}}\frac{f(z)-f(a)}{z-a} = f'(a).$$

Therefore:

- The integral $I(r)$ depends continuously on the parameter $r \in [r_1, r_2]$.

- The integrand is differentiable in each of the two open annuli in U

  $$A_1 := \{z \in U :\ r_1 < |z| < |a|\} \text{ and } A_2 := \{z \in U :\ |a| < |z| < r_2\}.$$

  Lemma 3.2 implies, that $I(r)$ remains constant when the parameter r varies in the interval

  $$[\ r_1, |a|\ [\ \text{or}\ ]\ |a|, r_2\ ].$$

As a consequence

$$I(r_1) = \lim_{r\uparrow a} I(r) = I(a) = \lim_{r\downarrow a} I(r) = I(r_2)$$

We obtain

$$\frac{1}{2\pi i}\int_{\partial A}\frac{f(z)-f(a)}{z-a}\,dz = I(r_2) - I(r_1) = 0,$$

which finishes the proof, q.e.d.

We are now ready to prove the converse of Theorem 2.3.

Corollary 3.6 (Differentiable functions are analytic). *Consider an open subset $U \subset \mathbb{C}$ and a differentiable function*

$$f : U \to \mathbb{C}.$$

For any point $a \in U$ and any finite radius $r > 0$ with $\overline{D}_r(a) \subset U$ the function f expands uniquely into a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

with coefficients

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta,\ n \in \mathbb{N}.$$

In particular, the function f is analytic.

Proof. W.l.o.g. $0 \in U$ and $a = 0$. Theorem 3.5 implies for all $z \in D_r(0)$

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

The geometric series of the Cauchy kernel

$$\frac{1}{\zeta - z} = \frac{1/\zeta}{1 - (z/\zeta)} = \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}$$

converges due to

$$\zeta \neq 0 \text{ and } |z/\zeta| < 1.$$

We obtain by rearrangement of integration and summation

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = r} \left(f(\zeta) \cdot \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}} \right) d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right) z^n$$

The uniqueness of the power series expansion follows from Proposition 1.11, q.e.d.

Corollary 3.7 (Differentiable functions have derivatives of arbitrary order). *A differentiable function*

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{C}$ has derivatives $f^{(n)}$ of arbitrary order $n \in \mathbb{N}$.

Proof. The claim follows from Corollary 3.6 and Theorem 2.3, q.e.d.

The results from Corollary 3.6 and 3.7 are in striking contrast to the properties of functions of a real variable: If a function

$$g : V \to \mathbb{R},$$

defined on an open set $V \subset \mathbb{R}$, is differentiable in the real sense, then the derivative

$$g' : V \to \mathbb{R}$$

is not necessarily continuous, and the second derivative of g does not necessarily exist. And even if g has derivatives of any order, the function does not necessarily expand into a convergent power series: A counter example is the function

$$g : \mathbb{R} \to \mathbb{R},\ g(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

All derivatives $g^{(n)}, n \in \mathbb{N}$, exist and satisfy $g^{(n)}(0) = 0$. But extending g into the complex plane creates an essential singulartiy at $z = 0$, see Definition 4.4.

3.2 The concept of holomorphy

We now combine the results obtained so far to prove the equivalence of the three approaches to complex analysis due to Cauchy, Weierstrass, and Riemann.

Theorem 3.8 (Equivalence of the approaches of Cauchy, Weierstrass, Riemann). *For a function*

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{C}$ the following properties of f are equivalent:

1. *The function is differentiable. (*Cauchy*)*

2. *The function is analytic. (*Weierstrass*)*

3. *The function has continuous partial derivatives which satisfy the Cauchy-Riemann differential equations. (*Riemann*)*

We emphasize, that each of the three properties from Theorem 3.8 does not refer to a single point but always considers a whole open set.

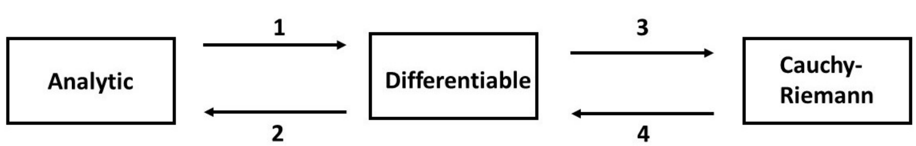


Fig. 3.3 Equivalence of the classical approaches to holomorphy

Proof. Figure 3.3 collects the results which imply the equivalences from Theorem 3.8:

- Implication 1: Theorem 2.3
- Implication 2: Corollary 3.6
- Implication 3: Theorem 2.6, part 1), and Corollary 3.7
- Implication 4: Theorem 2.6, part 2), q.e.d.

Definition 3.9 (Holomorphic function). A function

$$f : U \to \mathbb{C}$$

on an open set $U \subset \mathbb{C}$ is *holomorphic* if it satisfies the three equivalent properties from Theorem 3.8. A globally defined holomorphic function, i.e. $U = \mathbb{C}$, is named an *entire* function.

In the literature, *holomorphic* is often used as a synonym for *differentiable*. But the usage is not uniform. For more on the historical background see [20, Kap. Historische Einführung].

Definition 3.10 (Taylor series). Consider a holomorphic function

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open.}$$

For any $a \in U$ the uniquely determined convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

is the *Taylor series* of f with center $= a$. If $f \neq 0$ but $f(a) = 0$ with

$$c_0 = ... = c_{k-1} = 0 \text{ but } c_k \neq 0$$

then

$$ord(f;a) := k \in \mathbb{N}$$

is named the *order of f* at a or the *order of the zero of f* at a.

The Corollaries 3.6 and 3.16 prove two formulas to derive the coefficients of the Taylor series of f from respectively a path integral of f and from the derivatives $f^{(n)}$, $n \in \mathbb{N}$.

The three views onto holomorphy from Theorem 3.8 are on an equal footing. They should not be played off against each other like the different views onto an elephant in a well-known Indian simile, see Figure 3.4:

A group of blind men heard that a strange animal, called an elephant, had been brought to the town, but none of them were aware of its shape and form. Out of curiosity, they said: "We must inspect and know it by touch, of which we are capable". So, they sought it out, and when they found it they groped about it. In the case of the first person, whose hand landed on the trunk, said "This being is like a thick snake". For another one whose hand reached its ear, it seemed like a kind of fan. As for another person, whose hand was upon its leg, said, the elephant is a pillar like a tree-trunk. The blind man who placed his hand upon its side said the elephant, "is a wall". Another who felt its tail, described it as a rope. The last felt its tusk, stating the elephant is that which is hard, smooth and like a spear.

Fig. 3.4 The blind men and the elephant (Unknown illustrator)

Of course, the comparison is not exact: Holomorphy is not comparable with an elephant - at least, not in every respect .

Corollary 3.11 shows that one cannot choose arbitrary functions as real part or as imaginary part of a holomorphic function: Real part and imaginary part of a holomorphic function have to be harmonic.

Corollary 3.11 (Harmonic function). *Consider an open subset*

$$U \subset \mathbb{C} \simeq \mathbb{R}^2$$

and a holomorphic function

$$f : U \to \mathbb{C}.$$

with decomposition

$$f = u + i \cdot v.$$

Then real and imaginary part

$$u, v : U \to \mathbb{R}$$

are harmonic functions*, i.e. they are twice continuously differentiable and satisfy the* Laplace equation

$$\Delta\ u = \Delta\ v = 0$$

with Laplace operator

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. The proof applies the Cauch-Riemann differential equations from Theorem 2.6:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{v}}{\partial y}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial \mathrm{v}}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial \mathrm{v}}{\partial y}\right)$$

implies

$$\Delta\ u = 0,$$

and analogously

$$\Delta\ \mathrm{v} = 0,\ q.e.d.$$

The Cauchy-Riemann differential equations can be written in an elegant compact form by introducing two specific linear partial differential operators.

Definition 3.12 (Wirtinger operators). Consider an open subset

$$U \subset \mathbb{C} \simeq \mathbb{R}^2.$$

The following linear differential operators, named *Wirtinger operators*,

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

are defined on functions

$$f : U \to \mathbb{C}$$

with partial derivations.

Proposition 3.13 (Wirtinger test for differentiability). *Consider an open subset*

$$U \subset \mathbb{C}$$

and a function

$$f : U \to \mathbb{C}.$$

The following conditions are equivalent:

1. *The function f is holomorphic.*
2. *The function f has continous partial derivations and satisfies* $\dfrac{\partial f}{\partial \bar{z}} = 0.$

If these conditions are satisfied then

$$f' = \frac{\partial f}{\partial z}.$$

Proof. i) *Equivalence*: Because of Theorem 2.6 it suffices to show that the condition

$$\frac{\partial f}{\partial \bar{z}} = 0$$

is equivalent to the validity of the Cauchy Riemann differential equations. Using the decompositions into real- and imaginary part

$$z = x + iy \text{ and } f = u + i\mathrm{v}$$

the condition

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + i\mathrm{v}) =$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right) = 0$$

is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial \mathrm{v}}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial \mathrm{v}}{\partial x}.$$

ii) *Complex derivative*: If f is holomorphic then according to Proposition 2.4

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z}, \ q.e.d.$$

Remark 3.14 (Wirtinger calculus). The Wirtinger operators satisfy the usual rules for partial derivatives. In particular they satisfy the chain rule

$$f'(\gamma(t)) = f_z(\gamma(t)) \cdot \gamma'(t) + f_{\bar{z}}(\gamma(t)) \cdot \bar{\gamma}'(t).$$

Due to Theorem 3.8 we know that a holomorphic function f has derivatives $f^{(n)}$ of arbitrary order $n \in \mathbb{N}$. We now generalize Cauchy's integral formula to an integral representation of $f^{(n)}$.

Corollary 3.15 (Cauchy's integral formula for the derivatives). *Consider two radii*

$$0 \le r_1 < r_2 < \infty$$

and the closed annulus or disk respectively

$$A := \{z \in \mathbb{C} : \ r_1 \le |z| \le r_2\}.$$

Moreover, consider an open neighbourhood $A \subset U$ and a holomorphic function

$$f : U \to \mathbb{C}.$$

Then for any point $a \in \mathring{A}$ and all $n \in \mathbb{N}$ the derivatives satisfy Cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

Proof. The proof is by induction on $n \in \mathbb{N}$. It uses the theorem on interchanging integration and partial differentiation in case of an integrand with continuous partial derivatives.

Start of induction $n = 0$: Theorem 3.5.

Induction step $n - 1 \mapsto n$: Assume

$$f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{(\zeta - a)^n} \, d\zeta$$

We apply on both sides of the equation the Wirtinger operator $\dfrac{\partial}{\partial z}$ at $z = a$. Proposition 3.13 implies

$$f^{(n)}(a) = \frac{\partial}{\partial z} f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i} \int_{\partial A} f(\zeta) \cdot \frac{\partial}{\partial z}\left(\frac{1}{(\zeta - a)^n}\right) \, d\zeta.$$

Because

$$\frac{\partial}{\partial z}\left(\frac{1}{(\zeta - a)^n}\right) = (-n)(\zeta - a)^{-n-1}(-1)$$

we obtain

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta,$$

which finishes the induction step and proves the corollary, q.e.d.

Corollary 3.16 (Coefficients of the Taylor series). *Consider a holomorphic function*

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{C}$.

For any $a \in U$ and radius $r > 0$ with $\overline{D}_r(a) \subset U$ the Taylor series of f

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n,\ z \in D_r(a),$$

has coefficients

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta = \frac{f^{(n)}(a)}{n!}$$

Proof. Apply Corollary 2.3 and Corollary 3.15, q.e.d.

3.3 Principal theorems about holomorphic functions

Theorem 3.17 (Mean value property of holomorphic functions). *Consider an open set $U \subset \mathbb{C}$ and a holomorphic function*

$$f : U \to \mathbb{C}.$$

For any $a \in U$ and $r > 0$ with $\overline{D}_r(a) \subset U$ the function f satisfies the mean value formula

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\phi})\, d\phi.$$

Proof. The mean value formula equals Cauchy's integral formula from Theorem 3.5

$$f(a) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{\zeta - a}\, d\zeta$$

with the parametrization

$$\zeta = a + r \cdot e^{i\phi},\ d\zeta = ir \cdot e^{i\phi} d\phi,\ q.e.d.$$

Theorem 3.18 (Maximum principle). *Consider a domain $G \subset \mathbb{C}$ and a holomorphic function*

$$f : G \to \mathbb{C}.$$

If the modulus $|f|$ assumes its maximum at a point $a \in G$ then f is constant, equal to $f(a)$.

Proof. By assumption

$$|f(a)| = \sup\{|f(z)| : z \in G\}.$$

W.l.o.g.

$$f(a) \in \mathbb{R}^*_+.$$

Theorem 3.17 implies the existence of a radius $r_0 > 0$ such that for all $0 < r \leq r_0$ holds

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\phi})\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} Re\ f(a+re^{i\phi})\, d\phi.$$

Note that the integral of the imaginary part of the integrand vanishes, because the left-hand side of the equation is real.

The assumption $|f(a+re^{i\phi})| \leq f(a)$ implies

$$Re\ f(a+re^{i\phi}) \leq f(a).$$

If we had

$$Re\ f(a+re^{i\phi_0}) < f(a)$$

for a certain $\phi_0 \in [0,2\pi]$, then by continuity of f

$$\frac{1}{2\pi}\int_0^{2\pi} Re\ f(a+re^{i\phi})\, d\phi < \frac{1}{2\pi}\int_0^{2\pi} f(a)\, d\phi = f(a),$$

a contradiction. Hence for all $\phi \in [0,2\pi]$ and for all $0 \leq r < r_0$

$$Re\ f(a+re^{i\phi}) = f(a).$$

As a consequence, the real part $Re\ f$ is constant in $D_{r_0}(a)$. Corollary 2.8 implies that the holomorphic function f itself is constant in $D_{r_0}(a)$. The Identity Theorem 1.17 implies that f is constant in G, q.e.d.

Corollary 3.19 (Fundamental theorem of algebra). *Any polynomial*

$$p(z) \in \mathbb{C}[z]$$

of degree $k \geq 1$ *has at least one complex root.*

Proof. Because the polynomial p has positice degree we have

$$\lim_{z\to\infty} |p(z)| = \infty.$$

Hence $|p|$ assumes its minimum at a point $a \in \mathbb{C}$. In case $p(z) \neq 0$ for all $z \in \mathbb{C}$ the inverse function $1/p$ were holomorphic in $\mathbb{C}$ and $|1/p|$ would assume its maximum at a. Then Theorem 3.18 implies that $1/p$ and also p itself are constant. The contradiction to $deg\ p \geq 1$ proves the existence of at least one zero of p, q.e.d.

A direct consequence of the Identity Theorem and the Maximum Principle is the Open Mapping Theorem. A map

$$f : U \to \mathbb{C},\ U \subset \mathbb{C} \text{ open},$$

is *open* if for all open sets $V \subset U$ the image

$$f(V) \subset \mathbb{C}$$

is open, i.e. f maps open sets in the domain to open sets in the image.

Theorem 3.20 (Open mapping theorem). *Consider a domain $G \subset \mathbb{C}$ and a non-constant holomorphic function*

$$f : G \to \mathbb{C}.$$

Then f is an open map, and the image $f(G) \subset \mathbb{C}$ is also a domain.

Proof. To show the openess of $f(G)$ consider an arbitrary but fixed $w_0 \in f(G)$. We shall construct an open neighourhood of w_0 in $\mathbb{C}$, which is contained in $f(G)$.

i) Consequences of $f^{-1}(w_0)$ having only isolated points:

Because f is not constant, the Identity Theorem 1.17 implies that f attains in a neighbourhood $\overline{D}_r(z_0) \subset G$ of z_0 the value w_0 only at the point z_0. Hence a radius $r > 0$ exists with

$$\overline{D}_r(z_0) \cap f^{-1}(w_0) = \{z_0\}.$$

The compact boundary $\partial D_r(z_0)$ maps to the compact set $f(\partial D_r(z_0))$. The latter set is disjoint from $\{w_0\}$. Therefore boths sets have a positive distance: There exists $\varepsilon > 0$ with

$$z \in \partial D_r(z_0) \implies |f(z) - w_0| > 3\varepsilon.$$

ii) Claim $D_\varepsilon(w_0) \subset f(G)$:

For $z \in \partial D_r(z_0)$ and $w \in D_\varepsilon(w_0)$ we have

$$|f(z) - w)| \geq |f(z) - w_0| - |w - w_0| \geq 3\varepsilon - \varepsilon = 2\varepsilon,$$

and

$$|f(z_0) - w| = |w_0 - w| < \varepsilon,$$

in particular

$$|f(z_0) - w| < |f(z) - w|.$$

Now consider an arbitrary but fixed value

$$w \in D_\varepsilon(w_0) :$$

To obtain a contradiction, assume that

$$f - w : D_r(z_0) \to \mathbb{C}$$

has no zeros. Then the function

$$g := \frac{1}{f-w} : D_r(z_0) \to \mathbb{C}$$

is holomorphic. The Maximum Principle, see Theorem 3.18, implies

$$|g(z_0)| < \sup\{|g(z)| : z \in D_r(z_0)\} = max\{|g(z)| : z \in \overline{D}_r(z_0)\} = max\{|g(z)| : z \in \partial D_r(z_0)\}.$$

Hence

$$|f(z_0) - w| > min\{|f(z) - w| : z \in \partial D_r(z_0)\},$$

a contradiction to the previous estimate

$$|f(z_0) - w| < |f(z) - w|.$$

As a consequence

$$f - w : D_r(z_0) \to \mathbb{C}$$

has a zero in $D_r(z_0)$, i.e. f attains in $D_r(z_0)$ the value

$$w \in D_\varepsilon(w_0),$$

which finishes the proof.

iii) Connectedness of $f(G)$:

Part ii) implies that $f(G)$ is open. Continuity of f implies that $f(G)$ is connected.
q.e.d.

Theorem 3.21 (Cauchy inequalities). *Consider a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

with radius of convergence at least $r > 0$. Assume the existence of a constant M such that

$$|f(z)| \le M$$

for all $z \in D_r(a)$. Then for all $n \in \mathbb{N}$

$$|c_n| \le \frac{M}{r^n}$$

Proof. Corollary 3.6 implies for all $\rho < r$ and for all $n \in \mathbb{N}$

$$c_n = \frac{1}{2\pi\rho^n} \cdot \int_0^{2\pi} f(a + re^{i\phi}) \cdot e^{-in\phi}\ d\phi$$

Because $|e^{-in\phi}| = 1$ we have the estimate

$$|c_n| \le \frac{1}{2\pi\rho^n} \cdot M \cdot 2\pi = \frac{M}{\rho^n}$$

and

$$|c_n| \le \frac{M}{r^n},\ q.e.d.$$

Corollary 3.22 (Growth condition and boundary distance). *Consider an open subset $U \subset \mathbb{C}$ and a holomorphic function*

$$f : U \to \mathbb{C}$$

satisfying for all $z \in U$

$$|f(z)| \le M$$

Then for any point $z \in U$ with boundary distance at least $r > 0$, i.e.

$$D_r(z) \subset U,$$

holds for all $n \in \mathbb{C}$ the estimate

$$|f^{(n)}(z)| \le \frac{n!}{r^n} \cdot M$$

Proof. According to Corollary 3.16 the Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

has coefficients

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The Cauchy estimates from Theorem 3.21 imply

$$f^{(n)}(a) \le \frac{n!}{r^n} \cdot M,\ q.e.d.$$

Corollary 3.23 (Liouville's theorem). *A bounded entire function*

$$f : \mathbb{C} \to \mathbb{C}$$

is constant.

Proof. Assume the existence of a constant $M > 0$ such for all $z \in \mathbb{C}$

$$|f(z)| \leq M.$$

According to Corollary 3.6, for any radius $r > 0$ the function f expands for all $z \in D_r(0)$ into a unique convergent power series with center $= 0$

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n.$$

Theorem 3.21 implies for any $r \in \mathbb{R}$ and all $n \in \mathbb{N}$ the estimate

$$|c_n| \leq \frac{M}{r^n}$$

As a consequence $c_n = 0$ for all $n \geq 1$, q.e.d.

Remark 3.24 (Unboundedness of the complex sin*-function).* The *sin*-function is bounded for all real arguments $x \in \mathbb{R}$:

$$|sin(x)| \leq 1.$$

Because *sin* is not constant, Corollary 3.23 implies: The complex sinus-function

$$sin : \mathbb{C} \to \mathbb{C}$$

is not bounded.

The Cauchy kernel can be used to generate a holomorphic function from the path integral of a continuous function. Proposition 3.25 will be applied in the proof of Theorem 3.26.

Proposition 3.25 (Cauchy kernel). *Consider a piecewise continuously differentiable path*

$$\gamma : I \to \mathbb{C}$$

defined on a compact interval $I \subset \mathbb{R}$*, and denote by*

$$|\gamma| := \gamma(I) \subset \mathbb{C}$$

the image curve.

For any continuous function

$$\psi : |\gamma| \to \mathbb{C}$$

the function

$$f : \mathbb{C} \setminus |\gamma| \to \mathbb{C},\ f(z) := \frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{\zeta - z}\, d\zeta,$$

is holomorphic: For any point $a \in \mathbb{C}$ and radius $r > 0$ with

$$D_r(a) \cap |\gamma| = \emptyset$$

the function f expands for all $z \in D_r(a)$ into the convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z - a)^n$$

with

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

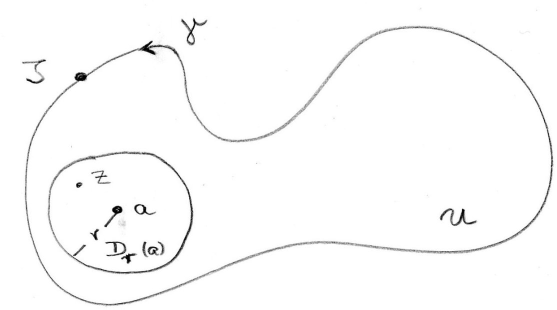


Fig. 3.5 Integration with the Cauchy kernel

Proof. For $z \in D_r(a)$ and $\zeta \in |\gamma|$ we have

$$|(z-a)/(\zeta - a)| < 1.$$

The Cauchy kernel expands into a convergent geometric series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \cdot \frac{1}{1 - (z-a)/(\zeta - a)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}.$$

For fixed $z \in D_r(0)$ the series on the right-hand side is uniformly convergent on the compact set $|\gamma|$. Interchanging summation and integration implies

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{z - \zeta}\, d\zeta = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left((z-a)^n \cdot \int_\gamma \frac{\psi(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta \right) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n,\ q.e.d.$$

Note that we did use only continuity of ψ in the proof of Proposition 3.25. The holomorphy of the result is due to the holomorphic dependency of the Cauchy kernel on z.

Theorem 3.26 (Weierstrass convergence theorem). *Consider an open set $U \subset \mathbb{C}$ and a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of holomorphic functions*

$$f_\nu : U \to \mathbb{C},$$

which is compact convergent towards a function

$$f : U \to \mathbb{C}.$$

Then f is holomorphic.

In addition, for all $n \in \mathbb{N}$ also the sequence of n-th derivatives

$$\left(f_\nu^{(n)} : U \to \mathbb{C}\right)_{\nu\in\mathbb{N}}$$

is compact convergent towards the n-th derivative $f^{(n)}$.

Proof. The limit f is continuous as compact limit of continuous functions.

i) *Holomorphy of the limit*: Consider an arbitrary but fixed point $a \in U$ and a radius $r > 0$ with $\overline{D}_r(a) \subset U$. Theorem 3.5 provides for any $z \in D_r(a)$ the Cauchy integral representation

$$f_\nu(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f_\nu(\zeta)}{\zeta - z}\,d\zeta.$$

The integral is taken along the compact boundary $\partial D_r(a)$. Here the sequence of integrands

$$\partial D_r(a) \to \mathbb{C},\ z \mapsto \frac{f_\nu(\zeta)}{\zeta - z},\ \nu \in \mathbb{N},$$

is uniformly convergent towards the function

$$\partial D_r(a) \to \mathbb{C},\ z \mapsto \frac{f(\zeta)}{\zeta - z},\ \nu \in \mathbb{N}.$$

Hence interchanging integration and limit is admissible. In the limit for $z \in D_r(a)$

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Due to Proposition 3.25 the restriction

$$f|D_r(a) : D_r(a) \to \mathbb{C}$$

is holomorphic. The arbitrary choice of a proves that f is holomorphic in U.

ii) *Compact convergence of the derivatives* $f_\nu^{(n)}$: Consider a given compact set $K \subset U$ and a given index $n \in \mathbb{N}$.

Compactness of K implies the existence of $\varepsilon > 0$ such that for all $a \in K$ also $D_\varepsilon(a) \subset U$. Choose a compact subset $K' \subset U$ such that for all $a \in K$

$$\overline{D}_{\varepsilon/2}(a) \subset K'.$$

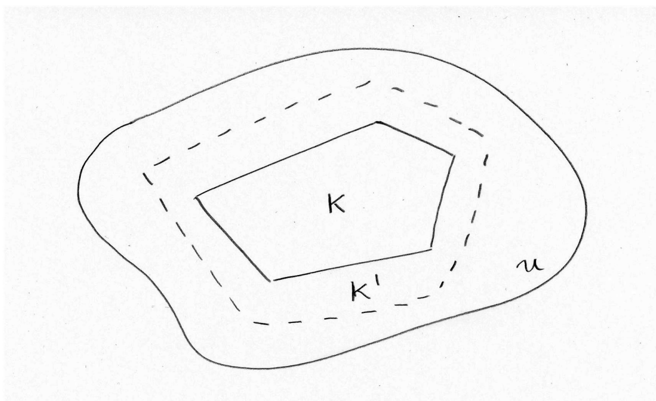


Fig. 3.6 Estimating derivatives on K by function values on K'

We obtain from Corollary 3.22 for all $z \in K$

$$\left| f_\nu^{(n)}(z) - f^{(n)}(z) \right| \leq \frac{n!}{(\varepsilon/2)^n} \cdot \|f_\nu - f\|_{K'}$$

Here the maximum-norm on the compact set K' is

$$\|f_\nu - f\|_{K'} := max\{|f_\nu(z) - f(z)| : z \in K'\}.$$

Because $(f_\nu)_{\nu\in\mathbb{N}}$ is uniformly convergent on K' towards f we conclude

$$\lim_{\nu\to\infty} \|f_\nu - f\|_{K'} = 0 \implies \lim_{\nu\to\infty} \|f_\nu^{(n)} - f^{(n)}\|_K = 0,$$

which finishes the proof of the compact convergence of $\left(f_\nu^{(n)}\right)_{\nu\in\mathbb{N}}$ towards $f^{(n)}$, q.e.d.

Remark 3.27 (Counter example). The analogue of Theorem 3.26 does not hold in the real context: Consider the family $(f_\nu)_{\nu\in\mathbb{N}}$ of differentiable functions

$$f_\nu : \mathbb{R} \to \mathbb{R},\ f_\nu(x) := \frac{1}{\nu} \cdot \sin(\nu x).$$

The sequence is uniformly convergent with limit $f = 0$. But the series of derivatives

$$(f'_\nu)_{\nu\in\mathbb{N}} \text{ with } f'_\nu(x) = \cos(\nu x)$$

is not convergent.

Remark 3.28 (Topological vector space). For any open set $U \subset \mathbb{C}$ the set

$$\mathscr{O}(U) := \{f : U \to \mathbb{C} |\ f \text{ holomorphic}\}$$

is a topological $\mathbb{C}$-vector space with respect to the *topology of compact convergence*. Defining for relatively compact open subsets

$$W \subset\subset U, \text{ i.e. } \overline{W} \subset U \text{ compact,}$$

and for any $f \in \mathscr{O}(U)$

$$\|f\|_W := max\{|f(z)| :\ z \in \overline{W}\}$$

shows: The topology of compact convergence on $\mathscr{O}(U)$ is the *Fréchet topology* defined by the seminorms corresponding to an exhaustion $(W_\nu)_{\nu\in\mathbb{N}}$ of U by relatively compact open subsets. Theorem 3.26 shows: The topological vector space $\mathscr{O}(U)$ is complete, see also [5, §2].

A useful exercise to become familiar with the present section is to investigate: In which way do the different results depend on Cauych's integral formula?

3.4 Outlook

Similar to Remark 3.28 one considers for an open set $U \subset \mathbb{C}$ the vector spaces of holomorphic, square-integrable functions

$$\mathscr{O}_{Hilb}(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic and } \int_U |f(x+iy)|^2\ dxdy < \infty\}$$

with the Hermitian scalar product

$$< -,- >: \mathscr{O}_{Hilb}(U) \times \mathscr{O}_{Hilb}(U) \to \mathbb{C},\ < f,g >:= \int_U f(x+iy) \cdot \overline{g}(x+iy)\ dxdy.$$

One checks that

$$(\mathscr{O}_{Hilb}(U), < -,- >)$$

is a Hilbert space, and one proves that the injection

$$\mathscr{O}_{Hilb}(U) \hookrightarrow \mathscr{O}(U)$$

is continuous and compact [11, Chap. VI, §3].

Analysis of several complex variables considers holomorphic functions defined on open sets $U \subset \mathbb{C}^n$. Cauchy's integral formula generalizes to holomorphic functions of several complex variables, see [5, §1]. As a consequence, a continuous function f of several complex variables with holomorphic partial derivatives is holomorphic. One can even dismiss the presupposition that f is continuous, see [14, Chap. 1, Sect. A].

For principal results about holomorphic functions of several complex variables see also [5, §2].

Chapter 4
Isolated Singularities of Holomorphic Functions

Isolated singularities of a holomorphic function f are isolated points of an open set $U \subset \mathbb{C}$ such that the function f is holomorphic in the complement. It is possible to expand f in a punctured disk around an isolated singularity a into a convergent Laurent series with summands

$$c_n \cdot (z-a)^n,\ c_n \in \mathbb{C},\ n \in \mathbb{Z}.$$

Apparently the Laurent series generalizes the Taylor series. The summands of the Taylor series have only non-negative powers, and the Taylor series is convergent in any disk where f is holomorphic.

The new fundamental concept of the present chapter is the concept of a *meromorphic function*.

We will often denote an open punctured disk with center $= a$ by

$$D_r^*(a) := D_r(a) \setminus \{a\}.$$

4.1 Laurent series and types of singularities

Definition 4.1 (Laurent series). A convergent *Laurent series* with center $a \in \mathbb{C}$ is a series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot (z-a)^n$$

which converges in an open annulus

$$\{z \in \mathbb{C} : r_1 < |z-a| \ < r_2\},\ 0 \le r_1 < r_2 \le \infty.$$

Here convergence of a series of complex summands

$$\sum_{n=-\infty}^{\infty} a_n,\ a_n \in \mathbb{C},$$

means that both series

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} a_{-n}$$

are convergent.

