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Annihilators of the ideal class group of a cyclic extension of a global function field

by

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1. Introduction. In [T88] F. Thaine studied the relation of the ideal class group $\mathrm{Cl}(L)$ of a totally real absolutely abelian number field L and a certain group of cyclotomic units introduced by W. Sinnott in [S80]. These can be used to produce annihilators of the p-Sylow subgroup $\mathrm{Cl}(L)_p$ of the ideal class group. Sinnott's methods were generalized by K. Rubin to abelian extensions of an imaginary quadratic base field K in [R87], where the cyclotomic units are replaced by elliptic units. These approaches are closely related to Kolyvagin's Euler system machinery; in fact, Rubin already works with so-called *special numbers* in a quite general setting and then specializes to the case of an imaginary quadratic base field. This method yields nice results when p does not divide $[L:\mathbb{Q}]$ (resp. $[L:K]$), but when p is a divisor of the degree of the extension, the annihilation statement obtained is not satisfying.

When $L/\mathbb{Q}$ is a cyclic extension of degree p^k, the ideal class group (considered as the Galois group of the Hilbert class field of L) splits into a genus part (corresponding to the extension F_I/L, where F_I denotes the genus field of L) and a so-called *non-genus part*. In order to study this non-genus part $(\sigma-1)\,\mathrm{Cl}(L)_p$, where σ denotes a generator of $\mathrm{Gal}(L/\mathbb{Q})$, C. Greither and R. Kučera [GK04], [GK06] extended Rubin's method, to so-called *semispecial numbers*. These satisfy weaker conditions but are still sufficient to produce annihilators. The source of semispecial numbers in this case are certain roots with respect to group-ring-valued exponents of Sinnott's cyclotomic units.

It was shown by D. Burns and A. Hayward that the annihilation result of Greither and Kučera can also be deduced from the equivariant Tamagawa

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number conjecture (see [BH07]); however, the proof of Greither and Kučera is constructive whereas the method of Burns and Hayward uses abstract arguments. In particular, the explicit construction of the roots of circular units enables Greither and Kučera to refine their method in [GK15] and weaken the conditions on L to cover even more cases. They use results on Sinnott's module from [GK14] which are formulated in an abstract way without specifying an extension of number fields. Hence, these results can also be used in other cases. This is done by H. Chapdelaine and R. Kučera in [CK19], where they prove an annihilation result for a cyclic extension of an imaginary quadratic field of prime power degree. They take roots of elliptic units studied by H. Oukhaba [O03] to obtain semispecial numbers and then adapt the methods of Greither and Kučera to this case.

In this article, we want to apply the methods described above to the case of global function fields. For this purpose, we explicitly construct elliptic units based on the torsion points of sign-normalized rank-1 Drinfeld modules as in [H85]. As in the case of cyclotomic units (see e.g. [K04]), there are several methods to construct elliptic units in an arbitrary real abelian extension of global function fields. We use the function field version of Sinnott's cyclotomic units and are hence able to prove an index formula for this subgroup of the units of L analogously to the ones in the rational case [S80] and in the imaginary quadratic case [O03]. There exist some other index formulae for elliptic units in function fields, e.g. by L. Yin [Y97a], [Y97b], who studied cyclotomic units in ray class fields in the sense of L. Washington [W97], or by H. Oukhaba [O92], [O95], [O97], who studied elliptic units in extensions where at most one prime ideal ramifies in L/K. However, there is no discussion of an index formula for a general abelian extension of global function fields known to the author. Moreover, we can use the methods of Greither and Kučera to extract roots of the elliptic units defined and obtain an annihilation result similar to the one in the number field case.

The article has the following structure: For the convenience of the reader, we first present a collection of the necessary notation and state the main results afterwards. Then we introduce the elliptic units and prove an index formula for them (Sections 2–4). This part will closely follow [O03]. The rest of the article (Sections 5–8) will deal with the desired annihilation result and will have the same structure as [CK19].

1.1. Notation and preliminaries. Let K be a global function field with constant field $\mathbb{F}_q$ and let ∞ be a fixed place of K of degree d_∞.

- $\mathcal{O}_K$ is the ring of functions in K which have no poles away from ∞.
- $h(K)$ (resp. $h := h_K$) is the class number of K (resp. $\mathcal{O}_K$), i.e. the cardinality of $\mathrm{Pic}(K)$ (resp. $\mathrm{Pic}(\mathcal{O}_K)$). Note that $h = h(K)d_\infty$.

- $w_\infty := q^{d_\infty} - 1$.
- ord_∞ is the valuation at ∞.
- K_∞ is the completion of K at ∞.
- $\mathbb{F}_\infty$ is the constant field of K_∞.
- For any prime $\mathfrak{p}$ of K set $N\mathfrak{p} := q^{\deg(\mathfrak{p})}$. This is the order of the residue class field at $\mathfrak{p}$.

