

Are law-invariant risk functions concave on distributions?

Beatrice Acciaio*

Gregor Svindland †

December 6, 2013

Abstract

While it is reasonable to assume that convex combinations on the level of random variables lead to a reduction of risk (diversification effect), this is no more true on the level of distributions. In the latter case, taking convex combinations corresponds to adding a risk factor. Hence, whereas asking for convexity of risk functions defined on random variables makes sense, convexity is not a good property to require on risk functions defined on distributions. In this paper we study the interplay between convexity of law-invariant risk functions on random variables and convexity/concavity of their counterparts on distributions. We show that, given a law-invariant convex risk measure, on the level of distributions, if at all, concavity holds true. In particular, this is always the case under the additional assumption of comonotonicity.

Keywords: convexity, law-invariant risk measure, convex order, comonotonicity

MSC 2010 Classification: 46N10, 60E15, 91B30

1 Introduction

The concept of *risk* is nowadays used to describe and capture many phenomena, different both by nature and for the context in which they arise. Therefore, when talking of risk measurement, one should first specify to which framework one refers. A *risk measure* is usually intended as a function f defined on the set \mathcal{X} of the accordingly identified risky elements, associating to each element $x \in \mathcal{X}$ a value $f(x)$ which expresses the riskiness of the “situation” described by x . Here we consider the case where \mathcal{X} is intended to model the set of all possible financial positions. The two most prominent approaches to describe these positions are either by random variables on some probability space, or by probability distributions, usually referred to as lotteries in decision theory. To be in line with the traditional notation, to indicate the elements in \mathcal{X} , we will use X, Y, \dots to denote random variables, and μ, ν, \dots for distributions. One property which is often required on risk measures defined on random variables is the so-called *law-invariance*, meaning that positions sharing the same distribution are equally risky. We have for example in mind expected losses, certainty equivalents, law-invariant coherent and convex risk measures as introduced by Artzner, Delbaen, Eber and Heath [2, 3] and by Föllmer and Schied [8] and Frittelli and Rosazza Gianin [11, 12], deviation measures in the sense of Rockafellar, Uryasev and Zabaranin [16], and quantile-based measures. Under the paradigm of law-invariance, there is a one-to-one relation between risk functions defined on some space

*Corresponding author. The London School of Economics and Political Science. E-mail: b.acciaio@lse.ac.uk

†University of Munich. E-mail: gregor.svindland@math.lmu.de

of random variables and risk functions defined on the corresponding space of distributions. Nevertheless, some care is needed when translating features from one setting to the other. In particular, we will illustrate how the properties of convexity and concavity are not transferable between the two settings. The reason is that the randomization $\alpha\mu_X + (1-\alpha)\mu_Y$ of the lotteries μ_X and μ_Y corresponding to the laws of the random variables X and Y under a probability measure generally differs from the lottery $\mu_{\alpha X + (1-\alpha)Y}$ corresponding to the law of the state-wise convex combination $\alpha X + (1-\alpha)Y$. A prominent example is the Value at Risk, which is not quasi-convex on random variables, while the corresponding risk measure on distributions is. On this matter we recall that Frittelli, Maggis and Peri [10] prove that convexity is not compatible with translation invariance on the space of probability distributions, which is the reason why they require quasi-convexity. However, note that concavity is compatible with translation invariance.

On the level of random variables it is well understood that convex combinations corresponding to a diversification in the portfolio should reduce the overall risk. However, on the level of distributions a convex combination corresponds to an additional randomization. Indeed, $\alpha\mu + (1-\alpha)\nu$ can be interpreted as the sampling of a lottery between μ and ν , depending on the outcome of a simultaneous independent toss with probabilities α and $(1-\alpha)$. Hence, we have an additional factor of risk coming from the toss. Thus requiring that a risk function ϕ on distributions be (quasi-) convex has not the diversification interpretation, and is not necessarily a natural property.

This paper is meant to analyse the interplay between convexity of law-invariant risk functions on random variables and convexity/concavity of their counterparts on distributions. We show that, given a law-invariant convex risk measure, on the level of distributions, if at all, concavity holds true. This is for example always the case under the additional assumption of comonotonicity; see Section 3. Under the assumption of translation invariance, Frittelli, Maggis and Peri [10] and Drapeau and Kupper [6] study quasi-convex risk measures over lotteries, as counterparts to convex risk measures on random variables, providing robust representations. Our analysis shows that replacing quasi-convexity on distributions by concavity (these properties do not exclude each other) could probably be more appropriate. Of course this would also provide nicer robust representations. Our results are illustrated by several examples using well-know risk functions.

The remainder of the paper is organized as follows. In Section 2 we specify our setting and prove how, in general, a risk measure ϕ defined on distributions is not convex (Proposition 2.1). To the contrary, under positive homogeneity we show a weak form of concavity for ϕ (Proposition 2.4). Moreover, we provide a dual characterization of concavity of ϕ . In Section 3 we work under the assumption of comonotonicity, and show that in this case concavity of ϕ is completely determined by the preservation of the convex order (Proposition 3.5).

2 Setup and first results

In all that follows, we work on a non-atomic standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that many of our results also hold on more general probability spaces. However, the assumption that the probability space is standard and non-atomic is in particular useful in Section 2.3 where we present a dual characterization of the concavity of ϕ based on results from [7, 9, 17] which require our assumptions on the probability space. For any random variables X, Y and a distribution μ , we write $X \stackrel{d}{=} Y$ to indicate that X and Y are equally distributed under

\mathbb{P} , and $X \sim \mu$ to indicate that the distribution of X under \mathbb{P} is μ . We do our analysis on $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$. It will be obvious from the results and proofs that most of them carry over to any law-invariant Banach spaces of (equivalence classes of) random variables, such as the $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ spaces for $p \in [1, \infty]$ equipped with the p -norm $\|\cdot\|_p := E[|\cdot|^p]^{1/p}$ for $p \in [1, \infty)$ and $\|\cdot\|_\infty := \inf\{m \in \mathbb{R} \mid \mathbb{P}(|\cdot| \geq m) = 0\}$ for $p = \infty$.

