Ludwig-Maximilians-Universität München
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Lecture in the summer term 2017

Riemannian geometry

Please note: These notes summarize the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. Some proofs were done in two lectures although they appear in a single lecture in these notes. Changes to this script are made without further notice at unpredictable times. If you find any typos or errors, please let me know.

1. Lecture on May, 4 – **Geodesics and the exponential map**

- Let (M, g) be Riemannian and ∇ the Levi-Civita connection. A smooth curve γ in M is a *geodesic* if $\frac{\nabla}{dt}\dot\gamma(t) = 0$ and γ is not constant. It is common to write $\nabla_{\dot\gamma}\dot\gamma = 0$.
- **Example:** Let $M \subset (\mathbb{R}^n, g_{standard})$ be a submanifold with the induced metric. Then a non-constant curve $\gamma : (a, b) \longrightarrow M$ is a geodesic if and only if $\ddot\gamma(t)$ is orthogonal to $T_{\gamma(t)}M$.
- **Fact:** If γ is a geodesic, then $|\dot\gamma|$ is constant.
- **Lemma:** Let $x^1, \ldots, x^n$ be coordinates on U and $\gamma : (a, b) \longrightarrow U \subset M$ be smooth. Then $\gamma = (\gamma^1, \ldots, \gamma^n)$ is a geodesic if and only if
$$\ddot\gamma^k(t) + \sum_{i,j} \dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)) = 0.$$
- **Proof:** Direct computation in coordinates.
- **Fact:** Standard ODE theory implies that this second order differential equation has maximal solutions $\gamma(t, p, v)$ which are uniquely determined by initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_pM$. The solutions depend smoothly on initial conditions and parameters.

  It is not true in general that γ exists for all times. However this is the case when g is positive definite and M is compact. A more general statement is the theorem of Hopf-Rinow below.
- **Definition/Theorem:** There is a unique vector field on TM whose trajectory through $v \in T_pM$ is
$$t \longmapsto (\gamma(t, p, v), \dot\gamma(t, p, v)).$$
  The flow of this vector field is the geodesic flow. Its flow preserves level sets of $\|\cdot\|^2$ on TM.
- **Remark:** If $\gamma : (-\delta, \delta) \longrightarrow M$ is a geodesic, then the same is true of $\gamma_a : (-\delta/2, \delta/a) \longrightarrow M$ defined by $\gamma_a(t) = \gamma(at)$ and $\gamma(t, p, av) = \gamma_a(t, p, v)$.
- **Definition/Theorem:** Let $p \in (M, g)$. Then there is a neighbourhood $U \subset T_pM$ of 0 such that
$$\begin{aligned} \exp_p : T_pM \supset U &\longrightarrow M \\ v &\longmapsto \gamma(1, p, v) \end{aligned}$$
  is well defined and smooth. This map is called the *exponential map* at p.

- **Theorem:** There is a neighbourhood $U \subset T_pM$ such that $\exp_p$ is defined on U and $\exp_p : U \longrightarrow \exp_p(U)$ is a diffeomorphism.
- **Proof:** By definition of $\exp_p$ the differential $D_0 \exp_p : T_0T_pM \simeq T_pM \longrightarrow T_pM$ is the identity. The theorem follows from the inverse function theorem.
- **Definition:** Let $A \subset \mathbb{R}^2$ be open and $f : A \longrightarrow M$ be smooth. A *vectorfield along* f is a smooth map $V : A \longrightarrow TM$ such that $V(s,t) \in T_{f(s,t)}M$. We will write $\frac{\nabla}{\partial s}V$ for the covariant derivative of the vector field $V(\cdot, t = t_0)$ along the curve $f|_{\{t=t_0\}}$ etc.
- **Lemma:** In the present situation

$$\frac{\nabla}{\partial s}\frac{\partial f}{\partial t} = \frac{\nabla}{\partial t}\frac{\partial f}{\partial s}. \tag{1}$$

- **Proof:** Near $f(s,t)$ one chooses local coordinates and expresses V and $f, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ in terms of these coordinates. Since ∇ is assumed to be torsion free, $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ for all i,j. Using this the claim follows.
- **Gauß-Lemma:** Let $v \in T_pM$ and $w \in T_vT_pM \simeq T_pM$ orthogonal to v. Then

$$g\left(\left(D_v \exp_p\right)(w), \left(D_v \exp_p\right)(v)\right) = 0. \tag{2}$$

- **Proof:** For $\varepsilon > 0$ small enough the map

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0,1] &\longrightarrow M \\ (s,t) &\longmapsto \exp_p(t(v + sw)) \end{aligned}$$

is well defined and smooth. We want to compute

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,1) = g\left(\left(D_v \exp_p\right)(w), \left(D_v \exp_p\right)(v)\right).$$

Because g is a metric connection, $\gamma_s(\cdot) = f(s,\cdot)$ is a geodesic and by (1)

$$\begin{aligned} \frac{\partial}{\partial t} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,t) &= g\left(\frac{\nabla}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,t) + g\Big(\frac{\partial f}{\partial s}, \underbrace{\frac{\nabla}{\partial t}\frac{\partial f}{\partial t}}_{\substack{=0 \\ \text{since } \gamma_s \text{ is geodesic}}}\Big)(0,t) \\ &= g\left(\frac{\nabla}{\partial s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) \\ &= 0 (\text{ when } s = 0) \end{aligned}$$

The last equality holds since $\|\dot{\gamma}_s\|^2 = (\|v\|^2 + s^2\|w\|^2)$. Hence $g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,t)$ is independent of t. For $t = 0$ we get

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,0) = 0.$$

2. Lecture on May, 8 – **Minimizing properties of geodesics**

- **Proposition:** Let $p \in M$ and $\delta > 0$ so that $\exp_p$ is defined on a neighbourhood of $\overline{B_\delta(0)}$. Let $v \in T_pM$ be a unit vector and

$$\begin{aligned} \gamma : [0,\delta] &\longrightarrow M \\ t &\longmapsto \exp_p(tv) = \gamma(1,p,tv). \end{aligned}$$

The for every other piecewise smooth path c in M from p to $\gamma(\delta)$

$$l(c) \geq l(\gamma) = \delta$$

with equality if and only of c is a reparametrisation of γ.

- **Proof:** It is enough to consider curves c whose image is contained in $\exp(B_\delta(0))$ and such that $c^{-1}(p) = 0$. For $t \neq 0$ there are unique functions which are smooth (because $\exp_p$ is a local diffeomorphism)

$$\begin{aligned} r : (0,1] &\longrightarrow (0,\delta] \\ \alpha : (0,1) &\longrightarrow \partial B_1(0) \subset T_pM \end{aligned}$$

with the property $c(t) = \exp_p(r(t)\alpha(t))$. Whenever c is smooth

$$\begin{aligned} \|\dot{c}(t)\|^2 &= \left\| D_{r(t)\alpha(t)} \exp_p\left(\dot{r}(t)\alpha(t) + r(t)\dot{\alpha}(t)\right)\right\|^2 \\ &= (\dot{r}(t))^2 + \left\| D_{r(t)\alpha(t)} \exp_p\left(r(t)\dot{\alpha}(t)\right)\right\|^2 \\ &\geq (\dot{r}(t))^2 \end{aligned}$$

with equality if and only if $\dot{\alpha}(t) = 0$. The second equality uses the Gauß-Lemma. For all $0 < \varepsilon < 1$

$$\begin{aligned} l(c|_{[\varepsilon,1]}) &= \int_\varepsilon^1 \|\dot{c}(t)\| dt \geq \int_\varepsilon^1 |\dot{r}(t)| dt \\ &\geq \int_\varepsilon^1 \dot{r}(t) dt = r(1) - r(\varepsilon) \to \delta = l(\gamma) \end{aligned}$$

when $\varepsilon \to 0$. It is now clear how to characterize the equality case.

- **Definition:** Let (M,g) be a connected Riemannian manifold. The metric induced by g is

$$d(p,q) = \inf\{l(c) \,|\, c \text{ is a piecewise smooth path from } p \text{ to } q\}.$$

The previous proposition implies that $d(p,q) = 0$ if and only if $p = q$. The other properties of a metric are easily verified.

- **Fact:** By the proposition a metric δ-ball around $p \in M$ is the diffeomorphic image of a δ-ball in T_pM under the exponential map for sufficiently small $\delta > 0$. This justifies the notation $B_\delta(p) = \exp_p(B_\delta(0))$ and shows that the topology induced by the metric coincides with the manifold topology of M. In particular, $d(p,\cdot)$ respectively $d(\cdot,\cdot)$ define continuous functions on M respectively $M \times M$.

  Moreover, for all $q \in B_\delta(p)$ with δ sufficiently small there is a unique piecewise smooth curve from p to q whose length is $d(p,q)$ and this curve is a geodesic.

  We did not yet show that curves with minimal length connecting two points in (M,g) are always reparametrisations of geodesics.

3. Lecture on May, 11 – **Convex neighbourhoods**

- **Theorem:** For each $p \in M$ there is $\delta > 0$ and a neigbourhood W such that for all $q \in W$ the exponential map $\exp_q$ is defined on $B_\delta(0) \subset T_qM$ and $\exp_q(B_\delta(0))$ contains W.

  For all p, q in W there is a unique shortest geodesic from p to q (it is shorter than δ) and it depends smoothly on p, q.
- **Proof:** Let $\delta' > 0$ so that $\exp_p$ is defined on $B_{2\delta'}(0)$. There is a neighbourhood U of p such that for all $q \in U$ the exponential map $\exp_q$ is defined on $B_{\delta'}(0)$. We denote $v \in T_qM$ by (q, v) to keep track of base points. One applies the inverse function theorem to the map

$$\psi : \bigcup_{q \in U} (B_{\delta'}(0) \subset T_qM) \longrightarrow M \times M$$
$$(q, v) \longmapsto (q, \exp_q(v)).$$

By definition $D_{(p,0)}\psi(v, 0) = (v, v)$ and $D_{(p,0)}\psi(0, w) = (0, w)$. Hence ψ is a local diffeomorphism of a neighbourhood $V \subset TM$ of $(p, 0)$ onto a neighbourhood of (p, p) in $M \times M$. There is a number $0 < \delta < \delta'$ and a neighbourhood V' of p in M such that

$$\bigcup_{q \in V'} (B_\delta(0) \subset T_qM) \subset V \subset TM.$$

Let W be neighbourhood of p so that $W \times W$ is contained in

$$\psi\left(\bigcup_{q \in V'} (B_\delta(0) \subset T_qM)\right).$$

Then δ and W have the desired properties.
- **Corollary:** Let c be a piecewise smooth curve from p to q in M which is parametrized by arc length so that $l(c) = d(p, q)$. Then c is a geodesic.
- **Proof:** If c is not smooth, then this curve can be shortened using the previous theorem. The same is true if c is not a geodesic.
- **Proposition:** Let $p \in M$. There is a number $\delta > 0$ such that for all $0 < \delta' < \delta$ and every geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ which is tangent to $\partial B_{\delta'}(p)$ in $\gamma(0)$ maps a neighbourhood of 0 to a set which meets $\overline{B_{\delta'}(p)}$ in $\gamma(0)$ only.
- **Proof:** Let $\delta'' > 0$ and W be a neighbourhood with the properties from the previous theorem and $0 < \delta' < \delta''$. We will assume that γ is a geodesic parametrized by arc length which is tangent to $\partial B_{\delta'}(p)$ in $\gamma(0)$. Let $F_\gamma(t) = \| \exp_p^{-1}(\gamma(t))\|^2$. Then

$$\frac{dF_\gamma}{dt}(t) = 2g_p\left(\frac{d}{dt}\exp_p^{-1}(\gamma(t)), \exp_p^{-1}(\gamma(t))\right)$$
$$= 0$$

by the Gauß-Lemma. Thus 0 is a critical point of F_γ when $\dot\gamma(0)$ is tangent to $\partial B_{\delta'}(p)$.

We now consider F_γ for all geodesics γ parametrized by arc length with $\gamma(0) \in B_\delta(p)$ and abbreviate $u_\gamma(t) = \exp_p^{-1}(\gamma(t))$. If γ is a geodesic with $\gamma(0) =$

p then $u_\gamma(0)$

$$\begin{aligned}\frac{d^2F_\gamma}{dt^2}(0) &= 2g_p\left(\frac{d^2}{dt^2}u_\gamma(0), u_\gamma(0)\right) + 2\left\|\frac{du_\gamma}{dt}(0)\right\|^2 \\ &= 2\left\|\frac{du_\gamma}{dt}(t)\right\|^2 = 2.\end{aligned}$$

Recall that the germ of a geodesic γ on (M, g) is uniquely determined $\gamma(0)$ and $\dot\gamma(0) \in T_{\gamma(0)}M$ and that the space of tangent vectors of length 1 in T_qM is compact for all q. Thus there is $0 < \delta < \delta'$ such that

$$\frac{d^2F_\gamma}{dt^2}(0) > 0$$

for all geodesics (parametrized by arc length) such that $d(p, \gamma(0)) < \delta$.

- **Corollary:** For all $p \in M$ there is $\delta > 0$ such that for $0 < \delta' < \delta$ the δ'-ball around p is strictly convex, i.e. for all p, q in $B_{\delta'}(p)$ there is a unique minimal geodesic from p to q which is contained in $B_{\delta'}(p)$.
- **Proof:** Let $p \in M$ and $\hat\delta > 0$ the number from the previous proposition. Then $\delta = \hat\delta/2$ has the desired property: Let $p, q \in B_{\delta'}(p)$ with $\delta' < \delta$ and $\gamma : [0, 1] \longrightarrow M$ the unique minimal geodesic from p to q. This geodesic is shorter than 2δ, so it cannot start at p, leave $B_{\delta'}(p)$ and return to q. Then the function $d(p, \gamma(t))^2$ has a global maximum at $t_0 \in (0, 1)$. This means that γ is tangent to the boundary of some ball B_0 around p which is contained in $B_{\delta'}(p)$ and γ is completely contained in B_0. This is a contradiction, thus γ never leaves $B_{\delta'}(p)$.