An extension of K is called *real* if it is contained in K_∞. Now let ρ be a sign-normalized rank-1 Drinfeld module with respect to a fixed sign-function sgn. Then we set $K_{(1)}$ to be the extension of K generated by all coefficients of ρ_x, $x \in \mathcal{O}_K$. Note that this extension is finite. For any integral ideal $\mathfrak{m} \subseteq \mathcal{O}_K$,

- $\rho_\mathfrak{m}$ is the generator of the principal ideal generated by the elements ρ_x for all $x \in \mathfrak{m}$,
- $\Lambda_\mathfrak{m}$ is the set of $\mathfrak{m}$-torsion points of ρ,
- $K_\mathfrak{m} := K_{(1)}(\Lambda_\mathfrak{m})$,
- $H_\mathfrak{m}$ is the maximal real subfield of $K_\mathfrak{m}$ and is called the *real ray class field* of K modulo $\mathfrak{m}$ (in particular $H = H_{(1)}$ is the *real Hilbert class field* of K),
- $H_{\mathfrak{m}^\infty} := \bigcup_{n\geq 1} H_{\mathfrak{m}^n}$.

For any finite extension L/K,

- $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L,
- $\mu(L)$ is the group of roots of unity in L,
- $w_L := |\mu(L)|$,
- h_L is the class number of $\mathcal{O}_L$,
- if $\mathfrak{p} \subseteq \mathcal{O}_K$ is a prime ideal, then $\mathfrak{p}_L$ is the product of all ideals of $\mathcal{O}_L$ above $\mathfrak{p}$,
- if L/K is abelian and $\mathfrak{m}$ is an integral ideal of K, set $L_\mathfrak{m} = L \cap H_\mathfrak{m}$.

Note that $w_K = q - 1$.

REMARK 1.1. It is shown in [H85, §3, §4] that

(i) $w_{H_\mathfrak{m}} = w_\infty$ for all $\mathfrak{m}$ (see [H85, §3]), so $\mathbb{F}_\infty$ is the constant field of $H_\mathfrak{m}$,
(ii) we have $[H_\mathfrak{m} : K] = \frac{h}{w_K}|(\mathcal{O}_K/\mathfrak{m})^\times|$ (see [H85, eq. (3.2)]) for $\mathfrak{m} \neq (1)$ and $[H : K] = h$,
(iii) $[K_\mathfrak{m} : H_\mathfrak{m}] = w_\infty$ for $\mathfrak{m} \neq 1$ (see [H85, §4]) and $[K_{(1)} : H] = w_\infty/w_K$ (see [H85, Cor. 4.8(2)]).

Now suppose that the extension L/K is Galois and $\mathfrak{p}$ is a prime of K. Then:

- $D_\mathfrak{P} \subseteq \mathrm{Gal}(L/K)$ is the decomposition group of a prime $\mathfrak{P}$ of L above $\mathfrak{p}$. If L/K is abelian, this subgroup does not depend on the choice of the prime $\mathfrak{P}$, hence we write $D_\mathfrak{p}$ in this case.
- $T_\mathfrak{P} \subseteq D_\mathfrak{P}$ is the inertia subgroup. If L/K is abelian, we again write $T_\mathfrak{p}$.

- $(\mathfrak{P}, L/K)$ (or $\sigma_{\mathfrak{P}}$ if the extension is clear) is a lift to $\mathrm{Gal}(L/K)$ of the corresponding Frobenius element in $D_{\mathfrak{P}}/T_{\mathfrak{P}}$. These elements form a conjugacy class in $\mathrm{Gal}(L/K)$ which will be denoted by $(\mathfrak{p}, L/K)$ (or $\sigma_{\mathfrak{p}}$). If L/K is abelian and $\mathfrak{p}$ is unramified, this conjugacy class contains only one element, which coincides with the Artin symbol.

For any abelian group G, we set

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$$

to be the group of characters of G. For any subset $U \subseteq G$, we define

$$s(U) := \sum_{\sigma \in U} \sigma \in \mathbb{Z}[G].$$

If U is a subgroup of G, we define the associated idempotent

$$e_U := \frac{1}{|U|} s(U) \in \mathbb{Q}[G].$$

To a character $\chi \in \widehat{G}$, we also assign an idempotent

$$e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathbb{C}[G].$$

By extension of scalars with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$, we can evaluate a character $\chi \in \widehat{G}$ at an element $a = \sum_{\sigma \in G} a_\sigma \sigma \in R[G]$, i.e. we set

$$\chi(a) = \sum_{\sigma \in G} a_\sigma \chi(\sigma) \in \mathbb{C}.$$

Finally, for any multiplicative abelian group A and a positive integer m, we set $A/m := A/(A^m)$.

