

# Limitations of law-invariant insurance pricing

FABIO BELLINI

*Department of Statistics and Quantitative Methods, University of Milano-Bicocca*

PABLO KOCH-MEDINA , COSIMO MUNARI

*Center for Finance and Insurance and Swiss Finance Institute, University of Zurich, Switzerland*

GREGOR SVINDLAND

*Mathematics Institute, LMU Munich, Germany*

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## Abstract

We show that a law-invariant pricing functional defined on a general Orlicz space is typically incompatible with frictionless risky assets in the sense that one and only one of the following alternatives can hold: Either every risky payoff has a strictly-positive bid-ask spread or the pricing functional is given by an expectation and, hence, every payoff has a zero bid-ask spread. By doing so we extend and unify a variety of “collapse to the mean” results from the literature and highlight the key role played by law invariance in causing the collapse. As a byproduct, we derive a number of applications to acceptance sets, risk measures, and Schur-convex functionals.

## 1 Introduction

In the actuarial literature it is customary to price insurance contracts by using law-invariant pricing functionals, i.e. pricing functionals whose outcomes are fully determined by the probability distribution of the contracts’ payoffs. Deprez and Gerber [13] refer to such functionals as classical premium principles. Much of the original actuarial literature focused on pricing functionals applied to insurance claims, i.e. pricing functionals that are defined on the positive cone of a suitable space of random variables. More recently, efforts have been made to extend the theory to encompass pricing functionals defined on the whole model space; see e.g. Venter [33], Gerber and Shiu [23], Wang et al. [35], Wang [34], Goovaerts et al. [24], Goovaerts and Laeven [25]. All of these papers study pricing functionals that, in addition to law invariance, satisfy a set of desirable properties such as convexity, sublinearity, comonotonicity, cash additivity, monotonicity with respect to stochastic orderings. We refer to Section 2 for the formal definitions of these and other properties.

In Castagnoli et al. [7], it was shown that requiring pricing functionals to satisfy law invariance along with some of the desirable properties mentioned above can be highly restrictive. More specifically, the focus of Castagnoli et al. [7] was essentially on law-invariant pricing functionals on the space of bounded random variables that are sublinear, nondecreasing, and comonotonic. The main result stated there is that, for such a pricing functional, one and only one of the following alternatives holds (see Theorem A.1 in the appendix):

- EITHER every payoff has zero bid-ask spread
- OR every risky payoff has a strictly-positive bid-ask spread.

In other words, either the underlying “market” is fully frictionless, or the only payoffs that can have zero bid-ask spread are risk-free payoffs. In the former case, the pricing functional collapses to a positive multiple of the expectation functional with respect to the reference probability measure. In the context of risk measures defined on the space of bounded random variables, the above result was extended by Frittelli and Rosazza Gianin [20]. They kept monotonicity but replaced sublinearity by convexity and comonotonicity by cash additivity (see Theorem A.2 in the appendix). It is worth noting that comonotonicity implies cash additivity and, in the space of bounded random variables, cash additivity implies continuity. Hence, in the above results the functionals under scrutiny are always continuous.

In this paper we extend the results in Castagnoli et al. [7] and Frittelli and Rosazza Gianin [20] in two directions: (1) We establish the minimal properties that cause a pricing functional to “collapse” to the mean. In particular, neither monotonicity nor cash additivity are needed and continuity can be relaxed to (order) lower semicontinuity. (2) We go beyond the setting of bounded positions and show that the above results hold on general Orlicz spaces. We note that even though we drop monotonicity, we continue to use the language of pricing functionals to highlight the original motivation of our research.

As a byproduct, we obtain a variety of interesting applications. First, we show that a coherent law-invariant acceptance set is always pointed unless it is generated by the expectation under the reference probability measure. Second, we provide a characterization of law invariance for a risk measure based on acceptance sets and eligible assets. These risk measures go back to the original contribution of Artzner et al. [2] and have been studied by Föllmer and Schied [16], Frittelli and Scandolo [21], Artzner et al. [3], and Farkas et al. [18] among others. In the convex case, we show that a multi-asset risk measure is never law invariant unless it is a negative multiple of the expectation under the reference probability measure. As a final application, we extend the quantile representation for Schur-convex functionals obtained in Dana [11] and Grechuk and Zabaranin [26] from  $L^p$  spaces to Orlicz spaces, and sharpen it by showing that one can always choose representing quantiles of bounded random variables.

The paper is structured as follows. Section 2 introduces terminology in the setting of  $L^1$ . In Section 3 we discuss some simple preliminary results on frictionless payoffs. These are needed in Section 4, which features our main results on sublinear and convex pricing functionals. In Section 5 we discuss applications to acceptance sets and risk measures. Finally, Section 6 is devoted to extending our results to general Orlicz spaces and highlighting applications to Schur-convex functionals. A brief review of the main results from the literature is provided in Section A and a general picture on some key results on (quantile) representations and extensions of law-invariant functionals is presented in Section B.

## 2 The underlying model

We consider a one-period market with dates  $t = 0$  and  $t = 1$  in which future uncertainty is modeled by a nonatomic probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . A random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$  is interpreted as the state-contingent payoff of a financial contract at time 1. We assume that payoffs at time 1 belong to

$$L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Each element of  $L^1$  is an equivalence class with respect to almost-sure equality under  $\mathbb{P}$  of random variables  $X : \Omega \rightarrow \mathbb{R}$  such that

$$\|X\|_1 := \mathbb{E}[|X|] < \infty,$$

where  $\mathbb{E}$  denotes the expectation with respect to  $\mathbb{P}$ . As usual, we do not distinguish between an element of  $L^1$  and any of its representatives. By abuse of notation, we will identify a real number with the random variable that is almost-sure identical to it. The space  $L^1$  is partially ordered by the standard almost-sure

ordering with respect to  $\mathbb{P}$  and is equipped with the linear topology induced by the lattice norm  $\|\cdot\|_1$ . We denote by  $L^\infty$  the subspace of  $L^1$  consisting of bounded random variables.

We assume that each payoff demands a certain price at time 0, which is represented by a functional

$$\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}.$$

More precisely, for every payoff  $X$  we interpret  $\pi(X)$  as an *ask price*, i.e. the price from a seller's perspective, expressed in a given numeraire. We adopt the standard convention according to which a positive value of  $\pi(X)$  means that the seller receives  $\pi(X)$  units of the numeraire when selling  $X$  and a negative value of  $\pi(X)$  means that the seller actually needs to pay  $-\pi(X)$  units of the numeraire to "sell"  $X$ . As usual, the corresponding *bid price*, i.e. the price from a buyer's perspective, is given by  $-\pi(-X)$ ; see Jouini [27]. The difference between the ask and the bid price for  $X$ , the so-called *bid-ask spread* associated to  $X$ , is therefore given by the quantity  $\pi(X) + \pi(-X)$ .

We say that a payoff is frictionless when it has zero bid-ask spread and strongly frictionless when, in addition, its price per unit does not depend on the transacted volume. This is made precise by the following definition.

**Definition 2.1.** Let  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ . We say that a payoff  $X \in L^1$  is:

- (1) *risk-free* if it is constant.
- (2) *risky* if it is not constant.
- (3) *frictionless (under  $\pi$ )* if  $\pi(-X) = -\pi(X)$ .
- (4) *strongly frictionless (under  $\pi$ )* if  $\pi(mX) = m\pi(X)$  for every  $m \in \mathbb{R}$ .

When there is no ambiguity we omit the explicit reference to  $\pi$  and speak simply of frictionless and strongly frictionless payoffs.

**Remark 2.2.** (i) Clearly, a payoff  $X$  is (strongly) frictionless if and only if  $-X$  is (strongly) frictionless. Moreover, for  $X$  to be frictionless we must have  $\pi(X) \neq \infty$ .

(ii) It is clear that if the payoff  $X$  is strongly frictionless, then  $mX$  is automatically frictionless for every  $m \in \mathbb{R}$ . The converse implication is, however, not true in general unless  $\pi$  is convex, see Proposition 3.5. To see this, take a nonzero  $Z \in L^1$  and consider the functional  $\pi : L^1 \rightarrow \mathbb{R}$  defined by

$$\pi(X) = \begin{cases} 1 & \text{if } X = mZ \text{ for some } m \in (0, \infty), \\ -1 & \text{if } X = mZ \text{ for some } m \in (-\infty, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $mZ$  is frictionless for every  $m \in \mathbb{R}$  but  $Z$  is not strongly frictionless under  $\pi$ .

For two payoffs  $X$  and  $Y$  we write  $X \sim Y$  whenever  $X$  and  $Y$  have the same probability law under  $\mathbb{P}$ . We say that  $X$  and  $Y$  are comonotone if there exists a payoff  $Z$  such that  $X = f(Z)$  and  $Y = g(Z)$  for suitable nondecreasing functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ . By convention we set  $0 \cdot \infty := 0$ . We recall the following terminology for functionals.