If a Laurent series converges pointwise for $z \in \mathbb{C}$ in an annulus, then it converges absolutely and uniformly in any compact smaller annulus.

Proposition 4.2 (Convergence of Laurent series). *Consider a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot (z-a)^n$$

which converges pointwise in the open annulus

$$\{z \in \mathbb{C} :\ r_1 < |z-a| < r_2\},\ 0 \le r_1 < r_2 \le \infty.$$

Assume two radii $\rho_1, \rho_2 > 0$ *satisfying*

$$r_1 < \rho_1 < \rho_2 < r_2.$$

Then the Laurent series converges absolutely and uniformly in the compact annulus

$$\{z \in \mathbb{C} :\ \rho_1 \le |z-a| \le \rho_2\}.$$

More specific: The series

$$\sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

converges absolutely and uniformly in the disc

$$\{z \in \mathbb{C} :\ |z-a| \le \rho_2\},$$

and the series

$$\sum_{n=-\infty}^{-1} c_n \cdot (z-a)^n$$

converges absolutely and uniformly in

$$\{z \in \mathbb{C} :\ \rho_1 \le |z-a|\}.$$

Proof. W.l.o.g. we may assume $a = 0$.

i) By assumption the series

$$\sum_{n=0}^{\infty} c_n \cdot z^n$$

converges for any point $z \in \mathbb{C}$ with

$$r_1 < |z| < r_2.$$

Therefore it converges absolutely and uniformly for

$$z \in \overline{D}_{\rho_2}(0).$$

ii) Concerning the series

$$\sum_{n=-\infty}^{-1} c_n \cdot z^n$$

we substitute

$$\zeta := 1/z$$

and consider the series

$$\sum_{n=1}^{\infty} c_{-n} \cdot \zeta^n$$

Then

$$r_1 < |z| < r_2 \implies \frac{1}{r_2} < |\zeta| < \frac{1}{r_1}$$

Therefore

$$\sum_{n=1}^{\infty} c_{-n} \cdot \zeta^n$$

converges for

$$\zeta \in D_{1/r_1}(0).$$

As a consequence, the series is absolutely and compactly convergent for

$$\zeta \in \overline{D}_{1/\rho_1}(0),$$

and the original series

$$\sum_{n=-\infty}^{-1} c_n \cdot z^n$$

is absolutely and compactly convergent for $z \in \mathbb{C}$ satisfying

$$\rho_1 \leq |z|, \ q.e.d.$$

Theorem 4.3 (Laurent expansion in an annulus). *Consider two radii*

$$0 \leq r_1 < r_2 \leq \infty$$

and the open annulus

$$G := \{z \in \mathbb{C} : \ r_1 < |z-a| < r_2\}.$$

Then any holomorphic function

$$f : G \to \mathbb{C}$$

expands uniquely into a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c \cdot (z-a)^n$$

which is convergent in G.

For all $n \in \mathbb{Z}$ the coefficients of the Laurent series can be obtained as

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

for any radius r with $r_1 < r < r_2$.

Proof. W.l.o.g assume $a = 0$.

i) *Uniqueness of the Laurent series*: Assume that f has a Laurent expansion satisfying for $z \in G$

$$f(z) = \sum_{m=-\infty}^{\infty} c_m \cdot z^m$$

Then

$$f(re^{i\phi}) = \sum_{m=-\infty}^{\infty} c_m \cdot r^m \cdot e^{i \cdot m\phi}.$$

Uniform convergence of the series on the circuit $|z| = r$ implies for arbitrary but fixed $n \in \mathbb{Z}$

$$\int_0^{2\pi} f(re^{i\phi}) \cdot e^{-in\phi} \, d\phi = \sum_{m=-\infty}^{\infty} c_m \cdot r^m \cdot \int_0^{2\pi} e^{i(m-n)\phi} \, d\phi = 2\pi \cdot c_n \cdot r^n.$$

As a consequence

$$c_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\phi}) \cdot e^{-in\phi} \, d\phi$$

is determined by the values of f. Moreover, making the substitution

$$\zeta = re^{i\phi}, \ d\zeta = ire^{i\phi} d\phi,$$

shows

$$\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(z)}{\zeta^{n+1}}\,d\zeta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i}\phi)}{r^{n+1}e^{i(n+1)\phi}}\cdot ire^{i\phi}\,d\phi = c_n.$$

ii) *Existence of the Laurent series*: We show the convergence of the two derived infinite series by dominating each of them by a suitable geometric series. Consider an arbitrary but fixed value $z \in G$. Choose two radii

$$r_1 < \rho_1 < |z| < \rho_2 < r_2.$$

Cauchy's integral formula, see Theorem 3.5, implies

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho_2}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{|\zeta|=\rho_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta. =: I(\rho_2) - I(\rho_1)$$

- For $I(\rho_2)$ we have
$$|\zeta| = \rho_2 > |z|.$$
Hence the Cauchy kernel expands as the convergent geometric series
$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\cdot\frac{1}{1-(z/\zeta)} = \sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}$$
and
$$I(\rho_2) = \frac{1}{2\pi i}\int_{|\zeta|=\rho_2}\frac{f(z)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{|\zeta|=\rho_2}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\right)\cdot z^n$$
- Analogously, for $I(\rho_1)$ we have
$$|\zeta| = \rho_1 < |z|.$$
Hence the Cauchy kernel expands as the convergent geometric series
$$\frac{1}{\zeta - z} = \frac{-1}{z}\cdot\frac{1}{1-(\zeta/z)} = -\sum_{n=0}^{\infty}\frac{\zeta^n}{z^{n+1}} = -\sum_{n=-\infty}^{-1}\frac{z^n}{\zeta^{n+1}}$$
and
$$-I(\rho_1) = -\frac{1}{2\pi i}\int_{|\zeta|=\rho_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}\left(\int_{|\zeta|=\rho_1}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\right)\cdot z^n.$$

Note that the integrand of the coefficient integrals is holomorphic within the whole annulus. Hence Lemma 3.2 implies that the values of the coefficient integrals in both representations do not change when taking

$$|\zeta| = r \text{ with arbitrary } r_1 < r < r_2$$

as common path of integration, i.e.

$$\int_{|\zeta|=\rho_2} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta = \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta = \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta$$

As a consequence

$$f(z) = I(\rho_2) - I(\rho_1) =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \right) \cdot z^n + \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \right) \cdot z^n =$$

$$\sum_{n=\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \right) \cdot z^n$$

Hence $f(z)$ has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot z^n$$

with the expected coefficients

$$c_n = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta, \ q.e.d.$$

The type of the Laurent series at an isolated singularity determines the type of the singularity.

Definition 4.4 (Classification of isolated singularities). Consider a point $a \in \mathbb{C}$, a radius $r > 0$, and a holomorphic function in the punctured disk

$$f : D_r^*(a) \to \mathbb{C}$$

with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot (z-a)^n.$$

Then exactly one of the following cases happens:

1. For all indices $n < 0$ holds $c_n = 0$. Then the Laurent series reduces to a power series convergent in the whole disc $D_r(a)$. The point a is named a *removable singularity* of f.

2. An index $k > 0$ exists with

   $$c_{-k} \neq 0 \text{ and } c_n = 0 \text{ for all } n < -k.$$

   The point a is named a *pole of order* k of f. Notation:

   $$ord(f;\, a) = -k.$$

Note. The order of a pole is a positive integer, while the order of the function f at the pole is the corresponding negative integer.

3. Infinitely many indices $n < 0$ exist with $c_n \neq 0$. The point a is named an *essential singularity* of f.

Theorem 4.5 (Riemann's theorem on removable singularities). *A holomorphic function*

$$f : D_r^*(a) \to \mathbb{C},\ a \in \mathbb{C},\ r > 0,$$

has a removable singularity at $a \in \mathbb{C}$ iff f is bounded in a punctured disk $D_{r_0}^*(a)$ for a suitable $r_0 > 0$.

In this case for $z \in D_r^*(a)$

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n,$$

and the function f extends to a holomorphic function

$$\hat{f} : D_r(a) \to \mathbb{C}$$

by the definition

$$\hat{f}(a) := c_0.$$

Proof. W.l.o.g. $a = 0$.

i) In case of a removable singularity at a the function f expands into a convergent power series with center $= a$, hence is a continuous function. As consequence f is bounded in a neighbourhood of a.

ii) Assume the existence of a radius r_0 with $0 < r_0 < r$ and a bound $M < \infty$ such that

$$|f(z)| < M$$

for

$$z \in D_{r_0}^*(a).$$

Theorem 4.3 implies for all $n \geq 1$ and all $0 < \rho \leq r_0$

$$c_{-n} = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{-n+1}}\, dz = \frac{\rho^n}{2\pi} \int_0^{2\pi} f(a + \rho \cdot e^{i\phi}) \cdot e^{in\phi}\ d\phi.$$

Hence

$$|c_{-n}| \leq \rho^n \cdot M.$$

Because ρ is arbitrary we obtain

$$c_{-n} = 0, \ i.e.$$

the Laurent series reduces to a power series, q.e.d.

A characterization of essential singularities and of poles will be given in Theorem 4.10 and Theorem 4.12 respectively.

4.2 Meromorphic functions and essential singularities

Definition 4.6 (Meromorphic function). Consider an open subset $U \subset \mathbb{C}$. A *meromorphic* function in U is a holomorphic function defined on an open subset $U' \subset U$

$$f : U' \to \mathbb{C}$$

such that

- the complement $U' \setminus U$ has only isolated points, and
- each point of $U' \setminus U$ is a pole of f.

Note. If $U' = U$ in Definition 4.6, then f is holomorphic. Hence a holomorphic function is also meromorphic.

Proposition 4.7 (Laurent expansion of meromorphic functions). *Each meromorphic function f on an open set $U \subset \mathbb{C}$ expands at a pole $a \in U$ of order $= k$ into a convergent Laurent series*

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-a)^n$$

with $c_{-k} \neq 0$.

1. *If a punctured disk*
$$D_r^*(a), \ r > 0,$$
contains no singularity of f, then the Laurent series is convergent for all $z \in D_r^*(a)$.

2. *In $D_r^*(a)$ the function f is the quotient*
$$f(z) = \frac{g}{h}$$
of the restriction of the two holomorphic functions

$$g, h : D_r(a) \to \mathbb{C}$$

with

$$h(z) := (z-a)^k \textit{ and } g := h \cdot f.$$

Proof. The claim is a Corollary of Theorem 4.3, q.e.d.

Corollary 4.8 (Field of meromorphic functions).

1. *The set $\mathscr{M}(G)$ of meromorphic functions on a domain $G \subset \mathbb{C}$ is a field with respect to addition and multiplication. It is the quotient field of the integral domain $\mathscr{O}(G)$. For two meromorphic functions $f, g \in \mathscr{M}(G),\ g \neq 0$, and $a \in G$ we have*
$$ord\left(\frac{f}{g};\ a\right) = ord(f;a) - ord(g;a).$$
2. *The quotient f/g of two polynomials*
$$f, g \in \mathbb{C}[z],\ g \neq 0,$$
is named a rational function. *The field $\mathbb{C}(z)$ of rational functions is the quotient field of the ring $\mathbb{C}[z]$ of polynomials.*

Proof. i) The sum and the product of two meromorphic functions is meromorphic: The singularities of respectively sum and product are contained in the union of the singularities of respectively the summands and the factors.

ii) In order to show that the inverse of a meromorphic function f is meromorphic, we apply Proposition 4.7. Connectedness of G implies that $g \neq 0$ has only isolated zeros. For a given point $a \in G$ exists a radius $r > 0$ such that in $D_r^*(a)$:

- $$f = \frac{g}{h}$$
  with the restrictions of the holomorphic functions
  $$g, h : D_r(a) \to \mathbb{C}$$
  $h(z) = (z-a)^k$ for $z \in D_r(a)$ for a suitable $k \in \mathbb{C}$.
- The function g has no zeros.

Hence for $z \in D_r^*(a)$

$$\frac{1}{f(z)} = \frac{(z-a)^k}{g(z)}$$

If

$$g(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

with

$$c_0 = ... = c_{m-1} = 0 \text{ and } c_m \neq 0,$$

then

$$g(z) = (z-a)^m \cdot g_1(z)$$

with

$$g_1 : D_r(a) \to \mathbb{C},\ g_1(z) := \sum_{n=0}^{\infty} c_{n+m} \cdot (z-a)^n,$$

a holomorphic function without zeros. We obtain

$$\frac{1}{g} = \frac{1}{(z-a)^m} \cdot \frac{1}{g_1}$$

In $D_r^*(a)$ the holomorphy of $1/g_1$, see Proposition 2.2, implies the holomorphy of $1/g$. Hence also the product

$$\frac{1}{f} = \frac{(z-a)^k}{g}$$

is holomorphic. Moreover

$$f = \frac{(z-a)^m}{(z-a)^k} \cdot g_1 \implies ord(f;a) = m-k$$

and

$$\frac{1}{f} = \frac{(z-a)^k}{(z-a)^m} \cdot \frac{1}{g_1} \implies ord(1/f;a) = k-m = -ord(f;a),\ q.e.d.$$

Remark 4.9 (Order function as group morphism). Corollary 4.8 shows that for a domain $G \subset \mathbb{C}$ and any point $a \in G$ the order function defines a group homomorphism

$$ord(-;\ a) : (\mathscr{M}(G)^*, \cdot) \to (\mathbb{Z}, +)$$

from the multiplicative group of non-zero meromorphic functions to the additive group of integers.

Theorem 4.10 (Casorati-Weierstrass). *Assume that the holomorphic function*

$$f : D_r^*(a) \to \mathbb{C},\ r > 0,$$

has an essential singularity at $a \in \mathbb{C}$. Then f comes near $a \in \mathbb{C}$ arbitrary close to any complex value, i.e. for any $c \in \mathbb{C}$ exists a sequence $(z_\nu)_{\nu \in \mathbb{N}}$ of points

$$z_\nu \in D_r^*(a)$$

with

$$\lim_{\nu\to\infty} z_\nu = a \text{ and } \lim_{\nu\to\infty} f(z_\nu) = c.$$

Proof. The proof is indirect. Assume the existence of a value $c \in \mathbb{C}$ which is not an accumulation point of $f(D_r^*(a))$. Then exist $\varepsilon > 0$ and $0 < \rho < r$ such that for all $z \in D_\rho^*(a)$

$$|f(z) - c| \geq \varepsilon.$$

The function

$$D_\rho^*(a) \to \mathbb{C},\ g(z) := \frac{1}{f(z) - c},$$

is holomorphic. Its modulus is bounded by $1/\varepsilon$. Theorem 4.5 implies that g extends holomorphically to the whole disc $D_\rho(a)$. For $z \in D_\rho^*(a)$ holds

$$f(z) = \frac{1}{g(z)} + c.$$

If g has at a a zero of order k, then f has at a a pole of order k. If $g(a) \neq 0$ then f is even holomorphic in $D_\rho(a)$. Both alternatives contradict the assumption about f, which proves the claim, q.e.d.

Remark 4.11 (Essential singularity).

1. The holomorphic function

   $$f : \mathbb{C}^* \to \mathbb{C}, f(z) := e^{1/z},$$

   has the Laurent series

   $$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$$

   Hence the point $a = 0$ is an essential singularity of f. Strengthening Theorem 4.10 we show that f assumes any value $c \in \mathbb{C}^*$ in any neighbourhood of $a = 0$:
   The existence of the logarithm according to Proposition 1.24 implies that the equation

   $$e^w = c$$

   has a solution w_0. Hence any solution w_ν has the form

   $$w_\nu = w_0 + \nu \cdot 2\pi i,\ \nu \in \mathbb{Z}.$$

   If $w_\nu \neq 0$ then set

   $$z_\nu := 1/w_\nu.$$

   As a consequence

$$f(z_\nu) = e^{w_\nu} = c$$

and

$$\lim_{\nu\to\infty,\ w_\nu\neq 0} z_\nu = 0.$$

2. The function

$$f(z) = e^{1/z}$$

exemplifies the claim of Picard's big theorem: Any holomorphic function assumes in any neighbourhood of an essential singularity any value $c \in \mathbb{C}$ with at most one exception infinitely often. For a proof see [21, Kap. 10, §4].

Theorem 4.12 (Characterization of a pole). *A holomorphic function*

$$f : D_r^*(a) \to \mathbb{C},\ a \in \mathbb{C},\ r > 0,$$

has a pole at $a \in \mathbb{C}$ iff

$$\lim_{z\to a,\ z\neq a} |f(z)| = \infty.$$

Proof. i) Assume that f has a pole of order $k \geq 1$ at a. Then for all $z \in D_r^*(a)$

$$f(z) = \frac{1}{(z-a)^k} \cdot g(z)$$

with a holomorphic function

$$g : D_r(a) \to \mathbb{C}.$$

As a consequence

$$\lim_{z\to a,\ z\neq a} |f(z)| = \infty.$$

ii) If

$$\lim_{z\to a,\ z\neq a} |f(z)| = \infty,$$

then f has at a neither an essential singularity, see Theorem 4.10, nor a removable singularity, see Theorem 4.5. Hence f has a pole at a, q.e.d.

4.3 The generating function of the Bernoulli numbers

Definition 4.13 (Generator of the Bernoulli numbers). The holomorphic function

$$f : D_r^*(0) \to \mathbb{C},\ f(z) := \frac{z}{e^z - 1},\ r := 2\pi,$$

has a removable singularity at $a = 0$, because

$$ord(f;a) = ord(id;a) - ord(e^z - 1;a) = 1 - 1 = 0.$$

The *Bernoulli numbers* $(B_n)_{n\in\mathbb{N}}$ are defined via the coefficients of the power series with center $a = 0$

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot z^n.$$

The series has radius of convergence $= 2\pi$. It is named the *generating function* of the Bernoulli numbers.

Remark 4.14 (Recursion formula for Bernoulli numbers).

1. One easily computes the first Bernoulli numbers:

$$\frac{z}{e^z - 1} = \frac{z}{z + z^2/2! + z^3/3! + O(4)} = \frac{1}{1 + z/2! + z^2/3! + O(3)} =$$

$$= 1 - (z/2 + z^2/3! + O(3)) + (z/2 + O(3))^2 + O(3) = 1 - z/2 - z^2/6 + z^2/4 + O(3) =$$

$$= 1 - z/2 + (1/12) \cdot z^2 = B_0 + B_1 \cdot z + (B_2/2!) \cdot z^2 + O(3)$$

Hence the first Bernoulli numbers are

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6.$$

2. A second method to compute Bernoulli numbers is the recursion formula: For all $N \in \mathbb{N}^*$

$$\sum_{n=0}^{N} \binom{N+1}{n} \cdot B_n = 0$$

with $B_0 = 1$.Proof: The defining equation

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot z^n$$

implies

$$1 = (e^z - 1) \cdot \frac{1}{z} \cdot \frac{z}{e^z - 1} = \left(\sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} \right) \cdot \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot z^n \right) =$$

$$= \sum_{N=0}^{\infty} \left(\sum_{n+m=N} \frac{1}{(m+1)!} \cdot \frac{B_n}{n!} \right) \cdot z^N = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} \frac{B_n}{(N+1-n)! \cdot n!} \right) \cdot z^N =$$

$$= \sum_{N=0}^{\infty} \left(\frac{1}{(N+1)!} \cdot \sum_{n=0}^{N} \binom{N+1}{n} \cdot B_n \right) \cdot z^N, \ q.e.d.$$

One checks

$$B_3 = 0, \ B_4 = -1/30, \ B_5 = 0, \ B_6 = 1/42$$

$$B_7 = 0, \ B_8 = -1/30, \ B_9 = 0, \ B_{10} = 5/66.$$

Proposition 4.15 (Vanishing of Bernoulli numbers with odd index ≥ 3). *For all $k \geq 1$ holds*

$$B_{2k+1} = 0.$$

Proof. To prove the claim we have to show that the Taylor series with center $= 0$ of

$$F(z) := \frac{z}{e^z - 1} + \frac{z}{2}$$

has only coefficents with even index, or

$$F(z) = F(-z).$$

We compute

$$F(z) = \frac{z}{e^z - 1} + \frac{z}{2} = \frac{2 \cdot z + z \cdot e^z - z}{2 \cdot (e^z - 1)} = \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1}$$

and

$$F(-z) = \left(-\frac{z}{2} \right) \cdot \frac{e^{-z} + 1}{e^{-z} - 1} = \left(-\frac{z}{2} \right) \cdot \frac{1 + e^z}{1 - e^z} = F(z), \ q.e.d.$$

Proposition 4.16 (Bernoulli numbers in the Taylor series of cotangent and tangent).

1. The cotangent function

$$cot\ z := \frac{cos\ z}{sin\ z}$$

is meromorphic in $\mathbb{C}$ with pole set

$$P = \mathbb{Z} \cdot \pi.$$

Each pole has order $= 1$. Its Laurent series with center $a = 0$ is

$$cot\ z = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot z^{2k-1}$$

The series is convergent for $0 < |z| < \pi$.

2. *The tangent function*

$$tan\ z := \frac{sin\ z}{cos\ z}$$

is meromorphic in $\mathbb{C}$ *with pole set*

$$P = (\pi/2) + \mathbb{Z} \cdot \pi.$$

Its Taylor series is

$$tan\ z = \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{2^{2k} \cdot (2^{2k} - 1) \cdot B_{2k}}{(2k)!} \cdot z^{2k-1}$$

The series has radius of convergence $= \pi/2$.

Proof. 1. We recall from Proposition 4.15 the function

$$F(z) := \frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} \cdot z^{2k}.$$

Moreover, we use the representation of the trigonometric functions by exponentials from Remark 1.22 and obtain

$$cot\ z = \frac{cos\ z}{sin\ z} = i \cdot \frac{e^{2iz} + 1}{e^{2iz} - 1}$$

hence

$$z \cot\ z = iz \cdot \frac{e^{2iz} + 1}{e^{2iz} - 1} = F(2iz) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (2iz)^{2k} =$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot z^{2k}.$$

2. We use the formula

$$tan\ z = cot\ z - 2 \cdot cot\ 2z,$$

which follows according to

$$cot\ z - 2 \cdot cot\ 2z = \frac{cos\ z}{sin\ z} - 2 \cdot \frac{cos\ 2z}{sin\ 2z} = \frac{cos\ z}{sin\ z} - \frac{cos^2\ z - sin^2\ z}{cos\ z \cdot sin\ z} = \frac{sin^2\ z}{sin\ z \cdot cos\ z}$$

$$= \frac{sin\ z}{cos\ z} = tan\ z$$

We obtain

$$tan\ z = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot z^{2k-1} - 2 \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \frac{2^{2k} \cdot B_{2k}}{(2k)!} \cdot (2z)^{2k-1} =$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{2^{2k} \cdot (2^{2k}-1) \cdot B_{2k}}{(2k)!} \cdot z^{2k-1}.$$

In both cases the value of the radius of convergence follows from Theorem 4.3, q.e.d.

4.4 Outlook

The singularities of holomorphic functions of several complex variables are no longer points, they do not have dimension zero. The singularities are higher-dimensional analytic sets. They carry their own complex structure, and have to be investigated independently from the embedding space. The global viewpoint leads to the study of complex spaces, see [12, Chap. 1 and 4].

The *lingua franca* used for the general theory of complex spaces is the language of sheaves, [5, §6].

Chapter 5
Mittag-Leffler Theorem and Weierstrass Product Formula

The Mittag-Leffler problems asks for meromorphic functions with prescribed negative parts of the Laurent series at a discrete set of points. While the Weierstrass problem deals with a similar problem in the context of holomorphic functions: The Weierstrass problem prescribes a discrete set of zeros and corresponding orders, and asks for holomorphic functions with these zeros.

The solution of both problems provides powerful existence theorems for both types of functions.

5.1 Meromorphic functions with prescribed principal parts

The current section formalizes the Mittag-Leffler problem and presents its solution.

Definition 5.1 (Principal part). Consider a meromorphic function f in $\mathbb{C}$. Because all poles of f are isolated singularities, the function f has at any pole $a \in \mathbb{C}$ a Laurent expansion with center $= a$

$$f(z) = \sum_{n=-k}^{\infty} c_n \cdot (z-a)^n,\ c_{-k} \neq 0.$$

The rational function

$$\sum_{n=-k}^{-1} c_n \cdot (z-a)^n$$

which sums up the finitely many summands with negative exponent, is the *principal part* of f at a.

The Mittag-Leffler problem considers a discrete set $P \subset \mathbb{C}$ and prescribes for each $a \in P$ a principal part

$$H_a(z) = \sum_{n=-k_a}^{-1} c_{a,n} \cdot (z-a)^n, \ c_{-k} \neq 0.$$

The problem asks:

Does there exists a meromorphic function f in $\mathbb{C}$ with poles exactly at the points from P and with the prescribed principal parts? If yes, how many solutions do exist? The solution of the problem is additive.

Remark 5.2 (Mittag-Leffler problem for finitely many principal parts). If the pole set is finite

$$P = \{a_1, ..., a_m\},$$

then the Mittag-Leffler problem has the trivial solution

$$f = H_{a_1} + ... + H_{a_m}$$

which just adds the finitely many principal parts.

The solution of the Mittag-Leffler problem for the general case will be given in Theorem 5.4. Note that the discreteness of $P \subset \mathbb{C}$ implies the countability of P.

Definition 5.3 (Compact convergence of meromorphic functions). Consider an open subset $U \subset \mathbb{C}$. A series

$$\sum_{\nu=0}^{\infty} f_\nu$$

of meromorphic functions $f_\nu, \ \nu \in \mathbb{N}$, on U is *compactly convergent*, if for any compact subset $K \subset U$ an index ν_0 exists such that

- the functions
$$f_\nu, \ \nu \geq \nu_0,$$
have no pole in K,
- and the series
$$\sum_{\nu=\nu_0}^{\infty} f_\nu$$
converges uniformly on K.

Theorem 3.26 implies: The limit of a compact convergent series of meromorphic functions is meromorphic.

Theorem 5.4 (Solution of the Mittag-Leffler problem). *Consider a sequence* $(a_\nu)_{\nu\in\mathbb{N}}$ *of pairwise distinct complex points* $a_\nu \in \mathbb{C}$ *satisfying*

$$\lim_{\nu\to\infty} |a_\nu| = \infty,$$

and an attached sequence of principal parts

$$H_\nu(z) = \sum_{n=-k_\nu}^{-1} c_{\nu,n} \cdot (z-a_\nu)^n,\ \nu \in \mathbb{N},\ k_\nu \in \mathbb{N}.$$

Then:

- *In* $\mathbb{C}$ *a meromorphic function* F *exists with exactly these principal parts.*
- *If two meromorphic functions* F_1 *and* F_2 *in* $\mathbb{C}$ *have the same principal parts, then their difference*

  $$F_2 - F_1$$

  is holomorphic in $\mathbb{C}$*, i.e. an entire function.*

Proof. i) *Polynomial approximation*: The idea of this step is to exhaust $\mathbb{C}$ by a sequence of relatively compact subsets

$$D_k(0) \subset\subset D_{k+1}(0),\ k \in \mathbb{N},$$

and to approximate the local solutions by polynomials, holomorphic on $\mathbb{C}$.

For each $k \in \mathbb{N}$ we collect the principal parts with center in the annulus

$$A_k := \{z \in \mathbb{C} :\ k \le |z| < k+1\}$$

by defining

$$F_k := \sum_{\nu:\ a_\nu \in A_k} H_\nu.$$

The sum is finite because $\overline{A}_k$ is compact and P is discrete. The function F_k is meromorphic in $\mathbb{C}$ with poles exactly in A_k. Because the function F_k is holomorphic in the disk $D_k(0)$, its approximation by a suitable polynomial Φ_k from its Taylor series in $D_k(0)$ satisfies

$$\|F_k - \Phi_k\|_{\overline{D}_{k-1}(0)} < (1/2)^k.$$

Note that the meromorphic function in $\mathbb{C}$

$$F_k - \Phi_k$$

has the same poles and principal parts as F_k. Define

$$F := \sum_{k=0}^{\infty} (F_k - \Phi_k).$$

To prove that the series is compact convergent, we note that the series

$$\sum_{\nu \geq k} \|F_\nu - \Phi_\nu\|_{\overline{D}_{k-1}(0)}$$

is dominated by the series

$$\sum_{\nu \geq k} (1/2)^\nu < \infty.$$

Hence F is meromorphic in $\mathbb{C}$. By construction, F has the prescribed poles and principal parts.

ii) *General solution*: Apparently the difference $F_2 - F_1$ is holomorphic, q.e.d.

There exist meromorphic functions which equal the infinite series of their principal parts, see Example 5.5. But in general, the series of principal parts is not convergent. The difference between the two meromorphic functions from Example 5.5 and from Proposition 5.6 is the pole order k of their principal parts. In the first example with $k = 2$ the series of principal parts is convergent. While in the second example $k = 1$, and the series of principal parts is not convergent. Here one has to introduce additional holomorphic summands - like Φ_k in the proof of Theorem 5.4 - to enforce convergence.

Example 5.5 (Sum of infinitely many principal parts). We investigate the meromorphic function in $\mathbb{C}$

$$f(z) := \frac{\pi^2}{sin^2(\pi z)}.$$

i) *Computing the principal parts*: The pole set P of f is the set of integers

$$P = \mathbb{Z}.$$

We compute the principal part $H_0(z)$ of f at $a = 0$:

$$\frac{\pi^2}{sin^2(\pi z)} = \frac{\pi^2}{(\pi z - (\pi^3 z^3/6) + O(5))^2} =$$

$$= \frac{\pi^2}{\pi^2 z^2 + O(4)} = \frac{\pi^2}{z^2 \pi^2 (1 + O(2))} = \frac{1}{z^2} \cdot (1 + O(2))$$

Hence

$$H_0(0) = \frac{1}{z^2}$$

with a pole of order $= 2$. Because f is peridodic with period $= 1$ the principal part of f at arbitrary $z = n \in \mathbb{Z}$ is

$$H_n(z) = \frac{1}{(z-n)^2}$$

ii) *Convergence of the series of principal parts*: Consider the series

$$\sum_{n\in\mathbb{Z}} H_n(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

of meromorphic functions in $\mathbb{C}$. Each summand H_n is meromorphic in $\mathbb{C}$ with a single pole at $z = n$, and each pole has order $= 2$. To prove the compact convergence of the series of meromorphic functions

$$\sum_{n\in\mathbb{Z}} H_n,$$

we choose an arbitrary but fixed radius $r > 0$. For all indices $|n| > r$ the function H_n has no pole on the compact set $\overline{D}_r(0)$.