4. Lecture on May, 15 – **Completeness, Theorem of Hopf-Rinow**

- **Theorem (Hopf-Rinow, first part)**: Let (M, g) be a connected Riemannian manifold and $p \in M$ such that

$$\exp_p : T_pM \longrightarrow M$$

  is well defined. Then for all q there is a geodesic γ from p to q whose length is $d(p, q)$.
- **Proof idea:** To understand the idea, assume that the theorem is true. Let $0 < \delta < d(p, q)$ be so small that $B_\delta(p) = \exp_p(B_\delta(0) \subset T_pM)$ is geodesically convex and γ a minimal geodesic from p to q. Then γ intersects $\partial B_\delta(p)$ exactly once, we call the intersection point p_0. Then p_0 has to be the point on $\partial B_\delta(p)$ which is closest to q. Thus, to find a candidate for a geodesic from p to q it is natural to consider the closest point p_0 on $\partial B_\delta(p)$ from q and to extend the unique geodesic from p to p_0 in the convex ball and then to show that this extension has the desired properties.
- **Proof:** Let δ be as above and $p_0 \in \partial B_\delta(p)$ such that $d(p_0, q) = d(B_\delta(p), q) = \min\{d(x, q) | x \in B_\delta(p)\}$. Then

$$d(p, q) = d(p, p_0) + d(p_0, q) \tag{3}$$

follows by considering a path from p to q, and decomposing this path at its first intersection point with $\partial B_\delta(p)$ (and the triangle inequality,choice of δ, choice of p_0).

Let γ be the radial unit speed geodesic from p to p_0. By assumption $\gamma(t)$ is defined for all t. Consider $I = \{t \in [0, d(p, q)] \,|\, d(\gamma(t), q) = d(p, q) - t\}$. Then

$\delta \in I$ since $\gamma(\delta) = p_0$. Let $0 < T = \sup I$. Then $T \in I$ by continuity and we assume $T < d(p,q)$ (otherwise we are done).

Let $p' = \gamma(T)$ and $0 < \delta' < d(p',q)$ so that $\exp_{p'}$ is defined in $B_{\delta'}(0)$. Consider p'_0 on $\partial B_{\delta'}(p')$ which is closest to q. Then in analogy to (3)

$$d(p',q) = d(p',p'_0) + d(p'_0,q). \tag{4}$$

Let γ' be the radial unit speed geodesic from p' to p'_0 of length $d(p',p'_0)$. By (4), the definition of T and the triangle inequality

$$\begin{aligned} d(p,q) &= d(q,\gamma(T)) + T \\ &= d(\gamma(T),p'_0) + d(p'_0,q) + T \\ &\geq d(p'_0,q) + d(p,p'_0) \\ &\geq d(p,q). \end{aligned}$$

Hence all inequalities are equalities and the concatination of $\gamma|_{[0,T]}$ with the minimal geodesic from $\gamma(T)$ to p'_0 has length $d(p,p'_0)$. Hence it is a geodesic and $p'_0 = \gamma(T+\delta')$ lies on γ and also in I. But then T is not maximal in I. This contradiction shows $T = d(p,q)$.

- **Definition:** (M,d) is *geodesically complete* if $\exp_p$ is defined on T_pM for all $p \in M$.
- **Theorem (Hopf-Rinow, part 2):** The following conditions are equivalent.
  (a) $\exp_p$ is defined on T_pM for some $p \in M$.
  (b) $K \subset M$ is compact if and only if K is closed and bounded.
  (c) M is complete as a metric space.
  (d) M is geodesically complete.
  (e) There are compact sets $K_n, n \in \mathbb{N}$ such that $\cup_n K_n = M$, $K_{n+1} \supset K_n$ and if $q_n \notin K_n$, then $\lim_{n\to\infty} d(q_n,p) = \infty$ for some $p \in M$.
- **Proof:** The proof of this is not very difficult, an important observation is that if (a) holds, then one can use the first part of Hopf-Rinow. $\overline{B_n(0)} \subset T_pM$ is compact, so its image under the exponential map is also compact. For notions like metric completeness and its relationship with compactness one can consult [**?**] or [Q].
- **Example:** In the open unit interval every pair of points is connected by a geodesic. But it is not complete with the standard metric.

5. Lecture on May, 18 – **Riemann curvature tensor**

- **Definition:** Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection. The *curvature tensor* of (M,g) is

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X,Y,Z) &\longmapsto R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \end{aligned}$$

- **Warning:** A sign convention which replaces R by $-R$ is common. Depending on which convention is used some of the definitions of objects induced by R have to be changed, too.
- **Lemma:** R is $C^\infty(M)$-linear in all three variables. In particular, it is a tensor.
- The proof of this is a direct computation. The lemma means that the value of $R(X,Y)Z$ at $p \in M$ depends only on $X(p), Y(p), Z(p)$.
- **Lemma:** The Riemann curvature tensor has the following algebraic properties:
  1. $R(X,Y)Z + R(Y,X)Z = 0$

2. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
3. $g(R(X,Y)Z,W) + g(R(X,Y)W,Z) = 0$
4. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.

- **Proof:** All identities can be proved by computation. It is convenient to assume that all vector fields commute with each other (this is possible since R is tensor). For a nicely organized proof of the last identity see Chapter 9 of [Mi].
- **Definition:** A trilinear map on a Euclidean vector space which satisfies the properties of the previous lemma is called a curvature tensor.
- **Definition:** Let (M,g) be a Riemannian manifold and $\sigma \subset T_pM$ a two-dimensional subspace. The *sectional curvature* $K(\sigma)$ of (M,g) is

$$K(\sigma) = \frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

- **Example:** Let $K \in \mathbb{R}$. Then the map

$$R(X,Y)Z = K(g(Y,Z)X - g(X,Z)Y)$$

is a curvature tensor whose sectional curvature is K. We did not yet show that R arises as the curvature tensor of a Riemannian metric, but this is the case (consider spheres of some radius R, Euclidean space and rescalings of hyperbolic space).
- **Lemma:** Let $f : \mathbb{R}^2 \supset A \longrightarrow M$ be smooth and V a vector field along f. Then

$$\frac{\nabla}{\partial t}\frac{\nabla}{\partial s}V(s,t) - \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}V(s,t) = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V(s,t). \tag{5}$$

- **Proof:** Choose local coordinates near $f(s,t)$ and expand all vector fields along f in terms of the coordinate basis. Then compute.

6. Lecture on May, 22 – **Jacobi fields**

- Let $\gamma = \exp_p(tv), t \in [0,1]$ be a geodesic. The goal is to better understand

$$D_v \exp_p : T_vT_pM \longrightarrow T_{\exp_p(v)}M.$$

If $v = 0$, then $D_0 \exp_p = \mathrm{Id}_{T_pM}$. Now consider a curve $w : (-\varepsilon, \varepsilon) \longrightarrow T_pM$ representing $w \in T_vT_pM$ and

$$\begin{aligned} f : (-\varepsilon,\varepsilon) \times [0,1] &\longrightarrow M \\ (s,t) &\longmapsto \exp_p(tw(s)). \end{aligned}$$

Because $f(s,\cdot)$ parametrizes a geodesic for fixed s we have $\frac{\nabla}{\partial t}\frac{\partial f}{\partial t} = 0$ for all s, hence

$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial f}{\partial t} = 0.$$

Using (5) and (1) this implies the *Jacobi equation* for $J(t) = \frac{\partial f}{\partial s}(0,t)$ along γ:

$$\frac{\nabla}{\partial t}\frac{\nabla}{\partial t}J(t) + R(J(t),\dot\gamma(t))\dot\gamma(t) = 0.$$

This is a second order linear differential equation for J, in particular J is determined by $J(0)$ and $\dot J(0)$, the domain of maximal solutions coincides with the domain of γ and the solutions of the Jacobi equation form a vector space whose dimension is $2\dim(M)$, its solutions are called *Jacobi fields*.
- **Note:** The proof of the Gauß-Lemma is based on a similar idea.

- **Remark:** It is important to note that in the derivation of the Jacobi equation we used only the fact that $f(s,\cdot)$ is a geodesic for fixed s, we did not use $f(s,0)=0$ for all s. Thus
- **Lemma:** Let $\gamma_s:[0,a]\longrightarrow M$ be a smooth family of geodesics. Then $J(t)=\frac{\partial f}{\partial s}(0,t)$ is a Jacobi field along γ_0.
- **Lemma:** Conversely, let $\gamma:[0,a]\longrightarrow M$ be a geodesic and J a Jacobi field along γ. Then there is a smooth family of geodesics γ_s such that $\gamma_0=\gamma$ and $J(t)=\frac{\partial\gamma_s}{\partial s}\big|_{s=0}(t)$.
- **Proof:** Let $U\subset M$ be a geodesically convex neighbourhood of p, i.e. for all $q,q'\in U$ there is a unique geodesic from q to q' which is contained in U (this geodesic has length $d(q,q')$). Choose $\delta>0$ so small that $\gamma([0,\delta])\subset U$ and curves α_0,α_δ in U representing the tangent vectors $J(0),J(\delta)$. Let γ_s be the unique geodesic such that $\gamma_s(0)=\alpha_0(s)$ and $\gamma_s(\delta)=\alpha_\delta(s)$. Then $\gamma_0=\gamma$ and this is a smooth family of geodesics (uniqueness of γ_s and the first theorem in Lecture 3). For $|s|$ sufficiently small γ_s can be extended to $[0,a]$ and $\gamma_0=\gamma$. The variation vector field $\frac{\partial\gamma_s}{\partial s}\big|_{s=0}(t)$ is a Jacobi field and it satisfies

$$\frac{\partial\gamma_s}{\partial s}\bigg|_{s=0}(0)=J(0)\qquad\qquad \frac{\partial\gamma_s}{\partial s}\bigg|_{s=0}(\delta)=J(\delta).$$

  We have shown that

$$\begin{aligned}\mathrm{ev}:\{\text{Jacobi fields along }\gamma\}&\longrightarrow T_{\gamma(0)}M\times T_{\gamma(\delta)}M\\ H&\longmapsto (H(0),H(\delta))\end{aligned}$$

  is surjective. For dimension reasons ev is also injective, hence $\frac{\gamma_s}{\partial s}(s=0,t)=J(t)$.
- **Example:** If γ is a geodesic, then $\dot\gamma(t)$ and $t\dot\gamma(t)$ are Jacobi fields along γ.
- **Fact:** $g(J(t),\dot\gamma(t))$ is an affine function in t when J is a Jacobi field along the geodesic γ. Adding multiples of $\dot\gamma(t),t\dot\gamma(t)$ to J one can obtain Jacobi fields which are everywhere orthogonal to γ.
- **Example:** Let M,g be a manifold of constant sectional curvature K (such space exist), γ a geodesic, W a parallel vector field along γ which is orthogonal to γ with $|W(t)|\equiv 1$. Then the following vector fields are Jacobi fields along γ:

$$J(t)=\begin{cases} tW(t) & \text{if } K=0\\ \frac{\sin(t\sqrt{K})}{\sqrt{K}}W(t) & \text{if } K>0\\ \frac{\sinh t\sqrt{-K}}{\sqrt{-K}}W(t) & \text{if } K<0.\end{cases}\tag{6}$$

7. Lecture on May, 29 – **Conjugate points, Theorem of Hadamard-Cartan**

- **Definition:** Let $\gamma:[0,a]\longrightarrow M$ be a geodesic and $0<t_0\le a$. Then $\gamma(t_0)$ is *conjugate* to $\gamma(0)$ along γ if there is a non-trivial Jacobi field J along γ such that $J(0)=J(t_0)=0$. The *multiplicity* of $\gamma(t_0)$ as a conjugate point is the dimension of the vector space of such Jacobi fields.
- **Remark:** Because $J(t)=t\dot\gamma(t)$ is a Jacobi field along γ which vanishes for $t=0$ but nowhere else, the multiplicity of conjugate points is bounded by $\dim(M)-1$. This bound is realized for the round metric on the sphere. If $\gamma(t_0)$ is not conjugate to $\gamma(0)$ along γ then $\exp_{\gamma(0)}$ is a local diffeomorphism near $t_0\dot\gamma(0)$.

- **Lemma:** Let $f : (M, g) \longrightarrow (N, h)$ be a smooth map between connected Riemannian manifolds such that
  (i) f is surjective,
  (ii) (M, g) is complete (as metric space),
  (iii) $\|Df(v)\| \geq \|v\|$ for all $v \in TM$.
  Then f is a covering.
- **Reminder:** If you are not familiar with coverings then you should read Chapter 5 from [Ma] or Chapter 9 from [J]. We will use covering spaces in several places.
- **Proof:** We show that f has the unique path lifting property for all starting points, i.e. for all smooth $\gamma : [0, 1] \longrightarrow N$ and $q \in f^{-1}(\gamma(0))$ there is a unique smooth map $\widetilde{\gamma} : [0, 1] \longrightarrow M$ such that $\widetilde{\gamma}(0) = q$ and $f \circ \widetilde{\gamma} = \gamma$.