**Definition 2.3.** A functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is:

- (1) *convex* if  $\pi(\lambda X + (1 - \lambda)Y) \leq \lambda\pi(X) + (1 - \lambda)\pi(Y)$  for all  $\lambda \in [0, 1]$  and  $X, Y \in L^1$ .
- (2) *positively homogeneous* if  $\pi(\lambda X) = \lambda\pi(X)$  for all  $\lambda \in [0, \infty)$  and  $X \in L^1$ .

- (3) *sublinear* if  $\pi$  is convex and positively homogeneous.
- (4) *nondecreasing* if  $\pi(X) \geq \pi(Y)$  for all  $X, Y \in L^1$  with  $X \geq Y$ .
- (5) *law-invariant* if  $\pi(X) = \pi(Y)$  for all  $X, Y \in L^1$  with  $X \sim Y$ .
- (6) *comonotonic* if  $\pi(X + Y) = \pi(X) + \pi(Y)$  for all comonotone  $X, Y \in L^1$ .
- (7) *(norm) continuous* if  $\pi(X) = \lim \pi(X_n)$  for all  $(X_n) \subset L^1$  and  $X \in L^1$  such that  $X_n \rightarrow X$  with respect to  $\|\cdot\|_1$ .
- (8) *(norm) lower semicontinuous* if  $\pi(X) \leq \liminf \pi(X_n)$  for all  $(X_n) \subset L^1$  and  $X \in L^1$  such that  $X_n \rightarrow X$  with respect to  $\|\cdot\|_\infty$ .
- (9) *Z-additive* (for  $Z \in L^1$ ) if  $\pi(X + mZ) = \pi(X) + m\pi(Z)$  for all  $X \in L^1$  and  $m \in \mathbb{R}$ .
- (10) *cash-additive* if it is 1-additive and  $\pi(1) = 1$ , i.e. if  $\pi(X + m) = \pi(X) + m$  for all  $X \in L^1$  and  $m \in \mathbb{R}$ .

**Remark 2.4.** For a functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  such that  $\pi(0) = 0$  the following statements hold:

- (1) If  $\pi$  is convex, then every payoff has a nonnegative bid-ask spread, i.e.  $\pi(X) + \pi(-X) \geq 0$  for every  $X \in L^1$ .
- (2) If  $\pi$  is convex and  $X \in L^1$  is frictionless, then  $\pi(mX) = m\pi(X)$  for every  $m \in [-1, 1]$ . In particular,  $mX$  is frictionless for every  $m \in [-1, 1]$ .
- (3) Every risk-free payoff is automatically frictionless under any of the following conditions:
  - (i)  $\pi(X + Y) = \pi(X) + \pi(Y)$  for all independent  $X, Y \in L^1$ .
  - (ii)  $\pi$  is comonotonic.
  - (iii)  $\pi$  is cash-additive.

### 3 Preliminary results

In this brief section we establish some useful results about frictionless payoffs and their link to  $Z$ -additivity. We start with the following simple preliminary lemma.

**Lemma 3.1.** *Let  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  and take  $Z \in L^1$ . If  $\pi(X + mZ) \leq \pi(X) + m\pi(Z)$  for all  $X \in L^1$  and  $m \in \mathbb{R}$ , then  $\pi$  is  $Z$ -additive.*

*Proof.* It suffices to observe that, for any arbitrary  $X \in L^1$  and  $m \in \mathbb{R}$ , we have

$$\pi(X) = \pi((X + mZ) - mZ) \leq \pi(X + mZ) - m\pi(Z) \leq \pi(X) + m\pi(Z) - m\pi(Z) = \pi(X)$$

where both inequalities follow by applying our assumption. □

We show that, under a sublinear pricing functional, the property of  $Z$ -additivity is equivalent to  $Z$  being frictionless or, equivalently, strongly frictionless.

**Proposition 3.2.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is sublinear. Then, for every  $Z \in L^1$  the following statements are equivalent:*

- (a)  $Z$  is frictionless.
- (b)  $Z$  is strongly frictionless.

(c)  $\pi$  is  $Z$ -additive.

*Proof.* Since  $\pi(0) = 0$ , it is immediate to see that (c) implies (b), which readily implies (a). To conclude the proof, assume that (a) holds and take  $X \in L^1$  and  $m \in \mathbb{R}$ . It follows from our assumption and from positive homogeneity that  $\pi(mZ) = m\pi(Z)$  for every  $m \in \mathbb{R}$ . Hence, sublinearity implies that

$$\pi(X + mZ) \leq \pi(X) + \pi(mZ) = \pi(X) + m\pi(Z).$$

That (c) holds now follows from Lemma 3.1.  $\square$

It is not difficult to show that the above equivalence does not hold if  $\pi$  is only required to be convex. For instance, given a payoff  $Z$ , consider the convex functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$\pi(X) = \begin{cases} m & \text{if } X = mZ \text{ for some } m \in [-1, 1], \\ \infty & \text{otherwise.} \end{cases}$$

Then,  $Z$  is easily seen to be frictionless, but not strongly frictionless. In addition,  $\pi$  fails to be  $Z$ -additive. However, in the convex case, we can still characterize when a payoff  $Z$  is strongly frictionless in terms of the  $Z$ -additivity of the recession functional associated with  $\pi$ .

**Definition 3.3.** Let  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ . The *recession functional* associated with  $\pi$  is the map  $\pi^\infty : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$\pi^\infty(X) := \sup_{\lambda \in (0, \infty)} \frac{\pi(\lambda X)}{\lambda}.$$

**Remark 3.4.** If  $\pi(0) = 0$ , then  $\pi^\infty$  is the smallest positively-homogeneous map dominating  $\pi$ , i.e. such that  $\pi(X) \leq \pi^\infty(X)$  for every  $X \in L^1$ . If, in addition,  $\pi$  is convex, then  $\pi^\infty$  is the smallest sublinear map dominating  $\pi$ .

**Proposition 3.5.** Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is convex and satisfies  $\pi(0) = 0$ . Then, for every  $Z \in L^1$  the following are equivalent:

- (a)  $Z$  is strongly frictionless under  $\pi$ .
- (b)  $mZ$  is frictionless under  $\pi$  for every  $m \in \mathbb{R}$ .
- (c)  $Z$  is frictionless under  $\pi^\infty$ .
- (d)  $Z$  is strongly frictionless under  $\pi^\infty$ .
- (e)  $\pi^\infty$  is  $Z$ -additive.

Under any of the above conditions we have  $\pi(mZ) = \pi^\infty(mZ)$  for every  $m \in \mathbb{R}$ . If, in addition,  $\pi$  is finite valued, the preceding statements are also equivalent to:

- (f)  $\pi$  is  $Z$ -additive.

*Proof.* After Definition 2.1, we already noted that (a) implies (b). Conversely, if (b) holds we have, by Remark 2.4, that  $\pi(\lambda mZ) = \lambda\pi(mZ)$  for every  $m \in \mathbb{R}$  and  $\lambda \in [-1, 1]$ . This implies that

$$\pi(Z) = \pi\left(\frac{1}{m}mZ\right) = \frac{1}{m}\pi(mZ),$$

for every  $m \in \mathbb{R}$  with  $|m| \geq 1$ . As a result,  $\pi(mZ) = m\pi(Z)$  for every  $m \in \mathbb{R}$ , showing that  $Z$  is strongly frictionless under  $\pi$ . Hence, (a) and (b) are equivalent. Moreover, (c) is equivalent to both (d) and (e) by Proposition 3.2. If (a) holds, then for every  $m \in \mathbb{R}$  we have that

$$\pi^\infty(mZ) = \sup_{\lambda \in (0, \infty)} \frac{\pi(\lambda mZ)}{\lambda} = \pi(mZ) = m\pi(Z).$$

This yields that (a) implies (c) and shows that  $\pi^\infty(mZ) = \pi(mZ)$  for every  $m \in \mathbb{R}$ . Finally, assume that (e) holds. Note that, since every payoff has a nonnegative bid-ask spread when  $\pi$  is convex and  $\pi(0) = 0$ , we have  $\pi(mZ) \geq -\pi(-mZ)$  for every  $m \in \mathbb{R}$  and, hence,

$$\pi^\infty(mZ) \geq \pi(mZ) \geq -\pi(-mZ) \geq -\pi^\infty(-mZ) = m\pi^\infty(Z) = \pi^\infty(mZ).$$

This yields  $\pi(mZ) = m\pi^\infty(Z)$  for every  $m \in \mathbb{R}$  and shows that (e) implies (a). This concludes the proof of the equivalence of (a) to (e)

Since  $\pi(0) = 0$ , it is clear that (f) always implies (a). Now, assume that  $\pi$  is finite valued and that (a) holds. For arbitrary  $X \in L^1$  and  $m \in \mathbb{R}$ , convexity implies that

$$\pi(X + mZ) \leq \lambda\pi\left(\frac{1}{\lambda}X\right) + (1 - \lambda)\pi\left(\frac{m}{1 - \lambda}Z\right) = \lambda\pi\left(\frac{1}{\lambda}X\right) + m\pi(Z)$$

for every  $\lambda \in (0, 1)$ . Since the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(\alpha) = \pi(\alpha X)$  is convex and, thus, continuous, we infer that  $\pi(X + mZ) \leq \pi(X) + m\pi(Z)$ . Lemma 3.1 implies that (e) holds.  $\square$

## 4 Main results

We are now able to provide our first generalizations of the results in Castagnoli et al. [7] and Frittelli and Rosazza Gianin [20].