In addition, we choose an index $n_0 > 2r$. Then for all $n \geq n_0$ and all $z \in \overline{D}_r(0)$:

$$|n-z| \geq n-r \geq n-(n/2) = n/2,$$

hence

$$\left|\frac{1}{(z-n)^2}\right| \leq \frac{1}{(n/2)^2}$$

The convergence of the dominating series

$$4 \cdot \sum_{n=1}^{\infty} (1/n)^2$$

implies the uniform convergence of

$$\sum_{|n|\geq n_0} f_n$$

on $\overline{D}_r(0)$. As a consequence, the limit

$$F(z) := \sum_{n\in\mathbb{Z}} H_n(z)$$

is meromorphic in $\mathbb{C}$.

iii) *A functional equation*: The two meromorphic functions $f(z)$ and $F(z)$ have the same principal parts. Their difference

$$g := f - F$$

is an entire function. We claim: Each of the three functions $\phi \in \{f, F, g\}$ satisfies the functional equation

$$\phi(z) + \phi(z + (1/2)) = 4 \cdot \phi(2z).$$

- $\phi = F$:
$$F(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \sum_{n\in\mathbb{Z}} \frac{4}{(2z-2n)^2}$$
and
$$F(z+(1/2)) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+(1/2)-n)^2} = \sum_{n\in\mathbb{Z}} \frac{4}{(2z-(2n-1))^2}$$
imply
$$F(z)+F(z+(1/2)) = \sum_{m\in\mathbb{Z}} \frac{4}{(2z-m)^2} = 4\cdot F(2z).$$
- $\phi = f$:
$$f(z)+f(z+(1/2)) = \frac{\pi^2}{sin^2(\pi z)} + \frac{\pi^2}{cos^2(\pi z)} = \frac{\pi^2}{sin^2(\pi z)\cdot cos^2(\pi z)} =$$
$$= \frac{4\pi^2}{sin^2(2\pi z)} = 4\cdot f(2z).$$
- $\phi = g$: The functional equation of
$$g = f - F$$
follows from the two previous functional equations.

iv) *Consequences of the functional equation*: If an entire function ϕ satisfies the functional equation
$$\phi(z)+\phi(z+(1/2)) = 4\cdot\phi(2z)$$
then $\phi = 0$.

For the proof set
$$M := \sup_{|z|=2} |\phi(z)|.$$
Then
$$M = \sup_{|z|=1} |\phi(2z)| \le 1/4\cdot\left(\sup_{|z|=1}|\phi(z)| + \sup_{|z|=1}|\phi(z+(1/2))|\right) \le$$
$$\le 1/4\cdot(M+M) = \frac{M}{2}.$$
Here the last estimate follows from the maximum principle Theorem 3.18. As a consequence
$$M = 0 \text{ and } \phi = 0$$
due to the Identity Theorem 1.17.

v) *Representation as sum of all principal parts*: As consequence of step i) - iv) we obtain the representation of the meromorphic function on the left-hand side

$$\frac{\pi^2}{sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

as the sum of its principal parts on the right-hand side.

Proposition 5.6 (Partial fraction expansion of the cot**-function).** *The cot-function is meromorphic in* $\mathbb{C}$ *and expands as the partial fraction*

$$\pi \cdot cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Proof. i) First we determine the principal parts of the meromorphic function

$$\pi \cdot cot(\pi z) = \pi \cdot \frac{cos(\pi z)}{sin(\pi z)}:$$

The pole set is $P = \mathbb{Z}$. At the center $a = 0$

$$\pi cot(\pi z) = \pi \cdot \frac{1 + O(2)}{\pi z + O(3)} = \pi \cdot \frac{1 + O(2)}{\pi z \cdot (1 + O(2))} =$$

$$= \frac{1}{z} \cdot (1 + O(2)) = \frac{1}{z} + O(1).$$

Hence $\pi \cdot \cot(\pi z)$ has at $a = 0$ the principal part

$$H_0(z) = \frac{1}{z}.$$

Because the function has period $= 1$ its principal part at the pole $n \in \mathbb{Z}$ is

$$H_n(z) = \frac{1}{z-n}.$$

All poles have order $= 1$.

ii) The infinite series of the principal parts H_n is not convergent. Therefore we subtract the holomorphic summands $-(1/n)$ to enforce convergence. We claim that the modified series of meromorphic functions

$$G(z) := \frac{1}{z} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(H_n(z) + \frac{1}{n} \right) = \frac{1}{z} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(\frac{1}{z-n} + \frac{1}{n} \right) =$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

is compact convergent. For the proof note

$$\frac{1}{z-n}+\frac{1}{n}=\frac{n+(z-n)}{n(z-n)}=\frac{z}{n(z-n)}.$$

For arbitrary but fixed radius $r>0$ choose $n_0>2r$. For $z\in\overline{D}_r(0)$ and $n\geq n_0$ holds

$$|z-n|\geq\frac{n}{2} \text{ and } \left|\frac{z}{n(z-n)}\right|=\frac{|z|}{n\cdot|z-n|}\leq\frac{r}{n(n/2)}=\frac{2r}{n^2}.$$

Hence

$$\sum_{n=n_0}^{\infty}\left\|\frac{1}{z-n}+\frac{1}{n}\right\|_{\overline{D}_r(0)}\leq 2r\cdot\sum_{n=n_0}^{\infty}\frac{1}{n^2}<\infty,$$

and similarly for

$$-n\leq -n_0.$$

Therefore

$$G(z)=\frac{1}{z}+\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{1}{z-n}+\frac{1}{n}\right)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

is a meromorphic function in $\mathbb{C}$.

iii) Due to part i) and ii) the two functions G and $\pi\cot(\pi z)$ have the same principal parts. We show that they are equal. Outside the poles on one hand,

$$\frac{d}{dz}\left(\pi cot(\pi z)\right)=-\frac{\pi^2}{sin^2(\pi z)}.$$

On the other hand, Theorem 3.26 implies

$$\frac{d}{dz}G(z)=-\frac{1}{z^2}-\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{(z-n)^2}=-\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}=-\frac{\pi^2}{sin^2(\pi z)}.$$

Here the last equality is a consequence of the result from Example 5.5. Hence

$$\frac{d}{dz}G(z)=\frac{d}{dz}(\pi cot(\pi z))$$

or

$$G(z)=\pi cot(\pi z)+const.$$

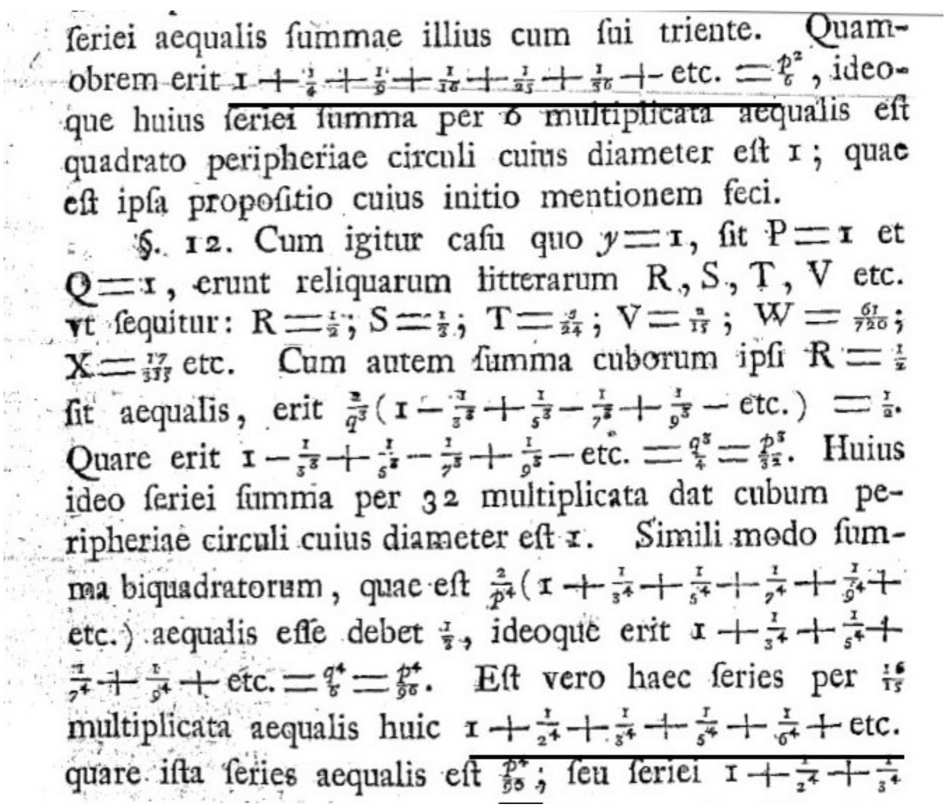
The function G is odd, i.e. $G(z)=-G(-z)$. Also the function

$$\pi\cdot cot(\pi z)$$

is odd because *cos* is even and *sin* is odd. Therefore the constant $=0$, i.e.

$$\pi \cdot cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \; q.e.d.$$

The partial fraction expansion of the cotangent function is the key to a summation problem, which occupied several European mathematicians in the second half of the 17th century. The first formulas from Proposition 5.7 are due to Leonhard Euler (1735 or 1740), a student of Johann Bernoulli (1667-1748), see Figure 5.1.


ſeriei aequalis ſummae illius cum ſui triente. Quamobrem erit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.} = \frac{p^2}{6}$, ideoque huius ſeriei ſumma per 6 multiplicata aequalis eſt quadrato peripheriae circuli cuius diameter eſt 1; quae eſt ipſa propoſitio cuius initio mentionem feci.

§. 12. Cum igitur caſu quo $y = 1$, ſit $P = 1$ et $Q = 1$, erunt reliquarum litterarum R, S, T, V etc. vt ſequitur: $R = \frac{1}{2}$; $S = \frac{1}{3}$; $T = \frac{5}{24}$; $V = \frac{2}{15}$; $W = \frac{61}{720}$; $X = \frac{17}{315}$ etc. Cum autem ſumma cuborum ipſi $R = \frac{1}{2}$ ſit aequalis, erit $\frac{2}{q^3}(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.}) = \frac{1}{2}$. Quare erit $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{q^3}{4} = \frac{p^3}{32}$. Huius ideo ſeriei ſumma per 32 multiplicata dat cubum peripheriae circuli cuius diameter eſt 1. Simili modo ſumma biquadratorum, quae eſt $\frac{2}{q^4}(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} +$ etc.) aequalis eſſe debet $\frac{1}{3}$, ideoque erit $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{q^4}{6} = \frac{p^4}{96}$. Eſt vero haec ſeries per $\frac{16}{15}$ multiplicata aequalis huic $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.}$ quare iſta ſeries aequalis eſt $\frac{p^4}{90}$; ſeu ſeriei $1 + \frac{1}{2^4} + \frac{1}{3^4}$

Fig. 5.1 Euler's first two formula from [3, p. 129] (emphasis added by J.W.)

Proposition 5.7 (Summae serierum reciprocarum and Bernoulli numbers). *For all $k \in \mathbb{N}^*$*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \cdot \frac{2^{2k-1} \cdot \pi^{2k} \cdot B_{2k}}{(2k)!}$$

in particular

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\ \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\ \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945},\ \sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450},$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{10}}=\frac{\pi^{10}}{93.555},\ \sum_{n=1}^{\infty}\frac{1}{n^{12}}=\frac{691\cdot\pi^{12}}{682.593.555}$$

Note. Proposition 5.7 implies that the Bernoulli numbers with even index change their sign from one number to the next.

Proof. Recall from Proposition 5.6 the partial fraction expansion of the cotangent function

$$\pi\cdot cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

For $|z|<1$ we expand each summand of the series into a convergent geometric series

$$\frac{2z}{z^2-n^2}=-\frac{1}{n^2}\cdot\frac{2z}{1-(z^2/n^2)}=-\frac{2z}{n^2}\cdot\sum_{k=0}^{\infty}\frac{z^{2k}}{n^{2k}}=-2\cdot\sum_{k=1}^{\infty}\frac{z^{2k-1}}{n^{2k}}$$

- On one hand, applying Theorem 1.5, part 2), to rearrange the double series we obtain

$$\pi\cdot cot(\pi z)=\frac{1}{z}-2\cdot\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{z^{2k-1}}{n^{2k}}\right)=\frac{1}{z}-2\cdot\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)\cdot z^{2k-1}$$

- On the other hand, the cotangent series from Proposition 4.16

$$cot\ z=\frac{1}{z}+\sum_{k=1}^{\infty}(-1)^k\cdot\frac{2^{2k}\cdot B_{2k}}{(2k)!}\cdot z^{2k-1}$$

 implies

$$\pi\cdot cot(\pi z)=\frac{1}{z}+2\cdot\sum_{k=1}^{\infty}(-1)^k\cdot\frac{2^{2k-1}\cdot\pi^{2k}\cdot B_{2k}}{(2k)!}\cdot z^{2k-1}$$

Comparing coefficients proves the claim, q.e.d.

The Riemann ζ-function generalizes the integer exponent $k\in\mathbb{N}^*$ from the formulas in Proposition 5.7 to a complex variable s. Recall

$$n^s:=e^{s\cdot ln(n)}.$$

Remark 5.8 (Riemann ζ-function). On the right half-plane

$$RH(1):=\{s\in\mathbb{C}:\ Re\ s>1\}$$

the *Riemann ζ-function*

$$\zeta : RH(1) \to \mathbb{C},\ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is holomorphic, see [25, Teil I, §1, Beispiel a) zu Satz 2].

Euler's result from Proposition 5.7 computes the values

$$\zeta(2k),\ k \in \mathbb{N}.$$

We will state some further results about the Riemann ζ-function in Remark 5.29.

5.2 Infinite products of holomorphic functions

The investigation of infinite products of holomorphic functions employs the concept of normal convergence. Normal convergence assures that an infinite product of holomorphic functions has a zero at $a \in \mathbb{C}$ if and only if at least one factor has a zero at the point a.

Normal convergence is strictly stronger than compact convergence. Theorem 5.11 and the example from Remark 5.12 show: Normal convergence is strictly stronger than compact convergence. Compact convergence does not assure the desirable property concerning the zeros of a product.

Definition 5.9 (Normal convergence). Consider an open subset $U \subset \mathbb{C}$ and a sequence of holomorphic functions

$$f_\nu : U \to \mathbb{C},\ \nu \in \mathbb{N}.$$

i) The series

$$\sum_{\nu=0}^{\infty} f_\nu$$

is *normally convergent* if for any compact subset $K \subset U$ the series of norms

$$\sum_{\nu=0}^{\infty} \|f_\nu\|_K$$

is convergent.

ii) The infinite product

$$\prod_{\nu=0}^{\infty} f_\nu$$

is *normally convergent* if the series

$$\sum_{\nu=0}^{\infty}(f_\nu - 1)$$

is normally convergent.

Lemma 5.10 (Some estimates for logarithm and exponential). *Consider a complex number $u \in \mathbb{C}$ with $|u| \leq 1/2$. Then*

$$|Log(1+u)| \leq 2|u| \text{ and } |e^u - 1| \leq 2|u|.$$

Proof. Estimating the geometric series

$$\sum_{n=0}^{\infty}|u|^n = \frac{1}{1-|u|} \leq 2$$

shows

$$|Log(1+u)| = \left|\sum_{n=1}^{\infty}(-1/n)^{n+1}\cdot u^n\right| = |u|\cdot\left|\sum_{n=0}^{\infty}(-1/n)^{n+2}\cdot u^n\right| \leq |u|\cdot\sum_{n=0}^{\infty}|u|^n \leq 2|u|,$$

and

$$|e^u - 1| = \left|\sum_{n=1}^{\infty}\frac{u^n}{n!}\right| \leq |u|\cdot\sum_{n=0}^{\infty}|u|^n \leq 2|u|,\ q.e.d.$$

Theorem 5.11 (Normal convergence). *Consider an open subset $U \subset \mathbb{C}$ and a sequence of holomorphic functions*

$$f_\nu : U \to \mathbb{C},\ \nu \in \mathbb{N}.$$

Assume that the infinite product

$$\prod_{\nu=0}^{\infty} f_\nu$$

is normally convergent. Then:

1. *The product is compact convergent towards a holomorphic function*

$$F : U \to \mathbb{C},$$

i.e.

$$F = \lim_{N\to\infty}\left(\prod_{\nu=0}^{N} f_\nu\right)$$

with respect to compact convergence.

2. *The limit F has a zero at $a \in U$ iff at least one factor f_ν has a zero at a. In this case*
$$ord(F;a) = \sum_{\nu=0}^{\infty} ord(f_\nu;a).$$

3. *The limit F is independent from the order of the factors in the product.*

Proof. 1. For each $\nu \in \mathbb{N}$ we set

$$\phi_\nu := f_\nu - 1.$$

Assume that
$$\prod_{\nu=0}^{\infty} f_\nu$$

is normally convergent. For any arbitrary but fixed compact $K \subset U$ the series

$$\sum_{\nu=0}^{\infty} \|\phi_\nu\|_K$$

converges. In particular, an index ν_0 exist such that for all $\nu \geq \nu_0$

$$\|\phi_\nu\|_K \leq 1/2.$$

Hence for all $\nu \geq \nu_0$ the function f_ν has no zeros on K and

$$Log\ f_\nu = Log(1 + \Phi_\nu)$$

is well-defined on K by the Log-series. For any $N \in \mathbb{N},\ N \geq n_0$. The functional equations of *exp* implies on K

$$\prod_{\nu=\nu_0}^{N} f_\nu = \prod_{\nu=\nu_0}^{N} (1 + \Phi_\nu) = \prod_{\nu=\nu_0}^{N} exp(Log(1 + \Phi_\nu)) = exp\left(\sum_{\nu=\nu_0}^{N} Log(1 + \Phi_\nu)\right)$$

Lemma 5.10 implies for $\nu \geq \nu_0$

$$\|Log(1 + \phi_\nu)\|_K \leq 2 \cdot \|\phi_\nu\|_K.$$

The convergence of
$$\sum_{\nu=0}^{\infty} \|\phi_\nu\|_K$$

implies: The series
$$\sum_{\nu=\nu_0}^{\infty} log(1 + \phi_\nu)$$

is uniformly convergent on K, and due to the continuity of the exponential

$$\prod_{\nu=0}^{\infty} f_\nu = exp\left(\sum_{\nu=0}^{\infty} Log(1+\Phi_\nu)\right)$$

is uniformly convergent on K.

2. In the compact set $K \subset U$ from part 1) the product

$$\prod_{\nu=\nu_0}^{\infty} f_\nu = exp\left(\sum_{\nu=\nu_0}^{\infty} Log(1+\Phi_\nu)\right)$$

has no zero because the exponential function has no zeros. As a consequence, the zeros in K of the infinite product

$$F = \prod_{\nu=\nu_0}^{\infty} f_\nu$$

are the zeros in K of the finite product

$$\prod_{0\le\nu<\nu_0} f_\nu,$$

and for all points $a \in K$

$$ord(F;a) = \sum_{0\le\nu<\nu_0} ord(f_\nu;a) = \sum_{\nu=0}^{\infty} ord(f_\nu;a)$$

3. The independence of the value of the product from the order of its factors follows from the analogous property of absolutely convergent series, q.e.d.

Remark 5.12 (Normal convergence and compact convergence). Due to Theorem 5.11 normal convergence of an infinite product implies compact convergence. But in general, the opposite direction does not hold: Consider the sequence of constant functions

$$f_\nu := 1/2, \nu \in \mathbb{N},$$

defined on $\mathbb{C}$. We have

$$F_n := \prod_{\nu=0}^{n} f_\nu = (1/2)^{n+1},$$

hence

$$F = \lim_{n\to\infty} F_n = 0,$$

but no factor f_ν has a zero. Apparently, the product

$$\prod_{\nu=0}^{n} f_\nu$$

does not satisfy the definition of normal convergence: The sequence $(f_\nu - 1)_{\nu \in \mathbb{N}}$ does not converge towards zero, hence

$$\sum_{\nu \in \mathbb{N}} (f_\nu - 1)$$

is not convergent.

5.3 Weierstrass product theorem for holomorphic functions

The Weierstrass problem ask's for a holomorphic function, which vanishes exactly at a given set of points with prescribed order. If such a solution exists the problem asks for the general solution. The solution of the Weierstrass problem is multiplicative.

Proposition 5.13 shows a specific case of the Weierstrass problem. All prescribed zeros are equidistant simple zeros: The solution is obtained by taking the product of linear polynomials vanishing at the given points.

Proposition 5.13 (Product representation of the ***sin*****-function).** *For the sequence of holomorphic functions*

$$f_n : \mathbb{C} \to \mathbb{C},\ f_n(z) := 1 - \frac{z^2}{n^2},\ n \geq 1.$$

the infinite product

$$\prod_{n=1}^{\infty} f_n$$

is normally convergent. It satisfies for all $z \in \mathbb{C}$

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin(\pi z)}{\pi z}.$$

Proof. i) *Normal convergence*: We show the normal convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^2}{n^2}.$$

Consider the compact set $K := \overline{D}_r(0)$ with arbitrary but fixed radius $r > 0$. For any $z \in K$

$$\left\| \frac{z^2}{n^2} \right\|_K = \frac{r^2}{n^2}.$$

Moreover

$$\sum_{n=1}^{\infty} \frac{r^2}{n^2} = r^2 \cdot \frac{\pi^2}{6} < \infty.$$

ii) *Properties of the product*: We consider the entire function

$$F(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Because $F(0) = 1$, there exists $0 < r < 1$ such that for $z \in D_r(0)$

$$|F(z) - 1| < 1.$$

Hence the logarithm

$$Log\ F : D_r(0) \to \mathbb{C}$$

is a well-defined holomorphic function. Taking the derivative shows

$$\frac{d}{dz} Log\ F(z) = \sum_{n=1}^{\infty} \frac{d}{dz} Log \left(1 - \frac{z^2}{n^2}\right) = \sum_{n=1}^{\infty} \frac{-2z}{1-(z^2/n^2)} \cdot \frac{1}{n^2} =$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cdot cot(\pi z) - \frac{1}{z} = \frac{d}{dz}\left(Log \frac{sin(\pi z)}{\pi z}\right).$$

Here we used Theorem 3.26 to interchange derivation and summation, the formula

$$\frac{d}{dz}(Log\ h) = \frac{h'}{h},$$

the *cot*-representation from Proposition 5.6, and the quotient rule for the holomorphic function

$$h(z) = \frac{sin\ \pi z}{\pi z}.$$

Hence

$$Log\ F(z) = Log \frac{sin(\pi z)}{\pi z} + const$$

or

$$F(z) = \frac{sin(\pi z)}{\pi z} \cdot const.$$

iii) *Product representation*: The functions on the left-hand side and on the right-hand side coincide for $z = 0$ with $const = 1$. Hence for all $z \in \mathbb{C}$

$$F(z) = \frac{sin(\pi z)}{\pi z},\ q.e.d.$$

Corollary 5.14 (Wallis product). *The number $\pi/2$ has the infinite product expansion*

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\ldots\cdot\frac{2n\cdot 2n}{(2n-1)(2n+1)}\cdot\ldots$$

Proof. Proposition 5.13 show for the argument $z = 1/2$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right).$$

The reciprocal value is the product formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty}\frac{2n\cdot 2n}{(2n-1)(2n+1)}, \ q.e.d.$$

Proposition 5.15 (Local logarithm of holomorphic functions without zeros). *Any holomorphic function without zeros*

$$f : D_r(0) \to \mathbb{C}^*, \ 0 < r \leq \infty,$$

has a holomorphic logarithm*, i.e. a holomorphic function*

$$g : D_r(0) \to \mathbb{C}$$

exists with

$$f = e^g.$$

Proof. By assumption the function

$$\frac{f'}{f} : D_r(0) \to \mathbb{C}$$

is well-defined and holomorphic. It expands into a convergent power series

$$\frac{f'}{f} = \sum_{n=0}^{\infty} c_n \cdot z^n.$$

Hence for any constant $c \in \mathbb{C}$ the power series

$$g(z) := c + \sum_{n=0}^{\infty}\frac{c_n}{n+1}\cdot z^{n+1}$$

with radius of convergence at least $= r$ is on $D_r(0)$ a primitive of the quotient f'/f. We choose the constant c such that

$$e^c = e^{g(0)} = f(0).$$

Then for all $z \in D_r(0)$

$$e^{g(z)} = f(z),$$

because for all $z \in D_r(0)$

$$\frac{d}{dz}\Big(e^{-g(z)} \cdot f(z)\Big) = -g'(z) \cdot e^{-g(z)} \cdot f(z) + e^{-g(z)} \cdot f'(z) =$$

$$= e^{-g(z)} \cdot \big(-g'(z) \cdot f(z) + f'(z)\big) = 0,$$

which implies for all $z \in D_r(0)$

$$e^{-g(z)} \cdot f(z) = 1, \ q.e.d.$$

Theorem 5.16 (Solution of the Weierstrass problem). *Consider a sequence of pairwise distinct points $(a_\nu)_{\nu \in \mathbb{N}}$ in $\mathbb{C}$ with*

$$\lim_{\nu \to \infty} |a_\nu| = \infty$$

and a sequence $(k_\nu)_{\nu \in \mathbb{N}}$ of natural numbers $k_\nu \geq 1, \ \nu \in \mathbb{N}$. Then a holomorphic function

$$F : \mathbb{C} \to \mathbb{C},$$

exists with zeros exactly the points $a_\nu, \ \nu \in \mathbb{N}$, and

$$ord(F;\ a_\nu) = k_\nu.$$

Any two holomorphic functions F_1, F_2 with these properties relate as

$$F_2 = e^g \cdot F_1$$

with a holomorphic function

$$g : \mathbb{C} \to \mathbb{C}.$$

Proof. i) *Quotient of two particular solutions*: The relation

$$F_2 = e^g \cdot F_1$$

follows from Proposition 5.15 because the quotient F_2/F_1 is holomorphic on $\mathbb{C}$ and has no zeros.

ii) *Polynomial approximation*: For any $n \in \mathbb{N}$ we consider the polynomial

$$P_n(z) := \prod_{n \leq |a_\nu| < n+1} (z - a_\nu)^{k_\nu}.$$

The polynomial has the prescribed zeros and orders in the annulus

$$A_n := \{z \in \mathbb{C} : \ n \leq |z| < n+1\},$$

and is holomorphic without zeros in the disc $D_n(0)$. Proposition 5.15 provides a holomorphic function

$$f_n : D_n(0) \to \mathbb{C}$$

with

$$P_n | D_n(0) = e^{f_n}.$$

A suitable Taylor polynomial g_n from the Taylor expansion of f_n is a polynomial approximation of f_n with

$$\|f_n - g_n\|_{\overline{D}_{n-1}(0)} < (1/2)^n.$$

iii) *Normal convergence*: We show that the infinite product

$$F(z) := \prod_{n=0}^{\infty} P_n(z) \cdot e^{-g_n(z)}$$

is normally convergent on $\mathbb{C}$. For the proof consider an arbitrary but fixed radius

$$0 < r < \infty$$

and choose an index

$$n_0 \geq r + 2.$$

Then for all $n \geq n_0$

$$\overline{D}_r(0) \subset D_{n-1}(0).$$

For $n \geq n_0$ and $z \in \overline{D}_r(0)$ holds

$$\left\|P_n \cdot e^{-g_n} - 1\right\|_{\overline{D}_r(0)} = \left\|e^{(f_n - g_n)} - 1\right\|_{\overline{D}_r(0)} \leq 2 \cdot \|f_n - g_n\|_{\overline{D}_r(0)} \leq (1/2)^{n-1}$$

using the estimate for the exponential from Lemma 5.10. As a consequence

$$\sum_{n=n_0}^{\infty} \|P_n \cdot e^{-g_n} - 1\|_{\overline{D}_r(0)} \leq \sum_{n=n_0}^{\infty} (1/2)^{n-1} < \infty$$

which finishes the proof of normal convergence. Due to Theorem 5.11 the function F has the prescribed zeros, q.e.d.

The general concept to formalize the input data of Theorem 5.16 is the concept of a *divisor*.

Remark 5.17 (Divisor).

1. A *divisor* on a non-empty set $U \subset \mathbb{C}$ is a map

$$D : U \to \mathbb{Z}$$

with support

$$supp\ D := \{z \in U : D(z) \neq 0\}$$

a discrete set, closed in U. Hence the support of a divisor on U is a countable set, discrete without any cluster point in U.

A divisor D is *non-negative*, denoted

$$D \geq 0,$$

if $D(z) \geq 0$ for all $z \in U$. A non-negative divisor is *positive*, denoted

$$D > 0,$$

if $D(z) > 0$ for at least one $z \in U$.

2. Any meromorphic function $f \in \mathscr{M}(U)$ defines on U the divisor $(f) := D$, named a *principal divisor*, with

$$D : U \to \mathbb{Z},\ D(a) := ord(f;\ a).$$

Concerning the opposite direction, Theorem 5.16 implies: Any divisor on $\mathbb{C}$ is a principal divisor, i.e. for suitable $f \in \mathscr{M}(\mathbb{C})$

$$D = (f).$$

For the proof one decomposes

$$D = D_1 - D_2$$

with two divisors $D_1, D_2 \geq 0$ on $\mathbb{C}$ and

$$supp\ D_1 \cap supp\ D_2 = \emptyset.$$

One chooses an enumeration

$$supp\ D_1 = (a_\nu)_{\nu \in \mathbb{N}}$$

and the sequence $(k_\nu)_{\nu \in \mathbb{N}}$ with

$$k_\nu := D(a_\nu),\ \nu \in \mathbb{N}.$$

Theorem 5.16 provides an entire function f_1 with $D_1 = (f_1)$. Analogously an entire function f_2 exists with $D_2 = (f_2)$. Then

$$f := \frac{f_1}{f_2} \in \mathscr{M}(\mathbb{C})$$

satisfies

$$(f) = (f_1) - (f_2) = D_1 - D_2 = D.$$

The question on the existence of holomorphic functions leads to the concept of a *domain of holomorphy*. This concept will show its far-reaching consequences in complex analysis of several variables.

Remark 5.18 (Domain of holomorphy).

1. A domain $G \subset \mathbb{C}$ is a *domain of holomorphy* of a holomorphic function

$$f : G \to \mathbb{C},$$

if f does not extend holomorphically across any point of the boundary ∂G, i.e. for any point $a \in G$ the convergence disk $D_r(a)$ of the Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

satisfies

$$D_r(a) \subset G.$$

As a consequence,

$$D_r(a) \cap \partial G = \emptyset.$$

2. A domain $G \subset \mathbb{C}$ is the *maximal domain of existence* for a holomorphic function

$$f : G \to \mathbb{C}$$

if there is no domain $G \subsetneq \hat{G} \subset \mathbb{C}$ with a holomorphic function

$$\hat{f} : \hat{G} \to \mathbb{C}$$

satisfying

$$\hat{f}|G = f.$$

3. The sliced complex plane $G := \mathbb{C}^-$ is the maximal domain of existence of the holomorphic function

$$Log : G \to \mathbb{C},$$

defined in Example 1.27.