  The assumptions imply that for all initial points $\gamma(0)$ of γ, the set $f^{-1}(\gamma(0))$ is not empty. Let $q \in f^{-1}(\gamma(0))$. Moreover, f is a local diffeomorphism ((iii) implies that Df is injective, since f is surjective it follows that Df is also surjective everywhere (see for example [BJ], Theorem 5.4). The set

$$I = \{t \in [0, 1] \,|\, \gamma\|_{[0,t]} \text{ has the unique path lifting property for initial point } q\}$$

  is not empty and open. We show that I is closed, hence $I = [0, 1]$. Let t_i be a monotone sequence converging to $\sup(I)$. Let $i \leq j$. Since $\gamma(t_i)$ converges to $\gamma(\sup(I))$ it is a Cauchy sequence. Because the path lifts with fixed starting point are unique the lift of $\gamma|_{[0,t_i]}$ is the restriction of the lift of $\gamma|_{[0,t_j]}$ to $[0, t_i]$. Hence we get a well defined sequence $\widetilde{\gamma}(t_i)$ in M. Then by (iii)

$$\begin{aligned} d(\widetilde{\gamma}(t_i), \widetilde{\gamma}(t_j)) &\leq \text{length}\left(\widetilde{\gamma}|_{[t_i,t_j]}\right) \\ &\leq \text{length}\left(\gamma|_{[t_i,t_j]}\right) \end{aligned}$$

  is a Cauchy sequence in M which converges because M is complete. Let $q_\infty = \lim_i \widetilde{\gamma}(t_i)$ (hence $f(q\infty) = \gamma(\sup(I)))$ and U a neighbourhood of q_∞ such that $f|_U$ is a diffeomorphism onto its image. For $\varepsilon > 0$ small enough $\gamma([\sup(I) - \varepsilon, \sup(I) + \varepsilon]) \subset f(U)$. Because $f|_U$ is a diffeomorphism onto $f(U)$ there is a unique lift of $\gamma|_{[\sup(I)-\varepsilon,\sup(I)+\varepsilon]}$ through a point $\widetilde{\gamma}(t_i)$ when for all $j \geq i$ $\widetilde{\gamma}(t_j)) \in U$. Thus I is closed. Since $[0, 1]$ is connected $I = [0, 1]$.
- **Definition:** Let (M, g) be a Riemannian manifold and $p \in M$. p is a *pole* of no geodesic γ through p contains a point which is conjugate to p along γ.
- **Theorem:** Let (M, g) be a complete Riemannian manifold and $p \in M$ a pole. Then $\exp_p : T_pM \longrightarrow M$ is a covering.
- **Proof:** By assumption $\exp_p$ is a local diffeomorphism. Therefore $h = \exp_p^* g$ is a Riemannian metric on T_pM. The geodesics through the origin are the rays through the origin since these get mapped to geodesics in M. In particular, the exponential map at $0 \in T_pM$ of the Riemannian manifold (T_pM, h) is well defined. By the theorem of Hopf-Rinow (T_pM, h) is complete! By definition $\exp_p : (T_pM, h) \longrightarrow (M, g)$ is a local isometry. By the previous lemma $\exp_p$ is a covering.
- **Lemma:** Let (M, g) be a complete Riemannian manifold such that $K(\sigma) \leq 0$ for all planes $\sigma \subset TM$. Then every $p \in M$ is a pole.
- **Proof:** Let $p \in M$, γ a geodesic with $\gamma(0) = p$ and J a non-trivial Jacobi field along γ with $J(0) = 0$. Because $\exp_p$ is a local diffeomorphism at $p \in 0$ there is a neighbourhood of $0 \in \mathbb{R}$ where 0 is the only zero of J. In particular, there is a sequence $0 < t_i \to 0$ such that $\frac{d}{dt} g(J(t_i), J(t_i)) > 0$. By the Jacobi equation

and since $K(\sigma) \leq 0$

$$\begin{aligned}\frac{d^2}{dt^2} g(J(t), J(t)) &= 2g(\ddot{J}(t), J(t)) + 2g(\dot{J}(t), \dot{J}(t)) \\ &\geq g(-R(J(t), \dot{\gamma}(t))\dot{\gamma}(t), J(t)) \\ &\geq 0.\end{aligned}$$

Hence $\frac{d}{dt} g(J(t), J(t)) > 0$ for all $t > 0$ and $J(t) \neq 0$ if $t \neq 0$.

- **Corollary:** Let (M, g) complete with non-positive sectional curvature and dimension n. Then $\exp_p : \mathbb{R}^n \simeq T_pM \longrightarrow M$ is a universal covering.
- **Corollary:** If (M, g) is complete, connected, simply connected and has non-positive sectional curvature, then for all p, q there is a unique geodesic from p to q (up to reparametrisation).

8. Lecture on May, 29 – **Cartan criterion for local isometries**

- We consider the following setting. Let $(M, g), (\overline{M}, \overline{g})$ be Riemannian manifolds of the same dimension n, $p \in M, \overline{p} \in \overline{M}$ and $I : T_pM \longrightarrow T_{\overline{p}}\overline{M})$ an isometry. R respectively $\overline{R}$ is the curvature tensor of g respectively $\overline{g}$.

$$\phi = \exp_{\overline{p}} \circ I \exp_p^{-1}$$

defines a local diffeomorphism of a neighbourhood of p to a neighbourhood of $\overline{p}$.

For a geodesic $\gamma : [0, a] \longrightarrow M$ we write P_γ for the parallel transport $T_{\gamma(0)}M \longrightarrow T_{\gamma(t)}M$ along γ, $P_{\overline{\gamma}}$ has the analogous meaning for geodesics $\overline{\gamma}$ in $\overline{\gamma}$. Given a geodesic γ through p we define $\overline{\gamma}$ as the unique geodesic through $\overline{p}$ with $\dot{\overline{\gamma}}(0) = I(\dot{\gamma}(0))$. For $t \in [0, a]$ let

$$I_\gamma = P_{\overline{\gamma}} \circ I \circ P_\gamma^{-1} : T_{\gamma(t)}M \longrightarrow T_{\overline{\gamma}(t)}\overline{M}.$$

Strictly speaking one should write $I_\gamma(t)$, but enough is enough.

- **Theorem (Cartan):** ϕ is a local isometry if and only if for all geodesics $\gamma : [0, t] \longrightarrow M$ starting at p and contained in a convex neighbourhood of p

$$g(R(X, Y)Z, W) = \overline{g}(\overline{R}(I_\gamma X, I_\gamma Y)I_\gamma Z, I_\gamma W) \tag{7}$$

for all $X, Y, Z, W \in T_{\gamma(t)}M$.

- **Proof:** Let γ be a geodesic through p parametrized by arc length and $e_1, \ldots, e_{n-1}, e_n = \dot{\gamma}$ a parallel orthonormal frame of TM along γ. We fix a Jacobi field J along γ with $J(0) = 0$. Since I_γ is a composition of isometries $\|J(t)\|^2 = \|I_\gamma J(t)\|^2$. Let $\overline{e}_i = I_\gamma e_i$. This is a parallel orthonormal frame along $\overline{\gamma}$. Then $J(t) = \sum_i y^i(t) e_i(t)$ and $I_\gamma J(t) = \sum_i y^i(t) \overline{e}_i(t)$. Using (7) one shows that I_γ is a Jacobi field which vanishes at $t = 0$.

Since e_i and $\overline{e}_i$ are parallel it follows that $I\dot{J}(0) = \frac{d}{dt}\big|_{t=0} (I_\gamma J(t))$. Now

$$\begin{aligned}J(t) &= \left(D_{te_n} \exp_p\right)(\dot{J}(0)) \\ I_\gamma J(t) &= \left(D_{t\overline{e}_n} \exp_{\overline{p}}\right)(I\dot{J}(0)).\end{aligned}$$

Hence $D_{\gamma(t)}\phi(J(t))\|^2 = \|J(t)\|^2$, i.e. ϕ is an isometry on the domain where it is a well defined diffeomorphism.

- **Corollary:** All manifolds with the same constant (sectional) curvature are locally isomteric to each other.

- **Corollary (Riemann):** A Riemannian manifold (M, g) is locally isometric to the Euclidian space of the same dimension if and only if $R \equiv 0$.

9. Lecture on June, 8 – **Cartan-Ambrose-Hicks theorem**

- We generalize the main theorem from the previous section to a more global statement. For this extend the above construction to broken geodesics.
- **Definition:** A piecewiese smooth curve $\gamma : [0, l] \longrightarrow M$ is a *broken geodesic* if there are $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = l$ such that $\gamma|_{[t_i,t_{i+1}]}$ is a geodesic for all $i = 0, \ldots, n-1$. We write ${}_i\gamma$ for $\gamma|_{[0,t_i]}$.
- We consider complete Riemannian manifolds $M, \overline{M}$ and an isometry $I : T_pM \longrightarrow T_{\overline{p}}\overline{M}$. For a broken geodesic γ in M with $\gamma(0) = p$ we define a broken geodesic $\overline{\gamma}$ in $\overline{M}$ inductively as follows: ${}_1\overline{\gamma}$ is the geodesic with domain $[t_0 = 0, t_1]$ and ${}_1\dot{\overline{\gamma}}(0) = I(\dot{\gamma}(0))$. We define
$$\begin{aligned} {}_1I : T_{\gamma(t_1)}M &\longrightarrow T_{\overline{{}_1\gamma}(t_1)} \\ v &\longmapsto P_{{}_1\overline{\gamma}} \circ I \circ P_{{}_1\overline{\gamma}}(v). \end{aligned}$$
If ${}_i\overline{\gamma}$ and ${}_iI$ are defined (with $i < n$), then let we set
$${}_{i+1}\overline{\gamma}(t) = \begin{cases} {}_i\overline{\gamma}(t) & t \leq t_i \\ \exp_{{}_i\overline{\gamma}(t_i)}\left((t - t_i)\,{}_iI(\dot{\gamma}(t_i))\right) & t_i \leq t \leq t_{i+1} \end{cases}$$
where $\dot{\gamma}(t_i)$ is the derivative of $\gamma|_{[t_i,t_{i+1}]}$. We define also
$$\begin{aligned} I_{{}_{i+1}\gamma} : T_{\gamma(t_{i+1})}M &\longrightarrow T_{{}_{i+1}\overline{\gamma}(t_{i+1})} \\ v &\longmapsto P_{{}_{i+1}\overline{\gamma}|_{[t_i,t_{i+1}]}} \circ {}_iI \circ P^{-1}_{\gamma|_{[t_i,t_i+1]}}(v). \end{aligned}$$
We will write $I(l)$ when we do not have to fix the number of breaking points of the broken geodesic.
- **Theorem (Cartan-Ambrose-Hicks):** Assume in addition that M is simply connected and for all broken geodesics $\gamma : [0, l] \longrightarrow M$
$$I(l)\big(R(X, Y)Z\big) = \overline{R}\big(I(l)(X), I(l)(Y)\big)I(l)(Z)$$
for all $X, Y, Z \in T_{\gamma(l)}M$. Then for any to broken geodesics $\gamma_0, \gamma_1 : [0, l] \longrightarrow M$ with $\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(l) = \gamma_1(l)$ the transferred geodesics $\overline{\gamma}_0$ and $\overline{\gamma}_1$ have the same endpoints, i.e.
$$\overline{\gamma}_0(l) = \overline{\gamma}_1(l).$$
- **Proof:** This is an adaptation of [CE], p. 37f. Let γ_0, γ_1 as above. By introducing artificial breaking points to γ_0, γ_1 where these curves are actually smooth we may assume that these two broken geodesics have the same break points $0 = t_0 < t_1 < \ldots < t_n = l$.

  We will first assume that γ_0, γ_1 are close to each other in the sense that for all $i = 0, \ldots, n$ the points $\gamma_0(t_{i-1}), \gamma_0(t_i), \gamma_0(t_{i+1}), \gamma_1(t_{i-1}), \gamma_1(t_i), \gamma_1(t_{i+1})$ together with the (unique) minimal geodesics connecting them lie in convex balls around $\gamma_0(t_i)$ (together with all minimal geodesic connecting these points). We make the analogous assumption for the break points of $\overline{\gamma}_0$ and $\overline{\gamma}_1$. If necessary one introduces additional (and artificial) break points to achieve this.

  We construct a chain of local isometries ${}_i\phi_0$ from neighbourhoods of $\gamma_0(t_i)$ to neighbourhoods of $\overline{\gamma}_0(t_i)$ such that the domains of ${}_{i-1}\phi_0$ intersect in convex sets containing the image of $\gamma_0|_{[t_{i-1},t_i]}$ and $\gamma_1|_{[t_{i-1},t_i]}$. We define ${}_i\phi_0$ to be the local isometry obtains in Cartan theorem such that ${}_i\phi_0(\gamma(t_i)) = \overline{\gamma}_0(t_i)$ and

$D\,{}_i\phi_0 =_i I$ (i.e. the differential at $\gamma_0(t_i)$. Analogously we construct a chain of local isometries ${}_i\phi_1$.

We can assume that the domain of ${}_i\phi_0$ and ${}_i\phi_1$ contains all minimal geodesic segment connecting any two of the points $\gamma_0(t_{i-1}), \gamma_0(t_i), \gamma_0(t_{i+1}), \gamma_1(t_{i-1}), \gamma_1(t_i), \gamma_1(t_{i+1})$ because all these points are supposed to lie in convex balls around $\gamma_0(t_i)$ and $\gamma_1(t_i)$. Recall that the proof of Cartan's theorem yields local isometries whose domain is any neighbourhood of a point m to a neighbourhood of a point $\overline{m}$ where exp_m and $\exp_{\overline{m}}$ are diffeomorphisms.

By induction we now show that ${}_i\phi_0$ and ${}_i\phi_1$ coincide on the intersection on their domain. For $i = 0$ we have $\gamma_0(t_0) = \gamma_1(t_0) = p$ and ${}_0I_0 = I =_0 I_1$. Therefore ${}_0\phi_0 =_0 \phi_1$ on the (convex) intersection of their domains.