### Sublinear pricing functionals

The result for sublinear pricing functionals is based on the quantile representation obtained by Dana [11] in the setting of bounded random variables. In what follows, for every payoff  $X$  we denote by  $q_X$  a fixed but arbitrary quantile function for  $X$ , i.e. a function  $q_X : (0, 1) \rightarrow \mathbb{R}$  such that

$$\inf\{m \in \mathbb{R}; \mathbb{P}(X \leq m) \geq \alpha\} \leq q_X(\alpha) \leq \inf\{m \in \mathbb{R}; \mathbb{P}(X \leq m) > \alpha\}$$

for every  $\alpha \in (0, 1)$ . Note that, since the cumulative distribution function of  $X$  has at most countably many discontinuity points, any two quantile functions for  $X$  coincide almost surely with respect to the Lebesgue measure on  $[0, 1]$ . We refer to Föllmer and Schied [17] for more on quantile functions.

**Lemma 4.1.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is sublinear, lower semicontinuous, and law invariant. Then, there exists a set  $\mathcal{D} \subset L^\infty$  such that for every  $X \in L^1$*

$$\pi(X) = \sup_{Y \in \mathcal{D}} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha.$$

*Proof.* Since  $L^\infty$  is the topological dual of  $L^1$ , it follows from a classical result in convex analysis, see e.g. Corollary 5.99 in Aliprantis and Border [1], that  $\pi$  is  $\sigma(L^1, L^\infty)$  lower semicontinuous. The desired representation is now a direct consequence of Proposition B.3.  $\square$

**Theorem 4.2.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is sublinear, lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

- (a) *There exists a frictionless risky payoff  $Z \in L^1$  with  $\mathbb{E}[Z] \neq 0$ .*
- (b) *There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^1$ .*
- (c) *Every payoff is frictionless.*

*Proof.* Clearly, we only need to prove that (a) implies (b). To this effect, assume that (a) holds so that  $\pi(Z) + \pi(-Z) = 0$ . Let  $\mathcal{D}$  be the subset of  $L^\infty$  from Lemma 4.1. Then, for every fixed  $Y \in \mathcal{D}$  we must have

$$\begin{aligned}
0 &\geq \int_0^1 q_Z(\alpha)q_Y(\alpha)d\alpha + \int_0^1 q_{-Z}(\alpha)q_Y(\alpha)d\alpha \\
&= \int_0^1 q_Z(\alpha)q_Y(\alpha)d\alpha - \int_0^1 q_Z(1-\alpha)q_Y(\alpha)d\alpha \\
&= \int_0^1 q_Z(\alpha)[q_Y(\alpha) - q_Y(1-\alpha)]d\alpha \\
&= \int_0^{1/2} q_Z(\alpha)[q_Y(\alpha) - q_Y(1-\alpha)]d\alpha + \int_{1/2}^1 q_Z(\alpha)[q_Y(\alpha) - q_Y(1-\alpha)]d\alpha \\
&= \int_0^{1/2} [q_Z(\alpha) - q_Z(1-\alpha)][q_Y(\alpha) - q_Y(1-\alpha)]d\alpha.
\end{aligned}$$

Since  $q_Z$  is nondecreasing, we have  $q_Z(\alpha) - q_Z(1-\alpha) \leq 0$  for almost every  $\alpha \in (0, 1/2]$ . The same holds for  $q_Y$ . Together with the above inequality, this implies that

$$\int_0^{1/2} [q_Z(\alpha) - q_Z(1-\alpha)][q_Y(\alpha) - q_Y(1-\alpha)]d\alpha = 0.$$

Since  $Z$  is risky, we find  $\beta \in (0, 1/2)$  such that  $q_Z(\alpha) - q_Z(1-\alpha) < 0$  for almost every  $\alpha \in (0, \beta]$ . Hence, the above identity can only hold if  $q_Y(\alpha) = q_Y(1-\alpha)$  for almost every  $\alpha \in (0, \beta]$ . Being nondecreasing,  $q_Y$  must therefore be almost-surely constant. It follows that in the representation in Lemma 4.1, the set  $\mathcal{D}$  must consist of constant random variables, i.e.  $\mathcal{D} \subset \mathbb{R}$ , and for every  $X \in L^1$

$$\pi(X) = \begin{cases} c_1\mathbb{E}[X] & \text{if } \mathbb{E}[X] < 0 \\ c_2\mathbb{E}[X] & \text{if } \mathbb{E}[X] \geq 0 \end{cases}$$

where  $c_1 = \inf \mathcal{D} \leq \sup \mathcal{D} = c_2$ . Assuming without loss of generality that  $\mathbb{E}[Z] > 0$  (otherwise replace  $Z$  by  $-Z$ ), we use that  $Z$  is frictionless to obtain

$$c_2\mathbb{E}[Z] = \pi(Z) = -\pi(-Z) = c_1\mathbb{E}[Z].$$

Recall that the fact that  $Z$  is frictionless also implies that  $\pi(Z)$  is finite, so that  $c_1$  and  $c_2$  must coincide and be finite. This establishes (b) and concludes the proof.  $\square$

**Remark 4.3.** The condition  $\mathbb{E}[Z] \neq 0$  in point (a) above is necessary for the equivalence to hold. To see this, consider the functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$\pi(X) = \begin{cases} \mathbb{E}[X] & \text{if } \mathbb{E}[X] \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that  $\pi$  is sublinear, law invariant, and lower semicontinuous. However, since a payoff  $X$  is frictionless if, and only if,  $\mathbb{E}[X] = 0$ , we see that condition (a) holds but conditions (b) and (c) are not satisfied.

The preceding theorem takes the following simpler form in the common situation where risk-free payoffs are assumed to be frictionless.

**Corollary 4.4.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is sublinear, lower semicontinuous, and law invariant. Moreover, assume that every risk-free payoff is frictionless (or equivalently, by sublinearity, some nonzero risk-free payoff is frictionless). Then, the following statements are equivalent:*

- (a) *There exist a frictionless risky payoff.*
- (b) *There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^1$ .*
- (c) *Every payoff is frictionless.*

*Proof.* In view of Theorem 4.2, it suffices to prove that (a) implies the existence of a frictionless risky payoff  $Z \in L^1$  such that  $\mathbb{E}[Z] \neq 0$ . To this end, assume that (a) holds. Let  $W \in L^1$  be a frictionless risky payoff and take any  $m \in \mathbb{R} \setminus \{-\mathbb{E}[W]\}$ . It is clear that  $Z = W + m$  is a risky payoff with nonzero expectation. Moreover,  $Z$  is frictionless. This follows from Proposition 3.2 once we note that

$$\pi(-Z) = \pi(0) - \pi(W) - \pi(m) = -\pi(Z + m) = -\pi(Z)$$

by  $W$ -additivity and  $m$ -additivity of  $\pi$ . This concludes the proof.  $\square$

**Remark 4.5.** The above results extend Theorem 1 in Castagnoli et al. [7] and Proposition 8 in Frittelli and Rosazza Gianin [20] beyond the setting of bounded payoffs and by getting rid of the assumptions of comonotonicity, monotonicity, and cash-additivity as well as of the implicit assumption of continuity (see Remark A.3). In doing so, they show that the “collapse to the mean” is a general phenomenon caused by the interaction between law invariance and the existence of frictionless risky payoffs.

### Convex pricing functionals

It is not difficult to verify that the preceding collapse to the mean does not generally hold if we consider a convex pricing functional instead of a sublinear one. This is illustrated by the following example.

**Example 4.6.** Consider the functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$\pi(X) = \inf\{m \in \mathbb{R}; X + m \geq -1\}.$$

It is clear that  $\pi$  is convex, lower semicontinuous, and law invariant. Moreover,  $\pi$  admits frictionless risky payoffs. For instance, taking  $E \in \mathcal{F}$  such that  $\mathbb{P}(E) = 1/2$  and setting  $X = 1_E - 1_{E^c}$ , one easily sees that  $\pi(X) = \pi(-X) = 0$  so that  $X$  is frictionless. However,  $\pi$  does not coincide with the expectation under  $\mathbb{P}$ .

In this section we show how to characterize when a convex pricing functional collapse to the mean by exploiting the properties of the associated recession functional. As a preliminary step, it is worth observing the following simple fact.

