Constructive proofs of negated statements

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Abstract

In constructive proofs of negated statements, case distinctions are permitted. We apply this well-known and useful fact in the context of convex analysis.

1 Introduction

Negated statements are often considered 'non-constructive'. When proving a negated statement $\neg b$ (for example, '$\sqrt{2}$ is irrational'), we assume b and derive a contradiction. Such a proof easily carries the label 'proof by contradiction' or 'indirect proof'. However, the proof itself may well be constructive (for example, '$\sqrt{2}$ is irrational' holds constructively). In this note, we discuss a related phenomenon. Suppose that our goal is to prove some negated statement $\neg b$. So we assume b and aim at deriving a contradiction. Let a be any statement. If we can show that, in presence of b, both a and $\neg a$ lead to a contradiction, we are done. This argument, which we call the $(*)$-rule, can be paraphrased as 'when proving a negated statement, finitely many case distinctions are allowed'. Working in the framework of Bishop-style constructive mathematics [3], we list up a few applications of the $(*)$-rule, in the context of convex analysis. Establishing new results of analysis by merely applying basic logic fits in well with the concept of *Proof Theory as Mathesis Universalis*.

2 Automatic continuity of convex functions

Definition 1. *Fix $a, b \in \mathbb{R}$ such that $a < b$. A function $f : [a, b] \to \mathbb{R}$ is*

(I) convex *if*

$$\forall s, t \in [a, b]\ \forall \lambda \in [0, 1]\, (f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda)f(t))\,,$$

(II) sequentially continuous *if* $t_n \to t$ *implies* $f(t_n) \to f(t)$ *for all* t *and* (t_n) *in* $[a,b]$,

(III) pointwise continuous at t *if*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall s \in [a,b] \, (|t-s| \leq \delta \Rightarrow |f(t) - f(s)| \leq \varepsilon) \, ,$$

(IV) pointwise continuous *if it is pointwise continuous at each* $t \in [a,b]$,

(V) uniformly continuous *if*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall s,t \in [a,b] \, (|t-s| \leq \delta \Rightarrow |f(t) - f(s)| \leq \varepsilon) \, .$$

and

(VI) Lipschitz continuous *if there exists* $\gamma \in \mathbb{R}$ *such that*

$$|f(t) - f(s)| \leq \gamma \, |t-s|$$

for all $s, t \in X$.

Note that (VI) $\Rightarrow$ (V) $\Rightarrow$ (IV) $\Rightarrow$ (II). The following lemma can be found in any textbook of convex analysis.

Lemma 1. *Fix real numbers* a, b, c *with* $a < b < c$. *If* $f : [a,c] \to \mathbb{R}$ *is convex, then*

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}.$$

Proof. Note that

$$b = \frac{c-b}{c-a} \, a + \frac{b-a}{c-a} \, c$$

and use the convexity of f. $\square$

Corollary 1. *Fix real numbers* a, b, c, d *with* $a < b \leq c < d$. *Let* $f : [a,d] \to \mathbb{R}$ *be convex. Then we have*

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(d) - f(c)}{d - c}.$$

The following lemma is very easy to prove, but the proof depends heavily on the $(*)$-rule.

Lemma 2. *For each* $f : [a,b] \to \mathbb{R}$, *the following are equivalent:*

a) f *is Lipschitz-continuous*

b) $\exists \alpha, \beta \in \mathbb{R}\, \forall s, t \in [a, b] \left(s < t \Rightarrow \alpha \leq \frac{f(t)-f(s)}{t-s} \leq \beta \right)$

Proof. Clearly, (a) implies (b). Assuming (b), set

$$\gamma := \max\left(|\beta|, |\alpha|\right).$$

For fixed $s, t \in [a, b]$, we can easily show

$$|f(t) - f(s)| \leq \gamma\, |t - s|$$

by case distinction: $s = t$, $s < t$, $s > t$. This is permitted in presence of the $(*)$-rule, since

$$|f(t) - f(s)| \leq \gamma\, |t - s|$$

is the negation of

$$|f(t) - f(s)| > \gamma\, |t - s|.$$

□

Proposition 1. *Fix real numbers* a, b, c, d *with* $a < b \leq c < d$. *Let* $f : [a, d] \to \mathbb{R}$ *be convex. Then* $f : [b, c] \to \mathbb{R}$ *is Lipschitz-continuous.*

Proof. Set $\alpha = \frac{f(b)-f(a)}{b-a}$ and $\beta = \frac{f(d)-f(c)}{d-c}$. For $s < t$ in $[b, c]$, Corollary 1 yields

$$\alpha \leq \frac{f(t) - f(s)}{t - s} \leq \beta.$$

By Lemma 2, $f : [b, c] \to \mathbb{R}$ is Lipschitz-continuous. □

Corollary 2.