For $i = 1$ we obtain local isometries. The domain of ${}_1\phi_0$ contains p and the differential of ${}_1\phi_0$ at that point is I since $p = \exp_{\gamma_0(t_1)}(-(t_1 - t_0)\dot\gamma_0(t_1))$ and the proof of Cartan's theorem computes the differential of ${}_1\phi_0$ at p as follows:

$$D_p\,{}_1\phi_0 = P^{-1}_{{}_1\overline{\gamma}_0} \circ_1 I_0 \circ P_{{}_1\gamma_0} = I.$$

For $D_p\,{}_1\phi_0$ we obtain the same result. Therefore these local isometries coincides on the intersection of their domains (recall that this intersection is convex, hence connected) which includes $\gamma_0(t_1)$. It follows that

$${}_2\overline{\gamma}_0(t_2) =_1 \phi_0(\gamma_0(t_2)) =_1 \phi_1(\gamma_0(t_2))$$

and that the differentials of ${}_1\phi_0$ and ${}_1\phi_1$ coincide at that point.

It now follows in the same way that ${}_2\phi_0$ and ${}_2\phi_1$ coincide at $\gamma_0(t_1)$ and that both local isometries have the same differential there (namely ${}_1I_0$). Therefore the coincide on the intersection of their domains. Hence

$${}_3\overline{\gamma}_0(t_3) =_2 \phi_0(\gamma_0(t_3)) =_2 \phi_1(\gamma_0(t_3))$$

and that the differentials of ${}_2\phi_0$ and ${}_2\phi_1$ coincide at that point. Now one iterates this argument.

We obtain two chains of local isometries, ${}_i\phi_0$ respectively ${}_i\phi_1$ centered around $\gamma_0(t_i)$ respectively $\gamma_1(t_i)$ for $i = 0, \ldots, n$. The domains of ${}_i\phi_0$ and ${}_i\phi_1$ intersect in a non-empty convex set and there these two isometries coincide. In particular, ${}_n\phi_0$ and ${}_n\phi_1$ coincide near $\gamma_0(t_n) = \gamma_1(t_n)$ and by definition ${}_n\phi_0(\gamma_0(t_n)) = \overline{\gamma}_0(t_n) = \overline{\gamma}_1(t_n) =_n \phi_1(\gamma_1(t_n))$.

In order to conclude the proof one has to eliminate the closeness argument. Let γ_0 and γ_1 be two broken geodesics with the same endpoints as in the theorem. Since $\pi_1(M) = \{1\}$, there is a continuous family γ_s interpolating between γ_0 and γ_1 which has fixed endpoints. For a sufficiently fine subdivision the points $\gamma_{s_i}(t_j)$ and $\gamma_{s_{i+1}}(t_j)$ satisfy the closeness assumption made above. Now the local isometry obtained as above near $\gamma_0(l)$ from $\gamma_0(0)$ is the same as the one obtained from γ_{s_1} on a neighbourhood of $\gamma_0(l)$. And the latter is the same as the local isometry obtained from γ_{s_2} on a neighbourhood near $\gamma_0(l)$ etc. Thus the germ at $\gamma_0(l)$ of the local isometry obtained as above is the same for γ_0 and γ_1.

- **Theorem:** Let $M, \overline{M}$ be complete Riemannian manifolds with constant sectional curvature K which are both simply connected. Then these manifolds are isometric. Moreover, the isometry group of M acts transitively on M and for all isometries $I : T_pM \longrightarrow T_{p'}M$ there is an isometry ϕ of M such that $\phi(p) = p'$ and $D_p\phi = I$.

- **Fact:** For $K \in \mathbb{R}$ the following Riemannian manifolds are simply connected, complete and have constant sectional curvature K.

$$M_K = \begin{cases} (\mathbb{R}^n, g_{\text{standard}}) & \text{if } K = 0 \\ \left(S^n_K = \{\|x\|^2 = \frac{1}{K}\} \subset \mathbb{R}^{n+1}, g = g_{\text{standard}}\big|_{S^n_K}\right) & \text{if } K > 0 \\ \left(\{\|x\|^2 < \frac{-4}{K}\} \subset \mathbb{R}^n, g(v,w) = \frac{\sum_i v_i w_i}{1+(K/4)\sum_i x_i^2}\right) & \text{if } K < 0 \end{cases}$$

- **Definition:** A complete Riemannian manifold with constant sectional curvature is called a *space form*.
- **Fact:** When M is a space form with sectional curvature K, then $\pi_1(M)$ acts properly discontinuously on the universal cover M_K by isometries. Conversely, any such action induces a space form on the quotient manifold.
- **Reminder:** A group action of G on M is *properly discontinuous* if for all $p \in M$ there is an open neighbourhood U such that $gU \cap U = \emptyset$ unless $g = 1$.
- **Proposition:** Assume that M^n is a space form with $K = +1$ and n is even. Then M is either the sphere S^n or the projective space $S^n/\{\pm 1\}$.
- **Proof:** If $g \in \text{Isom}(S_n) = \text{O}(2n+1)$ has $\det(g) = 1$, then 1 is an eigenvalue of g (the determinant is the product of all eigenvalues, a complex number ζ is an eigenvalue of g if and only if $\overline{\zeta}$ is an eigenvalue of g). The eigenvector is a fixed point of g. Because the action is properly discontinuous, $g = \text{id}$. If $\det(g) = -1$ then $g^2 = \text{id}$. Hence all eigenvalues of g are ± 1. Thus all eigenvalues have to be -1, and hence $g = -\text{id}$
- **Example:** For n odd there are many space forms with positive sectional curvature. For a prime p and $q \in \{1, \ldots, p-1\}$ consider

$$\begin{aligned} \mathbb{Z}/p\mathbb{Z} \times S^3 \subset \mathbb{C}^2 &\longrightarrow S^3 \\ ([k], (z_1, z_2)) &\longmapsto \left(e^{2\pi i/p} z_1, e^{2\pi q i/p} z_2\right). \end{aligned}$$

  This is a properly discontinuous action by isometries, the quotient is the *lens space* $L(p,q)$ which has a metric of constant positive sectional curvature and $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$. A comprehensive reference is [Wo].

10. Lecture on June, 12 – **First and second variation of energy**

- Let $\gamma : [a,b] \longrightarrow M$ be a piecewise smooth curve and $\alpha : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ a piecewise smooth variation of γ, i.e. α is continuous, here are $t_0 = a < t_1 < \ldots < t_k = b$ such that $\alpha|_{(-\varepsilon,\varepsilon)\times[t_i,t_{i+1}]}$ is smooth and $\alpha(0,t) = \gamma(t)$. The variation vectorfield is $W(t) = \frac{\partial}{\partial s}\big|_{s=0} \alpha(s,t)$. This is a continuous vector field along γ.
- **Definition:** The *energy* $E(\gamma)$ of γ is

$$E(\gamma) = \int_a^b \|\dot\gamma(t)\|^2 dt.$$

- **Remark:** Let $L(\gamma) = \int_a^b \|\dot\gamma(t)\| dt$ be the length of γ. By the Cauchy-Schwarz inequality $L(\gamma)^2 \leq (a-b)E(\gamma)$ with equality if and only if $\|\dot\gamma(t)\|$ is constant. Given p, q in a complete manifold there is a minimal geodesic γ_0 from p to q. All other piecewise smooth curves γ from p to q have $E(\gamma) \geq E(\gamma_0)$.

- **Theorem (First variation of energy):** In the situation above with $[a,b] = [0,1]$

$$\left.\frac{1}{2}\,\frac{\partial E\circ\alpha}{\partial s}\right|_{s=0} = -\sum_{i=1}^{n-1} g\left(W(t_i), \dot\gamma(t_i,+) - \dot\gamma(t_i,-)\right) - \int_0^1 g\left(W(t), \frac{\nabla}{dt}\dot\gamma(t)\right) dt. \tag{8}$$

- Here $\dot\gamma(t_i,+)$ respectively $\dot\gamma(t_i,+)$ is the derivative of γ at t_i from the right respectively the left.
- The proof is a direct computation.
- **Corollary:** If γ is not a smooth geodesic, then there is always a variation vector field W (vanishing at end points) such that a variation α of γ in direction W with fixed end points decreases the energy.
- We will now consider a smooth geodesic γ and a two-parameter variation $\alpha : (-\varepsilon,\varepsilon)^2 \times [0,1] \longrightarrow M$ which is piecewise smooth (defined as above). The variation vector field $W_j = \frac{\partial\alpha}{s_j}$ for $j = 1,2$, here s_1, s_2 are the cartesian coordinates on $(-\varepsilon,\varepsilon)^2$. W_j is a piecewise smooth vector field along the smooth curve γ.
- **Theorem (Second variation of energy):** In the situation above

$$\begin{aligned}\left.\frac{1}{2}\,\frac{\partial^2(E\circ\alpha)}{\partial s_1\partial s_2}\right|_{s_1=s_2=0} &= -\sum_{i=1}^{n-1} g\left(W_2(t), \frac{\nabla}{dt}W_1(t_i,+) - \frac{\nabla}{dt}W_1(t_i,-)\right)\\ &\quad - \int_0^1 g\left(W_2(t), \frac{\nabla^2}{dt^2}W_1(t) + R(W_1(t),\dot\gamma(t))\dot\gamma(t)\right) dt\end{aligned} \tag{9}$$

- The proof is a direct computation starting with (8). You should think that this is familiar, see for example the lecture on May, 22.

11. Lecture on June, 19 – **Minimizing geodesics, more on variation formulas**

- **Lemma:** Let $\gamma : [0,T] \longrightarrow M$ be a geodesic such that $\gamma(0)$ is conjugate to $\gamma(t_0)$ for $t_0 \in (0,T)$. Then γ is not a minimal geodesic from $\gamma(0)$ to $\gamma(T)$. In other words: Minimizing geodesics contain no conjugate points in their interior.
- **Remark:** Theorem 15.1 (Index Theorem) in [Mi] is a similar (but much stronger) statement.
- **Proof:** Let $J : [0,t_0] \longrightarrow TM$ be a non-trivial Jacobi field which vanishes at $\gamma(0)$ and $\gamma(t_0)$. Let $X_0 := \frac{\nabla}{dt}J(t_0)$.

  We extend J by 0 to a continuous, piecewise smooth vector field along γ and we choose a smooth vector field X along γ such that $X(t_0) = 0$ and X vanishes at the endpoints of γ. For $c > 0$ we apply the second variation formula to a variation α_s of γ with variation vector field $cJ - c^{-1}X$ and with fixed endpoints.

By the second variation formula and because J is a piecewise Jacobi field

$$\frac{1}{2}\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha(s)) = -g\left(cJ(t_0) - c^{-1}X(t_0), -c\underbrace{\frac{\nabla}{dt}J(t_0,-)}_{=X_0}\right)$$
$$- \int_0^T g\left(cJ(t) - c^{-1}X(t), -c^{-1}\left(\frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right)\right) dt$$
$$= -g(X_0, X_0) - \frac{1}{c^2}\int_0^T g\left(X, \frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right) dt$$
$$+ \int_0^{t_0} g\left(J, \frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right) dt$$
$$= -\|X_0\|^2 - \frac{1}{c^2}\int_0^T g\left(X, \frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right) dt$$
$$- \int_0^{t_0}\left(g\left(\frac{\nabla}{dt}J, \frac{\nabla}{dt}X\right) + g(R(J,\dot\gamma)\dot\gamma, X)\right) dt$$
$$= -\|X_0\|^2 - \frac{1}{c^2}\int_0^T g\left(X, \frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right) dt$$
$$- \int_0^{t_0} g\left(X, \underbrace{\frac{\nabla^2}{dt^2}J + R(J,\dot\gamma)\dot\gamma}_{=0}\right) dt - \left[g\left(\frac{\nabla}{dt}J, X\right)\right]_{t=0}^{t_0}$$
$$= -2\|X_0\|^2 - \frac{1}{c^2}\int_0^T g\left(X, \frac{\nabla^2}{dt^2}X + R(X,\dot\gamma)\dot\gamma\right) dt.$$

In this computation we (once) used the symmetry properties of the curvature tensor (cf. Lecture of May, 18) and (twice) the fact that ∇ is metric.

Since $J \not\equiv 0$ implies $X_0 \neq 0$ the first summand is negative. The second summand can then be ignored if c is big enough. Thus there is a variation of γ with fixed endpoints such that curves near γ have smaller energy than γ. Hence γ is not minimizing.

- **Remark:** So far we have considered only proper variations of curves/geodesics, i.e. variations with fixed end points. Similar computations as above lead to the following results.
- **First variation of energy:** α_s is a piecewise smooth, maybe not proper variation of a piecewise smooth curve $\gamma : [0,a] \longrightarrow M$ with variation vector field W along γ. Then

$$\frac{1}{2}\left.\frac{d}{ds}\right|_{s=0} E(\alpha_s) = -\int_0^a g\left(W, \frac{\nabla}{dt}\dot\gamma\right) dt - \sum_{t_i \text{ break points}} g\left(W(t_i), \dot\gamma(t_i,+) - \dot\gamma(t_i,-)\right)$$
$$- g(W(0), \dot\gamma(0)) + g(W(a), \dot\gamma(a))$$

- **Second variation of energy:** α_s is a piecewise smooth, maybe not proper variation of a geodesic $\gamma : [0,a] \longrightarrow M$ with variation vector field W along γ.