**Lemma 4.7.** *Consider a functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ . If  $\pi$  is convex, lower semicontinuous, and law invariant, then so is  $\pi^\infty$ .*

*Proof.* It suffices to observe that, under the above assumptions,  $\pi^\infty$  is the pointwise supremum of maps that are convex, lower semicontinuous, and law invariant and, as such, inherits all those properties.  $\square$

**Theorem 4.8.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is convex, lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

(a) *There exists a strongly frictionless risky payoff  $Z \in L^1$  with  $\mathbb{E}[Z] \neq 0$ .*

(b) *There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^1$ .*

(c) *Every payoff is frictionless.*

*Proof.* In view of Proposition 3.5, we only have to show that (a) implies (b). To this effect, assume that (a) holds and note that  $\pi(0) = \pi(0 \cdot Z) = 0\pi(Z) = 0$ . Since  $Z$  is frictionless under  $\pi^\infty$  by Proposition 3.5, we infer from Theorem 4.2 that  $\pi^\infty$  is a multiple of  $\mathbb{E}$ . Since

$$\pi^\infty(X) \geq \pi(X) \geq -\pi(-X) \geq -\pi^\infty(-X) = \pi^\infty(X)$$

for every  $X \in L^1$  by convexity of  $\pi$  and linearity of  $\pi^\infty$ , we conclude that  $\pi = \pi^\infty$  so that  $\pi$  is also a multiple of  $\mathbb{E}$ . This establishes (b) and concludes the proof.  $\square$

As above, the preceding theorem takes a simpler form in the case that risk-free payoffs are frictionless.

**Corollary 4.9.** *Assume that  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  is convex, lower semicontinuous, and law invariant. Moreover, assume that every risk-free payoff is frictionless. Then, the following statements are equivalent:*

(a) *There exist a strongly frictionless risky payoff.*

(b) *There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^1$ .*

(c) *Every payoff is frictionless.*

*Proof.* In view of Theorem 4.8, it suffices to prove that (a) implies the existence of a strongly frictionless risky payoff  $Z \in L^1$  such that  $\mathbb{E}[Z] \neq 0$ . To this effect, assume that (a) holds and let  $W \in L^1$  be a strongly frictionless risky payoff. Note that  $\pi(0) = \pi(0 \cdot W) = 0$ . Take any  $m \in \mathbb{R} \setminus \{-\mathbb{E}[W]\}$ . It is clear that  $Z = W + m$  is a risky payoff with nonzero expectation. Moreover, we have

$$\pi^\infty(-Z) = \pi^\infty(0) - \pi^\infty(W) - \pi^\infty(m) = -\pi^\infty(W + m) = -\pi^\infty(Z)$$

by Proposition 3.5. Hence,  $Z$  is frictionless under  $\pi^\infty$ . As a result, we may again apply Proposition 3.5 to infer that  $Z$  is strongly frictionless. This establishes the desired claim.  $\square$

**Remark 4.10.** In view of Proposition 3.5, the previous results extend Proposition 9 in Frittelli and Rosazza Gianin [20] beyond the setting of bounded payoffs and monotonic and cash-additive functionals. Besides delivering a more general result, our proof is more direct and, by exploiting the properties of recession functionals, avoids the duality argument used there.

## 5 Applications

In this section we discuss applications of our results to acceptance sets and to risk measures with respect to multiple eligible assets. We start by recalling some standard notions for sets.

**Definition 5.1.** We say that a set  $\mathcal{A} \subset L^1$  is:

- (1) *convex* if  $\lambda X + (1 - \lambda)Y \in \mathcal{A}$  for all  $\lambda \in [0, 1]$  and  $X, Y \in \mathcal{A}$ .
- (2) *conic* if  $\lambda X \in \mathcal{A}$  for all  $\lambda \in [0, \infty)$  and  $X \in \mathcal{A}$ .
- (3) *monotone* if  $X \in \mathcal{A}$  whenever  $X \geq Y$  for some  $Y \in \mathcal{A}$ .
- (4) *law invariant* if  $X \in \mathcal{A}$  whenever  $X \sim Y$  for some  $Y \in \mathcal{A}$ .

- (5) (*norm*) *closed* if  $X \in \mathcal{A}$  whenever  $X_n \rightarrow X$  with respect to  $\|\cdot\|_\infty$  for some sequence  $(X_n) \subset \mathcal{A}$ .
- (6) an *acceptance set* if  $\mathcal{A}$  is nonempty, monotone, and  $\mathcal{A} \neq L^1$ .
- (7) a *coherent acceptance set* if  $\mathcal{A}$  is a convex and conic acceptance set.

The basic construction principle for risk measures is captured in the next definition.

**Definition 5.2.** Consider a set  $\mathcal{A} \subset L^1$ , a vector space  $\mathcal{M} \subset L^1$  containing a nonzero positive payoff  $U \in L^1$ , and a linear functional  $\psi : \mathcal{M} \rightarrow \mathbb{R}$  such that  $\psi(U) > 0$ . The *risk measure* associated to  $(\mathcal{A}, \mathcal{M}, \psi)$  is the functional  $\rho : L^1 \rightarrow \mathbb{R} \cup \{\pm\infty\}$  defined by

$$\rho(X) := \inf\{\psi(Z); Z \in \mathcal{M}, X + Z \in \mathcal{A}\}.$$

In particular, if  $\dim(\mathcal{M}) = 1$ , then

$$\rho(X) := \psi(U) \inf\{m \in \mathbb{R}; X + mU \in \mathcal{A}\}.$$

If  $\dim(\mathcal{M}) = 1$ , we speak of a risk measure with respect to the single *eligible payoff*  $U$ . If  $\dim(\mathcal{M}) > 1$ , we speak of a risk measure with respect to the multiple *eligible payoffs* in  $\mathcal{M}$ .

The acceptance set  $\mathcal{A}$  represents the set of financial positions that are deemed to be acceptable, e.g. by a regulator. The subspace  $\mathcal{M}$  and the functional  $\psi$  represent, respectively, the marketed space and the pricing functional of a frictionless financial market where the Law of One Price holds. An element  $X$  of the subspace  $\mathcal{M}$  represents the payoff of a portfolio of traded securities and  $\psi(X)$  the (unique) value of any portfolio with payoff  $X$ . For every payoff  $X$ , the quantity  $\rho(X)$  is therefore interpreted as the minimal amount of capital that has to be raised and invested in a traded portfolio to ensure acceptability. Risk measures with respect to a single eligible payoff were introduced by Artzner et al. [2]. The generalization to multiple eligible payoffs has been studied in Föllmer and Schied [16], Frittelli and Scandolo [21], and Artzner et al. [3]; see also Farkas et al. [18] and Liebrich and Svindland [30] for recent results in this direction.

Before applying our results to acceptance sets and risk measures with respect to multiple eligible payoffs we make a simple observation about risk measures with respect to a single riskless eligible payoff.

**Lemma 5.3.** *Assume that  $\mathcal{A} \subset L^1$  is a closed acceptance set,  $\dim(\mathcal{M}) = 1$ , and  $U = 1$ . Then, the following statements are equivalent:*

- (a)  $\rho$  is law invariant.
- (b)  $\mathcal{A}$  is law invariant.

*Proof.* Since  $\mathcal{A}$  is closed, we clearly have

$$\mathcal{A} = \{X \in L^1; \rho(X) \leq 0\}.$$

This immediately shows that (a) implies (b). To establish the converse implication, take an arbitrary  $X \in L^1$  and note that

$$\rho(X) = \psi(1) \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\}$$

by our assumption. Since  $Y + m \sim X + m$  for every  $Y \in L^1$  with  $Y \sim X$  and every  $m \in \mathbb{R}$ , we see that (b) implies (a).  $\square$

## Acceptance sets

We show that every closed, coherent, law-invariant acceptance set is always pointed, unless it is the acceptance set generated by the expectation functional under  $\mathbb{P}$ .

**Proposition 5.4.** *Assume that  $\mathcal{A} \subset L^1$  is a closed, coherent, law-invariant acceptance set. Then, one of the following two alternatives holds:*

(i)  $\mathcal{A}$  is pointed, i.e.  $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ .

(ii)  $\mathcal{A} = \{X \in L^1; \mathbb{E}[X] \geq 0\}$ .