1.2. Main results. Let L/K be a finite real abelian extension with Galois group G. Then the elliptic units C_L of L are essentially the norms of torsion points in $K_{\mathfrak{m}}$ together with certain unramified units (for a precise definition see Section 2.3). These form a subgroup of $\mathcal{O}_L^\times$ which has finite index given by

Theorem A. *We have*

$$[\mathcal{O}_L^\times : C_L] = \frac{(h_K w_\infty)^{[L:K]-1} w_K h_L}{w_L h_K} \frac{\prod_{\mathfrak{p}} [L \cap H_{\mathfrak{p}^\infty} : L_{(1)}]}{[L : L_{(1)}]} \frac{[\mathbb{Z}[G] : U']}{d(L)}.$$

Remark. The Sinnott module $U' \subseteq \mathbb{Q}[G]$ is defined in [S80]. The Sinnott index $[\mathbb{Z}[G] : U']$ as well as the number $d(L)$ can be computed in certain cases (cf. Remark 4.6 and Proposition 4.5). This index formula is an analogue of [S80, Thm. 4.1] and [O03, Thm. 1]. It is proven as Theorem 4.4.

Now let p be an odd prime not dividing the class number h_K of $\mathcal{O}_K$, the characteristic of K or the number of roots of unity w_K of K. Suppose that

L/K is cyclic of degree p^k for some $k > 0$; then we can define a subgroup $\mathcal{C}_L \subseteq \mathcal{O}_L^\times$ satisfying $\mathcal{C}_L^{h_K} \cdot \mu(L) = C_L$. Let η be a top generator of $\mathcal{C}_L$ (precisely defined in Section 5) and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the primes of K which ramify in L. We assume $s \geq 2$. Let σ be a generator of $\mathrm{Gal}(L/K)$. Then there is a certain subextension $K \subseteq L' \subset L$ such that we get

THEOREM B. *Define* $y := \prod_{i=2}^{s-1}(1-\sigma^{n_i})$, *where* n_i *is the index of the decomposition group of* $\mathfrak{p}_i$ *in* $\mathrm{Gal}(L/K)$. *Then there exists a unique* $\alpha \in L$ *with* $\eta = \alpha^y$ *and* $N_{L/L'}(\alpha) = 1$.

REMARK. The element α of Theorem B is a semispecial number in the sense of Definition 7.3 (cf. Theorem 7.4). This result is an analogue of [GK15, Thm. 1.2] and [CK19, Thm. 4.2]. The field L' is defined right before Theorem 5.13.

Now we can extend $\mathcal{C}_L$ by α_j (taking a root for each subextension $K \subseteq L_j \subseteq L$) to obtain $\overline{\mathcal{C}_L}$. In this special case, the index formula of Theorem A simplifies significantly and we obtain

$$[\mathcal{O}_L^\times : \overline{\mathcal{C}_L}] = w_\infty^{p^k-1} \cdot \frac{h_L}{h_K} \cdot \varphi_L^{-1}$$

for a certain p-power φ_L (cf. Theorem 6.2). In particular, $[\overline{\mathcal{C}_L} : C_L] = p^\nu$ where ν is determined by the n_i (also see Theorem 6.2). Our main result then reads

THEOREM C. *There exists a number* $0 \leq r < k$ *such that*

$$\mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(L/K)]}((\mathcal{O}_L^\times/\overline{\mathcal{C}_L})_p) \subseteq \mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(L/K)]}((1-\sigma^{p^r})\,\mathrm{Cl}(\mathcal{O}_L)_p).$$

REMARK. This is an analogue of [GK15, Thm. 5.3] and [CK19, Thm. 7.5]. The number r has a concrete description given in Theorem 8.8.

2. Elliptic units in global function fields. Let Ω be the completion of the algebraic closure of K_∞ and let Γ be a lattice in Ω, i.e. a finitely generated projective $\mathcal{O}_K$-module. The *exponential function* associated to Γ is defined by

$$e_\Gamma : \Omega \to \Omega, \qquad z \mapsto z \prod_{\substack{\gamma \in \Gamma \\ \gamma \neq 0}} \left(1 - \frac{z}{\gamma}\right).$$

We say that Γ is *special* if the rank-1 Drinfeld module associated to Γ (see [H85, §5]) is sign-normalized with respect to the fixed sign-function sgn. For each Γ, there exists an invariant $\xi(\Gamma) \in \Omega^\times$ such that $\xi(\Gamma)\Gamma$ is special. This invariant is unique up to multiplication by an element of $\mathbb{F}_\infty$.

2.1. Unramified elliptic units. Following [O97, Sec. 2], we can fix a fractional ideal $\mathfrak{c}$ of K and a choice of the invariant $\xi(\mathfrak{c})$ such that the sign-normalized rank-1 Drinfeld module associated to $\Gamma := \xi(\mathfrak{c})\mathfrak{c}$ is exactly ρ. Let D be the differential of the twisted polynomial ring (see e.g. [H85, §4]). Then for any non-zero integral ideal $\mathfrak{a}$ of K, the rank-1 Drinfeld module associated to $D(\rho_{\mathfrak{a}})\mathfrak{a}^{-1}\Gamma$ is sign-normalized with respect to sgn, hence we can choose $\xi(\mathfrak{a}^{-1}\mathfrak{c}) = D(\rho_{\mathfrak{a}})\xi(\mathfrak{c})$. Any fractional ideal of K is of the form $\mathfrak{d} = \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}$ and setting $\tau := (\mathfrak{d}^{-1}\mathfrak{c}, K_{(1)}/K)$ we can define

$$\xi(\mathfrak{d}) = \frac{D(\rho_{\mathfrak{b}})}{D(\rho_{\mathfrak{a}})^{\tau}}\xi(\mathfrak{c}).$$