But G is not the domain of holomorphy of Log: For $a \in \mathbb{C}^-$ with $Re\ a < 0$ the Taylor expansion

$$Log(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

has radius of convergence $= |a|$, and

$$D_{|a|}(a) \subsetneq \mathbb{C}^-.$$

4. Any domain $G \subset \mathbb{C}$ is the maximal domain of existence of a holomorphic function
$$f : G \to \mathbb{C}.$$
For the proof one chooses a discrete subset $A \subset G$ with set of accumulation points
$$A' = \partial G.$$
A generalization of Theorem 5.16 provides a non-constant holomorphic function f with zero set A. Hence the Taylor series of f with center an arbitrary point $z_0 \in G$ is not convergent for any boundary point from ∂G.

5. Much deeper is the result that any domain $G \subset \mathbb{C}$ is a domain of holomorphy, see [21, Kap. 5, §2].

A meromorphic function is locally the quotient of two holomorphic functions, see Corollary 4.8. Theorem 5.19 states a global version of this property.

Theorem 5.19 (Global meromorphic function as quotient of entire functions). *Any meromorphic function $f \in \mathscr{M}(\mathbb{C})$ has the form*
$$f = \frac{g}{h}$$
with entire functions $g, h \in \mathscr{O}(\mathbb{C}),\ h \neq 0$.

Proof. W.l.o.g. $f \neq 0$. Consider the pairwise different poles of f
$$a_0, a_1, a_2, ... \text{ of order } k_0, k_1, k_2, ... \ .$$
Theorem 5.16 implies the existence of a holomorphic function $h \in \mathscr{O}(\mathbb{C})$ with zeros exactly at
$$a_0, a_1, a_2, ... \text{ of order } k_0, k_1, k_2, ... \ .$$
Then the function
$$g := f \cdot h \in \mathscr{O}(\mathbb{C})$$
is holomorphic, and satifies
$$f = \frac{g}{h}, \ q.e.d.$$

Proposition 5.13 exemplifies that a holomorphic function f can be represented as infinite product of its zeros. For a point $a \in \mathbb{C}, a \neq 0$, the polynomial
$$1 - \frac{z}{a}$$
has a zero exactly for $z = a$, and the zero has order $= 1$. To represent f one could therefore attempt to take the infinite product of such factors. But in general the

product is not convergent. To enforce normal convergence one multiplies each polynomial with a non-negative factor. An example of this idea has been given in the proof of Theorem 5.16.

We now present an explicit construction to enforce normal convergence of a solution of the Weierstrass problem, see Theorem 5.22.

Definition 5.20 (Weierstrass elementary factors and canonical product). Consider a natural number $p \in \mathbb{N}^*$. The holomorphic function

$$E_p : \mathbb{C} \to \mathbb{C},\ E_p(z) := (1-z) \cdot exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + ... + \frac{z^p}{p}\right),$$

is named the *Weierstrass elementary factor* of order p. For a sequence $(a_\nu)_{\nu \geq 1}$ of points $a_\nu \in \mathbb{C}^*$ the infinite product of elementary factors, a formal expression,

$$\prod_{\nu=1}^{\infty} E_p\left(\frac{z}{a_\nu}\right)$$

is named a *canonical product*.

A Weierstrass elementary factor $E_p(z/a)$ has exactly one zero, namely at $z = a$ with

$$ord(E_p(z/a); a) = 1$$

We now investigate under which assumptions a canonical product is normally convergent.

Lemma 5.21 (Weierstrass elementary factor). *The Weierstrass elementary factor E_p of order $= p$ satisfies for $|z| \leq 1$ the estimate*

$$|E_p(z) - 1| \leq |z|^{p+1}.$$

Proof. i) *Power series expansion*: The Weierstrass elementary factor expands into a convergent power series with center $= 0$

$$E_p(z) = 1 - \sum_{\nu=p+1}^{\infty} a_\nu \cdot z^\nu$$

with real coefficients $a_\nu \geq 0$. For the proof consider the expansion

$$E_p(z) = 1 - \sum_{\nu=1}^{\infty} a_\nu \cdot z^\nu.$$

On one hand, its derivative has the Taylor series

$$E_p{}'(z) = -\sum_{\nu=1}^{\infty} \nu \cdot a_\nu \cdot z^{\nu-1}.$$

On the other hand, by the product rule

$$E_p{}'(z) = -exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + ... + \frac{z^p}{p}\right) +$$

$$+(1-z)\cdot(1+z+...+z^{p-1})\cdot exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + ... + \frac{z^p}{p}\right) =$$

$$= -z^p \cdot exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + ... + \frac{z^p}{p}\right).$$

The coefficient comparison shows

$$a_\nu = 0 \text{ for } 1 \leq \nu \leq p$$

and

$$a_\nu \geq 0 \text{ for } p+1 \leq \nu.$$

ii) *Estimate*: Due to part i)

$$|E_p(z) - 1| = \left|\sum_{\nu=p+1}^{\infty} a_\nu \cdot z^\nu\right| \leq |z|^{p+1} \cdot \sum_{\nu=p+1}^{\infty} a_\nu \cdot |z|^{\nu-p-1}$$

The equation

$$0 = E_p(1) = 1 - \sum_{\nu=p+1}^{\infty} a_\nu$$

implies

$$\sum_{\nu=p+1}^{\infty} a_\nu = 1.$$

As a consequence for $|z| \leq 1$

$$|E_p(z) - 1| \leq |z|^{p+1} \cdot \sum_{\nu=p+1}^{\infty} a_\nu = |z|^{p+1},\ q.e.d.$$

Theorem 5.22 (Canonical products and Weierstrass product theorem). *Consider a sequence $(a_\nu)_{\nu\geq 1}$ of points $a_\nu \in \mathbb{C}^*$, not necessarily pairwise distinct. Then for any $p \in \mathbb{N}$ with*

$$\sum_{\nu=1}^{\infty} \frac{1}{|a_\nu|^{p+1}} < \infty$$

the canonical product

$$\prod_{\nu=1}^{\infty} E_p\left(\frac{z}{a_\nu}\right)$$

- *is normal convergent,*
- *is a holomorphic function F on $\mathbb{C}$ with zeros exactly at the points from the sequence,*
- *and order of each zero equal to the multiplicity of the point within the sequence.*

Proof. To prove normal convergence we consider an arbitrary but fixed radius $r > 0$. There exists an index n_0 such that for all $\nu \geq \nu_0$

$$|a_\nu| \geq r.$$

For $z \in \overline{D}_r(0)$ and $\nu \geq \nu_0$ we have

$$\left|\frac{z}{a_\nu}\right| \leq 1.$$

Hence Lemma 5.21 implies

$$\left|E_p\left(\frac{z}{a_\nu}\right) - 1\right| \leq \left|\frac{z}{a_\nu}\right|^{p+1} \leq r^{p+1} \cdot \frac{1}{|a_\nu|^{p+1}}$$

and

$$\sum_{\nu \geq \nu_0} \left|E_p\left(\frac{z}{a_\nu}\right) - 1\right| \leq r^{p+1} \cdot \sum_{\nu \geq \nu_0} \frac{1}{|a_\nu|^{p+1}} < \infty, \ q.e.d.$$

Example 5.23 (Canonical product). We consider the sequence of complex points

$$(a_\nu)_{\nu \in \mathbb{N}} = (0, \pm 1, \pm 2, \ldots).$$

With the choice $p = 1$ the series

$$\sum_{\nu=1}^{\infty} \frac{1}{|a_\nu|^{p+1}}$$

is convergent. Theorem 5.22 implies the normal convergence of the canonical product

$$\prod_{\nu=1}^{\infty} E_1\left(\frac{z}{a_\nu}\right) = \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} E_1\left(\frac{z}{n}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Hence Proposition 5.13 implies

$$\prod_{\nu=1}^{\infty} E_1\left(\frac{z}{a_\nu}\right) = \frac{sin\ \pi z}{\pi z}.$$

We now extend the Γ-function, originally defined for real $x > 0$, to a holomorphic function in the right half-plane.

For any $n \in \mathbb{Z}$ set

$$RH(n) := \{z \in \mathbb{C} : \; Re\, z > n\}.$$

Definition 5.24 (Γ-function). The function

$$\Gamma : RH(0) \to \mathbb{C}, \; \Gamma(z) := \int_0^\infty t^{z-1} \cdot e^{-t} \; dt,$$

is named Γ-*function*.

Proposition 5.25 (Holomorphy of the Γ-function). *For all $z \in RH(0)$*

$$\int_0^\infty t^{z-1} \cdot e^{-t} \; dt < \infty$$

The Γ-function is well-defined and holomorphic.

Proof. i) *Convergence of the integral*: For $z \in RH(0)$

$$t^{z-1} = e^{(z-1)\ln \, t}$$

implies with $x := Re\, z > 1$ the estimate

$$\left|t^{z-1}\right| = e^{(x-1)\ln(t)} = t^{x-1}.$$

Hence

$$\left|\int_{t_1}^{t_2} t^{z-1} \cdot e^{-t} \; dt\right| \le \int_0^\infty t^{x-1} e^{-t} \; dt = \lim_{\substack{R \uparrow \infty \\ \varepsilon \downarrow 0}} \int_\varepsilon^R t^{x-1} \cdot e^{-t} \; dt$$

if the limit on the right-hand side exists:

- *Convergence at the lower bound $t \downarrow 0$*: For all $x, \; t > 0$ we have

  $$|t^{x-1} e^{-t}| \le t^{x-1},$$

  and

  $$\int_0^1 t^{x-1} \; dt < \infty$$

  because $x > 0$.

- *Convergence at the upper bound $t \uparrow \infty$*: We have

  $$\lim_{t \to \infty} t^{x+1} e^{-t} = 0.$$

  Hence exists t_0 such that for all $t \ge t_0$

$$\left|t^{x-1}e^{-t}\right| = \left|\frac{1}{t^2}\cdot t^{x+1}e^{-t}\right| = \frac{1}{t^2}\cdot|t^{x+1}e^{-t}| \leq \frac{1}{t^2}$$

The integral is convergent because

$$\int_1^\infty \frac{1}{t^2}\,dt < \infty.$$

ii) *Holomorphy*: For arbitrary but fixed

$$0 < t_1 < t_2 < \infty$$

the function

$$RH(0) \to \mathbb{C},\ z \mapsto \int_{t_1}^{t_2} t^{z-1}\cdot e^{-t}\,dt,$$

is holomorphic, because

$$\frac{\partial}{\partial\bar{z}}\int_{t_1}^{t_2} t^{z-1}\cdot e^{-t}\,dt = \int_{t_1}^{t_2}\frac{\partial t^{z-1}}{\partial\bar{z}}\cdot e^{-t}\,dt$$

and

$$\frac{\partial t^{z-1}}{\partial\bar{z}} = \frac{\partial e^{(z-1)\cdot ln\ t}}{\partial\bar{z}} = 0.$$

To prove the holomorphy of Γ we show that the limit

$$\lim_{\substack{R\uparrow\infty\\ \varepsilon\downarrow 0}}\int_\varepsilon^R t^{z-1}\cdot e^{-t}\,dt$$

satisfies compact convergence with respect to z for any sequence of integrals indexed by R_ν and ε_μ. Then we apply Weierstrass convergence Theorem 3.26: A given compact subset $K \subset RH(0)$ is contained in a strip

$$\{z\in\mathbb{C}:\ x_1 \leq Re\,z \leq x_2\},\ 0 < x_1 < x_2 < \infty.$$

For $z \in K$ then

$$\left|\int_\varepsilon^1 t^{z-1}\cdot e^{-t}\,dt\right| \leq \int_\varepsilon^1 t^{x_1-1}\cdot e^{-t}\,dt \text{ and } \left|\int_1^R t^{z-1}\cdot e^{-t}\,dt\right| \leq \int_1^R t^{x_2-1}\cdot e^{-t}\,dt$$

For both estimates the integral on the right-hand side is independent from z, and convergent for $\varepsilon\downarrow 0$ and $R\uparrow\infty$. We obtain the compact convergence for $z\in RH(0)$

$$\lim_{\substack{R\uparrow\infty\\ \varepsilon\downarrow 0}}\int_\varepsilon^R t^{z-1}\cdot e^{-t}\,dt = \Gamma(z),$$

which finishes the proof of the holomorphy of the Γ-function, q.e.d.

Proposition 5.26 (Functional equation of the Γ-function). *The Γ-function satisfies the functional equations*

$$\Gamma(z+1) = z \cdot \Gamma(z),\ z \in RH(0),\ z \in RH(0),$$

$$\Gamma(n) = (n-1)!,\ n \in \mathbb{N}^*.$$

Proof. Because Γ is holomorphic, the Identity Theorem 1.17 implies: It suffices to prove the claim for real arguments $x > 0$ only. We apply partial integration

$$\int u' \cdot \mathrm{v}\ dt = [u \cdot \mathrm{v}] - \int u \cdot \mathrm{v}'\ dt :$$

Then

$$\Gamma(x+1) = \lim_{\substack{R\uparrow\infty \\ \varepsilon\downarrow 0}} \int_\varepsilon^R e^{-t} \cdot t^x\ dt = \lim_{\substack{R\uparrow\infty \\ \varepsilon\downarrow 0}} \left[-e^{-t} \cdot t^x\right]_{x=\varepsilon}^{x=R} + x \cdot \lim_{\substack{R\uparrow\infty \\ \varepsilon\downarrow 0}} \int_\varepsilon^R e^{-t} \cdot t^{x-1}\ dt = x \cdot \Gamma(x)$$

The first functional equation and the equation

$$\Gamma(1) = \int_0^\infty e^{-t}\ dt = \left[-e^{-t}\right]_{t=0}^{t=\infty} = 0 - (-1) = 1$$

imply by induction the second functional equation, q.e.d.

It is not unusual in complex analysis that a holomorphic function, at first defined only on a small open set U, extends to a meromorphic function on a much bigger open set. The best known example is the convergent geometric series

$$f(z) = \sum_{n=0}^\infty z^n$$

which has the convergence disk $U = D_1(0)$. The series is not convergent on the boundary ∂U, in particular it is not convergent for $z = 1$. The equality

$$f(z) = \frac{1}{1-z}$$

shows the reason: The function f has a pole at the boundary of U. Nevertheless, the function f extends the geometric series meromorphically to the whole plane. The meromorphic extension f has a single pole at $a = 1$. The pole has order $= 1$.

A similar meromorphic extension is possible for the Γ-function, see Theorem 5.27.

Theorem 5.27 (Meromorphic extension of the Γ-function). *The holomorphic function*

$$\Gamma : RH(0) \to \mathbb{C}$$

extends to a meromorphic function on $\mathbb{C}$, also denoted by Γ. The extended Γ-function has the pole set

$$P = \{-n : n \in \mathbb{N}\}.$$

Each pole has order $= 1$. The principal part of Γ at $z = -n$ is

$$H_{-n}(z) = \frac{(-1)^n}{n!} \cdot \frac{1}{z+n}$$

Proof. The functional equation of the Γ-function from Proposition 5.26 implies for each arbitrary but fixed $n \in \mathbb{N}$ and all $z \in RH(0)$

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1) \cdot \ldots \cdot (z+n-1)}.$$

For arbitrary but fixed $z \in \mathbb{C}$ we choose $n \in \mathbb{N}$ with $z+n \in RH(0)$ and define

$$\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1) \cdot \ldots \cdot (z+n-1)}$$

Due to the functional equation the definition of $\Gamma(z)$ is independent from the choice of n.

The right-hand side is a meromorphic function on the half-plane $RH(-n)$. Its pole set is

$$P_n = \{0, -1, \ldots, -n+1\}.$$

Because the choice of $n \in \mathbb{N}$ is arbitrary, the holomorphic function Γ extends to a meromorphic function on all of $\mathbb{C}$. It expands at $z = -n$ as

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1) \cdot \ldots \cdot (z+n)} = \frac{1}{z-(-n)} \cdot \Psi(z)$$

with

$$\Psi(z) := \frac{\Gamma(z+n+1)}{z(z+1) \cdot \ldots \cdot (z+n-1)}.$$

The function Ψ is holomorphic in a neighbourhood of $z = -n$ because

$$\Psi(-n) = \frac{\Gamma(1)}{(-n)(-n+1) \cdot \ldots \cdot -1} = (-1)^n \cdot \frac{1}{n!}$$

Therefore the principal part of Γ at $z = -n$ is

$$\frac{(-1)^n}{n!} \cdot \frac{1}{z-(-n)}, \ q.e.d.$$

Theorem 5.28 (Product representation of the Γ-function).

1. *The Γ-function has the product representation as a meromorphic function on $\mathbb{C}$*

$$\Gamma(z) = \frac{e^{-C\cdot z}}{z} \cdot \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+(z/n)}$$

Here

$$C := \lim_{N\to\infty}\left[\left(\sum_{n=1}^{N}\frac{1}{n}\right) - ln\,N\right]$$

is the Euler-Mascheroni constant.

2. *The Γ-function relates to the inverse sin-function as*

$$\Gamma(z)\cdot\Gamma(1-z) = \frac{z}{sin(\pi z)}$$

considered as an equality of meromorphic functions on $\mathbb{C}$. In particular, the Γ-function is meromorphic without zeros.

Proof. 1. According to Theorem 5.27 the function Γ-function has the pole set $-\mathbb{N}$. All poles have order $= 1$. Using a suitable canonical product we construct a function γ with the same poles and principal parts as Γ.

i) *Inverse γ of a canonical product*: Theorem 5.22 implies that the canonical product

$$z\cdot\prod_{n=1}^{\infty} E_1\left(1-\frac{z}{(-n)}\right) = z\cdot\prod_{n=1}^{\infty} E_1(1+(z/n))$$

is normal convergent and represents an entire function with zero set $-\mathbb{N}$. All zeros have order $= 1$. Recall

$$E_1(1+(z/n)) := (1+(z/n))\cdot exp(z/n).$$

Hence the inverse of the canonical product, multiplied with a non-zero constant,

$$\gamma(z) := \frac{e^{-C\cdot z}}{z}\cdot\prod_{n=1}^{\infty}\frac{e^{z/n}}{1+(z/n)}$$

has the same poles as the Γ-function.

Using the definition of the Euler-Mascheroni constant and the continuity of the exponential we now derive a second representation

$$\gamma(z) = \frac{1}{z}\cdot\lim_{N\to\infty}\left(exp\left[-z\cdot\left(\left(\sum_{n=1}^{N}\frac{1}{n}\right) - ln\,N\right)\right]\cdot\prod_{n=1}^{N}\frac{e^{z/n}}{1+(z/n)}\right) =$$

$$= \frac{1}{z} \cdot \lim_{N\to\infty} \left(exp(z \cdot ln\, N) \cdot \prod_{n=1}^{N} \frac{n}{n+z} \right) =$$

$$= \lim_{N\to\infty} \frac{N^z \cdot N!}{z \cdot (z+1) \cdot \ldots \cdot (z+N)}$$

ii) *Functional equation of γ and principal parts*: From the last formula of part i) one derives that γ satisfies the same functional equations as Γ:

$$\gamma(z+1) = z \cdot \gamma(z)$$

as well as

$$\gamma(1) = 1 \text{ and } \gamma(n) = (n-1)!$$

The last formula implies

$$\gamma(z) = \frac{\gamma(z+n)}{z \cdot (z+1) \cdot \ldots \cdot (z+n)},\ z \in \mathbb{C},$$

Hence γ has the same principal parts as Γ:

$$H_{\gamma,-n}(z) = \frac{(-1)^n}{n^1} \cdot \frac{1}{z+n},\ n \in \mathbb{N}.$$

iii) *Equality $\Gamma = \gamma$*: Part i) and ii) show: The two meromorphic functions Γ and γ have the same poles and the same principal parts. Therefore their difference

$$g := \Gamma - \gamma : \mathbb{C} \to \mathbb{C}$$

is an entire function. We claim $g = 0$: In the strip

$$B_{1,2} := \{z \in \mathbb{C} :\ 1 \le Re\, z \le 2\}$$

we have for $x := Re(z)$

$$|\Gamma(z)| \le \Gamma(x).$$

The latter function is continuous on the compact interval $[1,2] \subset \mathbb{R}$, hence bounded. For $x > 0$ we have

$$\left| \frac{N^z \cdot N!}{z \cdot (z+1) \cdot \ldots \cdot (z+N)} \right| \le \frac{n^x \cdot N!}{x \cdot (x+1) \cdot \ldots \cdot (x+N)}$$

which implies

$$|\gamma(z)| \le \gamma(x).$$

The latter function is also continuous, hence bounded on $[1,2] \subset \mathbb{R}$. As a consequence, the function g is bounded in $B_{1,2}$.

The functional equations of Γ and γ

$$\Gamma(z+1) = z \cdot \Gamma(z) \text{ and } \gamma(z+1) = z \cdot \gamma(z)$$

imply

$$g(z+1) = z \cdot g(z) \text{ i.e. } g(z) = \frac{g(z+1)}{z}$$

As a consequence, the holomorphic function g is also bounded in the strip

$$B_{0,1} := \{z \in \mathbb{C} : \ 0 \le Re\, z \le 1\}.$$

Eventually, we introduce the entire function

$$S : \mathbb{C} \to \mathbb{C}, \ S(z) := g(z) \cdot g(1-z).$$

Boundedness of g in $B_{0,1}$ implies boundedness of S in $B_{0,1}$. For all $z \in \mathbb{C}$

$$S(z+1) := g(z+1) \cdot g(-z) = z \cdot g(z) \cdot g(-z) = -g(z) \cdot (-z) \cdot g(-z) =$$

$$= -g(-z) \cdot g(-z+1) = -S(z).$$

As a consequence, the function S has period $= 2$, and due to

$$S(z+1) = -S(z)$$

the function S is even bounded in the strip

$$B_{0,2} := \{z \in \mathbb{C} : \ 0 \le Re\, z \le 2\}.$$

The periodicity of S implies that S is even bounded in $\mathbb{C}$, hence constant by Liouville's Theorem, see Corollary 3.23. The equality

$$S(z+1) = -S(z)$$

implies

$$S = 0.$$

The representation

$$S(z) = g(z) \cdot g(1-z)$$

of S as a product of two holomorphic functions shows $g = 0$, q.e.d.

2. To prove the claim, one first uses the equalities

$$\Gamma = \gamma \text{ and } \gamma(1-z) = (-z) \cdot \gamma(-z)$$

from part 1):

$$\gamma(z) \cdot \gamma(1-z) = (-z) \cdot \gamma(z) \cdot \gamma(-z) = (-z) \cdot \frac{e^{-Cz}}{z} \cdot \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+(z/n)} \cdot \frac{e^{Cz}}{(-z)} \cdot \prod_{n=1}^{\infty} \frac{e^{-z/n}}{1-(z/n)} =$$

$$= \frac{1}{z \cdot \prod_{n=1}^{\infty}(1-(z^2/n^2))} = \frac{\pi}{sin\ \pi z}$$

The last equality is the product representation of the *sin*-function from Example 5.23, q.e.d.

Remark 5.29 (ζ-function and Γ-function).

i) Similarly to the Γ-function also the ζ-function is holomorphic in the half-space $RH(1)$ and extends to a meromorphic function on $\mathbb{C}$. The extended ζ-function is holomorphic on $\mathbb{C}\setminus\{1\}$. The isolated singularity is a first order pole at $a = 1$.

ii) On the negative real axis the ζ-function satisfies for all $k \in \mathbb{N}^*$

$$\zeta(-2k) = 0 \text{ and } \zeta(1-2k) = -\frac{B_{2k}}{2k}$$

Riemann conjectured that all other complex zeros are located on the *critical line*

$$\frac{1}{2} + i \cdot \mathbb{R} \subset \mathbb{C}.$$

iii) The ζ-function satisfies a functional equation which compares the values at the complex arguments s and $1-s$, which are in mirror symmetry to the point $= 1/2$ on the critical line: If one defines on $\mathbb{C}$ the meromorphic Λ-function by

$$\Lambda(s) := \pi^{-(s/2)} \cdot \Gamma(s/2) \cdot \zeta(s)$$

then

$$\Lambda(s) = \Lambda(1-s).$$

Concerning these and other results consult [25, Teil I, §3 and §4].

iv) One may visualize the Riemann ζ-function as a landscape determined by the graph of the function: A point of the landscape with plane coordinate $s \in \mathbb{C}$ has the height $|\zeta(s)|$. The ζ-landscape has one single peak at $s = 1/2$ of infinite height, and infinitely many valleys at sea level.

The Riemann conjecture states: Besides the obvious valleys at sea level which are located at points with plane coordinate $s = -2k,\ k \in \mathbb{N}^*$, all other valleys at sealevel are located on the critical line.

5.4 Outlook

The problems of Mittag-Leffler and Weierstrass generalize from complex analysis to non-compact Riemann surfaces (manifolds depending on one complex variable) and to general Stein manifolds (several complex variables). In the context of several complex variables the two problems are distinguished as Cousin-I (additive) and Cousin-II (multiplicative) problem. The investigation of these problems triggered the development of analysis on complex manifolds in the midst of the 20th century. By means of sheaf theory the problems have been solved in a very satisfactory way, see [11, Kap. V, §2].

- What is a sheaf?
- How to generalize the Mittag-Leffler and Weierstrass problems to the context of manifolds by using sheaf theory?
- Which mathematical tools serve to solve the Cousin problems?

The Riemann ζ-function from Remark 5.8 is one of the main contributions of complex analysis to number theory. The study of number theory by analytic means is the subject of the mathematical field called *Analytical Number Theory*. A recommendable introduction to the Γ-function and the Riemann ζ-function is [25].

One of the mathematical *millenium problems* asks for a proof of Riemann's conjecture, see [1]. It is well-known that the conjecture is equivalent to a result about the zeros of a certain family of real polynomials

$$J^{d,n} \in \mathbb{R}[X]$$

which are well-defined for each degree $d \in \mathbb{N}^*$ and any index $n \in \mathbb{N}$. These polynomials, named *Jensen polynomials*, derive from the series expansion of the Λ-function defined in Remark 5.29. It suffices to prove the result for $n = 0$ only. Recently a new approach [13] was published, which proves the result for $J^{d,0}$ with $1 \leq d \leq 8$. Nevertheless, currently (June 2019) the conjecture is still open.

In Algebraic Geometry there is an analogue to Riemann's conjecture. The analogue is part of the *Weil conjectures*. They deal with the zero sets of polynomials with rational coefficients. The first non-trivial case investigates the points with rational coordinates on an elliptic curve, see [23, Chap. IV, §1]. The Weyl conjectures from the midth of the 20th century have been proven by Deligne in 1973, see [16, Appendix C]. His proof shows the strenght of Gothendieck's reformulation of Algebraic Geometry.

Chapter 6
Integral Theorems of Complex Analysis

Cauchy's general integral theorem is a result about the integration of a holomorphic function along the boundary of a relatively compact subdomain of its domain of definition. The theorem follows from converting the curve integral into a surface integral. Hence it is a specific case of the Green-Riemann integral formula. For a complex differentiable function the integrand of the surface integral vanishes due to the Cauchy-Riemann differential equations.

6.1 Cauchy's integral theorem and the residue theorem

The present section generalizes Chapter 3 concerning the domain of integration. Chapter 3 considered disks and annuli. Their boundary consist of one or two circles. The Cauchy integral theorem and the Cauchy integral formula for annuli follow from expanding the Cauchy kernel into a convergent geometric series and choosing the standard representation of the boundary circles. The result shows: The Cauchy integrals from Chapter 3 do not depend on the specific path of integration. They are dependent on the integrand being holomorphic or having singularities in the interior. But the impact of the singularity is confined to its residue.

In the more general context of the present chapter we integrate along the boundary of relatively compact open sets $A \subset\subset \mathbb{C}$. All integral formulas are special cases of Stoke's integral formula. Stoke's formula converts the surface integral about A into a path integral along the boundary ∂A. Therefore we have to presuppose that A has a smooth boundary.

A compact set $A \subset \mathbb{C}$ has a smooth boundary if ∂A is locally a curve with a continuously differentiable parameter representation. This concept is formalized in Definition 6.1.

Definition 6.1 (Smooth boundary). A set $A \subset \mathbb{R}^2$ has a *smooth boundary*, see Figure 6.1, if any point $a \in \partial A$ has an open neighbourhood $U \subset \mathbb{R}^2$ and a continuously differentiable function

$$\rho : U \to \mathbb{R}$$

such that:

- The set $A \cap U$ lies on one side of the zero set of ρ, i.e.

$$A \cap U = \{(x,y) \in U : \ \rho(x,y) \leq 0\}$$

- The zero set of ρ has a well-defined normal vector, i.e. for all $(x,y) \in U$

$$grad(\rho)(x,y) \neq 0.$$

The boundary is *piecewise smooth* if all but finitely many points $a \in \partial A$ satisfy the above definition.

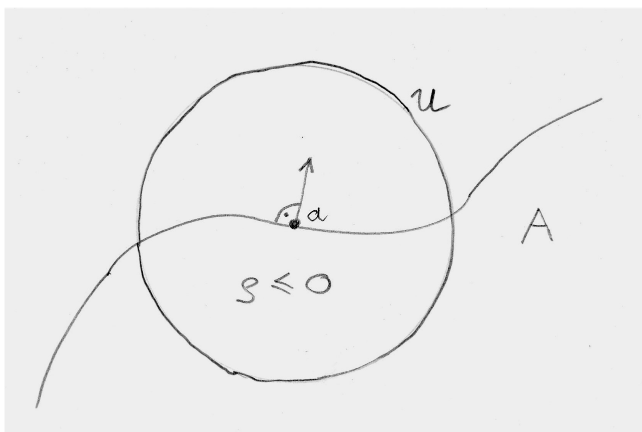


Fig. 6.1 Smooth boundary ∂A with exterior normal vector at $a \in \partial A$

Remark 6.2 (Smooth boundary).

1. In Definition 6.1 the zero set

$$V(\rho) := \{(x,y) \in U : \ \rho(x,y) = 0\}$$

   equals the local boundary, i.e.

$$V(\rho) = U \cap \partial A.$$

2. For all $(x,y) \in V(\rho)$

$$grad(\rho)(x,y) \neq 0.$$

Hence $V(\rho)$ is locally the graph of a continuously differentiable function

$$x = x(y) \text{ or } y = y(x).$$

This representation follows from the Implicit Function Theorem, see [7].

3. If $a \in \partial A$ is a smooth boundary point then the *exterior normal vector* at $a \in \partial A$ is well defined: It is the multiple of the gradient of unit lenght pointing to the exterior of A. Integration along ∂A with the induced orientation requires to parametrize the curve of integration in such a way that the interior $\mathring{A}$ is left-hand side when moving along the curve.