*Proof.* Consider the functional  $\pi : L^1 \rightarrow \mathbb{R} \cup \{\pm\infty\}$  defined by

$$\pi(X) = \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\}.$$

Note that  $\pi(0) = 0$  for otherwise  $\mathcal{A}$  would contain  $L^\infty$  and, hence,  $L^1$  by closedness. It follows from Lemma 5.3 that  $\pi$  is law invariant. Moreover, it is immediate to verify that  $\pi$  is sublinear and lower semicontinuous. As in the proof of Lemma 5.3, we have

$$\mathcal{A} = \{X \in L^1; \pi(X) \leq 0\}. \quad (5.1)$$

Since  $\pi(0) = 0$  and  $\pi$  is lower semicontinuous, it follows from Proposition 2.4 in Ekeland and Témam [15] that  $\pi(X) > -\infty$  for every  $X \in L^1$ . Note that for every  $m \in \mathbb{R}$  we have

$$-\pi(-m) = -\pi(0) - m = -m = \pi(0) - m = \pi(m), \quad (5.2)$$

showing that every risk-free payoff is frictionless. Now, assume that  $\mathcal{A}$  is not pointed so that  $Z \in \mathcal{A} \cap (-\mathcal{A})$  for some nonzero  $Z \in L^1$ . Then,  $Z$  must satisfy

$$0 = \pi(0) = \pi(Z - Z) \leq \pi(Z) + \pi(-Z) \leq 0$$

by sublinearity. In other words,  $Z$  is frictionless. In addition,  $Z$  must be risky because we would otherwise have

$$0 \leq -\pi(-Z) = -Z = \pi(Z) \leq 0$$

by (5.2), implying  $Z = 0$ . As a result, we infer from Corollary 4.4 that  $\pi(X) = -\mathbb{E}[X]$  for every  $X \in L^1$ , where we have used the fact that  $\pi(1) = \pi(0) - 1 = -1$ . In view of (5.1), this yields alternative (ii) and concludes the proof.  $\square$

**Remark 5.5.** The preceding result does not generally hold if  $\mathcal{A}$  fails to be either monotone or conic. To see this, consider first the set

$$\mathcal{A} = \{X \in L^1; X \text{ is risk-free}\}.$$

It is clear that  $\mathcal{A}$  satisfies all the assumptions of the proposition apart from monotonicity but neither (i) nor (ii) holds. Now, consider the set

$$\mathcal{A} = \{X \in L^1; \mathbb{E}[\min(X, 0)] \geq -1\}.$$

The set  $\mathcal{A}$  satisfies all the assumptions of the proposition apart from conicity but neither (i) nor (ii) holds.

## Risk measures with respect to multiple eligible assets

We characterize when a risk measure of the above type is law invariant in the case that  $\mathcal{M}$  contains a risky eligible payoff. In particular, we show that a risk measure with respect to multiple eligible assets is never law invariant unless it reduces to a multiple of the expectation under the reference probability measure. Here, we set  $\ker(\psi) := \{X \in \mathcal{M}; \psi(X) = 0\}$ .

**Proposition 5.6.** *Assume that  $\mathcal{A} \subset L^1$  is an acceptance set such that  $\mathcal{A} + \ker(\psi)$  is convex and closed and that  $\mathcal{M}$  contains a risky payoff. Moreover, assume that  $\rho(0) = 0$ . Then, the following statements are equivalent:*

(a)  $\rho$  is law invariant.

(b) There exists  $c \in (0, \infty)$  such that  $\rho(X) = -c\mathbb{E}[X]$  for every  $X \in L^1$ .

In this case, there exists  $c \in (0, \infty)$  such that  $\psi(X) = c\mathbb{E}[X]$  for every  $X \in \mathcal{M}$ .

*Proof.* To establish the equivalence, we only need to show that (a) implies (b). To this effect, assume that  $\rho$  is law invariant and note that  $\rho$  is convex and lower semicontinuous by Proposition 4 in Farkas et al. [18]. Since  $\rho(0) = 0$ , the lower semicontinuity of  $\rho$  implies that  $\rho(X) > -\infty$  for every  $X \in L^1$  by Proposition 2.4 in Ekeland and Témam [15]. Let  $U$  be the payoff as in Definition 5.2 and take a risky payoff  $W \in \mathcal{M}$ . Then, we find  $\lambda \in \mathbb{R}$  such that  $Z = U + \lambda W$  is risky and satisfies  $\mathbb{E}[Z] \neq 0$ . In particular, if  $U$  is risky, then it suffices to take  $\lambda = 0$  because  $U$  is nonzero and positive by definition. Now, note that

$$\rho(mZ) = \rho(0) - \psi(mZ) = -m\psi(Z) = m(\rho(0) - \psi(Z)) = m\rho(Z)$$

for every  $m \in \mathbb{R}$ , showing that  $Z$  is strongly frictionless under  $\rho$ . Then, by Theorem 4.8, there exists a constant  $c \in \mathbb{R}$  such that  $\rho(X) = c\mathbb{E}[X]$  for every  $X \in L^1$ . The desired statement follows by observing that  $c = \rho(1) \leq \rho(0) = 0$  and that  $c = 0$  is not possible because  $\psi$  is nonzero.

To conclude the proof, assume that (b) holds and observe that

$$\psi(X) = \rho(0) - \psi(-X) = \rho(-X) = -c\mathbb{E}[-X] = c\mathbb{E}[X]$$

for every  $X \in \mathcal{M}$ . □

**Remark 5.7.** The augmented acceptance set  $\mathcal{A} + \ker(\psi)$  is automatically convex whenever  $\mathcal{A}$  is convex. However, in general, the closedness of  $\mathcal{A}$  is not sufficient for  $\mathcal{A} + \ker(\psi)$  to be closed. We refer to Baes et al. [4] for a variety of conditions ensuring that  $\mathcal{A} + \ker(\psi)$  is closed when  $\mathcal{A}$  is closed.

## 6 Extension to Orlicz spaces

In this section we extend the preceding results to arbitrary Orlicz spaces. Here, payoffs at time 1 are elements of the space  $L^\Phi := L^\Phi(\Omega, \mathcal{F}, \mathbb{P})$  where  $\Phi : [0, \infty) \rightarrow [0, \infty]$  is a nonconstant convex function with  $\Phi(0) = 0$ . Each element of  $L^\Phi$  is an equivalence class with respect to almost-sure equality under  $\mathbb{P}$  of random variables  $X : \Omega \rightarrow \mathbb{R}$  such that

$$\|X\|_\Phi := \inf\{\lambda \in (0, \infty); \mathbb{E}[\Phi(|X|/\lambda)] \leq 1\} < \infty.$$

Note that  $L^\infty \subset L^\Phi \subset L^1$ . The space  $L^\Phi$  is partially ordered by the standard almost-sure ordering with respect to  $\mathbb{P}$  and is equipped with the linear topology induced by the lattice norm  $\|\cdot\|_\Phi$ . The standard space  $L^p$ ,  $p \in [1, \infty)$ , corresponds to the choice  $\Phi(t) = t^p$ . In this case, the function  $\Phi$  satisfies the  $\Delta_2$  condition, i.e. there exist  $s \in (0, \infty)$  and  $k \in \mathbb{R}$  such that  $\Phi(2t) < k\Phi(t)$  for all  $t \in [s, \infty)$ . Since our

underlying probability space is nonatomic, it follows from Theorem 2.1.17 in Edgar and Sucheston [14] that  $\Phi$  satisfies the  $\Delta_2$  condition if, and only if,  $L^\Phi$  coincides with its Orlicz heart

$$H^\Phi := \{X \in L^\Phi; \mathbb{E}[\Phi(|X|/\lambda)] < \infty, \lambda \in (0, \infty)\}.$$

The space  $L^\infty$  corresponds to the choice  $\Phi(t) = 0$  for  $t \in [0, 1]$  and  $\Phi(t) = \infty$  otherwise. In this case we have  $H^\Phi = \{0\}$ . Recall that, if  $\Phi$  is finite, then  $H^\Phi$  coincides with the closure of  $L^\infty$  with respect to the topology induced by  $\|\cdot\|_\Phi$  by Theorem 2.1.14 in Edgar and Sucheston [14].

In this section we assume that prices at time 0 are represented by a pricing functional

$$\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}.$$

All the notions recorded in Definitions 2.1 and 2.3 will be freely adapted to the space  $L^\Phi$ . Pricing functionals and risk measures on Orlicz spaces are discussed, for instance, in Biagini and Frittelli [6] and Cheridito and Li [8]. We also refer to the recent contributions by Gao et al. [22] and Liebrich and Svindland [30].

We start by showing that, in a general Orlicz space, the basic “collapse to the mean” established in Theorem 4.2 fails to hold.

**Example 6.1.** Assume that  $\Phi$  is finite and does not satisfy the  $\Delta_2$  condition so that  $L^\Phi \neq H^\Phi$  and set (see Theorem 1.2 in Gao et al. [22])

$$\mathcal{A} = \{X \in L^\Phi; \mathbb{E}[X] \geq 0, \min(X, 0) \in H^\Phi\}.$$

Clearly, the set  $\mathcal{A}$  is convex, conic, closed, and law invariant. The map  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\pm\infty\}$  given by

$$\pi(X) = \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\}$$

is therefore sublinear, lower semicontinuous, and law invariant. Note that, since  $\pi(0) = 0$ , lower semicontinuity of  $\pi$  implies that  $\pi(X) > -\infty$  for every  $X \in L^\Phi$  by Proposition 2.4 in Ekeland and Témam [15]. Now, taking any risky  $Z \in H^\Phi$  such that  $\mathbb{E}[Z] \neq 0$  it is easy to verify that

$$0 = \pi(0) = \pi(Z + (-Z)) \leq \pi(Z) + \pi(-Z) = -\mathbb{E}[Z] + \mathbb{E}[Z] = 0,$$

where the inequality is due to sublinearity. This shows that  $Z$  is frictionless. However,  $\pi$  is not a multiple of  $\mathbb{E}$  because, for instance,  $\mathbb{E}[X] < \infty = \pi(X)$  for every negative  $X \in L^\Phi \setminus H^\Phi$ .