Then

$$
\begin{aligned}
\frac{1}{2}\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha_s) = &- \int_0^a g\left(W(t), \frac{\nabla^2}{dt^2}W + R(W,\dot\gamma)\dot\gamma\right) dt \\
&- \sum_{t_i \text{ interior break points}} g\left(W(t_i), \frac{\nabla}{dt}W(t_i,+) - \frac{\nabla}{dt}W(t_i,-)\right) \\
&- g\left(\frac{\nabla}{\partial s}\frac{\partial \alpha_s}{\partial s}(0,0), \dot\gamma(0)\right) + g\left(\frac{\nabla}{\partial s}\frac{\partial \alpha_s}{\partial s}(0,a), \dot\gamma(a)\right) \\
&- g\left(W(0), \frac{\nabla}{dt}W(0)\right) + g\left(W(a), \frac{\nabla}{dt}W(a)\right).
\end{aligned} \tag{10}
$$

- **Reformulation of** (10)**:** Using $\frac{d}{dt}g(X,Y) = g\left(\frac{\nabla}{dt}X, Y\right) + g\left(X, \frac{\nabla}{dt}Y\right)$ (10) can be written in a more compact form:

$$
\begin{aligned}
\frac{1}{2}\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha_s) = &\int_0^a \left(g\left(\frac{\nabla W}{dt}, \frac{\nabla W}{dt}\right) - g(R(W,\dot\gamma)\dot\gamma, W)\right) dt \\
&- g\left(\frac{\nabla}{\partial s}\frac{\partial \alpha_s}{\partial s}(0,0), \dot\gamma(0)\right) + g\left(\frac{\nabla}{\partial s}\frac{\partial \alpha_s}{\partial s}(0,a), \dot\gamma(a)\right).
\end{aligned}
$$

- **Remark:** For proper variations the right hand side depends only on the variation vector field. If the variation is not proper, then the precise form of the variation at the end points of γ matters.
- **Definition:** Let $\gamma : [0,a] \longrightarrow M$ be a smooth geodesic and W a piecewise smooth variation vector field (maybe not vanishing at endpoints). Then we write

$$
I_a(W,W) := \int_0^a \left(g\left(\frac{\nabla W}{dt}, \frac{\nabla W}{dt}\right) - g(R(W,\dot\gamma)\dot\gamma, W)\right) dt.
$$

- **Definition:** Let (M,g) be a Riemannian manifold with $\dim(M) > 1$ and R its curvature tensor. Let $E_i, i = 1,\dots,n$ be an orthonormal basis of T_pM and $X, Y, V \in T_pM$.

$$
\begin{aligned}
\mathrm{Ric}(X,Y) &= \frac{1}{n-1}\sum_i g\left(R(X,E_i)E_i, Y\right) \\
\mathrm{Ric}(V) &= \frac{1}{n-1}\sum_i g\left(R(V,E_i)E_i, V\right)
\end{aligned}
$$

are both called the *Ricci curvature* of M. The *scalar curvature* is

$$
\begin{aligned}
\mathrm{scal}(p) &= \frac{1}{n}\sum_i \mathrm{Ric}(E_i) \\
&= \frac{1}{n(n-1)}\sum_{i\neq j} K(E_i, E_j).
\end{aligned}
$$

- **Remark:** $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$ by the symmetry properties of the curvature tensor. Hence the Ricci tensor can be compared with the metric tensor (which is also symmetric and bilinear).

12. Lecture on June, 22 – Theorems of Bonnet-Myers, Weinstein-Synge

- **Theorem (Bonnet-Myers):** Let M be a complete Riemannian manifold such that there is a constant $\rho > 0$ with

$$\mathrm{Ric}_p(v) \geq \frac{\|v\|^2}{\rho^2}. \tag{11}$$

Then $\mathrm{diam}(M) \leq \pi\rho$. In particular, M is bounded and hence compact.
- **Proof:** Let $p, q \in M$ we will show that $d(p,q) = l > \pi\rho$ leads to a contradiction. Let γ be a geodesic from p to q with length l, parametrized by arc length and $e_1, \ldots, e_{n-1}, e_n = \dot\gamma$ parallel orthonormal vector fields along γ. Let $W_i(t) = \sin(\pi t/l)e_i(t)$. Applying the second variation formula to a variation $\alpha_i(s,\cdot)$ with variation vector field W_i we get

$$\frac{1}{2}\left.\frac{d^2}{ds}\right|_{s=0} E(\alpha_j(s,\cdot)) = \int_0^l \sin^2(\pi t/l)\left(\pi^2/l^2 - K(e_n, e_j)\right) dt$$

Summing this over $j = 1, \ldots, n-1$ we get

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) &= \int_0^l \left(\sin^2(\pi t/l)\left((n-1)\frac{\pi^2}{l^2} - (n-1)\mathrm{Ric}_{\gamma(t)}(e_n(t))\right)\right) dt \\ &< (n-1)\int_0^l \sin^2(\pi t/l)\left(\frac{\pi^2}{(\pi\rho)^2} - \frac{1}{\rho^2}\right) dt = 0\end{aligned}$$

by (11) and our assumption $l < \pi\rho$. Then there is j such that $E_j''(0) < 0$ and γ is not the geodesic with minimal energy/length from p to q. This is a contradiction.
- **Corollary:** Let (M,g) be as in the theorem. Then $|\pi_1(M)| < \infty$.
- **Proof:** The universal cover of M with the the pull back metric is complete and satisfies (11). Hence the universal cover of M is compact, i.e. $\pi_1(M)$ is finite.
- **Remark:** The round sphere shows that the bound on the diameter is sharp.
- **Example:** No torus admits a Riemannian metric with positive Ricci-curvature.
- **Theorem (Weinstein-Synge):** Let M be compact, connected, orientable, Riemannian with $K(\sigma) > 0$ for all planes σ in TM. Then every isometry f of M has a fixed point provided f is
  orientation preserving if $\dim(M)$ is even
  orientation reversing if $\dim(M)$ is odd.
- **Proof:** Assume that f has no fixed point. By compactness, there is p_0 such that $0 < l = d(p_0, f(p_0)) \leq d(p, f(p))$ for all $p \in M$. There is a minimal geodesic $\gamma : [0,l] \longrightarrow M$ from p to $f(p)$.

  If $\dot\gamma(l) \neq Df(\dot\gamma(0))$ then the distance between $\gamma(l/2)$ and $f(\gamma(l/2))$ is smaller than l because there is a broken geodesic connecting these points whose length is l. Hence the map $Df : (\dot\gamma(0))^\perp \subset T_{\gamma(0)}M \longrightarrow (\dot\gamma(l))^\perp \subset T_{\gamma(l)}M$ is well defined. The parallel transport P_γ along the geodesic γ defines a linear map between the same spaces.

  By the orientation assumptions on M the map $P_\gamma^{-1} \circ Df$ has 1 as an eigenvalue, hence there is a unit vector V which is orthogonal to $\dot\gamma(0)$ such that $P_\gamma(V) = Df(V)$. Let $V(t)$ be the parallel extension of $V(0) = V$ and consider the variation

$$\begin{aligned}\alpha : (-\varepsilon, \varepsilon) \times [0, l] &\longrightarrow M \\ (s, t) &\longmapsto \exp_{\gamma(t)}(sV(t))\end{aligned}$$

of γ. By the second variation formula we get

$$\begin{aligned}\frac{1}{2} \frac{d^2}{ds^2}\bigg|_{s=0} E(\alpha(s, \cdot)) &= -\int_0^l g(R(V(t), \dot{\gamma}(t))\dot{\gamma}(t), V(t))dt \\ &= -\int_0^l K(V(t), \dot{\gamma}(t))dt < 0\end{aligned}$$

where $K(V(t), \dot{\gamma}(t))$ is the sectional curvature of the plane spanned by $V(t)$ and $\dot{\gamma}(t)$. For small $|s| \neq 0$ the energy of $\alpha(s, \cdot)$ is smaller than the energy of γ.

By the choice of V we have $Df(V(0)) = V(l)$ and hence

$$\alpha(s, l) = \exp_{\gamma(l)}(sDf(V(0))) = f\left(\exp_{\gamma(0)}(sV(0))\right) = f(\alpha(s, 0).$$

Since $L(\alpha(s, \cdot))^2 \leq lE(\alpha(s, \cdot)) < lE(\gamma) = L(\gamma)^2$ the path $\alpha(s, \cdot)$ is shorter than γ. This is a contradiction to the choice of p_0 (and γ).

13. Lecture on June, 26 – **Corollaries of Weinstein-Synge, Index Lemma**

- **Corollary:** Let M be an orientable, compact, connected Riemannian manifold of even dimension with positive sectional curvature. Then M is simply connected.
- **Proof:** The sectional curvature of M is bounded from below by $\kappa > 0$. Then $\mathrm{Ric}_p(v) \geq \kappa \|v\|^2$. The universal cover of M is complete, by the theorem of Bonnet-Myers, the universal cover $\widetilde{M} \longrightarrow M$ of M is compact and the deck transformations are orientation preserving. Hence they all have fixed points, i.e. M itself is simply connected.
- **Remark:** Compare with the Proposition in the lecture of June, 8 (on p. 13).
- **Corollary:** Let M be an orientable, compact, connected Riemannian manifold of odd dimension with positive sectional curvature. Then M is orientable.
- **Proof:** As before, the universal cover of M is compact. Because it is simply connected, it is orientable (one can use parallel transport to coherently orient all tangent spaces using some orientation of $T_p\widetilde{M}$ for some $p \in \widetilde{M}$). All deck transformations preserve the orientation, therefore the quotient of $\widetilde{M}$ by the deck group is orientable, but this quotient is M.
- **Remark:** All space forms with $K > 0$ and odd dimension are orientable.
- **Index-Lemma:** Let $\gamma : [0, a] \longrightarrow M$ be a geodesic such that $\gamma(t), t \in (0, a]$ is not conjugate to $\gamma(0)$ along γ. Let J be a Jacobi field along γ such that $J(0) = 0$ and $\frac{\nabla}{dt}J(0) \perp \dot{\gamma}(0)$ (hence $J(t) \perp \gamma(t)$ for all t). Let V be a piecewise smooth vector field along γ such that $V(0) = 0$ and $V(t_0) = J(t_0)$ for some $t_0 \in (0, a]$. Then

$$I_{t_0}(V, V) = \int_0^{t_0} \left(\left\| \frac{\nabla}{dt} V \right\|^2 - g\left(R(V, \dot{\gamma})\dot{\gamma}, V\right) \right) dt \geq I_{t_0}(J, J)$$

with equality if and only if $V(t) = J(t)$ for all $t \in [0, a]$.

14. Lecture on June, 29 – **Proof of the Index Lemma, Rauch Comparison theorem**

- **Proof of the Index Lemma:** Let $n = \dim(M)$. Pick a basis $J_1, \ldots, J_{n-1}$ of the space of Jacobi fields along γ which vanish at $\gamma(0)$ and are orthogonal to γ. Because there are no conjugate points to $\gamma(0)$ along γ, $J_1(t), \ldots, J_{n-1}(t)$ is a basis of $(\dot\gamma(t))^\perp \subset T_{\gamma(t)}M$ for $0 < t \leq a$. Hence there are functions $f_i : (0,a] \longrightarrow \mathbb{R}$ and constants α_i such that
$$V(t) = \sum_i f_i(t) J_i(t) \qquad\qquad J(t) = \sum_i \alpha_i J_i(t).$$
The functions f_i extend to piecewise smooth functions on the closed interval. For this notice that there are vector fields A_i along γ such that $tA_i(t) = J_i(t)$ and $A_i(0) = \frac{\nabla}{dt} J_i(0)$. $A_1(t), \ldots, A_{n-1}(t)$ is a basis of $\dot\gamma(t)^\perp$ for all t, i.e. $V(t) = \sum_i h_i(t) A_i(t)$ with $h_i(0) = 0$, i.e. there are functions g_i such that $tg_i(t) = h_i(t)$. Hence
$$V = \sum_i h_i A_i = \sum_i t g_i A_i = \sum_i g_i J_i.$$
Therefore g_i is the desired extension of f_i.
On an interval where V is smooth the following identity holds
$$\begin{aligned} &g\left(\frac{\nabla}{dt}V(t), \frac{\nabla}{dt}V(t)\right) - g\left(R(V(t),\dot\gamma(t))\dot\gamma(t), V(t)\right) = \\ &g\left(\sum_i f_i'(t)J_i(t), \sum_j f_j'(t) J_j(t)\right) + \frac{d}{dt} g\left(\sum_i f_i(t) J_i(t), \sum_j f_j(t) \frac{\nabla}{dt} J_j(t)\right). \end{aligned} \tag{12}$$
Integrating this identity (summing over smooth intervals), using $f_i(t_0) = \alpha_i$ and applying the same identity to J one obtains the desired inequality since
$$\int_0^{t_0} g\left(\sum_i f_i'(t) J_i(t), \sum_j f_j'(t) J_j(t)\right) dt \geq 0.$$
In the equality case one has $f_i' \equiv 0$ for $i = 1, \ldots, n-1$, hence $f_i = \alpha_i$ and $V(t) = J(t)$.
To prove (12), one expands $V(t) = \sum_i f_i(t) J_i(t)$ everywhere on both sides of the identity and apply the product rule. Most terms cancel, one of them can be treated using the symmetries of the curvature tensor and the Jacobi equation. Finally, one uses the the fact that for all i, j
$$h(t) = g\left(\frac{\nabla}{dt} J_i(t), J_j(t)\right) - g\left(J_i(t), \frac{\nabla}{dt} J_j(t)\right).$$
vanishes. To see this evaluate at zero and show that $h' \equiv 0$ (this uses again the Jacobi equation and the symmetries of the curvature tensor, c.f. Lecture on May, 18).
- **Rauch comparison theorem:** Let $(M,g), (\widetilde{M}, \widetilde{g})$ be Riemannian manifolds such that $\dim(\widetilde{M}) \geq \dim(M)$. Let $J : [0,a] \longrightarrow TM$ respectively $\widetilde{J} : [0,a] \longrightarrow T\widetilde{M}$ Jacobi fields along the geodesic $\gamma : [0,a] \longrightarrow M$ respectively $\widetilde{\gamma} : [0,a] \longrightarrow \widetilde{M}$ such that
  - $J(0) = 0, \widetilde{J}(0) = 0$,
  - $g(J'(0), \dot\gamma(0)) = \widetilde{g}\left(\widetilde{J}'(0), \dot{\widetilde{\gamma}}(0)\right)$ (we write J' instead of $\frac{\nabla}{dt} J$ etc.)