Let $\Phi : L^1 \rightarrow [-\infty, +\infty]$ be a law-invariant function, that is, $\Phi(X) = \Phi(Y)$ for all $X, Y \in L^1$ with $Y \stackrel{d}{=} X$. Then Φ induces a function $\phi : \mathcal{M}_1 \rightarrow [-\infty, +\infty]$ on the set \mathcal{M}_1 of all Borel probability measures on \mathbb{R} with finite first moment by

$$(2.1) \quad \phi(\mu) = \Phi(X) \quad \text{for any } X \text{ with } X \sim \mu.$$

Vice versa, any function $\phi : \mathcal{M}_1 \rightarrow [-\infty, +\infty]$ induces a law-invariant function on L^1 by (2.1). Our main aim is to study how convexity properties of Φ on L^1 are related to convexity properties of ϕ on \mathcal{M}_1 . We recall that a function $f : \mathcal{X} \rightarrow [-\infty, +\infty]$, where $\mathcal{X} = L^1$ or $\mathcal{X} = \mathcal{M}_1$, is quasi-convex if $f(\alpha x + (1 - \alpha)y) \leq f(x) \vee f(y)$ and convex if $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $\alpha \in (0, 1)$ and all $x, y \in \mathcal{X}$. The function f is (quasi-) concave if $-f$ is (quasi-) convex. Throughout this paper we will always treat the case that Φ is (quasi-) convex. This corresponds to the risk measurement point of view. The corresponding results for utility criteria, i.e. (quasi-) concave Φ , are then obvious. Φ is said to be lower semicontinuous (lsc) if the level sets $E_k := \{X \in L^1 \mid \Phi(X) \leq k\}$, $k \in \mathbb{R}$, are all closed in $(L^1, \|\cdot\|_1)$. This is equivalent to $\Phi(X) \leq \liminf_{n \rightarrow \infty} \Phi(X_n)$ whenever $(X_n)_{n \in \mathbb{N}} \subset L^1$ is a sequence converging to X in $(L^1, \|\cdot\|_1)$. It is shown in [7] that, for a law-invariant convex risk function Φ on L^1 , lsc (in contrast to continuity) is a natural property. Indeed, in most examples that we have in mind, such as convex risk measures, $\Phi|_{L^\infty}$ is lsc (even continuous) with respect to the convergence in the $\|\cdot\|_\infty$ -norm. Then it is shown in [7] that there exists a canonical lsc extension of $\Phi|_{L^\infty}$ to L^1 . Thus requiring lsc basically means that Φ should equal its canonical lsc extension from L^∞ to L^1 . As a lsc convex function which takes the value $-\infty$ cannot take any finite value, and as the value $-\infty$ does not make much sense from a risk perspective, we reduce our studies to the case $\Phi : L^1 \rightarrow (-\infty, +\infty]$. Note that an infinitely risky position makes sense, as it can be interpreted to be so bad that it cannot be hedged at any finite cost. Whereas to assume that there is a position which is infinitely good (risk $-\infty$) is clearly not reasonable since it would imply that we could withdraw any amount of money and still face no risk.

For the remainder of this paper $\Phi : L^1 \rightarrow (-\infty, +\infty]$ will always be a law-invariant function and the relation between the function Φ and $\phi : \mathcal{M}_1 \rightarrow (-\infty, +\infty]$ will be given by (2.1).

Prominent examples of law-invariant risk functions are

- expected losses $E[l(-X)]$ or certainty equivalents $l^{-1}(E[l(-X)])$ where the loss function $l : \mathbb{R} \rightarrow \mathbb{R}$ is convex and strictly increasing;
- convex risk measures, that is, convex functions which also are antitone (i.e. $X \geq Y$ \mathbb{P} -a.s. implies $\Phi(X) \leq \Phi(Y)$) and translation invariant: $\Phi(0) \in \mathbb{R}$ and $\Phi(X + x) = \Phi(X) - x$ for all $x \in \mathbb{R}$;
- coherent risk measures, that is, convex risk measures which are also positively homogeneous: $\Phi(tX) = t\Phi(X)$ for all $t \geq 0$;
- quantile-based risk measures;
- deviation measures, that is, convex, positively homogeneous functions which are positive on non-constants, and constant-absorbing: $\Phi(X + x) = \Phi(X)$ for all $x \in \mathbb{R}$.

2.1 In general ϕ is not convex

Let us recall the definition of the convex order \succeq : for $X, Y \in L^1$,

$$(X \succeq Y) \iff (E[c(X)] \geq E[c(Y)] \text{ for every convex function } c: \mathbb{R} \rightarrow \mathbb{R}).$$

Note that due to Jensen's inequality the expectations $E[c(X)]$ and $E[c(Y)]$ are well defined, possibly taking the value $+\infty$. It is proved in [4, Theorem 4.1] that if Φ is lsc, convex and law-invariant, then it is automatically *preserving the convex order* (see also [17]), which means that $\Phi(X) \geq \Phi(Y)$ whenever $X \succeq Y$. Hence, when considering convex Φ , it is also natural to assume that Φ is \succeq -preserving. As $X \succeq Y$ only depends on the distributions of X and Y , we may also see \succeq as an order on \mathcal{M}_1 . Clearly, Φ is \succeq -preserving if and only if ϕ is \succeq -preserving. Another natural request on a risk function (on discounted payoffs) is to be *invariant on constants*, that is, $\Phi(x) = -x$ for every $x \in \mathbb{R} \subset L^1$. Indeed, when there is no randomness involved, i.e., when we are facing a deterministic scenario $X \sim \delta_x$ (where δ_x denotes the Dirac-measure), it seems natural to assign it the value x , thus the risk $-x$. For instance translation invariant risk functions are, apart from the constant $\Phi(0)$, invariant on constants.

Having the mentioned typical properties of risk functions in mind, the message of the following proposition is that convexity is not a good property to require on ϕ .