In order to obtain the same “collapse to the mean” in a general Orlicz spaces we need to replace lower semicontinuity with a stronger property, namely the so-called Fatou property.

**Definition 6.2.** We say that  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  has the *Fatou property*, or is *order lower semicontinuous*, if  $\pi(X) \leq \liminf \pi(X_n)$  for every sequence  $(X_n) \subset L^\Phi$  that converges almost surely to  $X \in L^\Phi$  and admits  $M \in L^\Phi$  such that  $|X_n| \leq M$  for every  $n \in \mathbb{N}$ .

The following lemma highlights the link between the Fatou property and lower semicontinuity and shows that, for a convex and law-invariant functional on a general Orlicz space  $L^\Phi$ , the Fatou property is equivalent to  $\sigma(L^\Phi, L^\infty)$  lower semicontinuity.

**Lemma 6.3.** *For a functional  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  the following statements hold:*

- (1) *If  $\pi$  has the Fatou property, then  $\pi$  is lower semicontinuous.*
- (2) *If  $\pi$  is convex and law invariant, then the following statements are equivalent:*
  - (a)  *$\pi$  has the Fatou property.*

(b)  $\pi$  is  $\sigma(L^\Phi, L^\infty)$  lower semicontinuous.

If  $L^\Phi = H^\Phi$  (e.g. if  $\Phi$  satisfies the  $\Delta_2$  condition) or  $L^\Phi = L^\infty$  (or equivalently  $\Phi$  takes nonfinite values), the above are also equivalent to:

(c)  $\pi$  is lower semicontinuous.

*Proof.* (1) Take a sequence  $(X_n) \subset L^\Phi$  and  $X \in L^\Phi$  such that  $X_n \rightarrow X$  with respect to  $\|\cdot\|_\Phi$ . Since we also have  $X_n \rightarrow X$  with respect to  $\|\cdot\|_1$ , we can extract a subsequence  $(X_{n_k})$  such that  $X_{n_k} \rightarrow X$  almost surely. Without loss of generality we can assume that  $\|X_{n_k}\|_\Phi \leq 2^{-k}$  for every  $k \in \mathbb{N}$  (otherwise we extract a subsequence of  $(X_{n_k})$  that does so). As a result, we have  $M = \sum_{k \in \mathbb{N}} |X_{n_k}| \in L^\Phi$ . Moreover, we clearly have that  $|X_{n_k}| \leq M$  for every  $k \in \mathbb{N}$ . This implies that  $\pi$  is lower semicontinuous whenever it satisfies the Fatou property.

(2) If  $L^\Phi = L^\infty$ , the equivalence follows from point (1) and by combining Theorem 3.2 in Delbaen [10] and Proposition 1.1 in Svindland [32]. If  $L^\Phi = L^1$ , the equivalence between (a) and (c) follows from point (1) and from the dominated convergence theorem, and the equivalence between (b) and (c) follows from a classical result in convex analysis, see e.g. Corollary 5.99 in Aliprantis and Border [1]. In all the other cases, the equivalence follows from Theorems 1.1 and 1.2 in Gao et al. [22].  $\square$

Before we are able to extend our main results to a general Orlicz space we need the following additional preliminary lemma.

**Lemma 6.4.** *For every  $X \in L^\Phi$  there exists an increasing sequence  $(\mathcal{F}_n(X))$  of finitely-generated  $\sigma$ -fields over  $\Omega$  such that  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  with respect to  $\sigma(L^\Phi, L^\infty)$ .*

*Proof.* Fix  $X \in L^\Phi$  and take a sequence  $(\mathcal{F}_n(X))$  of finitely-generated  $\sigma$ -fields over  $\Omega$  such that the  $\sigma$ -field generated by  $\bigcup \mathcal{F}_n(X)$  coincides with the  $\sigma$ -field generated by  $X$ . Then, we have that  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  with respect to  $\|\cdot\|_1$  by Lévy's zero-one law. But this immediately implies that  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  also with respect to  $\sigma(L^\Phi, L^\infty)$ .  $\square$

The following lemma establishes a quantile representation for law-invariant functionals on Orlicz spaces, which extends the representation recorded in Lemma 4.1.

**Lemma 6.5.** *Let  $\varphi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  be a sublinear, law-invariant functional with the Fatou property. Then, there exists  $\mathcal{D} \subset L^\infty$  such that for every  $X \in L^\Phi$*

$$\varphi(X) = \sup_{Y \in \mathcal{D}} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha.$$

If  $L^\Phi = H^\Phi$  (e.g. if  $\Phi$  satisfies the  $\Delta_2$  condition) or  $L^\Phi = L^\infty$  (or equivalently  $\Phi$  takes nonfinite values), the Fatou property can be replaced by lower semicontinuity.

*Proof.* It follows from Lemma 6.3 that  $\pi$  is automatically  $\sigma(L^\Phi, L^\infty)$  lower semicontinuous. The above representation is then an immediate consequence of the general result recorded in Proposition B.3.  $\square$

In view of the above lemmas, all the results from Sections 4 and 5 can be easily extended from  $L^1$  to a general Orlicz space  $L^\Phi$  provided that we replace lower semicontinuity by the Fatou property whenever  $\Phi$  does not satisfy the  $\Delta_2$  condition.

**Theorem 6.6.** *Assume that  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  is convex, law invariant, and has the Fatou property. Then, the following statements are equivalent:*

(a) *There exists a strongly frictionless risky payoff  $Z \in L^\Phi$  with  $\mathbb{E}[Z] \neq 0$ .*

(b) There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^\Phi$ .

(c) Every payoff is frictionless.

If  $\pi$  is additionally sublinear, the above are equivalent to:

(d) There exists a frictionless risky payoff  $Z \in L^\Phi$  with  $\mathbb{E}[Z] \neq 0$ .

If  $L^\Phi = H^\Phi$  (e.g. if  $\Phi$  satisfies the  $\Delta_2$  condition) or  $L^\Phi = L^\infty$  (or equivalently  $\Phi$  takes nonfinite values), the Fatou property can be replaced by lower semicontinuity.

**Corollary 6.7.** Assume that  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  is convex, law invariant, and has the Fatou property. Moreover, assume that every risk-free payoff is frictionless. Then, the following statements are equivalent:

(a) There exist a strongly frictionless risky payoff.

(b) There exists  $c \in \mathbb{R}$  such that  $\pi(X) = c\mathbb{E}[X]$  for every  $X \in L^\Phi$ .

(c) Every payoff is frictionless.

If  $\pi$  is additionally sublinear, the above are equivalent to:

(d) There exists a frictionless risky payoff.

If  $L^\Phi = H^\Phi$  (e.g. if  $\Phi$  satisfies the  $\Delta_2$  condition) or  $L^\Phi = L^\infty$  (or equivalently  $\Phi$  takes nonfinite values), the Fatou property can be replaced by lower semicontinuity.

*Proof of Theorem 6.6 and Corollary 6.7.* It follows from Lemma 6.3 that  $\pi$  is  $\sigma(L^\Phi, L^\infty)$  lower semicontinuous. In view of Lemma 6.4, we can then apply Proposition B.4 to uniquely extend  $\pi$  to a convex,  $\sigma(L^1, L^\infty)$  lower semicontinuous, and law-invariant functional  $\bar{\pi} : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ . The desired assertions now follow by applying the results of Section 4 to the functional  $\bar{\pi}$ . Alternatively, one could start from the representation of  $\pi$  in Lemma 6.5 and follow the same arguments from Section 4 but replacing  $L^1$  with  $L^\Phi$ .  $\square$

**Remark 6.8 (Applications to Schur-convex functionals).** For all  $X, Y \in L^\Phi$  we say that  $X$  dominates  $Y$  in the *convex order*, written  $X \succeq_{cx} Y$ , whenever  $\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)]$  for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . A functional  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  is said to be *Schur-convex*, or *consistent with the convex order*, whenever  $\pi(X) \geq \pi(Y)$  for all  $X, Y \in L^\Phi$  such that  $X \succeq_{cx} Y$ . For more information about Schur-convex functionals we refer to Dana [11]. It follows from Lemma 6.3 and Proposition B.3 that every convex, Schur-convex functional  $\pi : L^\Phi \rightarrow \mathbb{R} \cup \{\infty\}$  with the Fatou property can be represented as

$$\varphi(X) = \sup_{Y \in L^\infty} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi^*(Y)$$

for every  $X \in L^\Phi$ , where

$$\pi^*(Y) = \sup_{X \in L^\Phi} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi(X)$$

for every  $Y \in L^\infty$ . If  $\pi$  is additionally sublinear, there exists  $\mathcal{D} \subset L^\infty$  such that

$$\varphi(X) = \sup_{Y \in \mathcal{D}} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha$$

for every  $X \in L^\Phi$ . If  $L^\Phi = H^\Phi$  (e.g. if  $\Phi$  satisfies the  $\Delta_2$  condition) or  $L^\Phi = L^\infty$  (or equivalently  $\Phi$  takes nonfinite values), the Fatou property can be replaced by lower semicontinuity. The above quantile representation extends the representation in Dana [11] and Grechuk and Zabaranin [26] beyond the  $L^p$  setting and sharpens it by showing that one can always use quantiles of bounded functions.