– $\|J'(0)\| = \|\widetilde{J}'(0)\|$
– $\|\dot\gamma(t)\| = \|\dot{\widetilde{\gamma}}(t)\| \neq 0$.

We assume moreover that no point $\widetilde{\gamma(t)}$ is conjugate to $\widetilde{\gamma}(0)$ along $\widetilde{\gamma}$ and that

$$K(V, \dot\gamma(t)) \leq \widetilde{K}(\widetilde{V}, \dot{\widetilde{\gamma}}(t))$$

for all $V \in \dot\gamma(t)^\perp$ and $\widetilde{V} \in \dot{\widetilde{\gamma}}(t)^\perp$. (Here K resp. $\widetilde{K}$ denote sectional curvatures of planes in TM resp. $T\widetilde{M}$.)

Then $\|\widetilde{J}(t)\| \leq \|J(t)\|$ and there are no conjugate points to $\gamma(0)$ along γ. If there is equality, then $K(J(t), \dot\gamma(t) = \widetilde{K}(\widetilde{J}(t), \dot{\widetilde{\gamma}}(t))$ for $t \in [0, a]$.

15. Lecture on July, 3 – **Proof of the Rauch Comparison theorem, simple applications**

- **Proof:** Using the assumptions on $J(0), \frac{\nabla}{dt}J(0), \widetilde{J}(0), \frac{\widetilde{\nabla}}{dt}\widetilde{J}(0)$ one can reduce to the case when both Jacobi fields are orthogonal to the geodesics they are associated with.

  We define $l(t) = g(J(t), J(t)), \tilde{l}(t) = \tilde{g}(\tilde{J}(t), \tilde{J}(t))$ and we will show l

  First, by l'Hopitals rule we extend $\frac{l(t)}{\tilde{l}(t)}$ to $t = 0$

$$\lim_{t\to 0}\frac{l(t)}{\tilde{l}(t)} = \lim_{t\to 0}\frac{2g(J'(t), J(t))}{2\tilde{g}(\tilde{J}'(t), \tilde{J}(t))} = \lim_{t\to 0}\frac{g(J'(t), J'(t)) + g(J''(t), J(t))}{\tilde{g}(\tilde{J}'(t), \tilde{J}'(t)) + \tilde{g}(\tilde{J}''(t), \tilde{J}(t))}$$
$$= 1$$

where we write J' for $\frac{\nabla}{dt}J$ etc.

We will show $\frac{d}{dt}\frac{l(t)}{\tilde{l}(t)} \geq 0$, or equivalently $\frac{l'(t)}{l(t)} \geq \frac{\tilde{l}'(t)}{\tilde{l}(t)}$. Let $t_0 \in (0, a]$, we put $U(t) = J(t)/\|J(t_0)\|$ (the notation U does not the reflect the dependence on t_0) and

$$\frac{l'(t_0)}{l(t_0)} = 2g(U'(t_0), U(t_0)) = \frac{d}{dt}g(U(t_0), U(t_0)) = \int_0^{t_0}\frac{d^2}{dt^2}g(U(t), U(t))dt$$
$$= 2\int_0^{t_0}\left(g(U'(t), U'(t)) - g(R(U, \dot\gamma)\dot\gamma, U)\right)dt = 2I_{t_0}(U, U). \tag{13}$$

In order to apply the index lemma we want to transplant U to $\widetilde{M}$ and compare the result to $\widetilde{U} = \tilde{J}/\|\tilde{J}(t_0)\|$. Let $E_1 = \dot\gamma/\|\dot\gamma\|, E_2, \ldots, E_n$ be a parallel orthonormal frame along γ such that $E_2(t_0) = U(t_0)$. We also fix $\widetilde{E}_1 = \widetilde{\gamma}/\|\dot{\widetilde{\gamma}}\|, E_2, \ldots, \widetilde{E}_{n+k}$ a parallel orthonormal frame along $\widetilde{\gamma}$ such that $\widetilde{E}_2(t_0) = \widetilde{U}(t_0)$. Let

$$\phi : \{\text{vector fields along } \gamma\} \longrightarrow \{\text{vector fields along } \widetilde{\gamma}\}$$
$$V(t) = \sum_i f_i(t)E_i(t) \longmapsto \sum_i f_i(t)\widetilde{E}_i(t).$$

This map is C^∞ linear, $\frac{\widetilde{\nabla}}{dt}\phi(V) = \phi\left(\frac{\nabla}{dt}V\right)$ and $\tilde{g}(\phi(W_1)(t), \phi(W_2)(t)) = g(W_1(t), W_2(t))$. By the curvature assumption

$$\begin{aligned} I_{t_0}(U,U) &= \int_0^{t_0} \left(g(U'(t), U'(t)) - g(R(U,\dot\gamma)\dot\gamma, U)\right) dt \\ &\geq \int_0^{t_0} \left(\tilde{g}\left(\phi(U)'(t), \phi(U)'(t)\right) - \tilde{g}\left(\widetilde{R}\left(\phi(U), \dot{\tilde\gamma}\right)\dot{\tilde\gamma}, \phi(U)\right)\right) dt \\ &= I_{t_0}(\phi(U), \phi(U)) \\ &\geq I_{t_0}(\widetilde{U}, \widetilde{U}) \end{aligned}$$

where in the last step we used the Index Lemma (by construction $\widetilde{U}$ is a Jacobi field such that $\widetilde{U}(0) = 0$ and $\widetilde{U}(t_0) = U(t_0)$). By the analogous computation to (13) for $\tilde{l}$ we obtain

$$\frac{l'(t_0)}{l(t_0)} \geq \frac{\tilde{l}'(t_0)}{\tilde{l}(t_0)}$$

for (arbitrary) $t_0 \in (0, a]$. This implies $\|J(t)\| \geq \|\widetilde{J}(t)\|$ for all t. In the case of equality $\|J(t_0)\| = \|\widetilde{J}(t_0)\|$ for $t_0 \in (0, a]$ we have $I_{t_0}(\widetilde{U}, \widetilde{U}) = I_{t_0}(\phi(U), \phi(U))$. Then $\phi(U)$ is a Jacobi field and $g(R(U,\dot\gamma)\dot\gamma, U) = \tilde{g}\left(\widetilde{R}\left(\phi(U), \dot{\tilde\gamma}\right)\dot{\tilde\gamma}, \phi(U)\right)$.

- **Applications:** The following two propositions are simple applications of Rauch's theorem.
- **Proposition:** Let M be a complete manifold with sectional curvature $K(\sigma)$ satisfying $0 < L \leq K(\sigma) \leq H$ for constants H, L. Let γ be a geodesic in M. Then the distance d (measured along γ) between $\gamma(0)$ and the next conjugate point along γ satisfies
$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$
- **Remark:** The upper bound can be obtained as in the theorem of Bonnet Myers. For the lower bound compare M with $\widetilde{M} = S^n(H)$, the sphere with sectional curvature H (cf. p. 13).
- **Proposition:** Let $M, \widetilde{M}$ be complete Riemannian manifolds of the same dimension such that for all planes $\sigma \subset TM$ and $\widetilde{\sigma} \subset T\widetilde{M}$ we have $K(\sigma) \leq \widetilde{K}(\widetilde{\sigma})$. Let $r > 0$ be such that
$$\begin{aligned} &\exp_p : B_r(0) \subset TM \longrightarrow M \text{ is a diffeo. onto its image} \\ &\exp_{\tilde{p}} : B_r(0) \subset T\widetilde{M} \longrightarrow \widetilde{M} \text{ is non-singular,} \end{aligned}$$
$c : [0,1] \longrightarrow \exp_p(B_r(0)) \subset M$ a smooth curve and $I : T_pM \longrightarrow T_{\tilde{p}}\widetilde{M}$ an isometry. Define $\widetilde{c}(t) = \exp_{\tilde{p}} \circ I \circ \exp_p^{-1}(c(t))$. Then $l(\widetilde{c}) \leq l(c)$.
- **Remark:** The previous proposition may help to verify that a given r works. Even more favorable circumstances are non-positive curvature and simple connectivity.

16. Lecture on July, 6 – Isometric immersions, Moore's theorem (non-existence of isom. immersions)

- **Remark:** Let $M \subset \overline{M}$ be a submanifold in a Riemannian manifold with Levi-Civita connection $\overline{\nabla}$. Then the Levi-Civita connection of the induced metric on M is

$$\nabla_X Y(p) = \text{pr}\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)$$

where $\overline{X}, \overline{Y}$ are extensions of X, Y to a neighbourhood of p and $\text{pr} : T_p\overline{M} \longrightarrow T_pM$ is the orthogonal projection (similar notation in what follows).

- **Notation:** $B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$.
- **Proposition:** $B(\cdot,\cdot)$ is well defined, symmetric, C^∞-linear.
- **Proof:** elementary, for symmetry use that the connections are torsion free.
- **Definition:** Let $N \in T_pM^\perp$. The *second fundamental form* II_N on T_pM is the quadratic form

$$II_N(X) := H_N(X,X) := \overline{g}(B(X,X),N).$$

  To H_N corresponds the symmetric endomorphism S_N of T_pM defined by $\overline{g}(S_N(X),Y) = H_N(X,Y)$.
- **Proposition:** Let $p \in M, X \in T_pM, N \in T_pM^\perp$. (We denote any extension of N to a local vectorfield orthogonal to M by the same letter.) Then

$$S_N(X) = -\text{pr}\left(\overline{\nabla}_X N\right).$$

- **Proof:** Computation:

$$\begin{aligned}\overline{g}\left(S_N(X),Y\right) &= \overline{g}\left(B(X,Y),N\right) = \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y, N\right)\\ &= \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{Y}, N\right) = \overline{g}\left(Y, -\overline{\nabla}_X N\right).\end{aligned}$$

- **Theorem (Gauß):** Let $X, Y \in T_pM$ be orthonormal. Then the sectional curvatures K and $\overline{K}$ of the X,Y-plane are related as follows:

(14) $$K(X,Y) - \overline{K}(X,Y) = \overline{g}(B(X,X),B(Y,Y)) - \|B(X,Y)\|^2.$$

- **Proof:** Computation:

$$\begin{aligned}K(X,Y) - \overline{K}(X,Y) &= \overline{g}\left(\nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \left(\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Y}\right), X\right)(p)\\ &\quad + \overline{g}\left(\nabla_{[X,Y]}Y - \overline{\nabla}_{[\overline{X},\overline{Y}]}Y, X\right)(p).\end{aligned}$$

  The second summand vanishes because the first entry in $\overline{g}$ is orthogonal to T_pM. Let $E_1, \ldots, E_m$ be a collection of pairwise orthonormal vector fields normal to M on a neighbourhood of p (with $m = \dim(\overline{M}) - \dim(M)$. Then

$$B(X,Y) = \sum_i H_{E_i}(X,Y)E_i.$$

  Express $\overline{\nabla}_{\overline{Y}}\overline{Y}$ in these terms, then express $\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Y}$ and proceed in the same way for the other summand.
- **Theorem (Moore):** Let $\overline{M}$ be complete, simply connected with sectional curvature $\overline{K}(\sigma) \leq b \leq 0$ for all planes $\sigma \subset T\overline{M}$.

  Let $M \subset \overline{M}$ be compact submanifold (maybe immersed) with sectional curvature K such that $K - \overline{K} \leq -b$. Then $\dim(\overline{M}) \geq 2\dim(M)$.
- **Discussion:** The Clifford torus in $\mathbb{R}^4$ is a flat T^2. Closed surfaces in $\mathbb{R}^3$ have positive curvature somewhere. The flat 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ contains a flat T^2, so the π_1-assumption is needed.
- **Proof:** By contradiction. Let $\overline{p} \in \overline{M} \setminus M$ and $p \in M$ such that

(15) $$l = d(\overline{p}, q) \geq d(\overline{p}, p) \text{for all } q \in M.$$

  There is a unique minimal geodesic from $\overline{p}$ to p with speed 1. By the first variation formula γ is perpendicular to T_pM.

Let $V \in T_pM$ of unit length and $c(s)$ a curve in M representing V. Let $\tilde{c}(s) = \exp_{\overline{p}}^{-1}(c(s))$ and

$$f : (-\varepsilon, \varepsilon) \times [0, l] \longrightarrow \overline{M}$$
$$(s,t) \longmapsto \exp_{\overline{p}}\left(\frac{t\tilde{c}(s)}{l}\right).$$

The variation vector field $\overline{V}(t)$ is a Jacobi field along γ since f is a variation through geodesics (with only one fixed endpoint). Moreover, $\overline{V}(0) = 0$ and $\overline{V}(l) = V$.