Lemma 2.1. *The element $\xi(\mathfrak{d})$ is well-defined, i.e. independent of the choice of $\mathfrak{a}$ and $\mathfrak{b}$. It depends on the choice of $\mathfrak{c}$ and $\xi(\mathfrak{c})$.*

Proof. Suppose that $\mathfrak{d} = \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c} = \mathfrak{a}'\mathfrak{b}'^{-1}\mathfrak{c}$. This implies $\mathfrak{a}\mathfrak{b}' = \mathfrak{a}'\mathfrak{b}$ and hence

$$\rho_{\mathfrak{a}\mathfrak{b}'} = \rho_{\mathfrak{a}'\mathfrak{b}}.$$

The ideal class group acts on the set of isomorphism classes of rank-1 Drinfeld modules and via this action we obtain (cf. [R02, Prop. 13.15])

$$\rho_{\mathfrak{a}\mathfrak{b}'}\rho_{\mathfrak{a}'}^{\sigma_{\mathfrak{a}\mathfrak{b}'}} = \rho_{\mathfrak{a}\mathfrak{a}'\mathfrak{b}'} = \rho_{\mathfrak{b}'}^{\sigma_{\mathfrak{a}\mathfrak{a}'}}\rho_{\mathfrak{a}\mathfrak{a}'}, \qquad \rho_{\mathfrak{a}'\mathfrak{b}}\rho_{\mathfrak{a}}^{\sigma_{\mathfrak{a}'\mathfrak{b}}} = \rho_{\mathfrak{a}\mathfrak{a}'\mathfrak{b}} = \rho_{\mathfrak{b}}^{\sigma_{\mathfrak{a}\mathfrak{a}'}}\rho_{\mathfrak{a}\mathfrak{a}'}.$$

Since $\mathfrak{a}\mathfrak{a}' \neq 0$ (we only consider non-zero ideals), we have $D(\rho_{\mathfrak{a}\mathfrak{a}'}) \neq 0$. Because of $\sigma_{\mathfrak{a}\mathfrak{b}'} = \sigma_{\mathfrak{a}'\mathfrak{b}} = \tau\sigma_{\mathfrak{a}\mathfrak{a}'}$, we get

$$\begin{aligned}\left(\frac{D(\rho_{\mathfrak{b}})}{D(\rho_{\mathfrak{a}})^{\tau}}\right)^{\sigma_{\mathfrak{a}\mathfrak{a}'}} &= \frac{D(\rho_{\mathfrak{b}}^{\sigma_{\mathfrak{a}\mathfrak{a}'}})}{D(\rho_{\mathfrak{a}}^{\sigma_{\mathfrak{a}'\mathfrak{b}}})} = \frac{D(\rho_{\mathfrak{a}'\mathfrak{b}})}{D(\rho_{\mathfrak{a}\mathfrak{a}'})}\\ &= \frac{D(\rho_{\mathfrak{a}\mathfrak{b}'})}{D(\rho_{\mathfrak{a}\mathfrak{a}'})} = \frac{D(\rho_{\mathfrak{b}'}^{\sigma_{\mathfrak{a}\mathfrak{a}'}})}{D(\rho_{\mathfrak{a}'}^{\sigma_{\mathfrak{a}\mathfrak{b}'}})} = \left(\frac{D(\rho_{\mathfrak{b}'})}{D(\rho_{\mathfrak{a}'})^{\tau}}\right)^{\sigma_{\mathfrak{a}\mathfrak{a}'}}. \ \blacksquare\end{aligned}$$

With these definitions, we obtain the following explicit form of the principal ideal theorem:

Lemma 2.2 ([O97, Lemma 3]). *Let $\mathfrak{d}_1, \mathfrak{d}_2$ and $\mathfrak{d}$ be fractional ideals of K. Then the ideal $\mathfrak{d}_2\mathfrak{d}_1^{-1}\mathcal{O}_{K_{(1)}}$ is principal, generated by $\xi(\mathfrak{d}_1)/\xi(\mathfrak{d}_2)$. Moreover,*

$$\left(\frac{\xi(\mathfrak{d}_1)}{\xi(\mathfrak{d}_2)}\right)^{(\mathfrak{d}, K_{(1)}/K)} = \frac{\xi(\mathfrak{d}_1\mathfrak{d}^{-1})}{\xi(\mathfrak{d}_2\mathfrak{d}^{-1})}.$$