Theorem 6.3 (Green-Riemann formula). *Consider an open subset $U \subset \mathbb{R}^2$ and a compact subset*

$$D \subset U$$

with piecewise smooth boundary ∂D. Let

$$P, Q : U \to \mathbb{C}$$

be two functions with continuous partial derivatives. Then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Figure 6.2 shows the set D with boundary

$$\partial D = C_1 \cup C_2 \cup C_3 \cup C_4.$$

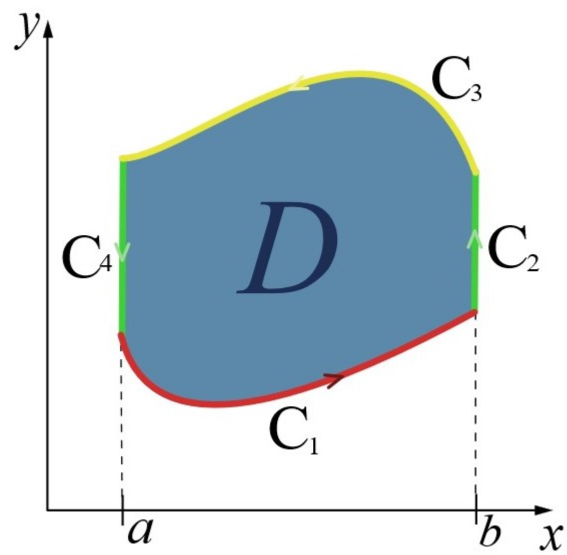


Fig. 6.2 Compact set D with piecewise smooth boundary $\partial D = C$ and oriented boundary curves

The Green-Riemann formula is a specific case of respectively Gauss' integral formula or - even more general - of Stoke's theorem

$$\int_{\partial D} \omega = \int_D d\omega$$

for a differential form ω which is continuously differentiable. Stokes' theorem states: The primitive ω integrated along the boundary ∂D equals the derivative $d\omega$ integrated along the bounded compact D.

To derive the Green-Riemann formula from Stokes' theorem one considers

$$\omega := P(x,y)\ dx + Q(x,y)\ dy$$

obtaining

$$d\omega = -\frac{\partial P(x,y)}{\partial y}\, dx \wedge dy + \frac{\partial Q(x,y)}{\partial x}\, dx \wedge dy$$

For integration on compact sets $D \subset \mathbb{R}^2$ the differential form $dx \wedge dy$ is the positive oriented surface element $d(x,y)$ and can be replaced by the symbol $dxdy$. For a proof of Gauss' and Stoke's theorem see [8].

Theorem 6.4 (Cauchy's integral theorem (general case)). *Consider an open set $U \subset \mathbb{C}$ and a compact set $A \subset A$ with piecewise smooth boundary. Any holomorphic function*

$$f : U \to \mathbb{C}$$

satisfies

$$\int_{\partial A} f(z)\, dz = 0.$$

Proof. We reduce the theorem to the Green-Riemann formula. The decomposition of arguments into real and imaginary part

$$z = x + iy$$

implies the decomposition of differential forms

$$dz = dx + idy.$$

Due to Theorem 6.3

$$\int_{\partial A} f(z)\, dz = \int_{\partial A} (fdx + i \cdot fdy) = \iint_A \left(i \cdot \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right)\, dx \wedge dy.$$

Due to the holomorphy of f the Cauchy-Riemann differential equations in the form of Proposition 2.4 imply

$$i \cdot \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

which finishes the proof, q.e.d.

Definition 6.5 (Residue). Consider a holomorphic function with an isolated singularity at a point $a \in \mathbb{C}$

$$f : D_r^*(a) \to \mathbb{C},\ r > 0.$$

If

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

denotes the Laurent expansion of f at a, then

$$res(f;\ a) := c_{-1} \in \mathbb{C}$$

is the *residue* of f at a.

Theorem 6.6 (Residue theorem). *Consider an open set $U \subset \mathbb{C}$ and a compact set $A \subset U$ with piecewise smooth boundary. For any finite set of interior points*

$$a_1, ..., a_m \in \mathring{A}$$

each holomorphic function

$$f : U \setminus \{a_1, ..., a_m\} \to \mathbb{C}$$

satisfies

$$\int_{\partial A} f(z)\ dz = 2\pi i \cdot \sum_{j=1}^{m} res(f;\ a_j).$$

Proof. For $\varepsilon > 0$ we set

$$A_\varepsilon := A \setminus \bigcup_{j=1}^{m} D_\varepsilon(a_j),$$

see Figure 6.3. For suitable $\varepsilon > 0$ we may assume $D_\varepsilon(a_j) \subset A$ for all $j = 1, ..., m$, and pairwise disjoint.

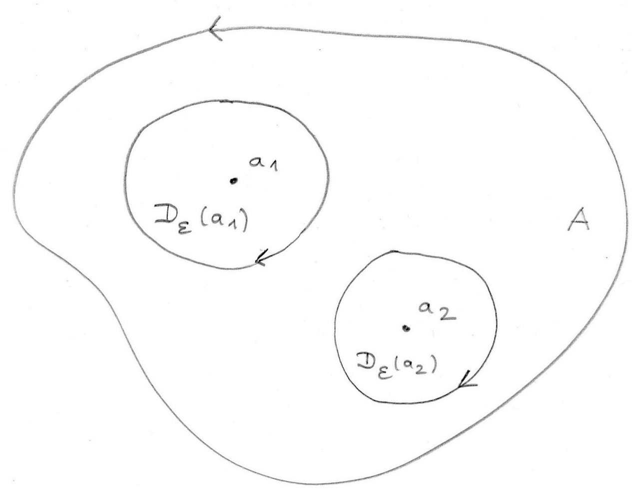


Fig. 6.3 Paths of integration

Cauchy's Integral Theorem 6.4 implies

$$\int_{\partial A_\varepsilon} f(z)\ dz = 0.$$

Hence

$$\int_{\partial A} f(z)\ dz = \sum_{j=1}^{m} \left(\int_{|z-a_j|=\varepsilon} f(z)\ dz \right),$$

which reduces the proof of the theorem to a local computation at a given isolated singularity $a \in U$ of f: Assume

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot (z-a)^n.$$

To compute

$$\int_{|z-a|=\varepsilon} f(z)\ dz$$

we choose the standard parametrization of the circuit $\partial D_\varepsilon(a)$

$$z = a + \varepsilon e^{i\phi},\ \phi \in [0, 2\pi],\ \text{with}\ dz = i\varepsilon e^{i\phi}\ d\phi.$$

Then

$$\int_{|z-a|=\varepsilon} f(z)\ dz = \sum_{n=-\infty}^{\infty} c_n \cdot \left(\int_0^{2\pi} \varepsilon^n e^{in\phi} \cdot i\varepsilon e^{i\phi}\ d\phi \right) =$$

$$= \sum_{n=-\infty}^{\infty} c_n \cdot i\varepsilon^{n+1} \left(\int_0^{2\pi} e^{i(n+1)\phi}\ d\phi \right) = 2\pi i \cdot c_{-1} = 2\pi i \cdot res(f;a),\ q.e.d.$$

Theorem 6.7 (Cauchy's integral formula (general case)). *Consider an open set $U \subset \mathbb{C}$ and a compact set $A \subset A$ with piecewise smooth boundary. Any holomorphic function*

$$f : U \to \mathbb{C}$$

satisfies for any $a \in \mathring{A}$ and arbitrary $n \in \mathbb{N}$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial A} \frac{f(z)}{(z-a)^{n+1}}\ dz.$$

Proof. The integrand

$$\frac{f(z)}{(\zeta - a)^{n+1}}$$

has a single pole at $z = a$. Theorem 6.6 implies

$$\int_{\partial A} \frac{f(z)}{(\zeta - a)^{n+1}}\ d\zeta = 2\pi i \cdot res\left(\frac{f(z)}{(z-a)^{n+1}};\ a \right)$$

To compute the residue we consider the Taylor expansion with center $= a$

$$f(z) = \sum_{k=0} c_k \cdot (z-a)^k \ \text{with}\ c_k = \frac{f^{(k)}(a)}{k!}$$

which implies

$$res\left(\frac{f(z)}{(z-a)^{n+1}};\ a\right) = c_n = \frac{f^{(n)}(a)}{n!},\ q.e.d.$$

Theorem 6.8 is a converse of Cauchy's integral theorem.

Theorem 6.8 (Morera's Theorem). *Consider a domain $G \subset \mathbb{C}$ and a continuous function*

$$f : G \to \mathbb{C}.$$

If for all rectangles $R \subset G$ which are parallel to the axes of $\mathbb{C} \simeq \mathbb{R}^2$ holds

$$\int_{\partial R} f(z)\ dz = 0$$

then the function f is holomorphic.

Proof. The claim is local. Hence we may assume $G = D_r(0)$ a disk with center $= 0$. Define

$$F : G \to \mathbb{C},\ F(z) := \int_\gamma f(\zeta)\ \zeta,$$

with a path γ in G from $z_0 = 0$ to z composed of two adjacent lines, one being parallel to the x-axis and the other being parallel to the y-axis. By assumption, the value $F(z)$ is independent from the choice of the two lines.

Claim: The function F satisfies the Cauchy-Riemann differential equations. For the proof we compute for $z \in G$

$$\frac{\partial}{\partial x}F(z) = \frac{\partial}{\partial x}F(x+iy) = \frac{\partial}{\partial x}\left(\int_0^{iy} f(\zeta)\ d\zeta + \int_{iy}^{x+iy} f(\zeta)\ d\zeta\right) =$$

$$= \frac{\partial}{\partial x}\left(\int_0^{y} f(it)\ dt + \int_0^{x} f(t+iy)\ dt\right) = \frac{\partial}{\partial x}\left(\int_0^{x} f(t+iy)\ dt\right) = f(x+iy) = f(z)$$

Similarly

$$\frac{\partial}{\partial y}F(z) = \frac{\partial}{\partial y}\left(\int_0^{x} f(\zeta)\ d\zeta + \int_{x}^{x+iy} f(\zeta)\ d\zeta\right) =$$

$$= \frac{\partial}{\partial y}\left(\int_0^{x} f(t)\ dt + \int_0^{y} f(x+it)\ i\cdot dt\right)$$

Note for the last integral the parametrization

$$\zeta := i\cdot t,$$

inducing

$$d\zeta = i\cdot dt.$$

Hence

$$\frac{\partial}{\partial y}F(z) = \frac{\partial}{\partial y}\left(\int_0^{y} f(x+it)\ i\cdot dt\right) = i\cdot f(x+iy) = i\cdot f(z)$$

Therefore F has continuous partial derivatives, which satisfy the Cauchy-Riemann differential equations. Hence due to Theorem 3.8 the function F is holomorphic, and F' is holomorphic too. Proposition 2.4 implies

$$F' = \frac{\partial F}{\partial x} = f, \ q.e.d.$$

6.2 Computation of integrals using the residue theorem

One of the main applications of the residue theorem in real analysis and mathematical physics is the computation of integrals.

Example 6.9 (Integration along the real axis). The meromorphic function on $\mathbb{C}$

$$f(z) := \frac{1}{1+z^2}$$

has the poles

$$z = \pm i.$$

Each pole has order $= 1$.

$$f(z) = \frac{1}{1+z^2} = \frac{1}{z-i} \cdot \frac{1}{z+i}$$

Hence the principal part

$$H_i(z) = \frac{1}{z-i} \cdot \frac{1}{2i}$$

and

$$res(f;\ i) = \frac{1}{2i}$$

To compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1+x^2}$$

we extend the path of integration from $-R$ to R on the real axis by the semi-circle

$$\gamma_R \subset \mathbb{C}$$

in the upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \ Im\, z > 0\},$$

see Figure 6.4.

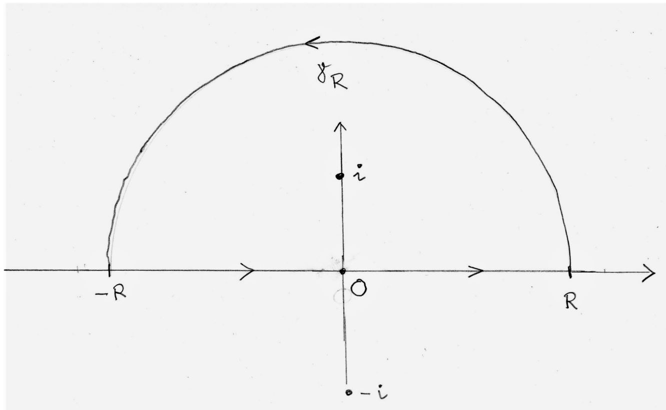


Fig. 6.4 Extended path of integration

Theorem 6.6 implies

$$2\pi i \cdot \frac{1}{2i} = \pi = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\gamma_R} \frac{dz}{1+z^2}$$

We have

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{1}{1+z^2}\, dz \right| \le \lim_{R\to\infty} \pi R \cdot \frac{1}{R^2-1}\, dz = 0$$

As a consequence

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

Example 6.9 is a specific case of the following Proposition 6.10.

Proposition 6.10 (Integration of rational functions). *Consider two polynomials*

$$p, q \in \mathbb{C}[z],$$

and assume

$$deg\ q \ge 2 + deg\ p$$

and the polynomial q without zeros on the real axis. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \cdot \sum_{\substack{a \in \mathbb{H}\ pole \\ of\ p/q}} res(p/q; a)$$

Proof. Theorem 6.6 implies

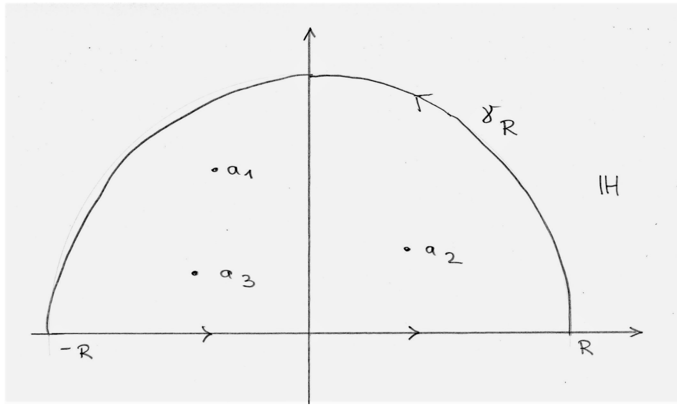


Fig. 6.5 Path of integration

$$2\pi i \cdot \sum_{j=1}^{m} res(p/q; a_j) = \int_{-R}^{R} \frac{p(x)}{q(x)}\, dx + \int_{\gamma_R} \frac{p(z)}{q(z)}\, dz$$

The assumption

$$deg\ q \geq 2 + deg\ p$$

implies the existence of constants $C, R_0 > 0$ such that for all $R \geq R_0$ and $|z| = R$

$$\left| \frac{p(z)}{q(z)} \right| \leq \frac{C}{R^2}$$

see Figure 6.5. W.l.o.g. all poles of p/q in $\mathbb{H}$

$$a_1, ..., a_m$$

have modulus less R_0. For $R \geq R_0$ we have

$$\left| \int_{\gamma_R} \frac{p(z)}{q(r)}\, dz \right| \leq \int_{\gamma_R} \left| \frac{p(z)}{q(z)} \right|\, dz \leq \pi R \cdot \frac{C}{R^2}.$$

As a consequence

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{p(z)}{q(r)}\, dz = 0$$

and

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{p(x)}{q(x)}\, dx = 2\pi i \cdot \sum_{\substack{a \in \mathbb{H} \\ \text{pole of } p/q}} res(p/q;\ a),\ q.e.d.$$

Lemma 6.11 (Computation of the residue). *Consider two functions f, g which are holomorphic in a neighbourhood of a point $a \in \mathbb{C}$. Assume*

$$ord(g;\ a) = 1.$$

Then

$$res\left(\frac{f}{g};\ a\right) = \frac{f(a)}{g'(a)}$$

Proof. The Taylor expansion of g around a

$$g(z) = 0 + g'(a)(z-a) + O(2).$$

implies the Laurent expansion of f/g around a:

$$\frac{f(z)}{g(z)} = \frac{1}{z-a} \cdot \frac{f(z)}{g'(a)+O(1)} = \frac{1}{z-a} \cdot \frac{f(z)}{g'(a)(1+O(1))} =$$

$$= \frac{1}{z-a} \cdot \frac{f(a)}{g'(a)}(1+O(1)).$$

Hence

$$res(f/g;\ a) = \frac{f(a)}{g'(a)},\ q.e.d.$$

Corollary 6.12 (Integration of a specific rational function).

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

Proof. i) The meromorphic function on $\mathbb{C}$

$$f(z) := \frac{1}{1+z^4}$$

has pole set

$$\{z \in \mathbb{C} :\ z^4 = -1\} = \{e^{i\pi\cdot((1/4)+(k/2))} :\ k = 0,1,2,3\}.$$

Each pole has order $= 1$. The two poles in $\mathbb{H}$ are

$$a_0 = e^{i\cdot\pi/4} \text{ and } a_1 = e^{i\cdot 3\pi/4}.$$

Due to Lemma 6.11 the corresponding residues are respectively

$$1/4 \cdot e^{-i(3/4)\pi} \text{ and } 1/4 \cdot e^{-i(1/4)\pi}.$$

Proposition 6.10 implies

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{i\pi}{2} \cdot (e^{-i(3/4)\pi} + e^{-i(1/4)\pi}) =$$

$$= \frac{i\pi}{2} \cdot \frac{\sqrt{2}}{2} \cdot (-1-i+1-i) = \frac{\pi}{\sqrt{2}}, \ q.e.d.$$

Proposition 6.13 (Integration of a trigonometric function).

$$\int_{-\infty}^{\infty} \frac{sin\ x}{x}\, dx = \pi.$$

Proof. The integrand is continuous along the real axis, in particular at the point $0 \in \mathbb{R}$. Therefore

$$\int_{-\infty}^{\infty} \frac{sin(x)}{x}\, dx = \lim_{R \to 0} \int_{\delta_R} \frac{sin(z)}{z}\, dz$$

with path of integration δ_R from Figure 6.6. To evaluate the integral on the right-hand side we recall

$$sin(z) = \frac{1}{2i} \cdot (e^{iz} - e^{-iz}),$$

and consider the meromorphic function on $\mathbb{C}$

$$f(z) := \frac{e^{iz}}{z}$$

Both functions

$$f(z) \text{ and } \frac{e^{-iz}}{z}$$

have a single pole at $z = 0$. The pole has order $= 1$ with residue $= 1$.

i) First we consider the integration

$$f(z) = \frac{e^{iz}}{z}$$

along the two paths

$$\delta_R \text{ and } \gamma_R^+$$

from Figure 6.6.

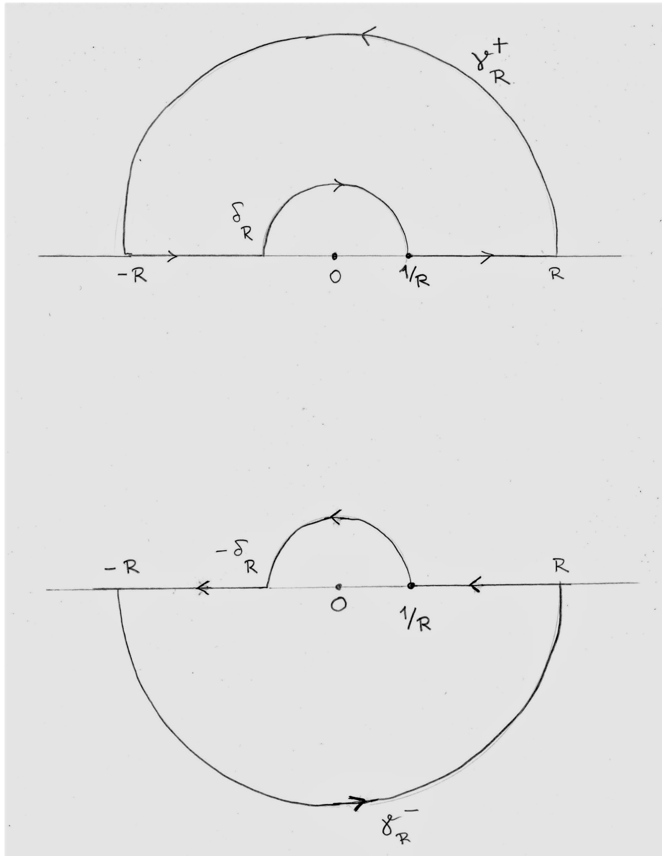


Fig. 6.6 Paths of integration

To estimate

$$\int_{\gamma_R^+} \frac{e^{iz}}{z}\,dz = \int_0^{\pi} exp(iR\cdot e^{i\phi})\cdot \frac{iR\cdot e^{i\phi}}{Re^{i\phi}}\,d\phi = i\cdot \int_0^{\pi} exp(iR\cdot e^{i\phi})\,d\phi$$

we use

$$exp(iR\cdot e^{i\phi}) = exp(iR\cdot(cos\ \phi + i\cdot sin\ \phi)) = exp(-R\sin\ \phi)\cdot exp(iR\cdot cos\ \phi)$$

We obtain

$$|exp(iR\cdot e^{i\phi})| = exp(-R\cdot \sin\ \phi)$$

and

$$\left|\int_{\gamma_R^+} \frac{e^{iz}}{z}\,dz\right| \le \int_0^{\pi} exp(-R\cdot sin\ \phi)\,d\phi = 2\cdot \int_0^{\pi/2} exp(-R\cdot sin\ \phi)\,d\phi$$

For $\phi \in [0, \pi/2]$ we estimate

$$0 \le \phi \cdot (2/\pi) \le \sin\ \phi.$$

Hence

$$\left| \int_{\gamma_R^+} exp(iR \cdot e^{i\phi}) \right| \le 2 \cdot \int_0^{\pi/2} \exp(-R \cdot (2/\pi) \cdot \phi)\ d\phi \le 2 \cdot \int_0^{\infty} exp\left(\frac{-2R}{\pi} \cdot \phi \right)\ d\phi =$$

$$= 2 \cdot \left[\frac{(-\pi)}{2R} \cdot exp\left(\frac{-2R}{\pi} \cdot \phi \right) \right]_{\phi=0}^{\phi=\infty} = \frac{-\pi}{R} \cdot \left[exp\left(\frac{-2R}{\pi} \cdot \phi \right) \right]_{\phi=0}^{\phi=\infty} = \frac{\pi}{R}$$

As a consequence

$$\lim_{R \to \infty} \int_{\gamma_R^+} \frac{e^{iz}}{z}\, dz = 0.$$

The function $f(z)$ has no poles in $\mathbb{H}$. Therefore Theorem 6.6 implies

$$0 = \int_{\delta_R} \frac{e^{iz}}{z}\, dz + \int_{\gamma_R^+} \frac{e^{iz}}{z}\, dz.$$

As a consequence

$$\lim_{R \to \infty} \int_{\delta_R} \frac{e^{iz}}{z}\, dz = 0.$$

ii) We now consider the integrand

$$\frac{e^{-iz}}{z}$$

along the two paths of integration $-\delta_R$ and γ_R^- from Figure 6.6. We parametrize

$$z := R \cdot e^{i\psi},\ dz = iR \cdot e^{i\psi}\ d\psi.$$

Then

$$\int_{\gamma_R^-} \frac{e^{-iz}}{z}\, dz = \int_{-\pi}^{0} exp(-iR \cdot e^{i\psi}) \cdot \frac{iR \cdot e^{i\psi}}{R \cdot e^{i\psi}}\, d\psi =$$

$$= i \cdot \int_{-\pi}^{0} exp(-iR \cdot e^{i\psi})\ d\psi.$$

We substitute

$$\psi = -\phi,\ d\psi = -d\phi$$

obtaining

$$\int_{\gamma_R^-} \frac{e^{-iz}}{z}\, dz = i \cdot \int_0^{\pi} exp(-iR \cdot e^{-i\phi}) d\phi$$

From

$$exp(-iR \cdot e^{-i\phi}) = exp(-iR \cdot (cos\ \phi - i \cdot sin\ \phi))$$

follows the estimate

$$|exp(-iR \cdot e^{-i\phi})| = exp(-R\sin(\phi))$$

Hence

$$\left| \int_{\gamma_r^-} \frac{e^{-iz}}{z} \, dz \right| \leq \int_0^{\pi} exp(-R \cdot sin\ \phi)\ d\phi.$$

Due to the estimate from part i)

$$\lim_{R\to\infty} \int_{\gamma_R^-} \frac{e^{-iz}}{z} \, dz = 0.$$

Theorem 6.6 implies - note the orientation of the integration paths -

$$2\pi i = -\int_{\delta_R} \frac{e^{-iz}}{z} \, dz + \int_{\gamma_R^-} \frac{e^{-iz}}{z} \, dz.$$

Hence

$$\lim_{R\to\infty} \int_{\delta_R} \frac{e^{-iz}}{z} \, dz = -2\pi i.$$

iii) As a consequence from part i) and ii) we get

$$\int_{-\infty}^{\infty} \frac{sin\ x}{x} \, dx = \lim_{R\to\infty} \left(\int_{\delta_R} \frac{sin\ z}{z} \, dz \right) = \frac{1}{2i} \cdot \lim_{R\to\infty} \left(\int_{\delta_R} \frac{e^{iz}\ z}{z} \, dz - \int_{\delta_R} \frac{e^{-iz}\ z}{z} \, dz \right) =$$

$$= \frac{1}{2i} \cdot (0 - (-2\pi i)) = \pi,\ q.e.d.$$

Lemma 6.14 (Gauss integral).

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

Proof. Set

$$I := \int_{-\infty}^{\infty} e^{-x^2} \, dx.$$

Introducing polar coordinates:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy = \int \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx\, dy = \int_{r=0}^{\infty} \left(\int_{\phi=0}^{2\pi} d\phi \right) r \cdot e^{-r^2} \, dr =$$

$$= 2\pi \cdot \int_0^{\infty} r \cdot e^{-r^2} \, dr = \pi \cdot \int_0^{\infty} e^{-t} \, dt = \pi \cdot \left[-e^{-t} \right]_{t=0}^{t=\infty} = \pi$$

Here we made the final substitution

$$r^2 = t,\ 2rdr = dt.$$

We obtain

$$I^2 = \pi \text{ and } I = \sqrt{\pi},\ q.e.d.$$

Proposition 6.15 (Fresnel integrals).

$$\int_0^\infty cos(x^2)\ dx = \int_0^\infty sin(x^2)\ dx = \sqrt{\frac{\pi}{8}}$$

Proof. The proof relates the Fresnel integrals to the Gauss integral from Lemma 6.14.

i) For $R > 0$ we choose the path γ_R from Figure 6.7 as path of integration.

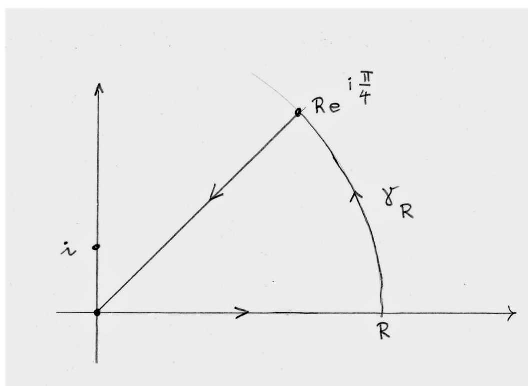


Fig. 6.7 Path of integration

$$\int_{\gamma_R} e^{-z^2}\ dz = \int_0^{\pi/4} exp(-R^2 \cdot e^{2i\phi}) \cdot iR \cdot e^{i\phi}\ d\phi$$

Hence

$$\left|\int_{\gamma_R} e^{-z^2}\ dz\right| \le \int_0^{\pi/4} exp(-R^2 \cdot cos(2\phi)) \cdot R\ d\phi = \frac{1}{2} \cdot \int_0^{\pi/2} exp(-R^2 \cdot cos\ \phi) \cdot R\ d\phi =$$

$$= \frac{1}{2} \cdot \int_0^{\pi/2} exp(-R^2 \cdot sin\ \phi) \cdot R\ d\phi \le \frac{1}{2} \cdot \int_0^\infty exp(-R^2 \cdot \frac{2}{\pi} \cdot \phi) \cdot R\ d\phi =$$

$$= \frac{1}{2} \cdot \left[-\frac{\pi}{2R^2} \cdot R \cdot exp(-R^2 \cdot \frac{2}{\pi} \cdot \phi) \right]_{\phi=0}^{\phi=\infty} = -\frac{\pi}{4R} \cdot \left[exp(-R^2 \cdot \frac{2}{\pi} \cdot \phi) \right]_{\phi=0}^{\phi=\infty} = \frac{\pi}{4R}$$

which implies

$$\lim_{R\to\infty} \int_{\gamma_R} e^{-z^2}\, dz = 0.$$

ii) The integrand e^{-z^2} is holomorphic. Theorem 6.6 implies

$$0 = \int_0^R e^{-x^2}\, dx + \int_{\gamma_R} e^{-z^2}\, dz + \int_{Re^{i\pi/4}}^0 e^{-z^2}\, dz$$

or

$$\int_0^\infty e^{-x^2}\, dx = \lim_{R\to\infty} \int_0^{Re^{i\pi/4}} e^{-z^2}\, dz.$$

iii) We parametrize the line of integration of the integral on the right-hand side as

$$z = te^{i\pi/4},\ t \in [0,R] \text{ and } dz = e^{i\pi/4}dt,\ z^2 = it^2.$$

$$\int_0^{Re^{i\pi/4}} e^{-z^2}\, dz = e^{i\pi/4} \cdot \int_0^R e^{-it^2}\, dt = e^{i\pi/4} \cdot \left(\int_0^R cos(t^2)\, dt - i \cdot \int_0^R sin(t^2)\, dt \right)$$

iv) As a consequence

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}\, dx = e^{i\pi/4} \cdot \int_0^\infty e^{-it^2}\, dt = e^{i\pi/4} \cdot \left(\int_0^\infty cos(t^2)\, dt - i \cdot \int_0^\infty sin(t^2)\, dt \right).$$

Computing

$$\frac{\sqrt{\pi}}{2} \cdot e^{-i\pi/4} = \frac{\sqrt{\pi}}{2} \cdot \frac{1-i}{\sqrt{2}} = (1-i) \cdot \sqrt{\frac{\pi}{8}}$$

and comparing real part and imaginary part proves

$$\int_0^\infty cos(t^2)\, dt = \sqrt{\frac{\pi}{8}} = \int_0^\infty sin(t^2)\, dt,\ q.e.d.$$

6.3 Applications of the residue theorem in complex analysis

Proposition 6.16 (Counting zeros and poles of a meromorphic function). *Consider a domain $G \subset \mathbb{C}$ and a non-constant meromorphic function f on G.*

If $A \subset U$ is a compact subset with piecewise smooth boundary such that f has neither zeros nor poles on ∂A, then the numbers N of zeros and P of poles of f in A

- counted with multiplicity - satisfy

$$N - P = \frac{1}{2\pi i} \cdot \int_{\partial A} \frac{f'(z)}{f(z)} \, dz.$$

Proof. The quotient f'/f is meromorphic on U. Consider a point $a \in A$ and set

$$k := ord(f;\, a).$$

The Laurent expansions with center $= a$ are

$$f(z) = \sum_{n=k}^{\infty} c_n \cdot (z-a)^n, \; c_k \neq 0,$$

$$f'(z) = \sum_{n=k}^{\infty} n \cdot c_n \cdot (z-a)^{n-1}.$$

- If $k \neq 0$ then

  $$ord(f;\, a) = k \text{ and } ord(f';\, a) = k-1,$$

  hence

  $$ord(f'/f;\, a) = ord(f';\, a) - ord(f;\, a) = -1.$$

  Hence the quotient f'/f has a pole of order $= 1$ at a. Its residuum follows from

  $$\frac{f'(z)}{f(z)} = \frac{k \cdot c_k (z-a)^{k-1} + \text{ terms of higher order}}{c_k (z-a)^k + \text{ terms of higher order}} = \frac{k}{z-a} + O(0)$$

  as

  $$res(f'/f;\, a) = k = ord(f;\, a).$$

- If $k = 0$ then $f(a) \neq 0$, and

  $$f,\; f', \text{ and } 1/f$$

  are holomorphic in a neigbourhood of a. Hence

  $$res(f'/f;\, a) = 0 = ord(f;\, a).$$

Theorem 6.6 implies

$$\int_{\partial A} \frac{f'(z)}{f(z)} \, dz = 2\pi i \cdot \sum_{a \in A} res(f'/f;\, a) = 2\pi i \cdot \sum_{a \in A} ord(f;\, a) = N - P, \; q.e.d.$$

Theorem 6.17 states that for a holomorphic map f the number of zeros within a compact set A remains constant under a perturbation ϕ, if the latter remains small on the boundary ∂A - and a posteriori within the compact set.