Proposition 2.1. *Let Φ be lsc, \succeq -preserving, invariant on constants (resp. translation invariant), and let ϕ be convex. Then $\Phi(X) = -E[X]$, i.e. $\phi(\mu) = -\int x d\mu$ (resp. $\Phi(X) = \Phi(0) - E[X]$).*

Proof. We only prove the case when Φ is invariant on constants. The proof in case of translation invariance similarly follows. Recall that $E[X] \preceq X$. As Φ is \succeq -preserving and invariant on constants, we have $\Phi(X) \geq \Phi(E[X]) = -E[X]$. For $x_1, \dots, x_n \in \mathbb{R}$, $\alpha_1, \dots, \alpha_n > 0$ such that $\sum_{i=1}^n \alpha_i = 1$, and a partition $A_1, \dots, A_n \in \mathcal{F}$ of Ω with $\mathbb{P}(A_i) = \alpha_i$, we have that $\sum_{i=1}^n x_i 1_{A_i} \sim \sum_{i=1}^n \alpha_i \delta_{x_i}$ and

$$\begin{aligned} \phi\left(\sum_{i=1}^n \alpha_i \delta_{x_i}\right) &= \Phi\left(\sum_{i=1}^n x_i 1_{A_i}\right) \geq -E\left[\sum_{i=1}^n x_i 1_{A_i}\right] \\ &= -\sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \alpha_i \Phi(x_i) = \sum_{i=1}^n \alpha_i \phi(\delta_{x_i}). \end{aligned}$$

As ϕ is convex, the inequality in the computation must be an equality. Hence, $\phi(X) = -E[X]$ for all simple random variables X . Now let $X \in L^1$ be arbitrary and choose a sequence of simple random variables $(X_n)_{n \in \mathbb{N}}$ converging to X in $(L^1, \|\cdot\|_1)$. Then

$$-E[X] = \lim_{n \rightarrow \infty} -E[X_n] = \liminf_{n \rightarrow \infty} \Phi(X_n) \geq \Phi(X) \geq -E[X],$$

where the first inequality follows by lsc of Φ . □

Corollary 2.2. *The only normalized ($\Phi(0) = 0$) lsc law-invariant convex risk measure which is convex on distributions is $\Phi(X) = -E[X]$.*

Note also that if Φ is a deviation measure, then $X \mapsto E[-X] + \Phi(X)$ is translation invariant. Hence, supposing that the deviation measure Φ is lsc, \succeq -preserving and that ϕ is convex, Proposition 2.1 implies that $\Phi(X) \equiv \Phi(0)$.

As mentioned in the introduction, a typical requirement when defining risk measures on \mathcal{M}_1 is quasi-convexity. The following Lemma 2.3 shows that starting from a quasi-convex and \succeq -preserving ϕ on \mathcal{M}_1 also Φ must be quasi-convex, and thus even convex if ϕ and thus also Φ is translation invariant (see e.g. [9, p. 178]). This further justifies our approach to study the convexity/concavity problem through the lens of a (quasi-) convex Φ . In order to prove this result we fix two independent sub- σ -algebras $\mathcal{G}_0, \mathcal{F}_0$ of \mathcal{F} such that $(\Omega, \mathcal{G}_0, \mathbb{P})$ and $(\Omega, \mathcal{F}_0, \mathbb{P})$ are non-atomic standard probability spaces. Then, for any $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ there exist \mathcal{G}_0 -measurable \tilde{X}, \tilde{Y} such that $(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y)$. Moreover, for any $\alpha \in (0, 1)$ there is $A \in \mathcal{F}_0$ such that $\mathbb{P}(A) = \alpha$, and thus if X and Y are \mathcal{G}_0 -measurable with $X \sim \mu, Y \sim \nu$, then

$$(2.2) \quad 1_A X + 1_{A^c} Y \sim \alpha \mu + (1 - \alpha) \nu \quad \text{and} \quad \alpha X + (1 - \alpha) Y = E[1_A X + 1_{A^c} Y \mid \mathcal{G}_0].$$

Relation (2.2) will turn out to be useful throughout the paper.

Lemma 2.3. *If ϕ is (quasi-)convex and \succeq -preserving, then Φ is (quasi-)convex.*

Proof. Let ϕ be quasi-convex. Fix $X, Y \in L^1$ and $\alpha \in [0, 1]$, and let (\tilde{X}, \tilde{Y}) be a \mathcal{G}_0 -measurable random vector such that $(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y)$, and thus also $\alpha X + (1 - \alpha) Y \stackrel{d}{=} \alpha \tilde{X} + (1 - \alpha) \tilde{Y}$. Moreover, let $A \in \mathcal{F}_0$ be such that $\mathbb{P}(A) = \alpha$. Using that for any random variable X we have $E[X \mid \mathcal{G}_0] \preceq X$, and that with ϕ also Φ must be \succeq -preserving, we obtain

$$\begin{aligned} \Phi(\alpha X + (1 - \alpha) Y) &= \Phi(\alpha \tilde{X} + (1 - \alpha) \tilde{Y}) = \Phi(E[1_A \tilde{X} + 1_{A^c} \tilde{Y} \mid \mathcal{G}_0]) \\ &\leq \Phi(1_A \tilde{X} + 1_{A^c} \tilde{Y}) = \phi(\alpha \mu + (1 - \alpha) \nu) \\ &\leq \phi(\mu) \vee \phi(\nu) = \Phi(X) \vee \Phi(Y). \end{aligned}$$

The case of ϕ convex follows in a similar way. □

In particular, this implies that the quasi-convex risk measures on distributions studied in [10] either correspond to quasi-convex risk measures on random variables or they are not \succeq -preserving.