Now let $\sigma \in \mathrm{Gal}(H/K)$ be arbitrary and let $\mathfrak{a} \subseteq \mathcal{O}_K$ be such that $(\mathfrak{a}^{-1}, H/K) = \sigma$. Let $x \in \mathcal{O}_K$ be a generator of the principal ideal $\mathfrak{a}^h$. Then we can define

$$\partial(\sigma) := (x\xi(\mathfrak{a})^h)^{w_\infty/w_K}.$$

Remark 2.3. (i) The element $\partial(\sigma)^{w_K}$ is well-defined, i.e. independent of the choice of $\mathfrak{a}$ and x. Indeed, it is even independent of the choice of $\mathfrak{c}$ and $\xi(\mathfrak{c})$: If $\mathfrak{c}'$ and $\xi'(\mathfrak{c}')$ were used to define the invariants $\xi'(\mathfrak{d})$ for any fractional ideal $\mathfrak{d}$, then $\xi'(\mathfrak{d})\mathfrak{d}$ would again correspond to a sign-normalized rank-1 Drinfeld module. Since these lattices only differ by an element of

$\mu(H)$ (see e.g. [O97, Sec. 2]), we obtain $\xi(\mathfrak{d}) = \zeta\xi'(\mathfrak{d})$ for some $\zeta \in \mu(H)$. Taking the w_∞th power kills the root of unity, so the element $\partial(\sigma)^{w_K}$ will be the same.

(ii) The above definition differs from the one given in [O97] by the factor $1/w_K$ in the exponent. This definition of $\partial(\sigma)$ still depends on the choice of the generator x and of the ideal $\mathfrak{c}$ and $\xi(\mathfrak{c})$. However, two different choices only differ by an element of $\mu(K)$. Since we are only interested in subgroups of the units containing $\mu(K)$, it suffices to define $\partial(\sigma)$ "up to roots of unity".

LEMMA 2.4. *Let $\sigma, \sigma_1, \sigma_2 \in \mathrm{Gal}(H/K)$. Then $\frac{\partial(\sigma_1)}{\partial(\sigma_2)} \in \mathcal{O}_H^\times$ and*

$$\left(\frac{\partial(\sigma_1)}{\partial(\sigma_2)}\right)^\sigma = \frac{\partial(\sigma_1\sigma)}{\partial(\sigma_2\sigma)}.$$

Proof. This follows directly from Lemma 2.2 and $[\mathcal{O}_{K_{(1)}}^\times : \mathcal{O}_H^\times] = w_\infty/w_K$ (see [Y97b, Lemma 1.5(1)]). ∎

2.2. Ramified elliptic units. Using the exponential function, we can define the element

$$\lambda_\mathfrak{m} := \xi(\mathfrak{m})e_\mathfrak{m}(1)$$

for each integral ideal $\mathfrak{m} \neq (1)$. It is shown in [H85, §5] that this element is a generator of the $\mathfrak{m}$-torsion points $\Lambda'_\mathfrak{m}$ of the sign-normalized rank-1 Drinfeld module ρ' associated to $\xi(\mathfrak{m})\mathfrak{m}$. The construction of $K_\mathfrak{m}$ does not depend on the chosen Drinfeld module but only on the sign-function, hence $\lambda_\mathfrak{m} \in K_{(1)}(\Lambda'_\mathfrak{m}) = K_\mathfrak{m}$ (cf. [H85, §4]). Indeed, if $\mathfrak{b}$ is an integral ideal of $\mathcal{O}_K$ such that $\mathfrak{b}$ is prime to $\mathfrak{m}$ and $(\mathfrak{b}, K_{(1)}/K) = (\mathfrak{m}^{-1}, K_{(1)}/K)$, one can show that $(\mathfrak{bc}, K_\mathfrak{m}/K)$ defines a bijection $\Lambda_\mathfrak{m} \to \Lambda'_\mathfrak{m}$ (note that $\xi(\mathfrak{m})\mathfrak{m}$ is associated to the Drinfeld module $\mathfrak{bc} * \rho$, then use [H85, Thm. 4.12]). It is also shown in [H85, Thm. 4.17] that

$$\alpha_\mathfrak{m} := -N_{K_\mathfrak{m}/H_\mathfrak{m}}(\lambda_\mathfrak{m}) = \lambda_\mathfrak{m}^{w_\infty} \in H_\mathfrak{m}$$

is a unit if $\mathfrak{m}$ is not a prime power and that $\alpha_{\mathfrak{p}^k}$ generates the ideal $\mathfrak{p}_{H_\mathfrak{m}}^{w_\infty/w_k}$.

REMARK 2.5. (i) The element $\lambda_\mathfrak{m}$ depends on the choice of $\mathfrak{c}$ which was used to define the invariants $\xi(\mathfrak{m})$. As already noted in Remark 2.3, changing $\mathfrak{c}$ would change $\xi(\mathfrak{m})$ by a root of unity in H, hence $\alpha_\mathfrak{m} = \lambda_\mathfrak{m}^{w_\infty}$ is independent of this choice.

(ii) Note that our definition of $\alpha_\mathfrak{m}$ differs from the one in [H85] by a sign. This is necessary for obtaining the correct norm relation; see Proposition 2.9 below.