Theorem 6.17 (Rouché's theorem). *Consider a compact set $A \subset \mathbb{C}$ with piecewise smooth boundary and an open neighbourhood $A \subset U$. If*

$$f,\ \phi : U \to \mathbb{C}$$

are two holomorphic functions satisfying for all $z \in \partial A$

$$|\phi(z)| < |f(z)|,$$

then

$$f \text{ and the disturbed funtion } f - \phi$$

have the same number of zeros in A - counted with multiplicity.

Proof. For each arbitrary but fixed real parameter $t \in [0,\ 1]$ we consider the holomorphic function

$$F_t := f - t \cdot \phi : U \to \mathbb{C}.$$

We have

$$f_0 = f \text{ and } f_1 = f - \phi.$$

By assumption each F_t has no zeros on ∂A. We denote by N_t the number of zeros of F_t in A. Proposition 6.16 implies

$$N_t = \frac{1}{2\pi i} \cdot \int_{\partial A} \frac{F_t'(z)}{F_t(z)}\, dz \in \mathbb{N}.$$

Because the integrand depends continuously on $t \in [0,\ 1]$ we obtain

$$N_0 = N_1,\ q.e.d.$$

Remark 6.18 (Fundamental theorem of algebra). Rouché's theorem provides a second proof of the fundamental theorem of algebra, cf. Corollary 3.19:

Consider a complex polynomial of degree $= n$

$$p(z) = z^n + a_{n-1} \cdot z^{n-1} + \ldots + a_1 \cdot z + a_0 \in \mathbb{C}[z].$$

The function $f(z) := z^n$ has the only zero $z = 0$; it has multiplicity $= n$. Define the polynomial of degree $\leq n - 1$

$$\phi(z) := -(a_{n-1} \cdot z^{n-1} + \ldots + a_1 \cdot z + a_0) \in \mathbb{C}[z].$$

For $R > 0$ sufficiently large we have for all $|z| = R$ due to their distinct degrees

$$|f(z)| > |\phi(z)|.$$

Theorem 6.17 implies that

$$p(z) = f(z) - \phi(z)$$

has the same number of zeros as $f(z)$, i.e. $p(z)$ has n zeros, q.e.d.

Example 6.19 (Counting zeros of a polynomial). To determine the number of zeros of the polynomial

$$p(z) = z^4 + 5z + 2$$

in the unit disk $D_1(0)$ we set

$$f(z) := 5z + 2$$

and

$$\phi(z) := -z^4.$$

The linear polynomial f has the single zero $z = -2/5$ in the unit disk; the zero has multiplicity $= 1$.

For $|z| = 1$ we have

$$|\phi(z)| = 1$$

and

$$|f(z)| = |5z + 2| \geq |5z| - |2| = 3,$$

in particular

$$|\phi(z)| = 1 < 3 \leq |f(z)|.$$

Theorem 6.17 implies that

$$p(z) = f(z) - \phi(z)$$

has a single zero in the unit disk, like $f(z)$.

Theorem 6.20 (Inverse function theorem). *Consider an open subset $U \subset \mathbb{C}$ and a holomorphic function*

$$f : U \to \mathbb{C}$$

with

$$f'(a) \neq 0$$

for a point $a \in U$. Then open neighbourhoods

$$V_1 \subset U \text{ of } a \text{ and } V_2 \subset \mathbb{C} \text{ of } b := f(a)$$

exist such that

$$f|V_1 : V_1 \to V_2$$

is bijective, and the inverse map

$$g := (f|V_1)^{-1} : V_2 \to V_1$$

is holomorphic, satisfying

$$g'(b) = \frac{1}{f'(a)}.$$

Proof. See Figure 6.8 for the notation.

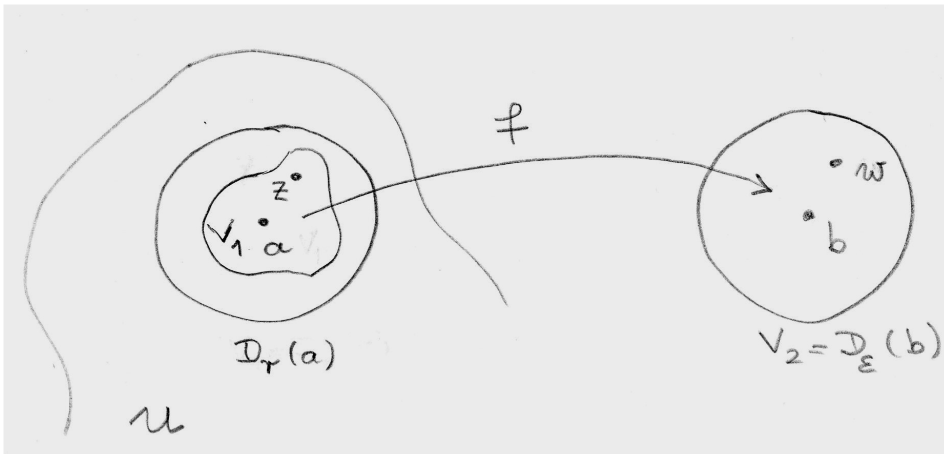


Fig. 6.8 Locally defined inverse function

i) *Bijectivity*: The function

$$f - b$$

has a zero of order $= 1$ at $z = a$ because $f'(a) \neq 0$. Lemma 1.10 implies the existence of a radius $r > 0$ such that

$$\overline{D}_r(a) \subset U,$$

and a is the only zero of $f - b$ in $\overline{D}_r(a)$. In particular

$$\varepsilon := \inf\{|f(z) - b| : \ |z - a| = r\} > 0.$$

For any arbitrary but fixed $w \in \mathbb{C}$ with

$$|w - b| < \varepsilon$$

we apply Rouché's Theorem 6.17 to the compact set

$$A := \overline{D}_r(a)$$

and the two holomorphic functions

$$f - b \text{ and } \phi := w - b :$$

For $|z-a| = r$ we have

$$|\phi(z)| = |w-b| < \varepsilon \leq |f(z)-b|.$$

Hence for exactly one point $z \in D_r(a)$

$$0 = (f(z)-b) - \phi = f(z) - b - (w-b) = f(z) - w.$$

We set

$$V_2 := D_\varepsilon(b) \text{ and } V_1 := D_r(a) \cap f^{-1}(V_2).$$

Then

$$f|V_1 : V_1 \to V_2$$

is bijective with inverse function

$$g : V_2 \to V_1,\ g(w) := z.$$

ii) *Holomorphy*: For arbitrary but fixed $w_0 \in V_2$ we claim

$$g(w_0) = \frac{1}{2\pi i} \cdot \int_{\partial V_1} \frac{z \cdot f'(z)}{f(z) - w_0}\, dz.$$

The integral on the right-hand side can be computed by the residue theorem: The integrand has in V_1 a single pole at

$$z_0 := g(w_0).$$

Due to part i) the pole has order $= 1$. Lemma 6.11 determines the residuum

$$res\left(\frac{z \cdot f'(z)}{f(z) - w_0};\ z_0\right) = \frac{z_0 \cdot f'(z_0)}{f'(z_0)} = z_0 = g(w_0).$$

Theorem 6.6 implies

$$\frac{1}{2\pi i} \cdot \int_{\partial V_1} \frac{z \cdot f'(z)}{f(z) - w_0}\, dz = res\left(\frac{z \cdot f'(z)}{f(z) - w};\ z_0\right) = g(w_0).$$

To prove that g is holomorphic we note: The function g has continuous partial derivatives and satisfies

$$\partial g / \partial \overline{w} = 0.$$

Both statements follow from interchanging integration and derivation in the integral-representation of g because the pole of the integrand is not located on the path of integration.

iii) *Derivative*: The claim

$$g'(w) = \frac{1}{f'(z)}, \ w := f(z),$$

follows from applying the chain rule to the composition

$$g \circ f = id, \ q.e.d.$$

Note. In the context of real analysis one cannot conclude that the inverse of an injective differentiable function is again differentiable. A counter example is the function $f(x) = x^3$ at the point $x = 0$.

Chapter 7
Homotopy

Homotopy means continuous deformation of continuous maps. Homotopy is a principle investigated in the mathematical field of *Algebraic Topology*. The first concept from Algebraic Topology which relies on homotopy is the concept of simple connectedness, see Section 7.2.

In the context of complex analysis the objects to deform are paths, considered as continuous maps from the closed unit interval to a domain in $\mathbb{C}$. The interplay of simple connectedness and holomorphy provides some of the most remarkable results from complex analysis, see also Section 8.2 in Chapter 8.

In the whole chapter $G \subset \mathbb{C}$ denotes a domain, because all results refer to connected open sets. The term *continuously differentiable path γ in G* means a continous function

$$\gamma : [0,1] \to G$$

which is piecewise continuously differentiable.

7.1 Integration along homotopic paths

If a function f has a holomorphic primitive (Deutsch: Stammfunktion) then integration of f along a path equals the difference of the primitive at the endpoints of the path. In particular, the integral is independent from the path chosen between the two points.

Proposition 7.1 (Integration and primitive). *Consider a holomorphic function*

$$f : G \to \mathbb{C}$$

which has a primitive F in G, i.e. having a holomorphic function

$$F : G \to \mathbb{C}$$

satisfying

$$f := F'.$$

Then for any piecewise continuously differentiable path

$$\gamma : [0,1] \to G$$

holds

$$\int_\gamma f(z)\, dz = F(\gamma(1)) - F(\gamma(0)).$$

In particular, the path integral is independent of the path choosen from the point $\gamma(0)$ *to the point* $\gamma(1)$.

Proof. By definition

$$\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Setting

$$\gamma_1 := Re\ \gamma \text{ and } \gamma_2 := Im\ \gamma$$

and identifying $G \subset \mathbb{C} \simeq \mathbb{R}^2$ we compute

$$\frac{d}{dt} F(\gamma(t)) = \frac{d}{dt} F(\gamma_1(t), \gamma_2(t)) = \frac{dF}{\partial x}(\gamma(t)) \cdot \gamma_1'(t) + \frac{dF}{\partial y}(\gamma(t)) \cdot \gamma_2'(t)$$

The holomorphy of F implies due to Proposition 2.4

$$\frac{d}{dt} F(\gamma(t)) = F'(\gamma(t)) \cdot \gamma_1'(t) + i \cdot F'(\gamma(t)) \cdot \gamma_2'(t) = F'(\gamma(t)) \cdot \gamma'(t)$$

Hence

$$\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 F'(\gamma(t)) \cdot \gamma'(t)\, dt =$$

$$= \int_0^1 \frac{d}{dt} F(\gamma(t))\, dt = F(\gamma(1)) - F(\gamma(0)),\ q.e.d.$$

In general, a holomorphic function does not have a primitive for arbitrary domain G, see Remark 7.2. The existence of a primitive may depend on a certain topological property of the domain G.

Remark 7.2 (Obstruction to a global branch of the logarithm function). Consider the function

$$f : \mathbb{C}^* \to \mathbb{C}^*,\ f(z) := 1/z.$$

The residue Theorem 6.6 implies

$$\int_{|z|=1} f(z)\,dz = \int_{|z|=1} \frac{dz}{z} = 2\pi i.$$

And Proposition 7.1 concludes that f has no primitive in the domain $G := \mathbb{C}^*$.

As a consequence, there is no holomorphic branch of the logarithm

$$F : \mathbb{C}^* \to \mathbb{C}$$

satisfying for all $z \in \mathbb{C}$

$$exp(F(z)) = z.$$

For an indirect proof, assume $exp(F(z)) = z$. Taking the derivative gives

$$F'(z) \cdot exp(F(z)) = 1$$

i.e.

$$F'(z) = exp(F(z))^{-1} = 1/z = f(z),$$

a contradiction, because f has no primitive in $\mathbb{C}^*$.

Despite Remark 7.2, locally any holomorphic function has a primitive and Proposition 7.1 applies.

Proposition 7.3 (Local existence of a primitive). *Consider a holomorphic function*

$$f : G \to \mathbb{C}$$

defined on a disk

$$G := D_r(z_0),\ r > 0.$$

Then f has a primitive in G.

Proof. For $z \in G$ the function f expands into a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n.$$

Therefore the power series

$$F : G \to \mathbb{C},\ F(z) := \sum_{n=0}^{\infty} \frac{c_n}{n+1} \cdot (z - z_0)^{n+1},$$

is convergent in G, and F is a primitive of f in G, q.e.d.

We now define and investigate the basic topological concept of the present chapter.

Definition 7.4 (Homotopy). Two paths

$$\gamma_0,\ \gamma_1 : [0,1] \to G$$

with

$$z_0 := \gamma_0(0) = \gamma_1(0) \text{ and } z_1 := \gamma_0(1) = \gamma_1(1)$$

are *homotopic* in G if a continuous map

$$\Phi : [0,1] \times [0,1] \to G$$

exists with

$$\Phi(-,0) = \gamma_0 \text{ and } \Phi(-,1) = \gamma_1$$

and

$$\gamma_s := \Phi(-,s) : [0,1] \to G,\ s \in]0,1[,$$

is a path with $\gamma_s(0) = z_0$ and $\gamma_s(1) = z_1$.

The map Φ is named a *homotopy* between the paths γ_0 and γ_1. The family of curves

$$(\gamma_s := \Phi(-,s))_{s\in[0,1]}$$

is a *deformation* of γ_0 to γ_1, see Figure 7.1.

In case both paths are continuously differentiable, then all paths $\gamma_s,\ s \in [0,1]$, have to be continuously differentiable too.

A homotopy $\Phi(t,s)$ has two variables. The second variable s parametrizes the intermediate paths γ_s, while the first variable t parametrizes the points along a given path γ_s. All intermediate paths in Definition 7.4 have the same start z_0 and the same end z_1.

Theorem 7.5 (Integration along homotopic paths). *Consider a holomorphic function*

$$f : G \to \mathbb{C}$$

and two homotopic, continuously differentiable paths

$$\gamma_0, \gamma_1 : [0,1] \to G$$

from a point $z_0 \in G$ to a point $z_1 \in G$. Then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof. It suffices to show: For any deformation $(\gamma_s = \Phi(-,s))_{s\in[0,1]}$ of γ_0 to γ_1 the integral

$$I(s) := \int_{\gamma_s} f(z)\,dz$$

does not depend on $s \in [0,1]$.

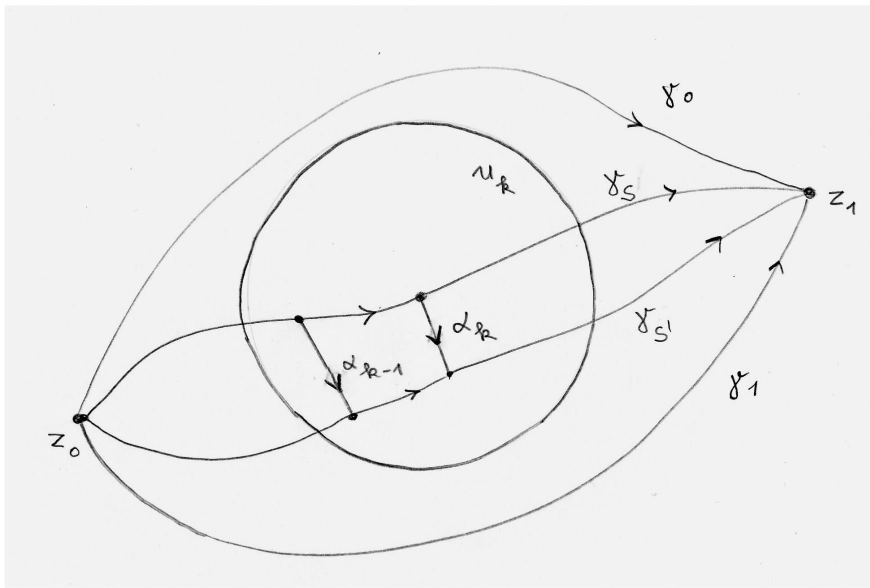


Fig. 7.1 Integration along homotopic paths

Choose an arbitrary but fixed $s \in [0,1]$. Compactness of $[0,1]$ and continuity of Φ imply the existence of

$$\varepsilon, \delta > 0$$

and of a finite decomposition

$$t_0 = 0 < t_1 < ... < t_n = 1$$

of the parameter interval $[0,1]$, such that the sets

$$U_k := D_\varepsilon(\gamma_s(t_k)) \subset G,\ k = 0,...,n,$$

satisfy: For all $|s-s'| < \delta$ and $k = 1,...,n$

$$\gamma_{s'}([t_{k-1},t_k]) \subset U_k.$$

Denote the restriction by

$$\gamma_{s',k} := \gamma_{s'}|[t_{k-1}, t_k],\ k = 1, ..., n.$$

We have

$$\int_{\gamma_s} f(z)\ dz = \sum_{k=1}^{n} \left(\int_{\gamma_{s,k}} f(z)\ dz \right).$$

For arbitrary but fixed s' with

$$|s - s'| < \delta$$

denote by

$$\alpha_k \subset U_k,\ k = 1, ..., n,$$

the line connecting $\gamma_s(t_k)$ with $\gamma_{s'}(t_k)$. Proposition 7.3 provides a primitive of f in U_k, and Proposition 7.1 implies

$$\int_{\gamma_{s,k}} f(z)\ dz = \int_{\alpha_{k-1}} f(z)\ dz + \int_{\gamma_{s',k}} f(z)\ dz - \int_{\alpha_k} f(z)\ dz$$

Summing up all summands for $k = 1, ..., n$ proves

$$\int_{\gamma_s} f(z)\ dz = \int_{\gamma_{s'}} f(z)\ dz,\ q.e.d.$$

7.2 Simply connectedness

The present section starts the investigation of simply connected domains. The investigation will be continued in Section 8.2 with the classification of all simply connected domains in $\mathbb{C}$ up to biholomorphic equivalence.

The concept of simply connectedness relies on closed paths and their homotopy. For closed paths the following Definition 7.6 of homotopy is slightly different from Definition 7.4.

Definition 7.6 (Closed paths and their homotopy). A path

$$\gamma : [0, 1] \to G$$

is *closed* if $\gamma(0) = \gamma(1)$. Two closed paths

$$\gamma_0,\ \gamma_1 : [0, 1] \to G$$

are *homotopic as closed paths* in G if a continuous map

$$\Phi : [0, 1] \times [0, 1] \to G$$

exists with

$$\Phi(-,0) = \gamma_0, \Phi(-,1) = \gamma_1$$

and all paths $\gamma_s := \Phi(-,s),\ s \in]0,1[$, are closed.

If γ_0, γ_1 are continuously differentiable, then also all paths $\gamma_s,\ s \in]0,1[$ are required to be continuously differentiable.

In Definition 7.6 all paths

$$\gamma_s := \Phi(-,s),\ s \in [0,1],$$

are closed, but - different from Definition 7.4 - each path γ_s may be attached to a different distinguished point $\gamma(0,s) = \gamma(1,s)$.

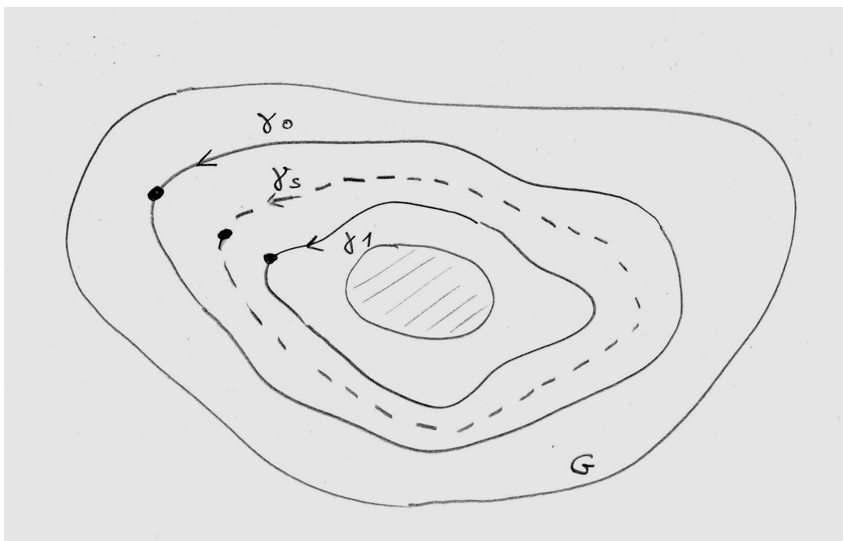


Fig. 7.2 Homotopic as closed paths

Definition 7.7 (Null-homotopic). Consider a closed path

$$\gamma : [0,1] \to G.$$

The path γ is a *constant path* in G if for suitable $z_0 \in G$

$$\gamma(t) = z_0$$

for all $t \in [0,1]$. The path γ is *null-homotopic* in G if γ is homotopic as closed path in G to a constant path in G.

Theorem 7.8 (Integration along homotopic closed paths). *Consider a holomorphic function*

$$f : G \to \mathbb{C}$$

and two closed, continuously differentiable paths

$$\gamma_0,\ \gamma_1 : [0,1] \to G$$

which are homotopic as closed paths in G. Then

$$\int_{\gamma_0} f(z)\ dz = \int_{\gamma_1} f(z)\ dz.$$

In particular,

$$\int_{\gamma_0} f(z)\ dz = 0$$

if γ_0 *is null-homotopic in G.*

The proof of Theorem 7.8 is analogous to the proof of Theorem 7.5. The second part of the theorem follows from the first part.

Definition 7.9 (Simply connectedness, star domain).

1. A domain $G \subset \mathbb{C}$ is *simply connected* if any closed path

$$\gamma : [0,1] \to G$$

   is null-homotopic.

2. The domain G is a *star domain* with respect to a point $a \in G$ if for any $z \in G$ also the line from z to a belongs to G.

Remark 7.10 (Star domains are simply connected). Any star domain G with respect to a point $a \in G$ is simply connected.

For the proof consider a closed path

$$\gamma : [0,1] \to G$$

Then

$$\Phi : [0,1] \times [0,1] \to G,\ \Phi(t,s) := as + (1-s)\gamma(t),$$

is a homotopy from γ to the constant curve at a, q.e.d.

In general, for three points $z_i \in G,\ i = 0,1,2,$ and two paths in G, a path α_1 from z_0 to z_1 and a path α_2 from z_1 to z_2, the *product*

$$\alpha_1 * \alpha_2$$

is the path in G from z_0 to z_2

$$\alpha_1 * \alpha_2 : [0,1] \to G,\ t \mapsto \begin{cases} \alpha_1(2t) & t \in [0,1/2] \\ \alpha_2(2t-1) & t \in [1/2,1] \end{cases}.$$

The inverse path α_1^{-1} is defined as the path in G from z_1 to z_0

$$\alpha_1^{-1}(t) := \alpha_1(1-t).$$

Theorem 7.11 (Path-independence in simply connected domains). *Consider a simply connected domain $G \subset \mathbb{C}$ and a holomorphic function*

$$f : G \to \mathbb{C}.$$

Then for any two points $z_0,\ z_1 \in G$ and continuously differentiable paths

$$\gamma_0, \gamma_1 : [0,1] \to G$$

from z_0 to z_1 holds

$$\int_{\gamma_0} f(z)\ dz = \int_{\gamma_1} f(z)\ dz.$$

Proof. Consider the closed path

$$\gamma := \gamma_0 * \gamma_1^{-1}.$$

According to Theorem 7.8

$$0 = \int_{\gamma} f(z)\ dz = \int_{\gamma_0} f(z)\ dz - \int_{\gamma_1} f(z)\ dz,\ q.e.d.$$

Theorem 7.12 (Global primitives in simply connected domains). *If the domain $G \subset \mathbb{C}$ is simply connected, then any holomorphic function*

$$f : G \to \mathbb{C}$$

has a primitive, i.e. a holomorphic function

$$F : G \to \mathbb{C}$$

with $F' = f$.

Proof. We choose a fixed point $z_0 \in G$ and define

$$F : G \to \mathbb{C},\ F(z) := \int_\gamma f(\zeta)\ d\zeta =: \int_{z_0}^{z} f(\zeta)\ d\zeta.$$

Here the integral is computed along a continuously differentiable path γ in G from z_0 to z. The result does not depend on the choice of γ, see Theorem 7.11.

In order to show that F is holomorphic we choose an arbitrary but fixed point $z_1 \in G$ and a disk

$$U := D_r(z_1) \subset G.$$

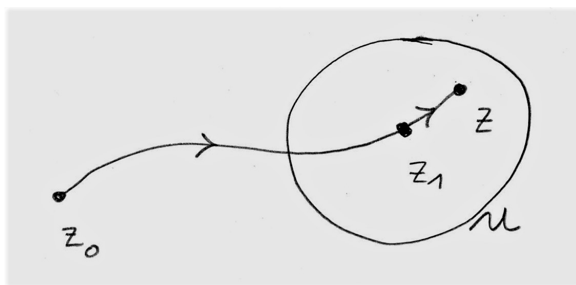


Fig. 7.3 Splitting the path of integration

Denote by

$$F_1 : U \to \mathbb{C}$$

a local primitive of f according to Proposition 7.3. Then for any $z \in U$

$$F(z) = \int_{z_0}^{z} f(\zeta)\ d\zeta = \int_{z_0}^{z_1} f(\zeta)\ d\zeta + \int_{z_1}^{z} f(\zeta)\ d\zeta =$$

$$= \int_{z_0}^{z_1} f(\zeta)\ d\zeta + [F_1(z) - F(z_1)] = const + F_1(z),$$

which proves the holomorphy of F. Moreover, for $z \in U$

$$F'(z) = F_1'(z) = f(z),\ q.e.d.$$

Corollary 7.13 globalizes the local statement of Proposition 5.15. It explains the result of Example 1.27.

Corollary 7.13 (Global branches in simply connected domains). *Consider a simply connected domain $G \subset \mathbb{C}$ and a holomorphic function without zeros*

$$f : G \to \mathbb{C}^*.$$

1. *In G exists a holomorphic branch of the logarithm of f, i.e. a holomorphic function*
$$F : G \to \mathbb{C}$$
satisfying
$$e^F = f.$$
All holomorphic branches F_k of the logarithm are the functions
$$F_k := F + k \cdot 2\pi i,\ k \in \mathbb{Z}.$$
2. *In G exists for any $n \in \mathbb{N}^*$ a holomorphic branch of the n-th root of f, i.e. a holomorphic function*
$$F : G \to \mathbb{C}^*$$
satisfying
$$F^n = f.$$
All holomorphic branches of the n-th root are the functions
$$F_k := e^{2\pi i \cdot (k/n)} \cdot F,\ k = 0, 1, ..., n-1.$$

Proof. 1. We choose a point $z_0 \in G$ and a logarithm w_0 of $f(z_0)$, i.e.
$$e^{w_0} = f(z_0) \neq 0.$$
Theorem 7.12 provides a primitive F of the holomorphic function f'/f with $F(z_0) = w_0$. The proof of the claim
$$e^F = f$$
follows in an analoguous manner as in the proof of Proposition 5.15.

2. We choose a holomorphic branch of the logarithm
$$log : G \to \mathbb{C}^*$$
and define
$$F := e^{(1/n) \cdot log\ f} : G \to \mathbb{C}.$$
Then
$$F^n = e^{n \cdot (1/n) \cdot log\ f} = e^{log\ f} = f.$$
If $\tilde{F}$ also satisfies $\tilde{F}^n = f$, then
$$\left(\frac{\tilde{F}}{F}\right)^n = 1.$$
As a consequence,
$$F_k := \frac{\tilde{F}}{F} = e^{2\pi i \cdot (k/n)}$$

with a fixed $k \in \{0, ..., n-1\}$ is an n-th root of unity, q.e.d.

7.3 Outlook

Maps like

$$exp : \mathbb{C} \to \mathbb{C}^*$$

and

$$\mathbb{C}^* \to \mathbb{C}^*, \ z \mapsto z^k,$$

are examples of *covering projections*. For a covering projection

$$f : X \to Y$$

between topological spaces X and Y the pre-image of small open subsets $V \subset Y$ splits into a disjoint union of sets which f maps homeomorphically to V. The number of pre-images is constant along Y.

The theory of covering projections belongs to the field of *algebraic topology*. Algebraic topology studies the relation between the category of topological spaces and the category of groups. A first set of relevant groups are the *fundamental groups* of topological spaces. The fundamental group of a topological space X classifies up to homotopy the closed curves in X. Moreover, the results from algebraic topology allow to prove statements about topological spaces by means of algebra. The fundamental group is a functor from the category of topological spaces to the category of groups.

Good textbooks on algebraic topology are [17], [22, Kap. III], and [24].