Consider the Value at Risk at level λ , i.e.

$$\Phi(X) = \text{VaR}_\lambda(X) = \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m \leq 0) \leq \lambda\}, \quad \text{where } \lambda \in (0, 1).$$

It is easily verified that the corresponding risk measure ϕ on distributions is quasi-convex (see [10]). Since it is also well-known that Φ is not quasi-convex, Lemma 2.3 implies that VaR_λ cannot preserve \succeq . So in particular there are payoff profiles X, Y such that every expected utility agent prefers X to Y , but under VaR_λ the profile Y is strictly less risky than X . Indeed, recall that a utility function is a concave and increasing function. Then note that $X \succeq Y$ implies $X \preceq_{uni} Y$, where the uniform order \preceq_{uni} is defined as follows: for $X, Y \in L^1$,

$$(X \preceq_{uni} Y) \iff (E[u(X)] \geq E[u(Y)] \text{ for every utility function } u : \mathbb{R} \rightarrow \mathbb{R}).$$

(This order is well defined since the expectations $E[u(X)], E[u(Y)]$ are well defined due to Jensen's inequality, possibly taking the value $-\infty$.) From this we see that since VaR_λ does not preserve \succeq , it cannot be \preceq_{uni} -reverting ($X \preceq_{uni} Y$ implies $\Phi(X) \geq \Phi(Y)$). The same is true for any other risk measure which does not preserve \succeq .

2.2 A weak form of concavity for ϕ

The next proposition shows that a weak form of concavity holds for ϕ when Φ is convex and positively homogeneous. In what follows we write $A \perp X$ to indicate that 1_A is independent of X .

Proposition 2.4. *Let Φ be lsc, convex, and positively homogeneous, then*

$$(2.3) \quad \frac{\Phi(1_A X)}{\mathbb{P}(A)} \geq \Phi(X) \quad \text{for all } X \in L^1 \text{ and } A \perp X \text{ with } \mathbb{P}(A) > 0.$$

In the positively homogeneous case (where automatically $\Phi(0) = \phi(\delta_0) = 0$), condition (2.3) can be seen as a weak form of concavity of ϕ since it implies that $\phi(\alpha\mu + (1-\alpha)\delta_0) \geq \alpha\phi(\mu)$ for all $\mu \in \mathcal{M}_1$ and $\alpha \in [0, 1]$. However, Example 2.8 shows that there are convex and positively homogeneous Φ which are not ‘truly’ concave.

Proof. Being law-invariant, lsc, convex, and positively homogeneous, Φ may be represented as

$$\Phi(X) = \sup_{Z \in \mathcal{Q}} E[ZX] = \sup_{Z \in \mathcal{Q}|X} E[ZX],$$

where $\mathcal{Q} \subset L^1$ is a $\sigma(L^1, L^\infty)$ -closed convex set (see e.g. [5]), and $\mathcal{Q} | X := \{E[Z | X] | Z \in \mathcal{Q}\}$. Note that by law invariance it follows that $\mathcal{Q} | X \subset \mathcal{Q}$; see [13, Lemma 4.2]. Thus, if $A \perp X$ we obtain

$$\Phi(1_A X) = \sup_{Z \in \mathcal{Q}} E[1_A ZX] \geq \sup_{Z \in \mathcal{Q}|X} E[1_A ZX] = \mathbb{P}(A) \sup_{Z \in \mathcal{Q}|X} E[ZX] = \mathbb{P}(A)\Phi(X).$$

□

From the proof of Proposition 2.4 it is also clear that ϕ is actually concave in case

$$Z1_A + \tilde{Z}1_{A^c} \in \mathcal{Q} \quad \text{for all } Z \in \mathcal{Q} | X \text{ and } \tilde{Z} \in \mathcal{Q} | Y \text{ with } X, Y \in L^1.$$

Note that (2.3) is a reasonable property because conditional on A the random variable $\hat{X} = X1_A$ has the same distribution as X . Thus the conditional risk $\frac{\Phi(1_A X)}{\mathbb{P}(A)}$ should be at least $\Phi(X)$. Clearly, for $\Phi(\cdot) = E[\cdot]$ equality holds in (2.3).

2.3 Dual characterization of the concavity of ϕ

In this section we provide an alternative way to check concavity of ϕ by looking at the dual side. This approach turns out to be useful in the case when the Fenchel-Legendre transform Φ^* of Φ is easier to study than Φ itself. In what follows, $q_X(s) := \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq s\}$, $s \in (0, 1)$, denotes the (left continuous) quantile function of a random variable X .

Proposition 2.5. *Let Φ be lsc and convex. Then ϕ is concave if and only if $\phi^* : \mathcal{M}_\infty \rightarrow (-\infty, \infty]$ is convex, where*

$$(2.4) \quad \phi^*(\mu) := \Phi^*(Z) = \sup_{X \in L^1} \int_0^1 q_X(t)q_Z(t)dt - \Phi(X) \quad \text{for } Z \sim \mu$$

and $\mathcal{M}_\infty := \{\mu \in \mathcal{M}_1 | \mu \text{ has compact support}\}$.

Proof. The equality in (2.4) for the dual function Φ^* of Φ holds by law-invariance, like in [9, Theorem 4.59]. In particular, Φ^* is itself a law-invariant convex function on L^∞ .

Being a law-invariant, lsc, convex function, Φ may be represented as

$$\begin{aligned}\Phi(X) &= \sup_{Z \in L^\infty} E[ZX] - \Phi^*(Z) = \sup_{Z \in L^\infty(\sigma(X))} E[ZX] - \Phi^*(Z) \\ &= \sup_{h \in M_b} E[h(X)X] - \Phi^*(h(X)),\end{aligned}$$

where M_b denotes the set of measurable and bounded $h : \mathbb{R} \rightarrow \mathbb{R}$. Here we used that $E[Z | X] \preceq X$ and that the lsc convex law-invariant function Φ^* is automatically \succeq -preserving. Similarly, using [7, Theorem 2.2] for the second equality, we also derive the following representation for Φ^* :

$$\begin{aligned}\Phi^*(Z) &= \sup_{X \in L^1} E[ZX] - \Phi(X) = \sup_{X \in L^\infty} E[ZX] - \Phi(X) \\ &= \sup_{X \in L^\infty(\sigma(Z))} E[ZX] - \Phi(X) = \sup_{h \in M_b} E[Zh(Z)] - \Phi(h(Z)), \quad Z \in L^\infty.\end{aligned}$$