2.3. The group of elliptic units in an arbitrary real abelian extension. Now let L be a finite real abelian extension of K of conductor $\mathfrak{m}$. Recall that for any integral ideal $\mathfrak{n} \subseteq \mathcal{O}_K$ we defined $L_\mathfrak{n} = L \cap H_\mathfrak{n}$. Set

$$\varphi_{L,\mathfrak{n}} := N_{H_\mathfrak{n}/L_\mathfrak{n}}(\alpha_\mathfrak{n})^h.$$

REMARK 2.6. Raising to the hth power is necessary to ensure compatibility with the unramified elliptic units for the desired index formula. If there are no unramified elliptic units (e.g. when L/K is a totally ramified extension), we can also work with the elements $\eta_{\mathfrak{n}} = \varphi_{L,\mathfrak{n}}^{1/h}$; see Section 5.

COROLLARY 2.7.

(i) *If* $\mathfrak{n}$ *is not a prime power, then* $\varphi_{L,\mathfrak{n}} \in \mathcal{O}_{L_{\mathfrak{n}}}^{\times}$.

(ii) *If* $\mathfrak{n} = \mathfrak{p}^k$, *then* $\varphi_{L,\mathfrak{n}}$ *generates the ideal* $\mathfrak{p}_{L_{\mathfrak{n}}}^{[H:L_{(1)}]hw_{\infty}/w_K}$.

Proof. This follows directly from [H85, Thm. 4.17]. ∎

DEFINITION 2.8.

(i) For $\sigma_1, \sigma_2 \in \mathrm{Gal}(L_{(1)}/K)$ define
$$\frac{\partial_L(\sigma_1)}{\partial_L(\sigma_2)} := N_{H/L_{(1)}}\left(\frac{\partial(\widehat{\sigma}_1)}{\partial(\widehat{\sigma}_2)}\right),$$
where $\widehat{\sigma}_i$ is any lift of σ_i to $\mathrm{Gal}(H/K)$.

(ii) The subgroup Δ_L of $\mathcal{O}_{L_{(1)}}^{\times}$ generated by $\mu(L)$ and the elements
$$\frac{\partial_L(\sigma_1)}{\partial_L(\sigma_2)}$$
for $\sigma_1, \sigma_2 \in \mathrm{Gal}(L_{(1)}/K)$ is the *group of unramified elliptic units* of L.

(iii) The elements $\varphi_{L,\mathfrak{n}}$ for $\mathfrak{n} \mid \mathfrak{m}$, $\mathfrak{n} \neq (1)$ are called the *ramified elliptic numbers* of L.

(iv) The $\mathrm{Gal}(L/K)$-submodule P_L of $L^{\times}$ generated by Δ_L and by the ramified elliptic numbers is called the *group of elliptic numbers* of L.

(v) The *group of elliptic units* C_L of L is defined by $C_L := P_L \cap \mathcal{O}_L^{\times}$.

PROPOSITION 2.9. *We have*
$$N_{L_{\mathfrak{n}\mathfrak{p}}/L_{\mathfrak{n}}}(\varphi_{L,\mathfrak{n}\mathfrak{p}}) = \begin{cases} \varphi_{L,\mathfrak{n}}, & \mathfrak{p} \mid \mathfrak{n}, \\ \varphi_{L,\mathfrak{n}}^{1-\sigma_{\mathfrak{p}}^{-1}}, & \mathfrak{p} \nmid \mathfrak{n}, \mathfrak{n} \neq (1), \\ x_{\mathfrak{p}}^{w_{\infty}/w_K[H:L_{(1)}]}\left(\frac{\partial_L(1)}{\partial_L(\sigma_{\mathfrak{p}}^{-1})}\right), & \mathfrak{n} = (1), \end{cases}$$
where $\sigma_{\mathfrak{p}} = (\mathfrak{p}, L_{\mathfrak{n}}/K)$ *and* $x_{\mathfrak{p}}$ *is a generator of* $\mathfrak{p}^h$. *The last equation should be read modulo roots of unity* (cf. Remark 2.3).

Proof. The first two cases can be deduced from the definition of the elliptic units and the norm relation in [O95, Prop. 2.3].

In the case $\mathfrak{n} = (1)$, we use [O97, Remark 1] where it is said that
$$N_{K_{\mathfrak{p}}/K_{(1)}}(\mu_{\mathfrak{p}}) = \frac{\xi(\mathfrak{p}^{-1}\mathfrak{c})}{\xi(\mathfrak{c})}$$
for a generator $\mu_{\mathfrak{p}}$ of $\Lambda_{\mathfrak{p}}$. As already discussed in Section 2.2, we can choose the generator $\mu_{\mathfrak{p}} = \lambda_{\mathfrak{p}}^{(\mathfrak{b}\mathfrak{c}, K_{\mathfrak{p}}/K)^{-1}}$, where $\mathfrak{b}$ is an integral ideal prime to $\mathfrak{p}$ such

that $(\mathfrak{b}, K_{(1)}/K) = (\mathfrak{p}^{-1}, K_{(1)}/K)$. Then via Lemma 2.2 we obtain

$$N_{K_\mathfrak{p}/K_{(1)}}(\lambda_\mathfrak{p}) = N_{K_\mathfrak{p}/K_{(1)}}(\mu_\mathfrak{p})^{(\mathfrak{bc}, K_{(1)}/K)} = \left(\frac{\xi(\mathfrak{p}^{-1}\mathfrak{c})}{\xi(\mathfrak{c})}\right)^{(\mathfrak{p}^{-1}\mathfrak{c}, K_{(1)}/K)} = \frac{\xi(\mathcal{O}_K)}{\xi(\mathfrak{p})}.$$

