Convexity and unique minimum points

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Abstract

We show constructively that every quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ with at most one minimum point has a minimum point, where C is a convex compact subset of a finite dimensional normed space. Applications include strictly quasi-convex functions, a supporting hyperplane theorem, and a short proof of the constructive fundamental theorem of approximation theory.

1 Introduction

Let (X, d) be a compact metric space. The *infimum* of a uniformly continuous function $f : X \to \mathbb{R}$ is given by

$$\inf f = \inf \{f(x) \mid x \in X\}.$$

An element x of X is a *minimum point* of f if

$$f(x) = \inf f.$$

The function f *has at most one minimum point* if

$$d(x, y) > 0 \;\Rightarrow\; \inf f < f(x) \vee \inf f < f(y)$$

for all $x, y \in X$. In Bishop's constructive mathematics [7, 8, 9], the framework of this paper, the following statements are equivalent:

(I) Brouwer's fan theorem for detachable bars.

(II) Every positive-valued, uniformly continuous function on a compact metric space has positive infimum.

(III) Every uniformly continuous function on a compact metric space which has at most one minimum point has a minimum point.

The equivalence of (I) and (II) was proved in [1, 10] and the equivalence of (I) and (III) was proved in [1, 2, 11]. In [3], we proved (II) for quasi-convex functions whose domain C is a convex compact subset of $\mathbb{R}^m$. *Quasi-convex* means that

$$f(\lambda x + (1-\lambda)y) \leq \max\left(f(x), f(y)\right)$$

for all $\lambda \in [0,1]$ and $x, y \in C$. That result was crucial for the constructive treatment of the *fundamental theorem of asset pricing* in [4] and corresponds to a constructively valid version of the fan theorem, see [5].

In this paper—which is a sequel of [3]—we show (III) for quasi-convex functions whose domain C is a convex compact subset of $\mathbb{R}^m$ (Theorem 1) and generalise this result to finite-dimensional normed spaces (Theorem 2). As applications we obtain a supporting hyperplane theorem (Proposition 1) and a result on strictly quasi-convex functions (Proposition 2). Moreover, we obtain a new short proof of the fundamental theorem of approximation theory (Proposition 3).

2 Unique minimum points

In this section, we prove the following theorem.

Theorem 1. *Let C be a convex compact subset of $\mathbb{R}^m$. Then every quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ with at most one minimum point has a minimum point.*

To this end, let C be a convex compact subset of $\mathbb{R}^m$. Fix a quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ which has at most one minimum point. Without loss of generality, we may assume that $\inf f = 0$. For a subset S of C let $f_{\upharpoonright S}$ denote the restriction of f to S.

Lemma 1. *Fix convex compact subsets A, B of C such that $d(a,b) > 0$ for all $a \in A$ and $b \in B$. Then $\inf(f_{\upharpoonright A}) > 0$ or $\inf(f_{\upharpoonright B}) > 0$.*

Proof. The set $A \times B$ is a convex compact subset of $\mathbb{R}^{2m}$, and the function

$$F : A \times B \to \mathbb{R},\ (a,b) \mapsto \max\left(f(a), f(b)\right),$$

is positive-valued, quasi-convex, and uniformly continuous. By [3, Theorem 1] we can conclude that $\varepsilon := \inf F > 0$. Since

$$\inf(f_{\upharpoonright A}) > 0 \ \vee\ \inf(f_{\upharpoonright A}) < \varepsilon$$

and

$$\inf(f_{\upharpoonright B}) > 0 \ \vee\ \inf(f_{\upharpoonright B}) < \varepsilon,$$

this implies $\inf(f_{\upharpoonright A}) > 0$ or $\inf(f_{\upharpoonright B}) > 0$. □

For $x \in \mathbb{R}^m$ and $j \in \{1, \ldots, m\}$ the j-th component of x is denoted by x_j.

Lemma 2. *For each $\varepsilon > 0$ there exists $C' \subseteq C$ such that*

(i) C' *convex and compact*

(ii) $\inf(f_{\restriction C'}) = 0$.

(iii) $\forall x, y \in C' \, \forall j \, (x_j - y_j < \varepsilon)$

Proof. Consider the first coordinate. The set $D = \{x_1 \mid x \in C\}$ is totally bounded. Set $\iota = \inf D$ and $\eta = \sup D$. If $\eta - \iota < \varepsilon$, set $C' = C$. Now assume that $\iota < \eta$. Define $s = \iota + \frac{1}{3}(\eta - \iota)$ and $t = \iota + \frac{2}{3}(\eta - \iota)$. By [3, Proof of Lemma 4], the sets

$$A = \{x \in C \mid x_1 \leq s\}$$

and

$$B = \{x \in C \mid x_1 \geq t\}$$

are convex and compact. For $a \in A$ and $b \in B$ we have $d(a, b) > 0$. By Lemma 1, we can conclude that

$$\inf(f_{\restriction A}) > 0 \quad \text{or} \quad \inf(f_{\restriction B}) > 0. \tag{1}$$

In the first case, set

$$C'' = \{x \in C \mid x_1 \geq s\}$$

and in the second case set

$$C'' = \{x \in C \mid x_1 \leq t\} .$$

The the set C'' fulfills the properties (i) and (ii). Iterating this, also over the coordinates, we eventually obtain a set C' which fulfills (iii) as well. □

The *diameter* of a compact subset S of X is defined by

$$\text{diam } S = \sup \{d(x, y) \mid x, y \in S\} .$$

By Lemma 2, we can construct a sequence (C_n) of compact subsets of C such that

(a) $\forall n \, (C_{n+1} \subseteq C_n))$

(b) $\lim_{n \to \infty} \text{diam } C_n = 0$

(c) $\forall n \, (\inf(f \restriction C_n) = 0)$.