Now suppose that ϕ is concave and let $\mu, \nu \in \mathcal{M}_\infty$ and $\alpha \in [0, 1]$. Choose the random variables X, Y and $A \in \mathcal{F}_0$ as in (2.2). Then

$$\begin{aligned}\phi^*(\alpha\mu + (1 - \alpha)\nu) &= \Phi^*(1_A X + 1_{A^c} Y) \\ &= \sup_{h \in M_b} E[(1_A X + 1_{A^c} Y)h(1_A X + 1_{A^c} Y)] - \Phi(h(1_A X + 1_{A^c} Y)) \\ &= \sup_{h \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yh(Y)] - \Phi(h(X)1_A + h(Y)1_{A^c}) \\ &\leq \sup_{h \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yh(Y)] - \alpha\Phi(h(X)) - (1 - \alpha)\Phi(h(Y)) \\ &\leq \alpha\Phi^*(X) + (1 - \alpha)\Phi^*(Y) = \alpha\phi^*(\mu) + (1 - \alpha)\phi^*(\nu),\end{aligned}$$

where we used the concavity of ϕ in the first inequality since

$$h(X)1_A + h(Y)1_{A^c} \sim \alpha \text{law}(h(X)) + (1 - \alpha) \text{law}(h(Y))$$

where $\text{law}(Z)$ denotes the distribution of a random variable Z .

Suppose that ϕ^* is convex and let $\mu, \nu \in \mathcal{M}_1$, $\alpha \in [0, 1]$, $X, Y \in L^1$ and $A \in \mathcal{F}_0$ be as in (2.2). In this case we have

$$\begin{aligned}\phi(\alpha\mu + (1 - \alpha)\nu) &= \Phi(1_A X + 1_{A^c} Y) \\ &\geq \sup_{h, g \in M_b} E[(1_A X + 1_{A^c} Y)(1_A h(X) + 1_{A^c} g(Y))] - \Phi^*(1_A h(X) + 1_{A^c} g(Y)) \\ &\geq \sup_{h, g \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yg(Y)] - \alpha\Phi^*(h(X)) - (1 - \alpha)\Phi^*(g(Y)) \\ &= \alpha\Phi(X) + (1 - \alpha)\Phi(Y) = \alpha\phi(\mu) + (1 - \alpha)\phi(\nu),\end{aligned}$$

where we used the convexity of ϕ^* in the second inequality. □

So if Φ is convex and ϕ is concave then the primal ordering is as follows:

$$\Phi(\alpha X + (1 - \alpha)Y) \leq \alpha\Phi(X) + (1 - \alpha)\Phi(Y) = \alpha\phi(\mu) + (1 - \alpha)\phi(\nu) \leq \phi(\alpha\mu + (1 - \alpha)\nu),$$

whereas for the dual we have

$$\begin{aligned}\Phi^*(\alpha X + (1 - \alpha)Y) &\leq \phi^*(\alpha\mu + (1 - \alpha)\nu) \leq \alpha\phi^*(\mu) + (1 - \alpha)\phi^*(\nu) \\ &= \alpha\Phi^*(X) + (1 - \alpha)\Phi^*(Y).\end{aligned}$$

If Φ is lsc, convex and positively homogeneous, then

$$\phi^*(\mu) = \delta(\mu \mid \text{dom } \phi^*) = \begin{cases} 0, & \text{if } \mu \in \text{dom } \phi^* \\ \infty, & \text{else,} \end{cases}$$

where $\text{dom } \phi^* = \{\mu \in \mathcal{M}_\infty \mid \exists Z \in \text{dom } \Phi^* : Z \sim \mu\}$; see for instance [9, Corollary 4.19]. Hence, we obtain:

Corollary 2.6. *If Φ is lsc, convex and positively homogeneous, then ϕ is concave if and only if $\text{dom } \phi^*$ is convex.*

In general, without any request on Φ , convexity of ϕ^* is clearly equivalent to convexity of

$$\text{epi } \phi^* := \{(\mu, a) \in \mathcal{M}_\infty \times \mathbb{R} \mid \phi^*(\mu) \leq a\},$$

which corresponds to $(1_A Z + 1_{A^c} \tilde{Z}, \alpha a + (1 - \alpha)b) \in \text{epi } \Phi^*$ whenever $(Z, a), (\tilde{Z}, b) \in \text{epi } \Phi^*$ and $A \perp (Z, \tilde{Z})$ with $\mathbb{P}(A) = \alpha$.

Examples 2.7. 1. The entropic risk measure: $\Phi(X) = \gamma \ln E[\exp(-X/\gamma)]$, $\gamma > 0$. In this case it is known that $\phi^*(\mu) = \gamma \int (-x) \log(-x) \mu(dx)$ whenever $\mu \in \mathcal{M}_\infty$ has support on \mathbb{R}_- and $\int x \mu(dx) = -1$ (i.e. μ is apart from the sign the distribution of a probability density), and $\phi^*(\mu) = \infty$ otherwise. Being linear on its convex support, ϕ^* is convex on \mathcal{M}_∞ , and thus ϕ is concave which is of course also easily verified directly.

2. The Average Value at Risk: $\Phi(X) = \text{AVaR}_\lambda(X) := -\frac{1}{\lambda} \int_0^\lambda q_X(t) dt$, $\lambda \in (0, 1]$. In this case $\text{dom } \phi^*$ is the set of all $\mu \in \mathcal{M}_\infty$ which have support on $[-1/\lambda, 0]$ and satisfy $\int x \mu(dx) = -1$. Clearly, $\text{dom } \phi^*$ is convex, so ϕ is concave.

3. The mean-variance evaluation principle: $\Phi(X) = -E[X] + \delta \text{Var}(X)$, $\delta > 0$. In this case $\phi^*(\mu) = \frac{1}{4\delta} (\int x^2 \mu(dx) - 1)$ whenever μ has support on \mathbb{R}_- , finite second moment and $\int x \mu(dx) = -1$, and $\phi^*(\mu) = \infty$ otherwise. Hence ϕ^* is convex, and so ϕ is concave. \diamond

In the following example we construct coherent risk measures Φ for which the corresponding functions ϕ are not concave.