Chapter 8
Holomorphic Maps

8.1 Montel's theorem

Lemma 8.1 (Bolzano-Weierstrass theorem: Analogue for a sequence of sequences). *Consider a sequence $(A_\nu)_{\nu\in\mathbb{N}}$ of sequences*

$$A_\nu = (a_{\nu n})_{n\in\mathbb{N}},\ \nu \in \mathbb{N},$$

of complex numbers $a_{\nu n} \in \mathbb{C}$.

Assume: For each $n \in \mathbb{N}$ exists a constant $M_n \in \mathbb{R}_+$ which bounds the n-th element of all sequences A_ν, i.e. for all $\nu \in \mathbb{N}$

$$|a_{\nu n}| \leq M_n.$$

Then a subsequence $(A_{\nu_k})_{k\in\mathbb{N}}$ of $(A_\nu)_{\nu\in\mathbb{N}}$ exists, such that for all $n \in \mathbb{N}$ the n-th elements of the subsequence converge, i.e.

$$a_n := \lim_{k\to\infty} a_{\nu_k n}$$

exists.

The original Bolzano-Weierstrass theorem considers a bounded sequence of numbers. It states the convergence of a suitable subsequence. Lemma 8.1 considers the more general case of a sequence of sequences, such that the elements with a fixed index from each sequence are bounded. The lemma implies the existence of a subsequence of sequences which converges component-wise. When considering a sequence of numbers as a vector $a \in \mathbb{C}^{\mathbb{N}}$ with infinitely many components, then Lemma 8.1 generalizes the Bolzano-Weierstrass theorem from numbers to infinite vectors.

Proof. First, the proof applies step by step the original Bolzano-Weierstrass theorem to select subsequences of $(A_\nu)_{\nu\in\mathbb{N}}$ with convergent components. Secondly, the diagonal sequence is defined.

i) *Iterative choice of subsequences*:

- The sequence of zero components

$$(a_{\nu 0})_{\nu \in \mathbb{N}}$$

is bounded by M_0. The Bolzano-Weierstrass theorem provides after step $= 0$ a subsequence

$$(A_{\nu_{k,0}})_{k \in \mathbb{N}} \text{ of } (A_\nu)_{\nu \in \mathbb{N}}$$

with limit

$$a_0 := \lim_{k \to \infty} a_{\nu_{k,0} 0}.$$

- The sequence of first components

$$(a_{\nu_{k,0} 1})_{k \in \mathbb{N}}$$

is bounded by M_1. The Bolzano-Weierstrass theorem provides after step $= 1$ a subsequence

$$(A_{\nu_{k,1}})_{k \in \mathbb{N}} \text{ of } (A_{\nu_{k,0}})_{k \in \mathbb{N}}$$

with additional limit

$$a_1 := \lim_{k \to \infty} a_{\nu_{k,1} 1}.$$

- Continuing in this manner, one obtains after step $= n$ a subsequence

$$(A_{\nu_{k,n}})_{k \in \mathbb{N}} \text{ of } (A_{\nu_{k,n-1}})_{k \in \mathbb{N}}$$

with additional limit

$$a_n := \lim_{k \to \infty} a_{\nu_{k,n} n}.$$

ii) *Diagonal sequence*: To finish the proof we define the subsequence

$$(A_{\nu_k})_{k \in \mathbb{N}} \text{ of } (A_\nu)_{\nu \in \mathbb{N}}$$

as follows: For $k \in \mathbb{N}$

$$A_{\nu_k} := A_{\nu_{k,k}}$$

the sequence which has been chosen during step $= k$ at position $= k$. Hence for all $n \in \mathbb{N}$ the n-th element of the sequence A_{ν_k} is the number $a_{\nu_{k,k} n}$. We claim: For all $n \in \mathbb{N}$

$$a_n = \lim_{k \to \infty} a_{\nu_{k,k} n}$$

The proof follows from step $= k$ considering $k \geq n$.

We now apply Lemma 8.1 to a sequence $(f_\nu)_{\nu \in \mathbb{N}}$ of uniformly bounded holomorphic functions: For given $\nu \in \mathbb{N}$ the sequence A_ν will the sequence of the Taylor coefficients of f_ν.

Proposition 8.2 (Uniform boundedness and compactly convergent subsequence). *Consider a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of holomorphic functions*

$$f_\nu : D_r(0) \to \mathbb{C},\ \nu \in \mathbb{N},$$

defined on a disk $D_r(0),\ r > 0$.

Assume the existence of a constant $M \in \mathbb{R}_+$ such that for all $z \in D_r(0)$ and all $\nu \in \mathbb{N}$

$$|f_\nu(z)| \le M.$$

Then $(f_\nu)_{\nu\in\mathbb{N}}$ has a subsequence

$$(f_{\nu_k})_{k\in\mathbb{N}}$$

which is compactly convergent to a holomorphic function

$$f : D_r(0) \to \mathbb{C}.$$

Proof. i) *Choosing a subsequence*: For each $\nu \in \mathbb{N}$ the Taylor coefficients of the holomorphic function

$$f_\nu(z) = \sum_{n=0}^{\infty} a_{\nu n} \cdot z^n$$

satisfy for all $n \in \mathbb{N}$ the Cauchy estimate

$$|a_{\nu n}| \le \frac{M}{r^n},$$

see Theorem 3.21. For each $\nu \in \mathbb{N}$ we consider the sequence of the Taylor coefficients of f_ν

$$A_\nu := (a_{\nu n})_{n\in\mathbb{N}}$$

Lemma 8.1 provides a subsequence $(A_{\nu_k})_{k\in\mathbb{N}}$ of $(A_\nu)_{\nu\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ the n-th elements $a_{\nu_k n}$ of the sequences A_{ν_k} are convergent:

$$a_n := \lim_{k\to\infty} a_{\nu_k n}$$

Apparently, for all $n \in \mathbb{N}$ also

$$|a_n| \le \frac{M}{r^n}.$$

For all $z \in D_r(0)$ the power series

$$f(z) := \sum_{n=0}^{\infty} a_n \cdot z^n$$

is convergent because

$$|a_n \cdot z^n| \le M \cdot \left(\frac{|z|}{r}\right)^n \text{ and } \frac{|z|}{r} < 1.$$

Hence the power series $f(z)$ is a holomorphic function

$$f : D_r(0) \to \mathbb{C}.$$

ii) *Compact convergence*: In order to show that $(f_{\nu_k})_{k \in \mathbb{N}}$ is compactly convergent to f we may assume that the selected subsequence equals the original sequence $(f_\nu)_{\nu \in \mathbb{N}}$. Consider an arbitrary but fixed radius $0 < \rho < r$ and $\varepsilon > 0$.

Claim: There exists an index ν_0 such that for all $z \in \overline{D}_\rho(0)$ and all $\nu \ge \nu_0$

$$|f(z) - f_\nu(z)| < \varepsilon.$$

The convergence of the geometric series

$$\sum_{n=0}^{\infty} M \cdot \left(\frac{\rho}{r}\right)^n$$

provides an index $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} M \cdot \left(\frac{\rho}{r}\right)^n < \frac{\varepsilon}{4}$$

As a consequence we obtain for all $z \in \overline{D}_\rho(0)$ and all $\nu \in \mathbb{N}$ the coarse estimate

$$\left|\sum_{n=N}^{\infty} a_n \cdot z^n - \sum_{n=N}^{\infty} a_{\nu n} \cdot z^n\right| \le \sum_{n=N}^{\infty} M \cdot \left(\frac{\rho}{r}\right)^n + \sum_{n=N}^{\infty} M \cdot \left(\frac{\rho}{r}\right)^n < \frac{\varepsilon}{2}$$

Due to step i) for all $n \in \mathbb{N}$ the limit

$$\lim_{\nu \to \infty} a_{\nu n} = a_n$$

provides an index ν_0 such that for all $z \in \overline{D}_\rho(0)$, for all $\nu \ge \nu_0$, and for the finitely many indices $n = 0, 1, ..., N-1$

$$|a_n \cdot z^n - a_{\nu n} \cdot z^n| = |a_n - a_{\nu n}| \cdot |z|^n \le |a_n - a_{\nu n}| \cdot \rho^n < \frac{\varepsilon}{2N}$$

Summing up we obtain for all $z \in \overline{D}_\rho(0)$ and all $\nu \ge \nu_0$

$$\left|\sum_{n=0}^{\infty} a_n \cdot z^n - \sum_{n=0}^{\infty} a_{\nu n} \cdot z^n\right| \le \left|\sum_{n=0}^{N-1} (a_n - a_{\nu n}) \cdot z^n\right| + \left|\sum_{n=N}^{\infty} (a_n - a_{\nu n}) \cdot z^n\right| \le$$

$$\leq N \cdot \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon, \ q.e.d.$$

Montel's Theorem 8.3 generalizes the Bolzano-Weierstrass theorem, valid for number sequences, to sequences of holomorphic functions. The condition on boundedness from the original theorem has to be replaced by boundedness of the sequence on compact subsets. The resulting subsequence of holomorphic functions is compactly convergent to a holomorphic limit.

Theorem 8.3 (Montel's theorem for locally bounded sequences of holomorphic functions). *Consider an open subset $U \subset \mathbb{C}$ and a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of holomorphic functions*

$$f_\nu : U \to \mathbb{C}, \ \nu \in \mathbb{N}.$$

Assume: For any compact subset $K \subset U$ exists a constant $M_K > 0$ such that for all $\nu \in \mathbb{N}$

$$\|f_\nu\|_K \leq M_K.$$

Then a subsequence

$$(f_{\nu_k})_{k\in\mathbb{N}}$$

exists which is compact convergent on U with limit a holomorphic function

$$f : U \to \mathbb{C}.$$

Proof. i) *Fixed compact subset*: Consider an arbitrary but fixed compact set $K \subset U$. Because K is compact there exists a constant $M > 0$ and finitely many disks

$$D_{r_i}(a_i) \text{ with closure } \overline{D}_{r_i}(a_i) \subset U, \ i = 1, ..., n,$$

such that

$$K \subset \bigcup_{i=1}^{n} D_{r_i}(a_i)$$

and

$$\|f_\nu\|_{\overline{D}_{r_i}(a_i)} \leq M.$$

Applying Proposition 8.2 successively for the finitely many indices $i = 1, ..., n$ provides a subsequence $(f_{\nu_k})_{k\in\mathbb{N}}$ which converges on K uniformly to a function

$$f_K : K \to \mathbb{C}.$$

ii) *Exhaustion by compact subsets*: We claim the existence of a sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of increasing compact subsets

$$K_0 \subset K_1 \subset ... \subset U$$

such that:

- The sequence $(K_\nu)_{\nu\in\mathbb{N}}$ is an exhaustion of U, i.e.

$$U = \bigcup_{\nu \in \mathbb{N}} K_\nu,$$

- and for each compact subset $K \subset U$ exists an index $\nu \in \mathbb{N}$ with $K \subset K_\nu$.

We construct K_ν by shrinking U to the subset of points with boundary distance at least $1/2^\nu$ and bound at most 2^ν: For each $\nu \in \mathbb{N}$ the set

$$K_\nu := \overline{\{z \in U : D_{1/2^\nu}(z) \subset U\}} \cap \overline{D}_{2^\nu}(0)$$

is bounded and closed, hence compact by the Heine-Borel theorem. Any compact subset $K \subset U$ is bounded and has finite distance from the boundary of U, see [7, § 3]. Hence $K \subset K_\nu$ for suitable $\nu \in \mathbb{N}$.

Due to part i) we now choose successively subsequences of $(f_\nu)_{\nu \in \mathbb{N}}$ which converge uniformly on

$$K_0, K_1, ..., K_n, ...$$

For the final subsequence of holomorphic functions

$$(f_{\nu_k})_{k \in \mathbb{N}} \text{ of } (f_\nu)_{\nu \in \mathbb{N}}$$

the function f_{ν_k} at position k is by definition the k-th function of that subsequence of functions which has been chosen for K_k. Then $(f_{\nu_k})_{k \in \mathbb{N}}$ converges uniformly on any $K_\nu,\ \nu \in \mathbb{N}$. Hence $(f_{\nu_k})_{k \in \mathbb{N}}$ converges compactly to f, and the function f is holomorphic, q.e.d.

8.2 Riemann Mapping Theorem

The Riemann mapping theorem deals with the classification of simply connected domains in the complex plane up to biholomorphic maps: Two domains G_1 and G_2 will be considered equivalent if a biholomorphic map

$$G_1 \xrightarrow{\simeq} G_2$$

exists. Elements from a given equivalence class are indistinguishable by means of complex analysis. Riemann's mapping theorem, Theorem 8.12, states: The equivalence relation has exactly two classes. One class contains i.a. the unit disk, the sliced plane and the upper half-plane. The other class has only one single member, the complex plane.

In the present section we denote by

$$D := D_1(0)$$

the unit disk with center $= 0$. We study holomorphic functions with domain or range in D. After proving some results about the group $Aut(D)$ of holomorphic automorphisms of D, the proof of Theorem 8.12 proceeds along the following steps: Consider a proper simply connected domain $G \subsetneq \mathbb{C}$.

- Study injective holomorphic maps defined on G, see Lemma 8.9 and 8.10.
- Investigate a stretching lemma for injective holomorphic maps $G \to D$, see Lemma 8.11.
- Verify that injective holomorphic maps with maximal stretching are surjective, see Theorem 8.12.

Theorem 8.4 (Schwarz Lemma about holomorphic endomorphism of D). *Consider a holomorphic function on the unit disk*

$$f : D \to D$$

satisfying $f(0) = 0$*. Then*

$$|f'(0)| \leq 1 \text{ and } |f(z)| \leq |z| \text{ for all } z \in D.$$

In addition: If

$$|f(z_0)| = |z_0|$$

for at least one $z_0 \in D^*$ *or if*

$$|f'(0)| = 1$$

then f *is a rotation, i.e. there exists* $\theta \in [0, 2\pi[$ *such that for all* $z \in D$

$$f(z) = z \cdot e^{i\theta}.$$

Proof. i) The function f expands into a convergent power series

$$f(z) = \sum_{n=1}^{\infty} c_n \cdot z^n = z \cdot g(z)$$

with g a holomorphic function on D. The derivative is

$$f'(z) = g(z) + z \cdot g'(z)$$

hence $f'(0) = g(0)$. For any radius $0 < r < 1$ and all $z \in D$ with $|z| = r$ holds

$$r \cdot |g(z)| = |f(z)| < 1$$

hence

$$|g(z)| < 1/r.$$

The maximum principle, Theorem 3.18, implies the same estimate in the whole disk $D_r(0)$, i.e. for all $z \in D_r(0)$:

$$|g(z)| < 1/r.$$

Taking the limit $lim_{r \to 1}$ proves for all $z \in D$

$$|g(z)| \leq 1.$$

As a consequence

$$|f\,'(0)| = |g(0)| \leq 1 \text{ and } |f(z)| = |z| \cdot |g(z)| \leq |z|.$$

ii) Each of the two additional assumptions imply that g assumes the maximum of its modulus at a point from D: If

$$|f\,'(0)| = 1$$

then $|g(0)| = 1$. And if for $z_0 \neq 0$

$$|f(z_0)| = |z_0|$$

then

$$|g(z_0)| = \frac{|f(z_0)|}{|z_0|} = 1.$$

Theorem 3.18 concludes that g is constant, i.e. for all $z \in D$

$$g(z) = e^{i\theta}$$

with a fixed $\theta \in [0, 2\pi[$. As a consequence for all $z \in D$

$$f(z) = z \cdot e^{i\theta},\ q.e.d.$$

Corollary 8.5 (Automorphisms of D fixing the origin). *Any holomorphic automorphism*

$$f : D \to D$$

with $f(0) = 0$ is a rotation.

Proof. Theorem 8.4 implies $|f'(0)| \leq 1$. The inverse function

$$g := f^{-1}$$

satisfies the same assumptions. Hence also

$$|g'(0)| \leq 1.$$

From $g \circ f = id$ follows

$$g'(0) = \frac{1}{f'(0)}$$

by the chain rule. Hence $|f'(0)| = 1$. Theorem 8.4 implies that f is a rotation, q.e.d.

Notation 8.6 (Automorphism group of D).

$$Aut(D) := \{\phi : D \to D \,|\phi \text{ biholomorphic}\}$$

denotes the group of holomorphic automorphims of the unit disk.

Proposition 8.7 shows: It is no restriction to assume $f(0) = 0$ in Theorem 8.4 and Corollary 8.5.

Proposition 8.7 (Transitive action of $Aut(D)$). *The group $Aut(D)$ of holomorphic automorphisms acts transitively on D, i.e. for any pair of points $a,\ b \in D$ exists*

$$\phi \in Aut(D) \text{ with } \phi(a) = b.$$

Proof. The group action is the canonical map

$$Aut(D) \times D \to D,\ (\phi, z) \mapsto \phi(z).$$

Because $Aut(D)$ is a group, it suffices to show: For each $a \in D$ exists $\phi \in Aut(D)$ with $\phi(a) = 0$.

We prove that the map

$$\phi_a : D \to \mathbb{C},\ \phi_a(z) := \frac{z-a}{\overline{a}z-1},$$

belongs to $Aut(D)$. Evidently

$$\phi_a(a) = 0 \text{ and } (\phi_a \circ \phi_a)(a) = a.$$

We even have

$$\phi_a \circ \phi_a = id$$

because for all $z \in D$

$$(\phi_a \circ \phi_a)(z) = \phi_a\left(\frac{z-a}{\overline{a}z-1}\right) = \frac{\dfrac{z-a}{\overline{a}z-1} - a}{\overline{a} \cdot \dfrac{z-a}{\overline{a}z-1} - 1} = \frac{z-a-a(\overline{a}z-1)}{\overline{a}z - \overline{a}a - \overline{a}z + 1} = \frac{z(1-\overline{a}a)}{1-\overline{a}a} = z$$

The map ϕ_a extends to the boundary ∂D. If

$$|z|^2 = z\bar{z} = 1$$

then

$$\phi_a(z) \cdot \overline{\phi_a(z)} = \frac{(z-a)(\bar{z}-\bar{a})}{(\bar{a}z-1)(a\bar{z}-1)} = \frac{1-a\bar{z}-z\bar{a}+a\bar{a}}{1-\bar{a}z-a\bar{z}+\bar{a}a} = 1$$

Hence $\phi_a(\partial D) \subset \partial D$, and $\phi_a \circ \phi_a = id$ implies $\partial D \subset \phi_a(\partial D)$, hence

$$\phi_a(\partial D) = \partial D.$$

Because $\phi_a(0) = a \in D$ we conclude

$$\phi_a(D) \subset D.$$

Hence $\phi_a \in Aut(D)$ which finishes the proof, q.e.d.

Theorem 8.8 (Structure of $Aut(D)$). *The group $Aut(D)$ of holomorphic automorphisms has the form*

$$Aut(D) = \{e^{i\alpha} \cdot \phi_a : \ \alpha \in [0, 2\pi[, \ a \in D\} \simeq [0, 2\pi[\times D.$$

Here

$$\phi_a(z) := \frac{z-a}{\bar{a}z-1}, \ z \in D.$$

In particular, holomorphic automorphisms of D depend on three real parameters.

Proof. i) *Reduction to automorphisms fixing the origin*: For a given holomorphic automorphism

$$f : D \to D$$

set

$$z_0 := f(0) \in D.$$

We recall from the proof of Proposition 8.7 that

$$\phi_{z_0} \in Aut(D) \text{ and } \phi_{z_0}(z_0) = 0.$$

Hence the composition

$$g := \phi_{z_0} \circ f \in Aut(D)$$

fixes the center $0 \in D$. Corollary 8.5 implies that g is a rotation, i.e. for suitable $\alpha \in [0, 2\pi[$ and all $z \in D$

$$g(z) = e^{i\alpha} \cdot z.$$

Because $\phi_{z_0}^{-1} = \phi_{z_0}$ we obtain

$$f = \phi_{z_0} \circ g,$$

or for all $z \in D$

$$f(z) = \phi_{z_0}(e^{i\alpha} \cdot z) = \frac{e^{i\alpha}z - z_0}{\overline{z_0}ze^{i\alpha} - 1} = e^{i\alpha} \cdot \frac{z - z_0e^{-i\alpha}}{\overline{z_0}ze^{i\alpha} - 1} = e^{i\alpha} \cdot \frac{z - a}{\overline{a}z - 1} = e^{i\alpha} \cdot \phi_a(z)$$

with

$$a := z_0e^{-i\alpha} \in D.$$

ii) *Parametrizing* $Aut(D)$: If

$$\phi := e^{i\alpha}\phi_a = e^{i\beta}\phi_b \in Aut(D)$$

then application to a and b shows

$$\phi(a) = 0 \text{ and } \phi(b) = 0,$$

hence $a = b$. While application to a point $z_0 \neq a$ shows

$$e^{i\alpha}\phi(z_0) = e^{i\beta}\phi(z_0) \text{ with } \phi(z_0) \neq 0,$$

hence $\alpha = \beta$, q.e.d.

In the following, Lemma 8.9 studies the image of simply connected domains G under injective holomorphic maps. And subsequently, Lemma 8.10 proves the existence of such maps under the assumption $G \neq \mathbb{C}$.

Lemma 8.9 (Injective holomorphic maps). *Consider a domain $G \subset \mathbb{C}$ and an injective holomorphic map*

$$f : G \to \mathbb{C}.$$

Then also $f(G) \subset \mathbb{C}$ is a domain and

$$f : G \to f(G)$$

is biholomorphic.

If G is even simply connected, then also $f(G)$ is simply connected.

Proof. i) *Homeomorphy of f*: Because f is not constant, Theorem 3.20 implies: The map f is an open map, and the image $f(G) \subset \mathbb{C}$ is also a domain. Hence the inverse map

$$f^{-1} : f(G) \to G$$

is continuous. As a consequence

$$f : G \to f(G)$$

is a homeomorphism, which implies that also $f(G)$ is simply connected.

ii) *Holomorphy of the inverse* f^{-1}: To prove the holomorphy of f^{-1} it suffices to show for all $z \in G$

$$f'(z) \neq 0$$

and to apply Theorem 6.20 about the inverse function. Consider an arbitrary but fixed $z_0 \in G$. After choosing translations in $\mathbb{C}$ we may assume $z_0 = 0$ and

$$f : D_r(0) \to \mathbb{C}$$

with $f(0) = 0$ but without zeros in $D_r(0)^*$. Being injective implies that f is not constant. Hence

$$k := ord(f;\, 0) \in \mathbb{N}^*$$

is well-defined, and for all $z \in D_r(0)$

$$f(z) = z^k \cdot f_1(z)$$

with a holomorphic function f_1 without zeros in $D_r(0)$. Proposition 5.15, or Corollary 7.13, imply the existence of a holomorphic root

$$\sqrt[k]{f_1}.$$

Therefore

$$g : D_r(0) \to \mathbb{C}, g(z) := z \cdot \sqrt[k]{f_1} = \sqrt[k]{f},$$

satisfies

$$g^k = f$$

and

$$k \cdot ord(g;\, 0) = ord(f;\, 0) = k.$$

As a consequence

$$ord(g;\, 0) = 1.$$

Theorem 6.20 implies that g is locally biholomorphic, i.e. for a suitable radius $0 < \rho \leq r$

$$g|D_\rho(0) : D_\rho(0) \xrightarrow{\simeq} g(D_\rho(0)) =: V$$

and the following diagram commutes

$$\begin{tikzcd}
D_\rho(0) \arrow[r, "f"] \arrow[d, "g|D_\rho(0)"'] & \mathbb{C} \\
V \arrow[ur, "z \mapsto z^k"'] &
\end{tikzcd}$$

Injectivity of f implies $k = 1$, hence

$$f\,'(0) \neq 0,\ q.e.d.$$

Lemma 8.10 (Embedding simply connected domains into D). *Consider a simply connected domain $G \subsetneq \mathbb{C}$. Then a subdomain*

$$G_0 \subset D$$

with a biholomorphic map

$$f : G \xrightarrow{\simeq} G_0$$

exists.

Proof. i) *Pushing G away from the origin to G_1*: Choose a point $a \in \mathbb{C} \setminus G$ and consider the translation

$$f_1 : \mathbb{C} \to \mathbb{C},\ f_1(z) := z - a.$$

Then

$$G_1 := f_1(G) \subset \mathbb{C}^*$$

is simply connected.

ii) *The "square root" G_2 of G_1*: Because $G_1 \subset \mathbb{C}^*$ is simply connected, Corollary 7.13 provides a holomorphic square root

$$g : G_1 \to \mathbb{C}$$

satisfying for all $z \in G_1$

$$g(z)^2 = z.$$

The root g is injective because

$$g(z_1) = g(z_2) \implies z_1 = g(z_1)^2 = g(z_2)^2 = z_2.$$

Lemma 8.9 implies that

$$G_2 := g(G_1) \subset \mathbb{C}^*$$

is a simply connected domain, and

$$g : G_1 \to G_2$$

is biholomorphic.

We claim: For any $w \in \mathbb{C}^*$

$$w \in G_2 \implies -w \notin G_2,$$

see Figure 8.1. For the indirect proof assume

$$w \in G_2 \text{ and } -w \in G_2.$$

Choose inverse images $z_1,\ z_2 \in G_1$ with

$$g(z_1) = w \text{ and } g(z_2) = -w.$$

Then

$$z_1 = g(z_1)^2 = w^2 = (-w)^2 = g(z_2)^2 = z_2.$$

Hence

$$w = g(z_1) = g(z_2) = -w$$

or $w = 0$, a contradiction.

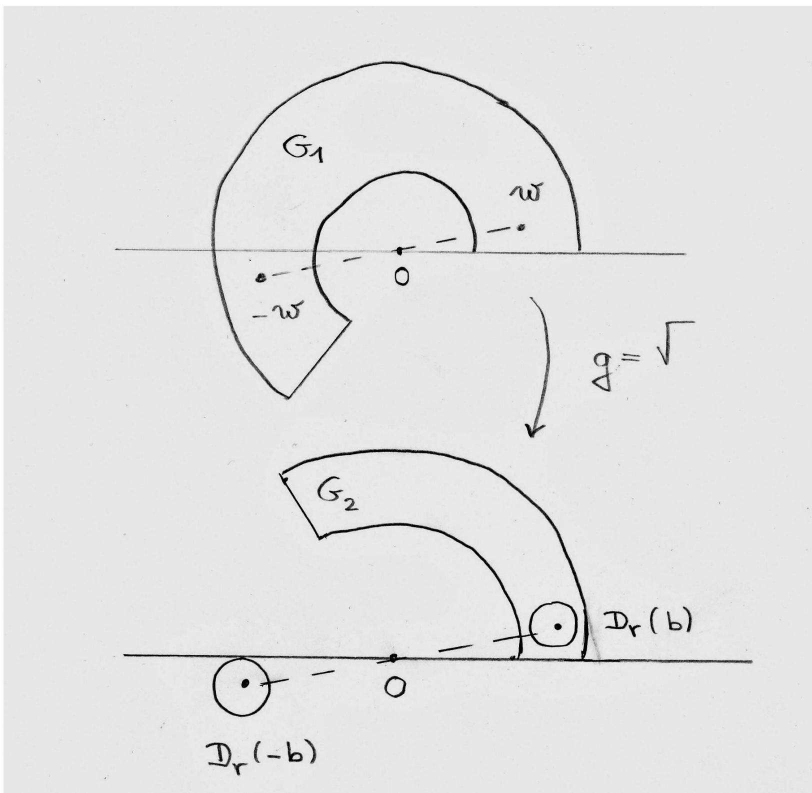


Fig. 8.1 The square root $G_2 = \sqrt{G_1}$

iii) *Embedding G_2 into D by reflection*: Because G_2 is an open set, we may choose a point $b \in G_2$ and a radius $r > 0$ such that

$$\overline{D}_r(b) \subset G_2.$$

Due to part ii)

$$\overline{D}_r(-b) \cap G_2 = \emptyset, \ i.e.$$

for all $z \in G_2$

$$|z+b| = |z-(-b)| > r.$$

The map

$$f_2 : G_2 \to \mathbb{C}, \ f_2(z) := \frac{r}{z+b},$$

is well-defined because $-b \notin G_2$. The map is injective and holomorphic, and satisfies

$$f_2(G_2) \subset D.$$

By Lemma 8.9 the injective holomorphic composition

$$f := f_2 \circ g \circ f_1 : G \to D$$

embedds G into D, q.e.d.

In Lemma 8.10 one cannot drop the assumption that the domain $G \subsetneq \mathbb{C}$ is simply connected: Consider the domain

$$G := \mathbb{C}^* \subsetneq \mathbb{C}$$

which is not simply connected. Any bounded holomorphic function f on G extends holomorphically to the origin $0 \in \mathbb{C}$ due to Theorem 4.5. Hence f is constant due to Liouvielle's Theorem, Corollary 3.23. As a consequence, G cannot be biholomorphically equivalent to a subset of the unit disk D, where the identity is a non-constant bounded holomorphic function.

Thanks to Lemma 8.10 the classification of proper, simply connected subdomains of $\mathbb{C}$ reduces to the classification of simply connected subdomains G of D. The final step will show: It is possible to stretch $G \subset D$ biholomorphically to the whole unit disk D. The main result is the "stretching lemma" 8.11.

Lemma 8.11 should be contrasted with Schwarz Lemma 8.4. The domain of the holomorphic map from Lemma 8.11 leaves out at least one point of D. The holomorphic map f from Schwarz Lemma, which is defined on all of D, satisfies

$$|f'(0)| \leq 1.$$

Lemma 8.11 (Stretching lemma for D). *Consider a simply connected proper subdomain*

$$G_0 \subsetneq D$$

and assume $0 \in G_0$. *Then a domain* $G \subset D$ *with* $0 \in G_1$ *and a biholomorphic map*

$$f : G_0 \xrightarrow{\simeq} G$$

exist with

$$f(0) = 0 \text{ and } |f'(0)| > 1.$$

Proof. Recall from Theorem 8.8 the automorphism $\phi_{z_0} \in Aut(D),\ z_0 \in D$,

$$\phi_{z_0}(z) := \frac{z - z_0}{\overline{z_0} z - 1}.$$

The maps are idempotent

$$\phi_{z_0} \circ \phi_{z_0} = id$$

and satisfy

$$\phi_{z_0}(z_0) = 0.$$

i) *Construction of f*: By assumption a point

$$a \in D \setminus G_0$$

exists. We choose $b \in D$ with $b^2 = a$.