We now compare $\overline{M}$ with the complete, simply connected Riemannian manifold $\widetilde{M}(b)$ with sectional curvature $b \leq 0$. There are no conjugate points along any geodesic. Since $\overline{K} \leq \widetilde{K} = b$

$$I_l(\overline{V}, \overline{V}) \geq I_l(\widetilde{V}, \widetilde{V})$$

where $\widetilde{V} = \phi(V)$ with ϕ from the proof of the Rauch comparison theorem. We can estimate $I_l(\widetilde{V}, \widetilde{V})$ from below using the unique Jacobi field $\widetilde{J}$ along $\widetilde{\gamma}$ which coincides with $\widetilde{V}$ at $t = 0$ and $t = l$. Recall (6) on p. 8

A computation shows $I_l(\widetilde{J}, \widetilde{J}) > \sqrt{-b}$ and by the Index Lemma we get

$$I_l(\overline{V}, \overline{V}) \geq I_l(\widetilde{V}, \widetilde{V}) \geq I_l(\widetilde{J}, \widetilde{J}) > \sqrt{-b}.$$

By the second variation formula

$$\frac{1}{2}\left.\frac{d^2}{ds^2}\right|_{s=0} E(f(s,\cdot)) = I_l(\overline{V}, \overline{V}) + \overline{g}\left(S_{\dot{\gamma}(l)}\overline{V}(l), \overline{V}(l)\right).$$

and since $E(\gamma_s) \leq E(\gamma_0 = \gamma)$ we know

$$0 \geq I_l(\overline{V}, \overline{V}) + \overline{g}\left(S_{\dot{\gamma}(l)}\overline{V}(l), \overline{V}(l)\right) > \sqrt{-b} + \overline{g}\left(S_{\dot{\gamma}(l)}\overline{V}(l), \overline{V}(l)\right)$$

Recall that $V = \overline{V}(l)$ was arbitrary (of unit length) and that $\overline{g}(S_{\dot{\gamma}(l)}(\overline{V}(l)), \overline{V}(l)) = \overline{g}(B(\overline{V}(l)), \overline{V}(l)), \dot{\gamma}(l))$. Thus

$$\|B(V,V)\| > \sqrt{-b} \tag{16}$$

Now let $V, W \in T_pM$ be orthonormal. Then by the Gauß-formula (14)

$$K(V,W) - \overline{K}(V,W) = \overline{g}(B(V,V), B(W,W)) - \|B(V,W)\|^2 \leq -b. \tag{17}$$

In the next lecture we will see how these inequalities interact with the dimension assumption to finish the proof.

17. Lecture on July, 10 – **Conclusion of the proof of Moore's theorem**

- The following Lemma shows that (16) and (17) contradict each other when $\dim(\overline{M}) < 2\dim(M)$. In the Lemma below $\mathbb{R}^n$ corresponds to T_pM and $\mathbb{R}^k$ corresponds to $T_pM^\perp$. The two quantitative conditions on B correspond to (16) and (17).
- **Lemma (Otsuki):** Let $B : (\mathbb{R}^n, g) \times \mathbb{R}^n \longrightarrow (\mathbb{R}^k, \overline{g})$ be a symmetric bilinear form such that for some $b \leq 0$:

$$\overline{g}(B(V,V), B(W,W)) - \|B(V,W)\|^2 \leq -b$$
$$\|B(V,V)\| > \sqrt{-b}$$

for all g-orthonormal pairs V, W. Then $k \geq n$.

- **Proof:** We assume $k < n$. The function $f : S^{n-1} \longrightarrow \mathbb{R}, f(V) = \|B(V,V)\|^2$ attains its minimum somewhere, we call that point V_0. Because of the second assumption on B in the Lemma we know $B(V_0, V_0) \neq 0$.

  Let $W \in S^{n-1}$ be orthogonal to V_0, so it can be thought of as element of $T_{V_0}S^{n-1}$. Because V is a critical point of f:

$$0 = (D_{V_0} f)(W) = 4\overline{g}(B(V_0, V_0), B(W, V_0)). \tag{18}$$

  Since V_0 is a minimum:

$$\begin{aligned} 0 &\leq (D^2_{V_0} f)(W,W) \\ &= 8\|B(W,V_0)\|^2 - 4\overline{g}(B(V_0,V_0), B(V_0,V_0)) + 4\overline{g}(B(W,W), B(V_0,V_0)). \end{aligned} \tag{19}$$

  Now assume that $k < n$. Then the linear map

$$\begin{aligned} T_{V_0}S^{n-1} &\longrightarrow (B(V_0,V_0))^{\perp} \subset \mathbb{R}^k \\ W &\longmapsto B(V_0, W) \end{aligned}$$

  is well defined by (18) and it has non-trivial kernel if $k < n$ since $(B(V_0,V_0))^{\perp}$ has dimension $k-1$ while $\dim(T_{V_0}S^{n-1}) = n-1$. Let W_0 be a unit vector in that kernel. Using (19) with $V = V_0, W = W_0$ and (18) we get

$$\begin{aligned} 0 &\leq 8\|B(W,V_0)\|^2 + 4\overline{g}(B(W_0,W_0), B(V_0,V_0)) - 4\overline{g}(B(V_0,V_0), B(V_0,V_0)) \\ &= 4\left(\left(\overline{g}(B(W_0,W_0), B(V_0,V_0)) - \underbrace{\|B(V_0,W_0)\|^2}_{=0}\right) - \overline{g}(B(V_0,V_0), B(V_0,V_0))\right) \\ &< -b - (-b) = 0 \end{aligned}$$

  using the assumptions on B in the lemma. Thus the assumption $k < n$ leads to a contradiction.

18. Lecture on July, 13 – **Focal points**

- Today $N \subset M$ is a submanifold.
- **Lemma:** Let $N \subset M$ be a submanifold, $p \in N$, $q \in M$ and $\gamma : [0,l] \longrightarrow M$ a geodesic from p to q parametrized by arc length. Let $f : (-\varepsilon, \varepsilon) \times [0,l] \longrightarrow M$ be a variation of γ such that
  - $\alpha(s) : f(s,0) \in N$,
  - $t \longmapsto f(s,t)$ is a geodesic,
  - $A(s) = \frac{\partial f}{\partial t}(s,0) \in T_{\alpha(s)}N^{\perp}$.

  Let $J(t) = \frac{\partial f}{\partial s}(0,t)$ be the variation vector field of f. Then J is a Jacobi field, $J(0) \in T_pN$ and $\frac{\nabla^M}{dt}J(0) + S_{\dot\gamma(0)}(J(0)) \in T_pN^{\perp}$.
- **Proof:** Only the last statement is non trivial. Let V a local vector field near p in M tangent to N at points in N. Then

$$\begin{aligned} g\left(\frac{\nabla^M}{dt}J(0), V\right) &= g\left(\frac{\nabla^M}{ds}\frac{d}{dt}f(0,0), V\right) = g\left((\nabla^M_{J(0)}A)(0), V\right) \\ &= g\left(\left((\nabla^M_{J(0)}A)(0)\right)^{tang}, V\right) = g\left(-S_{A(0)}J(0), V\right) \end{aligned}$$

  by the definition of S (on p. 22). $V(p) \in T_pN$ is arbitrary.
- **Lemma:** Let $\gamma : [0,l] \longrightarrow M$ be a geodesic and J a Jacobi field along γ such that
  - $\gamma(0) \in N$, $\dot\gamma(0) \in T_pN^{\perp}$,

– $J(0) \in T_pN$, and
– $J'(0) + S_{\dot\gamma(0)}J(0) \in T_pN^\perp$.

Then there is a variation f of γ as in the previous lemma.

- **Proof:** Pick a curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow N$ representing $J(0)$ and a vector field W along α such that $W(0) = \dot\gamma(0)$ and $\frac{\nabla W}{ds}(0) = J'(0)$. Decompose $W(s) = U(s) + V(s)$ with $U(s) \in TN$ and $V(s) \in TN^\perp$. Define

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0, l] &\longrightarrow M \\ (s,t) &\longmapsto \exp_{\alpha(s)}(tV(s)). \end{aligned}$$

This is a variation through geodesics perpendicular to N and $\frac{\partial f}{\partial s}(0,0) = J(0)$. Note that

$$\begin{aligned} \frac{\nabla}{\partial t}\frac{\partial f}{\partial s}(0,0) + S_{\dot\gamma(0)}\left(\frac{\partial f}{\partial s}(0,0)\right) &\in T_pN^\perp \\ \frac{\nabla J}{\partial t}(0) + S_{\dot\gamma(0)}\left(J(0)\right) &\in T_pN^\perp \end{aligned}$$

by the first lemma/assumption. Hence

$$\frac{\nabla J}{dt}(0) - \frac{\nabla}{\partial t}\frac{\partial f}{\partial s} = \frac{\nabla W}{ds}(0) - \frac{\nabla}{\partial s}\frac{\partial f}{\partial t} = \frac{\nabla U}{ds}(0) \in T_pN^\perp$$

U is tangent to N along α. Let X be a vector field orthogonal to N along α. Then

$$0 = \left.\frac{d}{ds}\right|_{s=0} g(U(s), X(s)) = g\left(\frac{\nabla U}{ds}(0), X(0)\right) + g\left(\underbrace{U(0)}_{=0}, \frac{\nabla X}{ds}(0)\right).$$

Therefore $\frac{\nabla}{\partial t}\frac{\partial f}{\partial s}(0,0) = J'(0)$ since X is arbitrary. Since $J(0) = \frac{\partial f}{\partial s}(0,0) = J(0)$ it follows that $\frac{\partial f}{\partial s}(s,t) = J(t)$ (both are Jacobi fields).

- **Definition:** Let $N \subset M$ be a submanifold of M. $q \in M$ is a *focal point* of N if there is a geodesic $\gamma : [0, l] \longrightarrow M$ such that $\gamma(l) = q$ and a non-trivial Jacobi field along γ with the properties as in the previous lemma.
- **Notation:** Let $TN^\perp = \{V \in T_pN^\perp, p \in N\}$ be the normal bundle in M. This is a smooth submanifold of the restriction of TM to N, its dimension is the dimension of M. There is a neighbourhood of the zero section of $TN^\perp$ on which the exponential map is defined. There is no harm in assuming that M is complete, then $\exp^\perp$ is defined on $TN^\perp$.
- **Proposition:** Let $q \in M$ and $N \subset M$ a submanifold and M complete. Then $q \in M$ is a focal point if and only of q is a critical value of $\exp^\perp$.
- **Proof:** Assume that q is a focal point. Let γ, J be a corresponding geodesic/non-trivial Jacobi field, f the variation provided by the previous lemma and $A(s) = \frac{\partial f}{\partial t}(s,0)$. Then the path $s \longmapsto w(s) = (f(s,0), lA(s))$ is a smooth path in $TN^\perp$ representing a tangent vector. $\exp^\perp(f(0,0), lA(0)) = q$ and $(D\exp^\perp)_{w(0)}(w'(0)) = (D\exp)_{lA(0)}(lJ'(0)) = J(l) = 0$. Since w represents a non-zero tangent vector (multiplication with real numbers induces an automorphism of $TN^\perp$ and $s \longmapsto (f(s,0), cA(s))$ maps to non-zero tangent vectors for many $c \in \mathbb{R}$ because $J(c)$ does not always vanish).

  Conversely one easily constructs a geodesic/Jacobi field as in the definition of focal points when q is a critical value of $\exp^\perp$.

19. Lecture on July, 17 – Rauch theorem for submanifolds, Toponogov's theorem, number of generators of $M_1(M)$ with lower curvature bounds

- **Definition:** Let $\gamma : [0,a] \longrightarrow M$ be a geodesic. Then γ is focal point free if there is ε such that γ contains no focal point for $\exp_{\gamma(0)}\left(B_\varepsilon(0) \subset (\dot\gamma(0))^\perp\right)$.
- **Index Lemma for focal points:** Let $\gamma : [0,l] \longrightarrow M$ be a focal point free geodesic and J a Jacobi field along γ which is orthogonal to $\dot\gamma$, $J'(0) = 0$, and V a piecewise smooth vector field along γ and $J(t_0) = V(t_0)$ for $t_0 \in (0,l]$. Then $I_{t_0}(V,V) \geq I_{t_0}(J,J)$ with equality if and only if $V \equiv J$ on $[0,t_0]$.
- **Rauch theorem, vers. 2:** Let $\gamma : [0,a] \longrightarrow M$ and $\overline{\gamma} : [0,a] \longrightarrow \overline{M}$ be two geodesics with Jacobi fields $J, \overline{J}$ such that
  - $\dim(M) \leq \dim(\overline{M})$, and $\|\dot\gamma\| = \|\dot{\overline{\gamma}}\|$,
  - $J'(0) = 0 = \overline{J}'(0)$, $\|J(0)\| = \|\overline{J}(0)\|$, and
  - $g(J(0),\dot\gamma(0)) = \overline{g}(\overline{J}(0),\dot{\overline{\gamma}}(0))$.