(a) Every convex function $f : [0, 1] \to \mathbb{R}$ *is pointwise continuous on* $(0, 1)$*.*

(b) Every convex function $f : \mathbb{R} \to \mathbb{R}$ *is pointwise continuous.*

(c) Every function $f : [0, 1] \to \mathbb{R}$ *which is convex and pointwise continuous at* 0 *and* 1 *is uniformly continuous.*

Proof. The statements (a) and (b) are immediate consequences of Proposition 1. In order to show (c), let $\varepsilon > 0$ and pick $\delta \in (0, 1/2)$ such that

$$|x| \leq \delta \quad \Rightarrow \quad |f(0) - f(x)| \leq \varepsilon/2 \tag{1}$$

and

$$|1 - x| \leq \delta \quad \Rightarrow \quad |f(1) - f(x)| \leq \varepsilon/2$$

for all $x \in [0,1]$. Let $a = \delta/4 > 0$ and $b = 1 - \delta/4 < 1$. By Proposition 1, f is uniformly continuous on $[a,b]$, thus there exists $\widetilde{\delta} > 0$ such that

$$\forall x, y \in [a,b] \ \Big(|x-y| \leq \widetilde{\delta} \quad \Rightarrow \quad |f(x) - f(y)| \leq \varepsilon \Big) .$$

Let $\theta = \min\{\widetilde{\delta}, \delta/4\}$. We prove that

$$\forall x, y \in [0,1] \ \left(|x-y| \leq \theta \quad \Rightarrow \quad |f(x) - f(y)| \leq \varepsilon \right) .$$

Fix $x, y \in [0,1]$. We either have $x < 1 - \delta$ or else $\delta < x$. Without loss of generality, we may assume the former.

<u>Case 1</u>: $x < 3/4 \cdot \delta$

Then $y < \delta$ and (1) yields $|f(x) - f(y)| \leq \varepsilon$.

<u>Case 2</u>: $x > 1/2 \cdot \delta$

Then both x and y are in $[a,b]$, thus $|f(x) - f(y)| \leq \varepsilon$ follows from the choice of $\widetilde{\delta}$.

□

Proposition 2. *Let $f : [0,1] \to \mathbb{R}$ be convex. Equivalent are:*

(a) $\lim_{n\to\infty} f(1/n) = f(0)$ *and* $\lim_{n\to\infty} f(1-1/n) = f(1)$

(b) f is sequentially continuous

(c) f is pointwise continuous

(d) f is uniformly continuous.

Proof. a) $\Rightarrow$ d): By part (c) of Corollary 2, is is sufficient to show that f is pointwise contnuous at 0 and 1. We show pointwise continuity at 0. Let $\varepsilon > 0$ and pick $n_0 \in \mathbb{N} \setminus \{0\}$ such that $|f(1/n) - f(0)| < \varepsilon/2$ for $n \geq n_0$. Let $\delta = 1/n_0$ and suppose $s \in [0, \delta]$. We prove $|f(s) - f(0)| \leq \varepsilon$. As this is the negation of $|f(s) - f(0)| > \varepsilon$ we may apply the $(*)$-rule and it thus suffices to consider the following cases: $s = 0$, $s = \delta$, $0 < s < \delta$. In the first, the assertion is trivial, in the second it holds by choice of n_0. In the third case $0 < s < \delta$ suppose s is rational. Compute $n \geq n_0$ such that $1/(n+1) < s \leq 1/n$. Then $1/(n+1) = \lambda s$ where

