Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 8 – 22.12.2022

Ex.1.1 (Partial integration)

Let $v \in H^1(\mathbb{R}^n)$, real valued and assume that $-\Delta v = f + g$, with $0 \geq f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. We want to show that

- for all $u \in H^1(\mathbb{R}^n)$, $u(\Delta v) \in L^1(\mathbb{R}^n)$
- $$-\int_{\mathbb{R}^n} u\Delta v = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v. \tag{1}$$

We assume that we already know that if $u \in H^1(\mathbb{R}^n)$ and $u(-\Delta v) \in L^1$ then (1) holds.

1. Justify that it is enough to prove it for u real and non-negative.
2. Assume that $u \geq 0$. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ sucht that $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ on $B(0,2)^c$ and define
$$u_j(x) = \chi(x/i)\min(u(x), j).$$
Show that $u_j \in H^1(\mathbb{R}^n)$ and that $u_j \to u$ in $H^1(\mathbb{R}^n)$ when $j \to \infty$.
3. Explain why (1) is true for u replaced by u_j.
4. Justify (= give a reason or theorem and check its assumptions are satisfied) that
   (a) $\int \nabla u_j \cdot \nabla v \underset{j\to\infty}{\longrightarrow} \int \nabla u \cdot \nabla v$
   (b) $\int u_j f \underset{j\to\infty}{\longrightarrow} \int uf$ (careful here)
   (c) $\int u_j g \underset{j\to\infty}{\longrightarrow} \int ug$
5. Explain why $u(-\Delta v) \in L^1(\mathbb{R}^n)$ and conlcude.

Ex.1.2

Let $\Omega = \mathbb{R}^n \setminus \{0\}$ and define for $\phi \in D(\Omega)$,

$$T(\phi) = \int_{\mathbb{R}^n} \frac{\phi(x)}{|x|^n}\mathrm{d}x.$$

1. Show that $T \in D'(\Omega)$.

2. Find a distribution $\widetilde{T} \in D'(\mathbb{R}^n)$ such that $\widetilde{T}(\phi) = T(\phi)$ for all $\phi \in D(\Omega)$. *Hint: one could consider changing the numerator in $\phi(x)/|x|^n$ to remove the divergence at* 0.

3. Using Theorem 6.14 in the Lieb-Loss, characterize all such $\widetilde{T} \in D(\mathbb{R}^n)$ that coincides with T on $D(\Omega)$.

Theorem 6.14
Let $T, S_1, \ldots, S_N \in D'(\Omega)$ such that

$$\bigcap_{i=1}^{N} \mathcal{N}_{S_i} \subset \mathcal{N}_T.$$

Then, there are $c_1, \ldots, c_N \in \mathbb{C}$ such that

$$T = \sum_{i=1}^{N} c_i S_i. \tag{2}$$

Ex.1.3

Let $(f_j) \subset H^1(\mathbb{R}^n)$ such that $f_j \rightharpoonup f$ in $L^2(\mathbb{R}^n)$ and $\nabla f_j \rightharpoonup g_i$ in $L^2(\mathbb{R}^n)$ ($\rightharpoonup$ means weakly).

1. Recall what $f_j \rightharpoonup f$ in L^2 and $\nabla f_j \rightharpoonup g_i$ in $L^2(\mathbb{R}^n)$ means.

2. Show that $f \in H^1(\mathbb{R}^3)$ and that $\nabla f = g$ in the distributional sense.

Ex.1.4

Let $f \in H^1(\mathbb{R}^n)$. We want to show that for $1 \leq j \leq n$,

$$\int_{\mathbb{R}^n} |\partial_j f|^2 = \lim_{t \to 0} \frac{1}{t^2} \int_{\mathbb{R}^n} |f(x + t\mathbf{e}_j) - f(x)|^2 \mathrm{d}x. \tag{3}$$

1. Justify that $\widehat{\partial_j f}$ makes sense and is in $L^2(\mathbb{R}^n)$. Rewrite

$$\int_{\mathbb{R}^n} |\widehat{\partial_j f}|^2$$

   in two ways.

2. Denote $g(x) = f(x + t\mathbf{e_j})$. Justify that $\widehat{g}$ makes sens and is in $L^2(\mathbb{R}^n)$. Compute $\widehat{g}(k)$ for $k \in \mathbb{R}^n$.

3. Using the Plancherel formula, rewrite

$$\int_{\mathbb{R}^n} |f(x + t\mathbf{e}_j) - f(x)|^2 \mathrm{d}x.$$

   in terms of $\widehat{f}$.

4. Show the limit (3) (give rigorous arguments, if you use a theorem, check the assumptions are satisfied).