Example 2.8. Let $\mu, \nu \in \mathcal{M}_\infty$ be nondegenerate such that $\int x \mu(dx) = -1$ and $\int x \nu(dx) = 0$. In particular, neither $\mu \succeq \nu$ nor $\nu \succeq \mu$, because either of the relations would imply equal expectation. Let $C(\mu) := \{Z \in L^\infty \mid \text{law}(Z) \preceq \mu\}$ and define $C(\nu)$ analogously. Recalling that the convex order is indeed an order on the distributions clarifies the definition of $C(\mu)$ and $C(\nu)$. Note that the convex sets $C(\mu)$ and $C(\nu)$ seen as subsets of L^1 are weakly compact (see e.g. [17]) and thus the convex hull $C := \text{co}(C(\mu) \cup C(\nu))$ is weakly closed (even weakly compact) in L^1 ; see [1, Lemma 5.29]. As C is a convex set it must also be closed in the norm topology on L^1 . This also implies that C is closed in $(L^\infty, \|\cdot\|_\infty)$ and thus, as a convex set, also in $\sigma(L^\infty, L^1)$. Now

$$\Phi(X) := \sup_{Z \in C} E[ZX], \quad X \in L^1$$

is lsc, law-invariant, convex and positively homogeneous. The law invariance follows from the law invariance of the set C ; see [9, Theorem 4.59]. Moreover, we have that $\Phi^* = \delta(\cdot | C)$, thus $\text{dom } \Phi^* = C$. In the following we will show that $\text{dom } \phi^* = \{\eta \in \mathcal{M}_\infty \mid \exists Z \in C : Z \sim \eta\}$ is not convex. Suppose it were, then there would exist a convex combination $\lambda Z + (1 - \lambda)\tilde{Z} \in C$, with Z and \tilde{Z} being elements of $C(\mu)$ or $C(\nu)$, such that $\hat{Z} := \lambda Z + (1 - \lambda)\tilde{Z} \sim \frac{1}{2}(\mu + \nu)$. Apparently it cannot happen that Z and \tilde{Z} are both in $C(\mu)$ or both in $C(\nu)$, as $\frac{1}{2}(\mu + \nu) \not\preceq \mu$ and $\frac{1}{2}(\mu + \nu) \not\preceq \nu$. Therefore, we may assume that $Z \in C(\mu)$ and $\tilde{Z} \in C(\nu)$. Computing the expectation of \hat{Z} and noting that $E[Z] = -1$ and $E[\tilde{Z}] = 0$, we deduce that $\lambda = 1/2$. Without loss of generality we may assume that Z, \tilde{Z} are \mathcal{G}_0 -measurable where \mathcal{G}_0 and \mathcal{F}_0 are as in (2.2). Otherwise we find a two dimensional \mathcal{G}_0 -measurable random vector with the same distribution as (Z, \tilde{Z}) and such that the corresponding convex combination has the same distribution as \hat{Z} . Then, for $A \in \mathcal{F}_0$ with $\mathbb{P}(A) = 1/2$, we have that $\hat{Z} = E[1_A Z + 1_{A^c} \tilde{Z} \mid \mathcal{G}_0]$, and Jensen's inequality for strict convex functions shows that $\hat{Z} \preceq 1_A Z + 1_{A^c} \tilde{Z}$ but not $\hat{Z} \succeq 1_A Z + 1_{A^c} \tilde{Z}$. This fact contradicts $\hat{Z} \sim \frac{1}{2}(\mu + \nu)$, since $1_A Z + 1_{A^c} \tilde{Z} \sim \frac{1}{2}(\mu + \nu)$. Hence $\text{dom } \phi^*$ is not convex, which in turn implies that ϕ is not concave by Corollary 2.6.

Similarly, we can also construct a lsc, translation invariant, convex, and positively homogeneous Φ , i.e. a coherent risk measure, such that ϕ is not concave. To this end, choose nondegenerate $\mu, \nu \in \mathcal{M}_\infty$ such that the support of μ and ν is contained in \mathbb{R}_- and such that $\int x\mu(dx) = \int x\nu(dx) = -1$, but neither $\mu \succeq \nu$ nor $\nu \succeq \mu$. Construct Φ as above, and suppose that there are elements Z and \tilde{Z} of $C(\mu)$ or $C(\nu)$, such that $\hat{Z} := \lambda Z + (1 - \lambda)\tilde{Z} \sim \frac{1}{2}(\mu + \nu)$. Again we may assume that $Z \in C(\mu)$ and $\tilde{Z} \in C(\nu)$. For any convex function $c : \mathbb{R} \rightarrow \mathbb{R}$ we obtain that

$$\begin{aligned} \frac{1}{2} \left(\int c(x)\mu(dx) + \int c(x)\nu(dx) \right) &= E[c(\hat{Z})] \leq \lambda E[c(Z)] + (1 - \lambda)E[c(\tilde{Z})] \\ &\leq \lambda \int c(x)\mu(dx) + (1 - \lambda) \int c(x)\nu(dx). \end{aligned}$$

As there must exist some convex functions $c_1, c_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int c_1(x)\mu(dx) < \int c_1(x)\nu(dx) \quad \text{and} \quad \int c_2(x)\mu(dx) > \int c_2(x)\nu(dx),$$

we conclude that $\lambda = 1/2$. Finally, the same arguments as above show that $\text{dom } \phi^*$ is not convex. \diamond

3 The comonotonic case

In many situations the risk of a combined position $X + Y$ turns out to be lower than the sum of the risks given by the individual positions. This is due to the fact that one position may serve as a hedge against unfavourable outcomes of the other. However, if this hedge is not possible because the two random variables are perfectly positively correlated, then the situation looks very different. This concept is what is captured by the so-called comonotonicity property. Φ is said to be comonotonic if it is linear on comonotone elements, that is $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ whenever X, Y satisfy $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ for all $(\omega, \omega') \mathbb{P} \times \mathbb{P}$ -a.s.