$$1 > \lambda = \frac{1}{(n+1)s} \geq \frac{n_0}{n_0+1} \geq \frac{1}{2}.$$

By convexity and $n \geq n_0$

$$f(0) - \varepsilon/2 \leq f(1/(n+1)) \leq \lambda f(s) + (1-\lambda) f(0)$$

and thus

$$f(s) \geq f(0) - \frac{\varepsilon}{2\lambda} \geq f(0) - \varepsilon.$$

Let $\mu = sn_0 \in [a,b]$ such that $s = \mu\delta$, then again by convexity and choice of n_0

$$f(s) \leq \mu f(\delta) + (1-\mu) f(0) \leq f(0) + \mu\frac{\varepsilon}{2} \leq f(0) + \varepsilon.$$

Hence, $|f(s) - f(0)| \leq \varepsilon$. By pointwise continuity of f on $(0,1)$ we conclude that $|f(s) - f(0)| \leq \varepsilon$ for all $s \in [0, \delta]$. □

3 Weak convexity of convex functions

We will use the following fact, see [3, Chapter 2, Proposition 4.6] for a proof.

Lemma 3. *For every uniformly continuous function $f : [a,b] \to \mathbb{R}$ the set $\{f(s) \mid s \in [a,b]\}$ has an infimum.*

A function $f : [a,b] \to \mathbb{R}$ is *weakly convex* if for all $t \in [a,b]$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either

$$\forall s \in [a,b]\,(s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon)$$

or else

$$\forall s \in [a,b]\,(t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon)\,.$$

The notion of weak convexity was introduced in [2] in order to relate convex functions to convex trees. See [1] for more on convex trees. In [2, Remark 3], we have shown that uniformly continuous, convex functions are weakly convex. In view of Proposition 1, which is based on the $(*)$-rule, we can do without uniform continuity.

Proposition 3. *Every convex function $f : [a,b] \to \mathbb{R}$ is weakly convex.*

First, we show a restricted version of Proposition 3.

Proposition 4. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Fix $t \in (a,b)$ and assume that $f(t) > 0$. Then there exists $\varepsilon > 0$ such that either*

$$\forall s \in [a,b]\,(s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon)$$

or

$$\forall s \in [a,b]\,(t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon)\,.$$

Proof. Set

$$r = t + \frac{1}{2}(b - t) \text{ and } \eta = \frac{1}{3}f(t).$$

Case 1: $f(r) < f(t)$

Then $\forall s \in [a,b]\,(s \le t \;\Rightarrow\; f(s) \ge f(t))$.

Case 2: $f(r) > 2\eta$

Then

$$\forall s \in [a,b]\,(r \le s \;\Rightarrow\; f(s) \ge \eta)\,.$$

By Proposition 1 and Lemma 3, we can define

$$\delta = \inf\left\{f(s) \mid t \le s \le r\right\}.$$

Case 2.1: $\delta > 0$

Then $\forall s \in [a,b]\,(t \le s \;\Rightarrow\; f(s) \ge \min(\eta, \delta))$.

Case 2.2: $\delta < f(t)$

Then $\forall s \in [a,b]\,(s \le t \;\Rightarrow\; f(s) \ge f(t))$.

□

Proof of of Proposition 3. We may assume that $a = 0$ and $b = 1$. Fix $t \in [0,1]$ and assume that $f(t) > 0$. We either have $0 < t$ or else $t < 1$. Without loss of generality, we may assume the latter. If $f(1) < f(t)$, we can conclude that

$$\forall s \in [0,1]\,(s \le t \quad \Rightarrow \quad f(s) \ge f(t))\,.$$

So assume that $f(1) > 0$. Without loss of generality, we may assume that $f(1) = 1$ (otherwise, consider the function $g(s) := \frac{f(s)}{f(1)}$). Fix n such that $3/n < f(t)$. If $t > 0$, apply Proposition 4. Now assume that $t < 1/n$.

Case 1: $f(1/n) < 3/n$. Then

$$\forall s \in [0,1]\,(s \le t \quad \Rightarrow \quad f(s) \ge f(t))\,.$$

Case 2: $f(1/n) > 2/n$. Then

$$\forall s \in [0,1]\,(s \le t \quad \Rightarrow \quad f(s) \ge 1/n)\,.$$

□

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