## A Results from the literature

For easy reference and comparability we reproduce here the key results from literature. In Castagnoli et al. [7] the focus is on pricing functionals given by Choquet integrals on  $L^\infty$ . Their main result is the following “collapse to the mean”.

**Theorem A.1** (Theorem 1 in [7]). *Consider a map  $c : \mathcal{F} \rightarrow [0, 1]$  satisfying the following properties:*

- (1)  $c(\Omega) = 1$  and  $c(E) = 0$  for every  $E \in \mathcal{F}$  such that  $\mathbb{P}(E) = 0$ .
- (2)  $c(E) \leq c(F)$  for all  $E, F \in \mathcal{F}$  such that  $E \subset F$ .
- (3)  $c(E \cup F) \leq c(E) + c(F) - c(E \cap F)$  for all  $E, F \in \mathcal{F}$ .
- (4)  $c(E_n) \rightarrow 0$  for every decreasing sequence  $(E_n) \subset \mathcal{F}$  such that  $\bigcap E_n = \emptyset$ .

Consider the functional  $\pi : L^\infty \rightarrow \mathbb{R}$  defined by

$$\pi(X) = \int_{-\infty}^0 (c(X > x) - 1)dx + \int_0^\infty c(X > x)dx.$$

If  $\pi$  is law invariant, then the following statements are equivalent:

- (a) There exists a frictionless risky payoff.
- (b)  $\pi(X) = \mathbb{E}[X]$  for every  $X \in L^\infty$ .

The previous result was generalized by Frittelli and Rosazza Gianin [20] in the context of risk measures. The reformulation in terms of pricing functionals is as follows.

**Theorem A.2** (Propositions 8 and 9 in [20]). *Let  $\pi : L^\infty \rightarrow \mathbb{R}$  be convex, nondecreasing, law invariant, and cash additive. Then, the following statements are equivalent:*

- (a) There exists a strongly frictionless risky payoff.
- (b)  $\pi(X) = \mathbb{E}[X]$  for every  $X \in L^\infty$ .

If  $\pi$  is additionally sublinear, the above are equivalent to:

- (c) There exists a frictionless risky payoff.

**Remark A.3.** (i) Note that, by Schmeidler [31], the assumptions on  $c$  in Theorem A.1 imply that  $\pi$  is sublinear, nondecreasing, and cash additive. Hence, Theorem A.2 is a true generalization of Theorem A.1.

(ii) The functional  $\pi$  in Theorem A.2 is automatically continuous. This follows from Lemma 4.3 in Föllmer and Schied [17].

## B Law-invariant functionals

We consider a dual pairing  $(\mathcal{X}, \mathcal{X}^*)$  where  $\mathcal{X}$  and  $\mathcal{X}^*$  are law-invariant subspaces of  $L^1$  such that  $\mathbb{E}[XY]$  is finite for all  $X \in \mathcal{X}$  and  $Y \in \mathcal{X}^*$ . As usual, we denote by  $\sigma(\mathcal{X}, \mathcal{X}^*)$  the coarsest topology on  $\mathcal{X}$  that ensures the continuity of all the linear functionals  $\varphi_Y : \mathcal{X} \rightarrow \mathbb{R}$  defined by  $\varphi_Y(X) = \mathbb{E}[XY]$ .

Let  $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ . The conjugate of  $\pi$  is the functional  $\pi^* : \mathcal{X}^* \rightarrow \mathbb{R} \cup \{\pm\infty\}$  defined by

$$\pi^*(Y) := \sup_{X \in \mathcal{X}} \mathbb{E}[XY] - \pi(X).$$

It follows from classical convex duality that, if  $\pi$  is convex and  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous, then

$$\pi(X) = \sup_{Y \in \mathcal{X}^*} \mathbb{E}[XY] - \pi^*(Y)$$

for every  $X \in \mathcal{X}$ . If  $\pi$  is also sublinear, the only finite value that  $\pi^*$  can take is 0 and, hence, there exists a set  $\mathcal{D} \subset \mathcal{X}^*$  such that

$$\pi(X) = \sup_{Y \in \mathcal{D}} \mathbb{E}[XY]$$

for every  $X \in \mathcal{X}$ . In what follows we adhere to the notation for quantile functions introduced at the beginning of Section 4. The following characterization of the so-called convex order in terms of quantiles can be found e.g. in Dana [11].

**Lemma B.1** (Lemma 2.2 in [11]). *For all  $X, Y \in L^1$  the following statements are equivalent:*

- (a)  $\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)]$  for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$ .
- (b) For every nondecreasing function  $g : (0, 1) \rightarrow \mathbb{R}$  we have

$$\int_0^1 q_X(\alpha)g(\alpha)d\alpha \geq \int_0^1 q_Y(\alpha)g(\alpha)d\alpha.$$

The following Hardy-Littlewood-type result can be found in Chong and Rice [9]. For convenience, for every  $X \in L^1$  we set  $\mathcal{C}(X) := \{Y \in L^1; Y \sim X\}$ .

**Lemma B.2** (Theorem 13.4 in [9]). *For all  $X, Y \in L^1$  such that  $\mathbb{E}[XY]$  is finite we have*

$$\sup_{X' \in \mathcal{C}(X)} \mathbb{E}[X'Y] = \sup_{Y' \in \mathcal{C}(Y)} \mathbb{E}[XY'] = \int_0^1 q_X(\alpha)q_Y(\alpha)d\alpha.$$

The next proposition extends to our general setting the quantile representation obtained by Dana [11] in  $L^\infty$  and by Grechuk and Zabaranin [26] in the setting of  $L^p$  spaces (for sublinear functionals). The notions of law invariance and Schur-convexity are defined as above.

**Proposition B.3.** *For a convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous map  $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$  the following statements are equivalent:*

- (a)  $\pi$  is law invariant.
- (b)  $\pi$  is Schur-convex.
- (c)  $\pi^*$  is law invariant.
- (d)  $\pi^*$  is Schur-convex.

In any of the above cases, for every  $X \in \mathcal{X}$  we have

$$\pi(X) = \sup_{Y \in \mathcal{X}^*} \int_0^1 q_X(\alpha)q_Y(\alpha)d\alpha - \pi^*(Y)$$

and, similarly, for every  $Y \in \mathcal{X}^*$  we have

$$\pi^*(Y) = \sup_{X \in \mathcal{X}} \int_0^1 q_X(\alpha)q_Y(\alpha)d\alpha - \pi(X).$$

If  $\pi$  is also sublinear, we find  $\mathcal{D} \subset \mathcal{X}^*$  such that for every  $X \in \mathcal{X}$

$$\pi(X) = \sup_{Y \in \mathcal{D}} \int_0^1 q_X(\alpha)q_Y(\alpha)d\alpha.$$

*Proof.* To establish the equivalence, we clearly only have to prove that (a) and (b) are equivalent and imply (c). It is immediate to see that (b) implies (a). To show the converse, assume that  $\pi$  is law invariant. Then, it follows from classical convex duality that

$$\pi(X) = \sup_{X' \in \mathcal{C}(X)} \pi(X') = \sup_{X' \in \mathcal{C}(X)} \sup_{Y \in \mathcal{X}^*} \mathbb{E}[X'Y] - \pi^*(Y) = \sup_{Y \in \mathcal{X}^*} \sup_{X' \in \mathcal{C}(X)} \mathbb{E}[X'Y] - \pi^*(Y)$$

for every  $X \in \mathcal{X}$ , where we have used law invariance in the first equality. As a result, we infer from Lemma B.2 that

$$\pi(X) = \sup_{Y \in \mathcal{X}^*} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi^*(Y)$$

for every  $X \in \mathcal{X}$ . A direct application of Lemma B.1 now shows that  $\pi$  is Schur-convex. Hence, (a) implies (b).

To prove that (a) implies (c), assume that  $\pi$  is law invariant and note that

$$\pi^*(Y) = \sup_{X \in \mathcal{X}} \sup_{X' \in \mathcal{C}(X)} \mathbb{E}[X'Y] - \pi(X') = \sup_{X \in \mathcal{X}} \sup_{X' \in \mathcal{C}(X)} \mathbb{E}[X'Y] - \pi(X)$$

for every  $Y \in \mathcal{X}^*$ , where we have used law invariance in the second equality. Using Lemma B.2 we conclude that

$$\pi^*(Y) = \sup_{X \in \mathcal{X}} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi(X)$$

for every  $Y \in \mathcal{X}^*$ . This shows that  $\pi^*$  is law invariant and proves the desired implication.  $\square$

We conclude this appendix by highlighting two key results about law-invariant functionals that are convex and  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous. On the one side, they are uniquely determined by the values they take on  $L^\infty$ . On the other side, they can always be extended to convex, (norm) lower semicontinuous, and law-invariant functionals defined on the entire space  $L^1$ . For this to hold, every element of  $\mathcal{X}$  has to be the limit of a suitable sequence of conditional expectations. Here, for every functional  $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$  and every subset  $\mathcal{Y} \subset \mathcal{X}$  we denote by  $\pi|_{\mathcal{Y}}$  the restriction of  $\pi$  to  $\mathcal{Y}$ .