  Assume that $K(X,\dot\gamma) \leq \overline{K}(\overline{X},\dot{\overline{\gamma}})$ for all $X \in T_{\gamma(t)}M$ and $\overline{X} \in T_{\overline{\gamma}(t)}\overline{M}$. Then
  $$\|\overline{J}(t)\| \leq \|J(t)\|.$$
  Equality for some $t_0 \in (0,a]$ implies $\overline{K}(\overline{J}(t),\dot{\overline{\gamma}}(t)) = K(J(t),\dot\gamma(t))$ for all $t \in [0,t_0]$.
- **Definition:** A *geodesic triangle* in M is a set of three geodesics $\gamma_1,\gamma_2,\gamma_3$ parametrized by arc-length such that $\gamma_i(l_i) = \gamma_{i+1}(0)$ (indices are taken mod 3) and $l_i + l_{i+1} \geq l_{i+2}$. The angle between $-\dot\gamma_{i+1}(l_{i+1})$ and $\dot\gamma_{i+2}(0)$ is denoted by $0 \leq \alpha_i \leq \pi$. The length of γ_i is l_i.
- **Notation:** $\overline{M}(H)$ is the complete, simply connected Riemannian manifold of dimension 2 and constant sectional curvature (unique up to isometry).
- **Theorem (Toponogov):** Let M be a complete Riemannian manifold with sectional curvature $K_M \geq H \in \mathbb{R}$. If $H \leq 0$ the conditions pertaining to $\pi/\sqrt{H}$ are vacuous.
  - (A) Let $(\gamma_1,\gamma_2,\gamma_3)$ be a geodesic triangle in M such that γ_1,γ_3 are minimal and $l_i \leq \pi/\sqrt{H}$. In $\overline{M}(H)$ there is a geodesic triangle $(\overline{\gamma}_1,\overline{\gamma}_2,\overline{\gamma}_3)$ such that $l(\overline{\gamma}_i) = l_i$ and $\overline{\alpha}_i \leq \alpha_i$ for $i = 1,3$. This triangle in $\overline{M}(H)$ is unique up to isometry except when $l_i = \pi/\sqrt{H}$ for some i.
  - (B) Let γ_1,γ_2 be geodesic segments (param. by arclength) in M such that $\gamma_1(l_1) = \gamma_2(0)$. We denote the angle between $-\dot\gamma_1(l_1)$ and $\dot\gamma_2(0)$ by α (the configuration $(\gamma_1,\gamma_2,\alpha)$ is called a *geodesic hinge*. Assume that γ_1 is minimal and $l_2 \leq \pi/\sqrt{H}$. Let $(\overline{\gamma}_1,\overline{\gamma}_2,\alpha)$ be a geodesic hinge in $\overline{M}(H)$ with $l(\overline{\gamma}_i = l_i$ for $i = 1,2$. Then
  $$d(\gamma_1(0),\gamma_2(l_2)) \leq d(\overline{\gamma}_1(0),\overline{\gamma}_2(l_2)).$$
- The proof of this theorem is lengthy but not very difficult, see for example Chapter 2 of [CE]. Instead of dealing with the proof we present an application of part (B) of Toponogov's theorem (following [Me]).
- **Consequence of Rauch/Toponogov:** Let (a,b,c) be a non-degenerate geodesic triangle. If $K < 0$, then the sum of the interior angles is in $(0,\pi)$.
- **Theorem (Gromov):** Let M be a complete manifold with non-negative sectional curvature and dimension n. There is a number $C(n)$ such that $\pi_1(M)$ can be generated by $C(n)$ elements.

- **Proof:** Consider the universal covering $\mathrm{pr} : \widetilde{M} \longrightarrow M$, the action of $\pi_1(M)$ on $\widetilde{M}$ (by isometries) and $x_0 \in \widetilde{M}$. Define the displacement function
$$\begin{aligned} |\cdot| : \pi_1(M) &\longrightarrow \mathbb{R} \\ g &\longmapsto d(x_0, gx_0). \end{aligned}$$
Since $\widetilde{M}$ is complete, closed balls in M are compact. Hence if there were infinitely many $g \in \pi_1(M)$ with $|g| < R$ for fixed R this would contradict the fact that pr is a covering. Thus we can pick generators of $\pi_1(M)$ according to the following rules.
  1. Pick g_1 so that $0 < |g_1| \leq |g|$ for all $g \neq 1$.
  2. Assume $g_1, \ldots, g_k$ are already fixed. If the smallest subgroup $\langle g_1, \ldots, g_k \rangle \subset \pi_1(M)$ containing $g_1, \ldots, g_k$ is not the entire group, pick g_{k+1} so that $|g_{k+1}| \leq |g|$ for all $g \in \pi_1(M) \setminus \langle g_1, \ldots, g_k \rangle$. Otherwise stop (it is not yet clear that this algorithm really stops).

  Then for $i < j$ we have $l_i = |g_i| \leq |g_j| = l_j$ and
$$l_{ij} = d(g_i x_0, g_j x_0) \geq l_j.$$
Otherwise, $|g_i^{-1} g_j| < l_j$ and we should have chosen $g_i^{-1} g_j$ instead of g_j as j-th generator. For all i fix a minimal geodesic γ_i (parametrized by arc length) in $\widetilde{M}$ from x_0 to $g_i x_0$. Let α_{ij} be the angle between $\dot{\gamma}_i(0)$ and $\dot{\gamma}_(0)$.

  We apply part (B) of Toponogov's theorem with $H = 0$. By the law of cosines $l_{ij}^2 \leq l_i^2 + l_j^2 - 2 l_i l_j \cos(\alpha_{ij})$. Hence if $i < j$
$$\cos(\alpha_{ij}) \leq \frac{l_i^2 + l_j^2 - l_{ij}^2}{2 l_i l_j} \leq \frac{l_i^2 + l_j^2 - l_j^2}{2 l_i^2} = \frac{1}{2} \tag{20}$$
Therefore $\alpha_{ij} \geq \frac{\pi}{3}$. Thus the $\pi/6$-balls around $\dot{\gamma}_i(0), i = 1, \ldots$ are pairwise disjoint. $\dot{\gamma}_i(0)$ are unit vectors in $T_{x_0}\widetilde{M}$. Hence the number of generators is bounded from above by
$$\frac{\mathrm{vol}(S^{n-1})}{2\mathrm{vol}(\pi/6 - \text{ball around a point in } S^{n-1})}$$
(the additional factor in the denominator comes from considering $\pi/6$-balls around $\pm\dot{\gamma}_i(0)$. The denominator can be roughly estimated from below by the volume of the Euclidean $\sin(\pi/6)$-ball in $n-1$-dimensional Euclidean space. The volume of the r-ball in Euclidean $n-1$-space is $r^{n-1} \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}$, the volume of the 1-sphere in Euclidean n-space is $\frac{n r \pi^{n/2}}{\Gamma((n+2)/2)}$, see [Wa] on p. 254f.

  Hence the algorithm above stops after at most $\frac{n\pi^{n/2}\Gamma((n+1)/2)}{2\Gamma((n+2)/2)\pi^{(n-1)/2}(1/2)^{n-1}} = \sqrt{\pi} n 2^{n-1} \frac{\Gamma((n+1)/2)}{2\Gamma((n+2)/2)} = C(n)$ steps.
- **Theorem (Gromov):** Let M be complete, n-dimensional with diameter $\mathrm{diam}(M) = D$ and sectional curvature bounded from below by $-\lambda^2$. Then there is a function $C(n, D, \lambda)$ such that $\pi_1(M)$ can be generated by $C(n, D, \lambda)$ elements.
- **Remark:** Since M is complete and bounded in the above theorem, M is compact (unlike in the previous theorem).
- **Proof:** The following modifications in the proof of the previous theorem are needed:

1. The law of cosines in the plane with sectional curvature $-\lambda^2$ in the above notation is

$$\cosh(\lambda l_{ij}) = \cosh(\lambda l_i)\cosh(\lambda l_j) - \sinh(\lambda l_i)\sinh(\lambda l_j)\cos(\alpha_{ij}).$$

2. Let $\varepsilon > 0$ be arbitrary. Then $\pi_1(M)$ is generated by elements g_i with $|g_i| < 2D + \varepsilon$. To see this consider a loop σ representing some element h of $\pi_1(M)$. Decompose γ into consecutive segments σ_i of length $\leq \varepsilon$. For each segment σ_i pick a path β_i from x_0 to the starting point of σ_i such that the length of β_i is at most D. For the first segment choose β_1 to be the constant path at x_0. Then $\beta_i * \sigma_i * \overline{\beta_{i+1}}$ ($\overline{\beta_{i+1}}$ denotes orientation reversal of β_{i+1}, $*$ the concatination of paths) has length at most $2D+\varepsilon$. The product of these loops represents h. Hence $l_i \leq 2D$ since ε was arbitrary.

A computation similar to (20) above shows that $\alpha_{ij}, i < j$, is bounded from below (independent of i, j, of course) by $\alpha(D, \lambda) > 0$. At the end of the previous proof replace $\pi/3$ by this $\alpha(D, \lambda)$.

20. Lecture on July, 20 – **Closed geodesics, translations**

- **Fact:** Let γ_0, γ_1 represent elements of $\pi_1(M, x_0)$. Then γ_0 and γ_1 are freely homotopic (i.e. homotopic without fixed base point) if and only if γ_0 is conjugate to γ_1 in $\pi_1(M, x_0)$. In particular, γ_0 is freely null homotopic if and only if $\gamma_0 = 1 \in \pi_1(M, x_0)$.
- **Theorem:** Let M be compact and $\gamma_0 : S^1 \longrightarrow M$ represents a nontrivial element of $pi_1(M, x_0)$. Then there is a closed geodesic $\gamma : S^1 \longrightarrow M$ (i.e. $\dot\gamma_1$ is parallel) which is freely homotopic to γ_0.
- **Proof:** Let $d := \inf\{l(\gamma) \,|\, \gamma \text{ a loop freely homotopc to } \gamma_0\}$. Show that there is a sequence of piecewise smooth geodesics γ_j which are parametrized by arclength such that $\lim_j l(\gamma_j) = d$. Use the Arzelà-Ascoli Theorem to extract a convergent subsequence and show that the limit γ is a closed geodesic which is freely homotopic to γ_0.
- **Reference:** For the theorem of Arzelà-Ascoli see Theorem II.3.4 in [We]. For the relationship between the characteristic subgroup of a covering and the group of decktransformations see Chapter 9.7 of [J].
- **Definition:** Let $f : \widetilde{M} \longrightarrow \widetilde{M}$ be an isometry without fixed points. f is a *translation* if there is a geodesic $\gamma : \mathbb{R} \longrightarrow \widetilde{M}$ such that $f(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$. (Then f is a translation along γ and γ is the axis of f.)
- **Proposition:** Let M be compact, $\widetilde{M}$ a covering and $f : \widetilde{M} \longrightarrow \widetilde{M}$ a non-trivial covering transformation. Then f is a translation along a geodesic.
- **Proof:** Let $\widetilde{x} \in \widetilde{M}$ be a basepoint and γ a closed path from $\widetilde{x}$ to $f(\widetilde{x})$. γ projects to a closed loop in M which is homotopically non-trivial and therefore freely homotopic to a closed geodesic. f is then a translation along a certain lift of γ to $\widetilde{M}$.

21. Lecture on July, 24 – **Preissman's theorem, topology of compact manifolds with $K < 0$**

- **Lemma:** Let $\widetilde{M}$ be complete, simply connected with $K < 0$ and f a fixed point free isometry along γ. Then γ is the unique axis of f.

- **Proof:** By contradiction. If there are two axis one finds a rectangle with two opposite sides on the two axis and interior angle sum $= 2\pi$. This leads to a contradiction to the fact that the angle sum of a non-degenerate geodesic triangle is in $(0, \pi)$ when $K < 0$.
- **Lemma:** Let $\widetilde{M}$ be as above, f, g two commuting isometries along axis. Then their axis coincide.
- **Theorem (Preissman):** Let M be a compact manifold with $K < 0$ and A an Abelian subgroup of $\pi_1(M)$. Then A is trivial or infinite cyclic.
- **Proof:** Consider the action of $\pi_1(M)$ on the universal covering. The elements of A act by isometries and properly discontinuously on a geodesic $\gamma(\mathbb{R})$. Moreover, two deck transformations coincide when the coincide on a point. Hence A is cyclic.
- **Corollary:** Let M be compact with $\pi_1(M) \neq \{1\}$. Then $M \times S^1$ does not admit a metric of negative sectional curvature.
- **Lemma:** Let M be a complete manifold with $K < 0$ and $\widetilde{\gamma}$ an axis in the universal cover such that all elements of the Deck group preserve $\widetilde{\gamma}$. Then M is not compact.
- **Proof:** Let x_0 be a point on $\widetilde{\gamma}$ and $\widetilde{\beta}$ a unit speed geodesic through $\widetilde{\beta}(0) = x_0$, orthogonal to $\widetilde{\gamma}$ and parametrized by arc length. Let α_t be a minimal geodesic in M connecting $\mathrm{pr}(\widetilde{\beta}(t))$ to $\mathrm{pr}(x_0)$. Then $l(\alpha_t) \leq t$. If one lifts α_t to $\widetilde{M}$ with starting point $\widetilde{\beta}(t)$ one arrives at a point $g(x_0)$ on $\widetilde{\gamma}$ with g a deck transformation (because all decktransformations preserve $\widetilde{\gamma}$. If $g \neq \mathrm{id}$, then one obtains a non-degenerate geodesic triangle in $\widetilde{M}$ and since $K < 0$

  $$t^2 \geq l(\text{lift of } \alpha_t)^2 > l(\beta([0,t]))^2 + d(x_0, g(x_0))^2 \geq t^2.$$

  Hence $g(x_0) = x_0$ and the lift of α_t is the unique geodesic from $\widetilde{\beta}(t)$ to x_0. Then $l(\alpha_t) = t$ and M cannot be bounded.
- **Corollary:** If M is compact and $K < 0$, then $\pi_1(M)$ is not Abelian, i.e. $\pi_1(M) \not\cong \mathbb{Z}$.
- **Definition:** Let H be a group. Then H is *solvable* if there are subgroups $H = H_0 \supset H_1 \supset \ldots H_{k-1} \supset H_k = \{1\}$ such that H_i is a normal subgroup in H_{i-1} and H_{i-1}/H_i is Abelian.
- **Theorem (Byers):** Let M be as in Preissman's theorem and H a solvable subgroup of $\pi_1(M)$. Then H is trivial or infinite cyclic.
- **Proof:** H_{k-1} is Abelian, hence cyclic and there is an axis γ in the universal covering on which H_{k-1} acts. Let $1 \neq a \in H_{k-2}$ and $1 \neq b \in H_{k-1}$. Then $aba^{-1}b^{-1} \in H_{k-1}$. Hence $aba^{-1}b^{-1}$ preserves γ as set, b also preserves γ, hence aba^{-1} preserves γ as set. Then b preserves $a^{-1}(\gamma)$. Since b has a unique axis $a^{-1}(\gamma) = \gamma$. Thus H_{k-2} acts freely by isometries on γ. Hence H_{k-2} is Abelian, etc.

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