The main message of this section is that under comonotonicity, concavity of ϕ is completely determined by the preservation of the convex order (Proposition 3.5).

Lemma 3.1. *If Φ is comonotonic and \succeq -preserving, then ϕ is concave.*

Proof. Suppose that Φ is comonotonic and \succeq -preserving. Let $\mu, \nu \in \mathcal{M}_1$ and $\alpha \in [0, 1]$. Recall the sub- σ -algebras \mathcal{G}_0 and \mathcal{F}_0 from (2.2). Let U be a \mathcal{G}_0 -measurable $(0, 1)$ -uniform random variable, and set $X := q_\mu(U) \sim \mu$ and $Y := q_\nu(U) \sim \nu$, so that X, Y are \mathcal{G}_0 -measurable and comonotone. Here $q_\eta(s) := \inf\{x \in \mathbb{R} \mid \eta(-\infty, x] \geq s\}$, $s \in (0, 1)$, is the quantile function of the distribution η . Moreover, take $A \in \mathcal{F}_0$ such that $\mathbb{P}(A) = \alpha$. Then we obtain

$$\begin{aligned} \phi(\alpha\mu + (1 - \alpha)\nu) &= \Phi(1_A X + 1_{A^c} Y) \geq \Phi(E[1_A X + 1_{A^c} Y \mid \mathcal{G}_0]) \\ &= \Phi(\alpha X + (1 - \alpha)Y) = \alpha\Phi(X) + (1 - \alpha)\Phi(Y) \\ &= \alpha\phi(\mu) + (1 - \alpha)\phi(\nu) \end{aligned}$$

where the comonotonicity of Φ enters in the second but last equality. \square

Corollary 3.2. *Every lsc law-invariant comonotonic convex risk measure is concave on distributions.*

Kusuoka's representation [15] of law-invariant convex risk measures shows what goes wrong in the general non-comonotone case:

$$(3.1) \quad \rho(X) = \sup_{\mu \in \mathcal{M}(0,1]} \int_{(0,1]} \text{AVaR}_\lambda(X) \mu(d\lambda) - \rho^*(\mu),$$

where $\mathcal{M}(0, 1]$ is the set of all probability measures on $(0, 1]$, and ρ^* is the Fenchel-Legendre transform of ρ seen as a function on $\mathcal{M}(0, 1]$. On the level of distributions we see that the building blocks $\int_{(0,1]} \text{AVaR}_\lambda(X) \mu(d\lambda)$ in (3.1) are all comonotonic and \succeq -preserving, so the corresponding risk measures on distributions are concave according to Lemma 3.1. However, when taking the supremum, the concavity of the risk measure on distributions that corresponds to ρ does not necessarily follow.

The following lemma translates convexity of Φ to property (3.2) of ϕ . This will prove to be useful later on.

Lemma 3.3. *Let Φ be \succeq -preserving. Then the following are equivalent:*

(i) Φ is convex;

(ii) For all $\mu \in \mathcal{M}_1$, $f, g : \mathbb{R} \rightarrow \mathbb{R}$ increasing such that $\mu \circ f^{-1}, \mu \circ g^{-1} \in \mathcal{M}_1$, and $\alpha \in [0, 1]$, the following holds:

$$(3.2) \quad \phi(\mu \circ (\alpha f + (1 - \alpha)g)^{-1}) \leq \alpha\phi(\mu \circ f^{-1}) + (1 - \alpha)\phi(\mu \circ g^{-1}).$$

Proof. (i) \Rightarrow (ii): Let the random variable $X \in L^1$ have distribution μ . Moreover, let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be measurable functions such that $\mu \circ f^{-1}, \mu \circ g^{-1} \in \mathcal{M}_1$, i.e. $f(X), g(X) \in L^1$ since $f(X) \sim \mu \circ f^{-1}$ and $g(X) \sim \mu \circ g^{-1}$, and let $\alpha \in [0, 1]$. Then $\alpha f(X) + (1 - \alpha)g(X) \sim (\mu \circ (\alpha f + (1 - \alpha)g)^{-1})$ which also shows that $\mu \circ (\alpha f + (1 - \alpha)g)^{-1} \in \mathcal{M}_1$. Now it is clear how (3.2) follows from the convexity of Φ .

(ii) \Rightarrow (i): Let $X, Y \in L^1$ and $\alpha \in [0, 1]$. Recall the sub- σ -algebras \mathcal{G}_0 and \mathcal{F}_0 from (2.2). Let

U be a \mathcal{G}_0 -measurable $(0, 1)$ -uniformly distributed random variable and call $\mu = \text{law}(U)$. Then we have $X \stackrel{d}{=} q_X(U)$ and $Y \stackrel{d}{=} q_Y(U)$, thus

$$\begin{aligned} \alpha\Phi(X) + (1 - \alpha)\Phi(Y) &= \alpha\phi(\mu \circ q_X^{-1}) + (1 - \alpha)\phi(\mu \circ q_Y^{-1}) \\ &\geq \phi(\mu \circ (\alpha q_X + (1 - \alpha)q_Y)^{-1}) \\ &= \Phi(\alpha q_X(U) + (1 - \alpha)q_Y(U)) \\ &\geq \Phi(\alpha X + (1 - \alpha)Y), \end{aligned}$$

where the last inequality follows from the fact that

$$\alpha X + (1 - \alpha)Y \preceq \alpha X^c + (1 - \alpha)Y^c$$

for all X^c, Y^c comonotone such that $X^c \stackrel{d}{=} X, Y^c \stackrel{d}{=} Y$; see [14, Theorem 6] and references therein. \square

From the proof of Lemma 3.3 it is clear that if one replaces convexity with quasi-convexity, then (3.2) needs to be replaced by

$$\phi(\mu \circ (\alpha f + (1 - \alpha)g)^{-1}) \leq \phi(\mu \circ f^{-1}) \vee \phi(\mu \circ g^{-1}).$$

The Value at Risk is an example which shows that the request of being \succeq -preserving in Lemma 3.3 cannot be removed. Indeed, Var_λ is comonotone and thus satisfies (3.2) according to the following Lemma 3.4. However Var_λ is not \succeq -preserving and not convex.