First we map G_0 away from the origin by considering

$$G_1 := \phi_a(G_0) \subset \mathbb{C}^*.$$

Denote by

$$g : G_1 \to \mathbb{C}$$

the branch of the square root with $g(a) = b$, which exists due to Corollary 7.13. The map g is injective. We consider the composition

$$f := \phi_b \circ g \circ \phi_a : G_0 \to D$$

and define

$$G := f(G_0).$$

By construction f is holomorphic, injective and satisfies $f(0) = 0$.

ii) *The derivative $f\,'(0)$*: By the chain rule the derivative of f computes as

$$f'(0) = \phi_b'(b) \cdot g'(a) \cdot \phi_a'(0).$$

The derivative of a fractional linear transformation

$$\phi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

computes as

$$\phi'(z) = \frac{\alpha(\gamma z + \delta) - \gamma(\alpha z + \beta)}{(\gamma z + \delta)^2} = \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2}$$

- We obtain

$$\phi_a'(z) = \frac{-1+\bar{a}a}{(\bar{a}z-1)^2},$$

hence

$$\phi_a'(0) = -1+|a|^2$$

- and similarly

$$\phi_b'(b) = \frac{-1+|b|^2}{(|b|^2-1)^2} = \frac{1}{|b|^2-1}$$

- Moreover

$$g(z)^2 = z \implies 2g(z)\cdot g'(z) = 1$$

i.e.

$$g'(z) = \frac{1}{2g(z)},\ g'(a) = \frac{1}{2g(a)} = \frac{1}{2b}$$

As a consequence

$$f'(0) = \left(|a|^2-1\right)\cdot\frac{1}{2b}\cdot\frac{1}{|b|^2-1}$$

and

$$|f'(0)| = (1-|b|^4)\cdot\frac{1}{2|b|}\cdot\frac{1}{1-|b|^2} = \frac{1+|b|^2}{2|b|} > 1.$$

The last estimate follows from

$$|b| < 1$$

which implies

$$0 < (1-|b|)^2 = 1+|b|^2-2|b|$$

i.e.

$$2|b| < 1+|b|^2,\ q.e.d.$$

Theorem 8.12 (Riemann mapping theorem). *For each simply connected, proper subdomain of the plane*

$$G \subsetneq \mathbb{C}$$

exists a biholomorphic map

$$f : G \xrightarrow{\simeq} D$$

onto the unit disc.

In Theorem 8.12 the assumption $G \neq \mathbb{C}$ is necessary, i.e. the domain G has to be a proper subdomain of the complex plane. The plane $\mathbb{C}$ and the unit disk D are not biholomorphic equivalent, because each bounded holomorphic function on $\mathbb{C}$ is constant according to Liouville's theorem, Corollary 3.23, while the identity is a non-constant bounded holomorphic function on D.

Proof. Due to Lemma 8.10 we may assume $G \subset D$. For the proof we need to consider only the case $G \neq D$. W.l.o.g. $0 \in G$ due to Proposition 8.7.

i) *Set of embeddings*: We consider the set of injective holomorphic maps to the unit disk which fix the origin

$$\mathscr{F} := \{f : G \to D : \ f \text{ holomorphic, injective, and } f(0) = 0\}.$$

The set is not empty because $id_G \in \mathscr{F}$.

For a suitable radius $r > 0$ we have $D_r(0) \subset G$. The Cauchy inequalities from Theorem 3.21 imply for each $f \in \mathscr{F}$

$$|f'(0)| \leq \frac{1}{r}.$$

Hence

$$M := \sup\{|f'(0)| : \ f \in \mathscr{F}\} < \infty,$$

and $id_G \in \mathscr{F}$ implies

$$M \geq 1.$$

In part ii) we will show that the supremum M is attained by an element $f \in \mathscr{F}$. If the function f is not a biholomorphic map to the unit disk, then part iv) will construct a contradiction to the maximality of f by applying Lemma 8.11.

ii) *Constructing an extremal* $f \in \mathscr{F}$: We choose a sequence $(f_\nu)_{\nu \in \mathbb{N}}$ of functions $f_\nu \in \mathscr{F}$, $\nu \in \mathbb{N}$, with

$$\lim_{\nu \to \infty} |f_\nu'(0)| = M.$$

Due to Montel's Theorem 8.3 we may even assume that the sequence

$$(f_\nu)_{\nu \in \mathbb{N}}$$

is compactly convergent to a holomorphic function

$$f : G \to \mathbb{C}.$$

Theorem 3.26 about the convergence of the derivatives implies

$$\lim_{\nu \to \infty} f_\nu'(0) = f'(0).$$

In particular

$$|f'(0)| = M \geq 1.$$

Moreover for all $z \in G$

$$|f(z)| \leq 1.$$

Assume

$$|f(z_0)| = 1$$

for a point $z_0 \in G$. Then f assumes the maximum of its modulus at z_0. Theorem 3.18 implies that f is constant, contradicting

$$|f'(0)| \geq 1.$$

Hence

$$f : G \to D.$$

iii) *Injectivity of f*: By indirect proof. We assume the existence of two distinct points $z_1 \neq z_2 \in G$ with

$$w := f(z_1) = f(z_2).$$

Then the function

$$g := f - w : G \to \mathbb{C}$$

has the isolated zeros $z = z_1$ and $z = z_2$. We choose two disjoint disks

$$D_r(z_1) \text{ and } D_r(z_2)$$

such that g has no zeros in

$$\overline{D_r(z_1)} \setminus \{z_1\} \text{ and in } \overline{D_r(z_2)} \setminus \{z_2\},$$

see Figure 8.2.

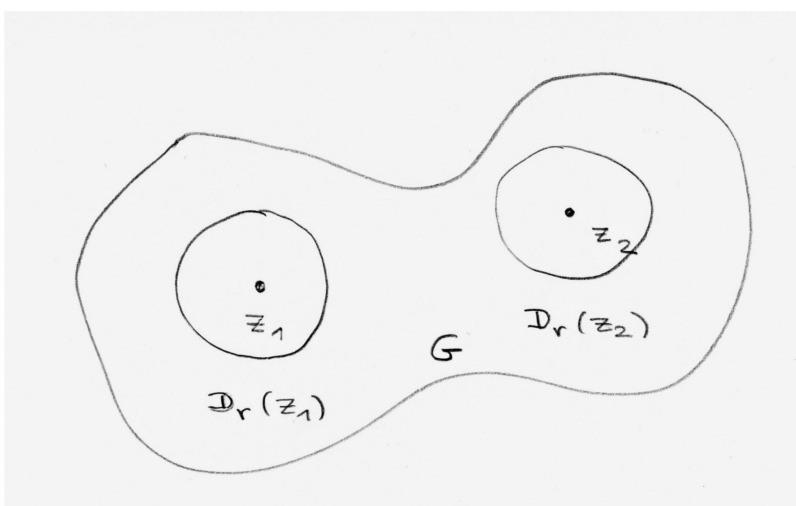


Fig. 8.2 Injectivity of f

The functions
$$g_\nu := f_\nu - w,\ \nu \in \mathbb{N},$$
are compact convergent to g. For $\nu \in \mathbb{N}$ suitable large the function g_ν has the same number of zeros like g in each of both closed disks, see Theorem 6.17. Hence g_ν has zeros in each of the two discs. As a consequence f_ν is not injective, a contradiction.

iv) *Surjectivity of* f: By indirect proof. Assume
$$G_0 := f(G) \subsetneq D.$$
By Lemma 8.9 the domain G_0 is simply connected because f is injective due to part iii). Lemma 8.11 implies the existence of an injective map
$$g : G_0 \to D$$
with
$$g(0) = 0 \text{ and } |g'(0)| > 1.$$
Hence the composition
$$\tilde{f} := g \circ f : G \to D$$
is holomorphic and injective, satisfying
$$\tilde{f}(0) = 0.$$
Therefore
$$\tilde{f} \in \mathscr{F}.$$
But
$$\tilde{f}\,'(0) = g'(0) \cdot f\,'(0)$$
implies
$$|\tilde{f}\,'(0)| > |f\,'(0)| = M,$$
a contradiction to the maximality of $|f\,'(0)|$, q.e.d.

8.3 Projective space and fractional linear transformation

Until now, all holomorphic functions under considerations were defined on open subsets of the complex plane $\mathbb{C}$. In the following, we extend the domain of definition and introduce the compactification $\hat{\mathbb{C}}$ of $\mathbb{C}$ by adding one single point ∞. The concept of holomorphy extends to $\hat{\mathbb{C}}$. As a benefit, meromorphic functions can be considered as holomorphic maps to $\hat{\mathbb{C}}$.

The stereographic projection identifies $\hat{\mathbb{C}}$ with the unit sphere $S^2 \subset \mathbb{R}^3$. Provided with the complex structure of $\hat{\mathbb{C}}$ the sphere is named the Riemann sphere. The Riemann sphere is biholomorphic equivalent to the complex projective space $\mathbb{P}^1$, which is the most simple example of a compact Riemann surface. As a consequence, the following different approaches lead to the same complex manifold:

- The extended plane $\hat{\mathbb{C}}$,
- the Riemann sphere $S^2 \subset \mathbb{R}^3$, and
- the complex projective space $\mathbb{P}^1$.

The group of fractional linear transformations on $\hat{\mathbb{C}}$ is isomorphic to the group of holomorphic automorphisms of $\mathbb{P}^1$.

We start with the extended plane and the transformations from Theorem 8.8.

Definition 8.13 (Fractional linear transformation). A *fractional linear transformation* is a meromorphic function f on $\mathbb{C}$ of the form

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}$$

with a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

We distinguish two cases of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from Definition 8.13:

- If $c = 0$ then
$$a \neq 0 \text{ and } d \neq 0.$$
Hence for all $z \in \mathbb{C}$
$$f(z) = \frac{a}{d} \cdot z + \frac{b}{d}$$
and f is an entire function.

- If $c \neq 0$ then the denominator has a zero at
$$z_0 = -\frac{d}{c}$$
with numerator
$$a \cdot z_0 + b = -\frac{ad}{c} + b = -\frac{1}{c} \cdot (ad - bc) \neq 0.$$
Hence f has a pole at z_0 of order $= 1$, and z_0 is the only pole of f.

To handle meromorphic functions with a pole at a point z_0, we will introduce a new value

$$\infty = f(z_0)$$

such that f becomes a holomorphic map into the extended plane, the one-point-compactification of $\mathbb{C}$,

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

Definition 8.14 (Topology of $\hat{\mathbb{C}}$).

1. The *extended complex plane* is the set

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

2. If $a \in \hat{\mathbb{C}}$ then a subset $U \subset \hat{\mathbb{C}}$ is a *neighbourhood* of a if

   - *Case* $a \neq \infty$: For suitable $\varepsilon > 0$

$$D_\varepsilon(a) \subset U.$$

   - *Case* $a = \infty$: For suitable $\varepsilon > 0$

$$\{z \in \mathbb{C} : \ |z| > 1/\varepsilon\} \cup \{\infty\} \subset U.$$

3. A subset $U \subset \hat{\mathbb{C}}$ is *open* if U is a neighbourhood of each point $a \in U$.

4. Consider a sequence $(z_\nu)_{\nu \in \mathbb{N}}$ of points from $\hat{\mathbb{C}}$ and a point $a \in \hat{\mathbb{C}}$. Then

$$\lim_{\nu \to \infty} z_\nu := a$$

   if any neighbourhood of a contains for all but finitely many $\nu \in \mathbb{N}$ the point z_ν.

5. Consider a subset $M \subset \hat{\mathbb{C}}$. A map

$$f : M \to \hat{\mathbb{C}}$$

   is *continuous at a point* $a \in M$ if any sequence $(z_\nu)_{\nu \in \mathbb{N}}$ of points of M with

$$\lim_{\nu \to \infty} z_\nu = a$$

   satisfies:

$$\lim_{\nu \to \infty} f(z_\nu) = f(a).$$

   The map f is *continuous* if f is continuous at any point $a \in M$.

Example 8.15 (Fractional linear transformations are continuous mappings). The meromorphic function of a fractional linear transformation

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}$$

extends to a continuous map

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}},$$

defined as follows:

- If $c \neq 0$ then
$$z \mapsto \begin{cases} f(z) & z \in \mathbb{C} \setminus \{-(d/c)\} \\ \infty & z = -(d/c) \\ a/c & z = \infty \end{cases}$$
  Continuity of f at $z_0 = -(d/c)$ follows from the characterization of a pole, see Theorem 4.12. To show continuity at $z_0 = \infty$ compute
$$\lim_{\substack{z\to\infty \\ z\neq\infty}} \frac{a \cdot z + b}{c \cdot z + d} = \lim_{\substack{z\to\infty \\ z\neq\infty}} \frac{a + (b/z)}{c + (d/z)} = \frac{a}{c}.$$
- If $c = 0$ then
$$z \mapsto \begin{cases} f(z) & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$$
  Continuity of f at $z_0 = \infty$ follows from $a \neq 0$, $d \neq 0$, and
$$\lim_{\substack{z\to\infty \\ z\neq\infty}} \frac{a \cdot z + b}{c \cdot z + d} = \lim_{\substack{z\to\infty \\ z\neq\infty}} \frac{a}{d} \cdot z + \frac{b}{d} = \infty.$$

Definition 8.16 (The standard open covering of the extended plane). We consider the two open subsets of the extended plane

$$D_0 := \hat{\mathbb{C}} \setminus \{\infty\} \text{ and } D_1 := \hat{\mathbb{C}} \setminus \{0\}$$

together with the two homeomorphism

$$p_j : D_j \to \mathbb{C},\ j = 0, 1,$$

defined respectively as

$$p_0(z) := z \text{ and } p_1(z) := \begin{cases} 1/z & z \neq \infty \\ 0 & z = \infty \end{cases}$$

Then $(D_j)_{j=0,1}$ is an open covering of $\hat{\mathbb{C}}$, named its *standard covering*.

The family

$$p_j : D_j \to \mathbb{C},\ j = 0, 1,$$

is the basic means to define holomorphy of maps on open subsets of the extended plane. For $j = 0, 1$ the notation

$$p_j : D_j \to \mathbb{C}$$

will always refer to the standard covering from Definition 8.16.

Definition 8.17 (Holomorphic maps and the extended plane). Let $U \subset \hat{\mathbb{C}}$ be an open subset.

1. A function
$$f : U \to \mathbb{C}$$
is *holomorphic*, if for $j = 0, 1$ and
$$\tilde{U}_j := p_j(U \cap D_j) \subset \mathbb{C}$$
the composition
$$f \circ (p_j^{-1} | \tilde{U}_j) : \tilde{U}_j \to \mathbb{C},\ j = 0, 1,$$
is holomorphic in the sense of Definition 3.9. Note that $\tilde{U}_j$ is an open subset of $\mathbb{C}$.

2. A continuous map
$$f : U \to \hat{\mathbb{C}}$$
is *holomorphic* if for $j = 0, 1$, and
$$U_j := U \cap f^{-1}(D_j)$$
the composition
$$p_j \circ (f | U_j) : U_j \to \mathbb{C}$$
is holomorphic in the sense of part 1). Note that U_j is an open subset of $\hat{\mathbb{C}}$.

Of course, the important point in Definition 8.17, part 1) is the holomorphy on $U \cap D_1$: One considers the reciprocal of the arguments of f

$$\tilde{U}^1 \to \mathbb{C},\ z \mapsto \frac{1}{z}$$

And part 2) uses in addition the reciprocal of the values of f. Both cases rely on the idea, to investigate the point at infinity and its small neighbourhoods by reflecting the point $\infty \in \hat{\mathbb{C}}$ to the origin $0 \in \mathbb{C}$, and then to consider neighbourhoods of zero.

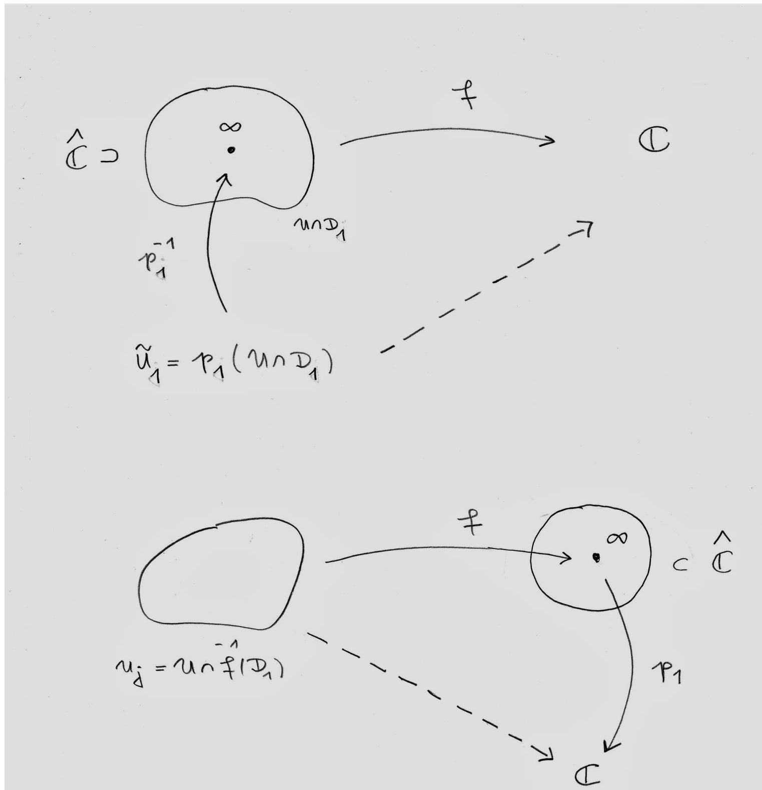


Fig. 8.3 Holomorphic maps and the extended plane

Proposition 8.18 (Meromorphic functions are holomorphic maps). *Any meromorphic function f on an open subset $U \subset \mathbb{C}$ defines a holomorphic map in the sense of Definition 8.17*

$$\hat{f} : U \to \hat{\mathbb{C}},\ z \mapsto \begin{cases} f(z) & f \text{ is holomorphic at } z \\ \infty & f \text{ has a pole at } z \end{cases}$$

Proof. Consider a pole $a \in U$ of f. Theorem 4.12 implies

$$\lim_{\substack{z \to a \\ z \neq a}} f(z) = \infty.$$

Hence $\hat{f}$ is continuous. For $j = 0, 1$, consider the open set

$$U_j := U \cap \hat{f}^{-1}(D_j) \subset U.$$

The set $U_0 \subset U$ does not contain any pole of f. Hence

$$[p_0 \circ (\hat{f}|U_0) : U_0 \to \mathbb{C}] = [f|U_0 : U_0 \to \mathbb{C}]$$

is holomorphic. The other set U_1 does not contain any zero of f. The function

$$p_1 \circ (\hat{f}|U_1) : U_1 \to \mathbb{C}, z \mapsto \begin{cases} \dfrac{1}{f(z)} & f \text{ is holomorphic at } z \\ \\ 0 & f \text{ has a pole at } z \end{cases}$$

has a removable singularity at any pole of f, hence is holomorphic, q.e.d.

In the sequel, we will skip the notation $\hat{f}$ and denote by f also the extended holomorphic map to the Riemann sphere.

Corollary 8.19 (Fractional linear transformations are holomorphic maps on the extended plane). *Any fractional linear map*

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

is a holomorphic map

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ with } f(\infty) := \begin{cases} a/c & c \neq 0 \\ \infty & c = 0 \end{cases}$$

Proof. If $c = 0$ then

$$f(\infty) = \infty.$$

For $D_r(0)$ with suitable $r > 0$ we check

$$p_1 \circ f \circ p_1^{-1} : D_r(0) \to \mathbb{C}, \ z \mapsto 1/f(1/z).$$

The function

$$1/f(1/z) = \frac{d}{a \cdot (1/z) + b} = \frac{d \cdot z}{a + b \cdot z}$$

is holomorphic on $D_r(0)$ because $a \neq 0$.

For $c \neq 0$ we have

$$f(\infty) \in U_0.$$

For $D_r(0)$ with suitable $r > 0$ we check

$$p_0 \circ f \circ p_1^{-1} : D_r(0) \to \mathbb{C}, \ z \mapsto f(1/z).$$

The function

$$f(1/z) = \frac{a \cdot (1/z) + b}{c \cdot (1/z) + d} = \frac{a + b \cdot z}{c + d \cdot z}$$

is holomorphic on $D_r(0)$ because $c \neq 0$, q.e.d.

Proposition 8.20 explains how the extended plane $\hat{\mathbb{C}}$ relates to a sphere.

Proposition 8.20 (Stereographic projection). *Denote by*

$$S^2 := \{\xi \in \mathbb{R}^3 : \|\xi\| = 1\}$$

the unit sphere *in* $\mathbb{R}^3$ *with respect to the Euclidean norm*

$$\|(\xi_1, \xi_2, \xi_3)\| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$

The stereographic projection

$$\varphi : S^2 \to \hat{\mathbb{C}},\ \xi = (\xi_1, \xi_2, \xi_3) \mapsto \begin{cases} \dfrac{\xi_1 + i\xi_2}{1 - \xi_3} & \text{if } \xi_3 \neq 1 \\ \\ \infty & \text{if } \xi_3 = 1 \end{cases}$$

is a continuous map, when $S^2 \subset \mathbb{R}^3$ is provided with the subspace topology induced by the Euclidean space $\mathbb{R}^3$. The stereographic projection has the continuous inverse

$$\psi : \hat{\mathbb{C}} \to S^2, z \mapsto \begin{cases} \dfrac{1}{x^2 + y^2 + 1} \cdot (2x, 2y, x^2 + y^2 - 1) & \text{if } z = x + iy \in \mathbb{C} \\ \\ (0, 0, 1) & \text{if } z = \infty \end{cases}$$

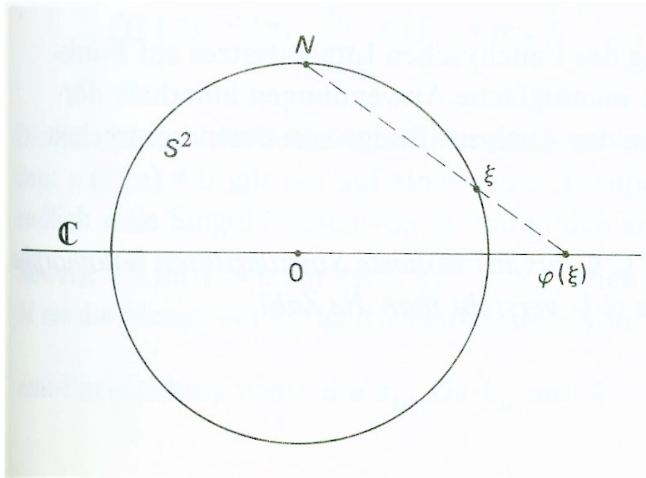


Fig. 8.4 Stereographic projection φ of the unit sphere, from [4, Bild 29]

Figure 8.4 shows the stereographic projection φ from the north pole of the sphere bijectively onto the extended plane. Because φ has the continuous inverse ψ the map

$$\varphi : S^2 \xrightarrow{\simeq} \hat{\mathbb{C}}$$

is a homeomorphism between the unit sphere and the extended plane.

The homeomorphic stereographic projection from Proposition 8.20 allows to transfer the concept of a holomorphic map to the sphere, obtaining the Riemann sphere.

Definition 8.21 (The Riemann sphere). Consider the stereographic projection

$$\phi : S^2 \to \hat{\mathbb{C}}$$

and its inverse

$$\psi : \hat{\mathbb{C}} \to S^2.$$

A function

$$f : U \to \hat{\mathbb{C}},\ U \subset S^2 \text{ open},$$

is *holomorphic* if the composition

$$f \circ \psi : \phi(U) \to \hat{\mathbb{C}}$$

is holomorphic in the sense of Definition 8.17. The *Riemann sphere* is S^2 provided with the transferred concept of holomorphy.

Eventually, we define the projective space $\mathbb{P}^1$ as a compact complex manifold.

Definition 8.22 (Projective space, homogeneous coordinates). On the set

$$\mathbb{C}^2 \setminus \{0\} = \left\{ \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} : \ z_1, z_0 \in \mathbb{C} \text{ and } (z_1, z_0) \neq (0,0) \right\}$$

we introduce the equivalence relation

$$\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \sim \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \iff \exists \lambda \in \mathbb{C}^* \text{ with } z_j = \lambda \cdot w_j, \, j = 0, 1.$$

The set of equivalence classes

$$(\mathbb{C}^2 \setminus \{0\}) / \sim$$

provided with the quotient topology with respect to the canonical projection

$$\pi : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$$

is the *projective space*. The class of a point

$$z = \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}$$

is denoted

$$(z_1 : z_0) := \pi(z) \in \mathbb{P}^1,$$

named the *homogeneous coordinates* of the class. By definition,

$$(z_1 : z_0) = (w_1 : w_0) \iff \exists \lambda \in \mathbb{C}^* \text{ with } z_1 = \lambda \cdot w_1 \text{ and } z_0 = \lambda \cdot w_0.$$

Note. A given class $(z_1 : z_0) \in \mathbb{P}^1$ represents the inverse image

$$q^{-1}((z_1 : z_0)) = \mathbb{C} \cdot \begin{pmatrix} z_1 \\ z_0 \end{pmatrix},$$

i.e. the line passing through the origin and the point

$$\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \in \mathbb{C}^2.$$

Accordingly, the projective space $\mathbb{P}^1$ is sometimes named the parameter set of all lines in $\mathbb{C}^2$ passing through the origin. Figure 8.5 shows: The subset $U_0 \subset \mathbb{P}^1$ parametrizes all lines except the line $\{z_0 = 0\}$, while $U_1 \subset \mathbb{P}^1$ parametrizes all lines except the line $\{z_1 = 0\}$.

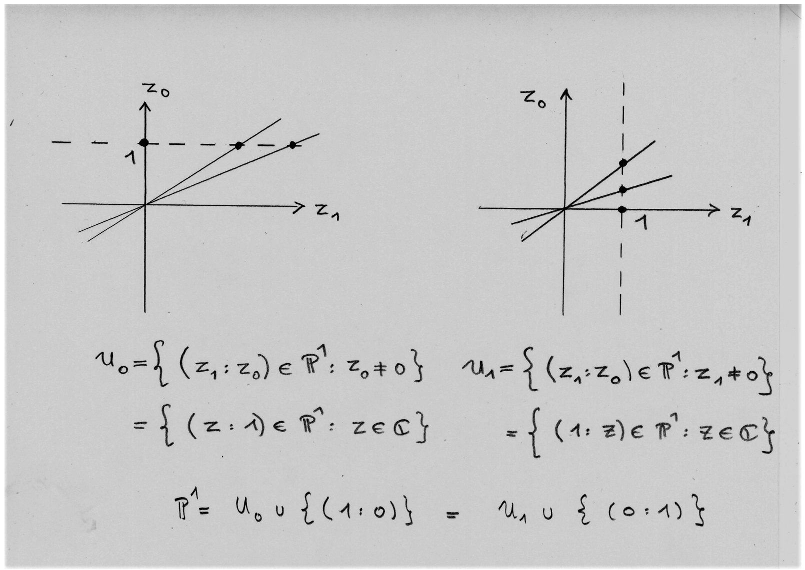


Fig. 8.5 Parametrizing lines in $\mathbb{C}^2 \setminus \{0\}$

Proposition 8.23 (Projective space and extended plane). *The canonical map*

$$q : \mathbb{P}^1 \to \hat{\mathbb{C}},\ (z_1 : z_0) \mapsto \begin{cases} \dfrac{z_1}{z_0} & \text{if } z_0 \neq 0 \\ \\ \infty & \text{if } z_0 = 0 \end{cases}$$

is a homeomorphism. For $j = 0, 1$

$$U_j := q^{-1}(D_j) = \{(z_1 : z_0) \in \mathbb{P}^1 : z_j \neq 0\}.$$

In particular,

$$j : \mathbb{C} \hookrightarrow \mathbb{P}^1,\ j(z) := (z : 1),$$

is a holomorphic embedding, and

$$\mathbb{P}^1 \setminus j(\mathbb{C}) = \{(1 : 0)\} = q^{-1}(\infty).$$

Remark 8.24 (The Riemann surface $\mathbb{P}^1$*).*

1. By definition of the quotient topology on $\mathbb{P}^1$ a subset $U \subset \mathbb{P}^1$ is open iff

$$\pi^{-1}(U) \subset \mathbb{C}^2 \setminus \{0\}$$

is open. The quotient topology is Hausdorff. For $j = 0, 1$ the maps

$$\phi_j : U_j \to \mathbb{C}, (z_1 : z_0) \mapsto \begin{cases} \dfrac{z_1}{z_0} & j = 0 \\[2ex] \dfrac{z_0}{z_1} & j = 1 \end{cases}$$

are homeomorphisms. They are called *complex charts* of $\mathbb{P}^1$. On the intersection

$$U_{01} := U_{10} := U_1 \cap U_0 = \{(z_1 : z_0) \in \mathbb{P}^1 : z_0, z_1 \neq 0\}$$

one switches between the two complex charts by means of two *transition functions*. The first transition function is

$$g_{01} := \phi_0 \circ \phi_1^{-1} : \phi_1(U_{01}) \to \phi_0(U_{01}).$$

$$\begin{tikzcd} & U_{01} \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_0"] & \\ \phi_1(U_{01}) \arrow[rr, dashed, "g_{01}"] & & \phi_0(U_{01}) \end{tikzcd}$$

Because

$$\phi_0(U_{01}) = \phi_1(U_{01}) = \mathbb{C}^*$$

the transition function

$$g_{01} : \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto 1/z,$$

is holomorphic in the sense of Definition 3.9. Its inverse is the holomorphic second transition function

$$g_{10} := \phi_1 \circ \phi_0^{-1} : \phi_0(U_{10}) \to \phi_1(U_{10}), z \mapsto 1/z.$$

The family of charts

$$\mathscr{A} := (\phi_j : U_j \to \mathbb{C})_{j=0,1}$$

is a named a *complex atlas*. The atlas provides the topological space $\mathbb{P}^1$ with the structure of a *Riemann surface*.

2. Holomorphy is a local property. One defines: A function

$$f : U \to \mathbb{C}$$

defined on an open set $U \subset \mathbb{P}^1$ is *holomorphic* if for $j = 0, 1$ the restrictions $f|U_j$ are holomorphic; and $f|U_j$ is *holomorphic* if the composition with the chart

$$f \circ (\phi_j | U \cap U_j)^{-1}$$

is holomorphic in the sense of Definition 3.9. In case

$$U \subset (U_0 \cap U_1)$$

the definition is independent from the choice of $j \in \{0,1\}$ because the transition functions g_{01} and g_{10} are holomorphic.

As a consequence, the function

$$f : U \to \mathbb{C}$$

is holomorphic iff

$$f \circ q^{-1} : q(U) \to \mathbb{C}$$

is holomorphic in the sense of Definition 8.17 .

8.4 Outlook

The complex projective space $\mathbb{P}^1$ is the most simple compact Riemann surface. The theory of Riemann surfaces generalizes complex analysis of one variable from the plane $\mathbb{C}$ to complex manifolds of complex dimension $= 1$, see [9].

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Meromorphic functions are holomorphic maps to $\hat{\mathbb{C}}$ (Prop. 8.18)

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