For each n, fix $x_n \in C_n$ with $f(x_n) < 1/n$. The sequence (x_n) is a Cauchy sequence and its limit is a minimum point of f.

This concludes the proof of Theorem 1.

3 Applications

3.1 Finite-dimensional normed spaces

A normed space V is *finite-dimensional* if there exist $b_1, \ldots, b_m \in V$ such that the linear mapping

$$\kappa : \mathbb{R}^m \to V, \lambda \mapsto \sum_{i=1}^{m} \lambda_i b_i$$

is bijective. (Injective in the sense that $\|\lambda\| > 0$ implies $\|\kappa(\lambda)\| > 0$.) In this case, both κ and its inverse κ^{-1} are uniformly continuous. See [7, 8, 9] for more information on finite-dimensional normed spaces.

In view of the definition of a finite-dimensional normed space, we obtain a straightforward generalisation of Theorem 1.

Theorem 2. *Let C be a convex compact subset of a finite-dimensional normed space. Then every quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ with at most one minimum point has a minimum point.*

3.2 Supporting hyperplanes

A subset C of a normed space X is *strictly convex* if

$$\lambda a + (1 - \lambda) b \in C^\circ$$

for all $a, b \in C$ with $d(a, b) > 0$ and all $\lambda \in \,]0, 1[$. The set C°, the *interior* of C, is defined as usual:

$$x \in C^\circ \Leftrightarrow \exists \varepsilon > 0 \, \forall y \in X \, (d(y, x) < \varepsilon \Rightarrow y \in C) .$$

Lemma 3. *Fix a subset C of X.*

(a) If C is convex and open, then it is strictly convex.

(b) If C is strictly convex and closed, then it is convex.

Proposition 1. *Let C be a compact, strictly convex subset of a finite dimensional normed space V. Let $g : V \to \mathbb{R}$ be a linear function, and v an element of V with $g(v) > 0$. Then the restriction of g to C has a minimum point z.*

Proof. Let f denote the restriction of g to C. Note that linear functions are quasi-convex. Fix a, b with $d(a, b) > 0$. Set $c = (a + b)/2$. Since C is strictly convex, there exists $\delta > 0$ such that $c - \delta \cdot v \in C$. We obtain

$$f(c - \delta \cdot v) < f(c) \leq \max(f(a), f(b)).$$

Thus f has at most one minimum point. By Theorem 2, f has a minimum point. □

In the situation of Proposition 1, the set

$$\{x \in V \mid g(x) = g(z)\}$$

is called a *supporting hyperplane of* C.

3.3 Strictly quasi-convex functions

Let C be a convex subset of a normed space. A function $f : C \to \mathbb{R}$ is *strictly quasi-convex* if

$$f(\lambda x + (1 - \lambda)y) < \max(f(x), f(y))$$

for all $\lambda \in]0, 1[$ and $x, y \in C$ such that $\|x - y\| > 0$.

Lemma 4. *Every strictly quasi-convex function is quasi-convex.*

Proof. Assume that f is strictly quasi-convex. Fix $x, y \in C$ and $\lambda \in [0, 1]$. We have to show

$$f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y)). \tag{2}$$

This is the negation of

$$f(\lambda x + (1 - \lambda)y) > \max(f(x), f(y)).$$

If $\|x - y\| = 0$ or $\lambda \in \{0, 1\}$, the statement (2) holds anyway. In the case $\lambda \in]0, 1[$ and $\|x - y\| > 0$, the statement (2) holds by the quasi-convexity of f. □

Since strictly quasi-convex functions have at most one minimum point, Theorem 2 yields the following proposition.

Proposition 2. *Let C be a convex compact subset of a a finite-dimensional normed space. Then every strictly quasi-convex, uniformly continuous function $f : C \to \mathbb{R}$ has a minimum point.*

3.4 Approximation theory

Let Y be a subset of a normed linear space X. For $a \in X$ let f_a^Y be the function

$$f_a^Y : Y \ni y \mapsto d(y,a).$$

The set Y is *quasiproximinal* if for every $a \in X$ the implication

$$f_a^Y \text{ has at most one minimum point} \quad \Rightarrow \quad f_a^Y \text{ has a minimum point}$$

is valid.

As an immediate consequence of Theorem 2, we obtain the the *constructive fundamental theorem of approximation theory* from [6].

Proposition 3. *Every finite-dimensional subspace V of a normed space X is quasiproximinal.*

Proof. Fix $a \in X$. Set $f = f_a^V$ and suppose that f has at most one minimum point. Fix $b \in V$ such that $d(a,b) > 0$. Set

$$V_0 = \{v \in V \mid d(v,b) \leq 3 \cdot d(a,b)\}.$$

Then V_0 is compact, see [9, Corollary 4.1.7], and convex. The function $f \restriction V_0$ is uniformly continuous, quasi-convex and has at most one minimum point. Theorem 2 implies that $f \restriction V_0$ has a minimum point v_0. Fix $v \in V$. We show

$$f(v_0) \leq f(v),$$

which implies that v_0 is a minimum point of f.
Case 1. If $d(v,b) \leq 3 \cdot d(a,b)$ then $v \in V_0$ and therefore $f(v_0) \leq f(v)$.
Case 2. If $2 \cdot d(a,b) \leq d(v,b)$, we obtain

$$2 \cdot d(a,b) \leq d(v,b) \leq d(v,a) + d(a,b),$$

and therefore

$$f(v_0) = d(v_0,a) \leq d(a,b) \leq d(v,a) = f(v).$$

□

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