Lemma 3.4. *Let Φ be positively homogeneous. Then, Φ is comonotonic if and only if ϕ satisfies (3.2) with equality.*

Proof. The ‘only if’ implication is clear from the first part of the proof of Lemma 3.3 since for increasing f and g the random variables $f(X)$ and $g(X)$ are comonotone. Now, let $X, Y \in L^1$ be comonotone and U be a random variable which is uniformly distributed on $(0, 1)$ such that $q_X(U) = X$ and $q_Y(U) = Y$. If ϕ satisfies (3.2) with equality, then

$$\begin{aligned} \Phi(\alpha X + (1 - \alpha)Y) &= \Phi(\alpha q_X(U) + (1 - \alpha)q_Y(U)) = \phi(\mu \circ (\alpha q_X + (1 - \alpha)q_Y)^{-1}) \\ &= \alpha\phi(\mu \circ q_X^{-1}) + (1 - \alpha)\phi(\mu \circ q_Y^{-1}) = \alpha\Phi(X) + (1 - \alpha)\Phi(Y). \end{aligned}$$

\square

Note that if Φ is comonotonic, then automatically $\Phi(rX) = r\Phi(X)$ for all rational numbers $r \geq 0$. Thus, under some continuity condition on Φ , positive homogeneity automatically follows.

Proposition 3.5. *Let Φ be comonotonic, positively homogeneous, and \succeq -preserving. Then Φ is convex and ϕ is concave.*

Proof. Convexity of Φ follows by Lemmas 3.3 and 3.4. Concavity of ϕ follows by Lemma 3.1. \square

Remark 3.6. Proving that the Average Value at Risk is convex is typically not an easy task; see e.g. [9, Theorem 4.52]. However, by applying Proposition 3.5 we can deduce the convexity of the AVaR quickly. Indeed, AVaR is \succeq_{uni} -reverting by [9, Theorem 2.57: (a) \Leftrightarrow (e)],

hence it preserves the convex order. Moreover, it is continuous, comonotonic, and positively homogeneous. Thus convexity follows by Proposition 3.5. Therefore, it admits dual representation, and the definition of the Fenchel-Legendre transform easily excludes measures with Radon-Nikodym derivatives greater than $1/\lambda$. In this way the well-known dual representation $\text{AVaR}_\lambda(X) = \sup\{E_{\mathbb{Q}}[-X] : d\mathbb{Q}/d\mathbb{P} \leq 1/\lambda\}$ readily follows (cf. [9, Theorem 4.52]). \diamond

Acknowledgements. The authors thank two anonymous referees for helpful comments. The first author acknowledges the Vienna Science and Technology Fund (WWTF) under Grant MA09-003.

References

- [1] C. D. ALIPRANTIS AND K. C. BORDER. Infinite Dimensional Analysis, 3rd edition, Springer, (2006).
- [2] P. ARTZNER AND F. DELBAEN AND J. M. EBER AND D. HEATH. Thinking coherently. *Risk* **10**, 68-71, (1997).
- [3] P. ARTZNER AND F. DELBAEN AND J. M. EBER AND D. HEATH. Coherent measures of risk. *Math. Finance* **9**, 203-228, (1999).
- [4] R.-A. DANA. A representation result for concave Schur concave functions. *Math. Finance* **15**, 613-634, (2005).
- [5] F. DELBAEN. Coherent risk measures. Lectures notes, Scuola Normale Superiore di Pisa, (2001).
- [6] S. DRAPEAU AND M. KUPPER. Risk Preferences and Their Robust Representation. *Math. Oper. Res.* **38/1**, 28-62, (2013).
- [7] D. FILIPOVIĆ AND G. SVINDLAND. The Canonical Model Space for Law-invariant Convex Risk Measures is L^1 . *Math. Finance* **22**, 585-589, (2012).
- [8] H. FÖLLMER AND A. SCHIED. Convex measures of risk and trading constraints. *Finance Stoch.* **6**, 429-447, (2002).
- [9] H. FÖLLMER AND A. SCHIED. Stochastic finance: An introduction in discrete time, 3rd Edition, De Gruyter, (2011).
- [10] M. FRITTELLI AND M. MAGGIS AND I. PERI. Risk Measures on $\mathcal{P}(\mathbb{R})$ and Value At Risk with Probability/Loss function. *Math. Finance*, forthcoming, (2013).
- [11] M. FRITTELLI AND E. ROSAZZA GIANIN. Putting order in risk measures. *Journal of Banking and Finance* **26**, 1473-1486, (2002).
- [12] M. FRITTELLI AND E. ROSAZZA GIANIN. Law-invariant convex risk measures. *Adv. Math. Econ.* **7**, 33-46, (2005).
- [13] E. JOUINI AND W. SCHACHERMAYER AND N. TOUZI. Law invariant risk measures have the Fatou property. *Adv. Math. Econ.* **9**, 49-71, (2006).

- [14] R. KAAS AND J. DHAENE AND D. VYNCKE AND M.J. GOOVAERTS AND M. DENUIT. A simple geometric proof that comonotonic risks have the convex-largest sum. *Astin Bull.* **32/1**, 71-80, (2002).
- [15] S. KUSUOKA. On law-invariant coherent risk measures. *Adv. Math. Econ.* **3**, 83-95, (2001).
- [16] R. T. ROCKAFELLAR AND S. URYASEV AND M. ZABARANKIN. Generalized Deviations in Risk Analysis. *Finance Stoch.* **10**, 51-74, (2006).
- [17] G. SVINDLAND. Dilatation monotonicity and convex order. *Math. Financ. Econ.*, available online at <http://link.springer.com/article/10.1007%2Fs11579-013-0112-y>, (2013)