**Proposition B.4.** *Assume that  $L^\infty \subset \mathcal{X}$  and that for every  $X \in \mathcal{X}$  there exists an increasing sequence  $(\mathcal{F}_n(X))$  of finitely-generated  $\sigma$ -fields over  $\Omega$  such that  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  with respect to  $\sigma(\mathcal{X}, \mathcal{X}^*)$ . Then, the following statements hold:*

(1) *For every convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous, law-invariant map  $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$  we have*

$$\pi(X) = \lim \pi|_{L^\infty}(\mathbb{E}[X | \mathcal{F}_n(X)])$$

*for every  $X \in \mathcal{X}$ . In particular,  $\pi$  is uniquely determined by its restriction to  $L^\infty$ .*

(2) *Assume that  $L^\infty \subset \mathcal{X}^*$ . Then, every convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous, law-invariant map  $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$  is  $\sigma(\mathcal{X}, L^\infty)$  lower semicontinuous and admits a convex, lower semicontinuous, and law-invariant map  $\bar{\pi} : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  satisfying  $\bar{\pi}|_{\mathcal{X}} = \pi$ . In particular, for every  $X \in L^1$  we have*

$$\bar{\pi}(X) = \sup_{Y \in L^\infty} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi^*(Y).$$

*Proof.* (1) Take  $X \in \mathcal{X}$ . Note that for every  $n \in \mathbb{N}$  and for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we have

$$\mathbb{E}[f(\mathbb{E}[X | \mathcal{F}_n(X)])] \leq \mathbb{E}[\mathbb{E}[f(X) | \mathcal{F}_n(X)]] = \mathbb{E}[f(X)]$$

by Jensen's inequality. Hence,  $\pi(\mathbb{E}[X | \mathcal{F}_n(X)]) \leq \pi(X)$  by the Schur-convexity established in Proposition B.3, so that

$$\limsup \pi(\mathbb{E}[X | \mathcal{F}_n(X)]) \leq \pi(X).$$

Moreover,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuity implies that

$$\pi(X) \leq \liminf \pi(\mathbb{E}[X | \mathcal{F}_n(X)]).$$

This yields  $\pi(X) = \liminf \pi(\mathbb{E}[X | \mathcal{F}_n(X)])$ , proving the desired claim.

(2) The restriction of  $\pi$  to  $L^\infty$  is clearly convex and law invariant. Moreover, it is lower semicontinuous with respect to  $\|\cdot\|_\infty$ . This is because for every sequence  $(X_n) \subset L^\infty$  and every  $X \in L^\infty$  such that  $X_n \rightarrow X$  with respect to  $\|\cdot\|_\infty$  we automatically have  $X_n \rightarrow X$  with respect to  $\sigma(\mathcal{X}, \mathcal{X}^*)$ . Then, it follows from Proposition 1.1 in Svindland [32] that  $\pi|_{L^\infty}$  is even  $\sigma(L^\infty, L^\infty)$  lower semicontinuous. In particular, for every  $X \in L^\infty$  we have

$$\pi(X) = \sup_{Y \in L^\infty} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi^*(Y)$$

by Proposition B.3. Now, define a map  $\bar{\pi} : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$  by setting

$$\bar{\pi}(X) = \sup_{Y \in L^\infty} \int_0^1 q_X(\alpha) q_Y(\alpha) d\alpha - \pi^*(Y).$$

It is clear that  $\bar{\pi}$  is convex, lower semicontinuous, and law invariant. Note that the restriction of  $\bar{\pi}$  to  $\mathcal{X}$  is automatically  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous because  $L^\infty$  is contained in  $\mathcal{X}^*$ . Since  $\bar{\pi}$  and  $\pi$  have the same restrictions to  $L^\infty$ , it follows from point (1) that  $\bar{\pi}$  is the unique desired extension of  $\pi$ . Since, being lower semicontinuous,  $\bar{\pi}$  is automatically  $\sigma(L^1, L^\infty)$  lower semicontinuous by Corollary 5.99 in Aliprantis and Border [1], we infer that  $\pi$  is  $\sigma(\mathcal{X}, L^\infty)$  lower semicontinuous.  $\square$

We conclude this appendix by highlighting an interesting result about law-invariant sets. For every  $\mathcal{A} \subset \mathcal{X}$  it is useful to consider the functional  $\delta_{\mathcal{A}} : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$\delta_{\mathcal{A}}(X) := \begin{cases} 0 & \text{if } X \in \mathcal{A}, \\ \infty & \text{otherwise.} \end{cases}$$

**Lemma B.5.** *Assume that  $\mathcal{A} \subset \mathcal{X}$  is convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  closed, and law invariant. Then, for every  $X \in \mathcal{A}$  and every sub- $\sigma$ -field  $\mathcal{G}$  of  $\mathcal{F}$  we have  $\mathbb{E}[X | \mathcal{G}] \in \mathcal{A}$ .*

*Proof.* Note that  $\delta_{\mathcal{A}}$  is convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  lower semicontinuous, and law invariant. Then, it follows from Proposition B.3 that  $\delta_{\mathcal{A}}$  is also Schur-convex. Let  $\mathcal{G}$  be a sub- $\sigma$ -field of  $\mathcal{F}$  and note that  $\mathbb{E}[f(\mathbb{E}[X | \mathcal{G}])] \leq \mathbb{E}[f(X)]$  for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by Jensen's inequality. This yields  $\delta_{\mathcal{A}}(\mathbb{E}[X | \mathcal{G}]) \leq \delta_{\mathcal{A}}(X) = 0$ , which in turn implies that  $\delta_{\mathcal{A}}(\mathbb{E}[X | \mathcal{G}]) = 0$  and, thus, that  $\mathbb{E}[X | \mathcal{G}] \in \mathcal{A}$ .  $\square$

Here, for every set  $\mathcal{A} \subset \mathcal{X}$  we denote by  $\text{cl}_{\sigma(\mathcal{X}, \mathcal{X}^*)}(\mathcal{A})$  the closure of  $\mathcal{A}$  with respect to  $\sigma(\mathcal{X}, \mathcal{X}^*)$  and by  $\text{cl}_{\|\cdot\|_1}(\mathcal{A})$  the closure of  $\mathcal{A}$  with respect to  $\|\cdot\|_1$ .

**Proposition B.6.** *Assume that  $L^\infty \subset \mathcal{X}$  and that for every  $X \in \mathcal{X}$  there exists an increasing sequence  $(\mathcal{F}_n(X))$  of finitely-generated  $\sigma$ -fields over  $\Omega$  such that  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  with respect to  $\sigma(\mathcal{X}, \mathcal{X}^*)$ . Then, for every convex,  $\sigma(\mathcal{X}, \mathcal{X}^*)$  closed, and law-invariant set  $\mathcal{A} \subset \mathcal{X}$  the following statements hold:*

- (1)  $\mathcal{A} = \text{cl}_{\sigma(\mathcal{X}, \mathcal{X}^*)}(\mathcal{A} \cap L^\infty)$ .
- (2) If  $L^\infty \subset \mathcal{X}^*$ , then  $\mathcal{A} = \text{cl}_{\|\cdot\|_1}(\mathcal{A}) \cap \mathcal{X}$ .

*Proof.* (1) Clearly, we only need to show that  $\mathcal{A} \subset \text{cl}_{\sigma(\mathcal{X}, \mathcal{X}^*)}(\mathcal{A} \cap L^\infty)$ . To this effect, take  $X \in \mathcal{A}$  and note that  $\mathbb{E}[X | \mathcal{F}_n(X)] \in \mathcal{A} \cap L^\infty$  for every  $n \in \mathbb{N}$  by Lemma B.5. Since  $\mathbb{E}[X | \mathcal{F}_n(X)] \rightarrow X$  with respect to  $\sigma(\mathcal{X}, \mathcal{X}^*)$  by assumption, we conclude that  $X \in \text{cl}_{\sigma(\mathcal{X}, \mathcal{X}^*)}(\mathcal{A} \cap L^\infty)$ .

(2) Clearly, we only need to show that  $\text{cl}_{\|\cdot\|_1}(\mathcal{A}) \cap \mathcal{X} \subset \mathcal{A}$ . To this end, take  $X \in \mathcal{X}$  and assume that  $X_n \rightarrow X$  with respect to  $\|\cdot\|_1$  for a suitable sequence  $(X_n) \subset \mathcal{A}$ . Since  $\delta_{\mathcal{A}}$  is  $\sigma(\mathcal{X}, L^\infty)$  lower semicontinuous by Proposition B.4, we infer that  $\mathcal{A}$  is  $\sigma(\mathcal{X}, L^\infty)$  closed. Since  $X_n \rightarrow X$  with respect to  $\sigma(\mathcal{X}, L^\infty)$ , we conclude that  $X \in \mathcal{A}$ .  $\square$

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