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Introduction to the Calculus of Variations
Homework 3, due date 15.06

Ex. 1 (Banach Alaoglu for (separable) Hilbert spaces)

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space with a Hilbert basis $(e_k)_{k\in\mathbb{N}}$. This means that for any $x \in \mathcal{H}$, we have

$$x = \sum_{k\in\mathbb{N}} \langle e_k, x\rangle \, e_k, \qquad \langle e_k, e_j\rangle = \delta_{k,j}.$$

We want to show that if $\{x_k\} \subset \mathcal{H}$ is a bounded sequence, then it has an accumulation point for the weak convergence.

1. Justify that for all $k \geq 1$, the sequence of numbers $\{\langle e_k, x_n\rangle\}_n$ is bounded.
2. Justify that there exists $\varphi_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $\{\langle e_1, x_{\varphi_1(n)}\rangle\}_n$ converges.
3. Justify that there exists $\varphi_2 : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $\{\langle e_k, x_{\varphi_1\circ\varphi_2(n)}\rangle\}_n$ converge for $k \in \{1, 2\}$.
4. We construct by induction, $\{\varphi_j\}_{1\leq j\leq k}$ strictly increasing, such that $\{\langle e_k, x_{\Phi_k(n)}\rangle\}_n$ converge for all $1 \leq j \leq k$, where $\Phi = \varphi_1 \circ \cdots \circ \varphi_k$. Show that $\{x_{\Phi_n(n)}\}$ has a weak limit in $\mathcal{H}$. One will be careful in proving that the weak limit actually belongs to $\mathcal{H}$.

Ex. 2 (The dual of $L^p(\Omega)$, for $1 < p < \infty$) Here we consider $\Omega \subset \mathbb{R}^n$ a measurable set and we will denote $L^p = L^p(\Omega, \mathbb{R})$ for $1 < p < \infty$. For simplicity we consider real valued function but the proof adapts easily when $\mathbb{R}$ is replaced by $\mathbb{C}$. We want to show that $(L^p)'$ can be identified to $L^{p'}$ where $1/p + 1/p' = 1$. We assume to know that

- bounded sequences in L^p are precompact for the weak-convergence (meaning they have weak limits up to extracting a subsequence)
- and that L^p is uniformly convex, that is that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f, g \in L^p$, with $\|f\|_p \leq 1$, $\|g\|_p \leq 1$ and $\|f - g\|_p > \delta$ then $\|(f + g)/2\|_p \leq 1 - \varepsilon$.

1. Let $\Phi \in (L^p)'$, that is $\Phi : L^p \to \mathbb{R}$, linear and bounded ($\sup_{f\in L^p\setminus\{0\}} |\Phi(f)|/\|f\|_p < +\infty$). Define $K = \{f \in L^p, \quad \Phi(f) = 0\}$. Show that K is a (strongly) closed and vector space.
2. Let $f \in L^p \setminus K$ and define $d = \inf\{\|f - g\|_p, g \in K\}$. Justify that $d > 0$.
3. Let $\{f_n\} \subset K$, such that $\|f - f_n\|_p \to d$ as $n \to \infty$. Define $g_n = (f - f_n)/\|f - f_n\|_p$. Show that $\frac{1}{2}\|g_n + g_m\|_p \to 1$ as $n, m \to \infty$.
4. Deduce from this that $\{g_n\}$ is Cauchy and that there exists $f_0 \in K$ such that $\|f - f_\infty\|_p = d$.

5. Justify that for any $g \in K$, $t \in [-1, 1]$

$$\int |f - f_0|^p \leq \int |f - f_0 + tg|^p$$

and show that for all $g \in K$,

$$\int |f - f_0|^{p-2}(f - f_0)g = 0.$$

6. Let $g \in L^p$ and denote $g = g_1 + g_2$, with

$$g_1 = \frac{\Phi(g)}{\Phi(f - f_0)}(f - f_0),$$

Show that $g_2 \in K$ and deduce from it that there exits $u \in L^{p'}$, independent of g, such that

$$\Phi(g) = \int ug.$$