With the definitions of $\varphi_{L,\mathfrak{p}}$ and $\partial_L(\sigma)$ the desired result follows directly. ■

2.4. L-functions and the analytic class number formula for function fields. Let L be an arbitrary finite abelian extension of K and set $G := \mathrm{Gal}(L/K)$. Let χ be a character of G and let $\mathfrak{p}$ be a prime of K with decomposition group $D_\mathfrak{p}$ and inertia group $T_\mathfrak{p} \subseteq G$. Recall that $\sigma_\mathfrak{p} \in G$ is a lift of the Frobenius element in $D_\mathfrak{p}/T_\mathfrak{p}$. We set

$$\chi(\mathfrak{p}) = \chi(\sigma_\mathfrak{p} e_{T_\mathfrak{p}}).$$

Note that $\chi(\mathfrak{p}) \neq 0$ if and only if $T_\mathfrak{p} \subseteq \ker(\chi)$.

For a finite set S of primes of K we define the S-truncated L-function $L_S(\chi, s)$ associated to χ as the Euler product

$$\prod_{\mathfrak{p}\notin S} (1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1,$$

where the product runs over all primes of K which are not contained in S.

If $S = \emptyset$, we simply write

$$L(\chi, s) = L_\emptyset(\chi, s).$$

If $\chi = 1$, we find that

$$L_S(1, s) = \zeta_{K,S}(s)$$

is the S-truncated Dedekind ζ-function of K.

We summarize some results on L-functions:

Proposition 2.10.

(i) *$L_S(\chi, s)$ has a meromorphic continuation to the whole complex plane, which will also be denoted by $L_S(\chi, s)$. If the extension $K_\chi = L^{\ker(\chi)}$ is not a constant field extension, this continuation is holomorphic.*

(ii) *We have*

$$\zeta_K(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $P(x) \in \mathbb{Z}[x]$ satisfies $P(0) = 1$ and $P(1) = h(K)$.

(iii) *If $L' \supseteq L$ is a finite abelian extension of K with Galois group G' and if ψ is the inflation of χ to G', we have*

$$L_S(\chi, s) = L_S(\psi, s),$$

i.e. the L-function is invariant under inflation.

(iv) *We have*

$$\zeta_L(s) = \zeta_K(s) \cdot \prod_{\chi \neq 1} L(\chi, s),$$

where the product runs over all non-trivial characters of G.

Proof. (i) is [R02, Thm. 9.25]; (ii) is [R02, Thm. 5.9]; (iii) is [N06, Ch. VII, Thm. (10.4)(iii)]; and (iv) is [N06, Ch. VII, Cor. (10.5)(iii)].

Note that the proofs in [N06] do not use the fact that the L-functions considered there are defined over number fields. ∎

NOTATION 2.11. Let $L_S^*(\chi, 0)$ be the leading term of the Taylor expansion of $L_S(\chi, s)$ at $s = 0$.

Now suppose that L is a *real* abelian extension of K. Define $S_\infty(L)$ to be the set of all primes of L lying over ∞ (if the extension is clear, we will simply write S_∞). Since L/K is real, these are exactly $[L : K]$ many primes and each has norm $N\infty = q^{d_\infty}$. Note that $\mathcal{O}_L^\times/\mu(L)$ is a free $\mathbb{Z}$-module of rank $|S_\infty(L)| - 1$ and hence we can choose units $u_1, \ldots, u_{[L:K]-1}$ which project to a basis. Choosing a place $w_0 \in S_\infty(L)$, we can define a matrix

$$(-d_\infty \operatorname{ord}_w(u_i))_{\substack{w \in S_\infty(L)\setminus\{w_0\} \\ i \in \{1,\ldots,[L:K]-1\}}} \in \mathbb{Z}^{([L:K]-1)\times([L:K]-1)}.$$

Then we define the *regulator* R_L of L as the absolute value of the determinant of this matrix. Note that the regulator R_L^{Ros} defined in [R02, Ch. 14] can be obtained from our definition by

$$R_L^{\mathrm{Ros}} = (\log(q))^{[L:K]-1} R_L. \tag{2.1}$$

Hence we obtain

THEOREM 2.12 (Analytic class number formula). *We have*

$$\zeta_{L,S_\infty}^*(0) = -(\log(q))^{[L:K]-1} \frac{h_L R_L}{w_L}.$$

Proof. This is [R02, Thm. 14.4] together with (2.1). ∎

COROLLARY 2.13. *We have*

$$\zeta_{K,\{\infty\}}^*(0) = -\frac{h}{w_K}.$$

Proof. Since $\mathcal{O}_K^\times = \mu(K)$, we get $R_K = 1$. ∎

2.5. Kronecker's limit formulae. We fix a prime $w_0 \in S_\infty(H_\mathfrak{m})$. Then for each subfield L of $H_\mathfrak{m}$, there is a unique prime in $S_\infty(L)$ below w_0. Since ∞ splits completely in $H_\mathfrak{m}$, the valuations of these primes are compatible. By abuse of notation, we denote each of these valuations by $\operatorname{ord}_\infty$, i.e. for an element $x \in H_\mathfrak{m}$ we implicitly set

$$\operatorname{ord}_\infty(x) := \operatorname{ord}_{w_0}(x)$$

and analogously for each subfield L of $H_{\mathfrak{m}}$. The same convention will be used for absolute values.

For $\mathfrak{n} \neq (1)$ let $S_{\mathfrak{n}} := \{\mathfrak{p} \subseteq \mathcal{O}_K \mid \mathfrak{p} \text{ prime}, \mathfrak{p} \mid \mathfrak{n}\}$ be the support of $\mathfrak{n}$. Now we can state Kronecker's second limit formula:

PROPOSITION 2.14.

(i) *Let* $(1) \neq \mathfrak{n} \mid \mathfrak{m}$ *and let* $\chi \in \widehat{\mathrm{Gal}(H_{\mathfrak{n}}/K)}$. *Then*

$$L_{S_{\mathfrak{n}}}(\chi, 0) = \frac{1}{w_\infty} \sum_{\sigma \in \mathrm{Gal}(H_{\mathfrak{n}}/K)} \mathrm{ord}_\infty(\alpha_{\mathfrak{n}}^{\sigma}) \chi(\sigma).$$

(ii) *For any non-trivial character* $\chi \in \widehat{\mathrm{Gal}(H/K)}$, *we have*

$$L(\chi, 0) = \frac{1}{w_\infty h} \sum_{\sigma \in \mathrm{Gal}(H/K)} \mathrm{ord}_\infty(\partial(\sigma)) \chi(\sigma).$$

Proof. Part (i) is exactly the last equation in [H85], whereas (ii) follows directly from [O97, proof of Prop. 3] and Remark 2.3. ∎

REMARK 2.15. By the proposition above, we can regard the ramified elliptic units as Stark units. Indeed, if $\mathfrak{n} \neq (1)$ then the set $S := S_{\mathfrak{n}} \cup \{\infty\}$ contains all places which ramify in $H_{\mathfrak{n}}/K$ and $|S| \geq 2$. Moreover, S contains the completely split prime ∞. Then Stark's conjecture (cf. [T84, Ch. IV, Conj. 2.2]) predicts the existence of an element ε such that

$$L'_S(\chi, 0) = -\frac{1}{w_{H_{\mathfrak{n}}}} \sum_{\sigma \in \mathrm{Gal}(H_{\mathfrak{n}}/K)} \log(|\varepsilon^{\sigma}|_\infty) \chi(\sigma)$$

for all $\chi \in \widehat{\mathrm{Gal}(H_{\mathfrak{n}}/K)}$. By definition of the L-function, we obtain

$$L_S(\chi, s) = (1 - \chi(\infty) N\infty^{-s}) L_{S_{\mathfrak{n}}}(\chi, s) = (1 - N\infty^{-s}) L_{S_{\mathfrak{n}}}(\chi, s)$$

and hence

$$L'_S(\chi, 0) = \log(N\infty) L_{S_{\mathfrak{n}}}(\chi, 0) = -\frac{1}{w_\infty} \sum_{\sigma \in \mathrm{Gal}(H_{\mathfrak{n}}/K)} \log(N\infty^{-\mathrm{ord}_\infty(\alpha_{\mathfrak{n}}^{\sigma})}) \chi(\sigma)$$

$$= -\frac{1}{w_\infty} \sum_{\sigma \in \mathrm{Gal}(H_{\mathfrak{n}}/K)} \log(|\alpha_{\mathfrak{n}}^{\sigma}|_\infty) \chi(\sigma).$$

3. Sinnott's module. Let L/K be a fixed finite real abelian extension of conductor $\mathfrak{m}$ (as in Section 2.3). Remember that for a prime $\mathfrak{p}$ of K the element $\sigma_{\mathfrak{p}} \in G = \mathrm{Gal}(L/K)$ is the lift of an associated Frobenius element in $D_{\mathfrak{p}}/T_{\mathfrak{p}}$. Define $\tau_{\mathfrak{p}} := \sigma_{\mathfrak{p}}^{-1} e_{T_{\mathfrak{p}}} \in \mathbb{Q}[G]$.

DEFINITION 3.1.

(i) We define $\rho'_{\mathfrak{n}} := s(\mathrm{Gal}(L/L_{\mathfrak{n}})) \prod_{\mathfrak{p} \mid \mathfrak{n}} (1 - \tau_{\mathfrak{p}})$ for any integral ideal $\mathfrak{n}$, where the product runs over all prime ideals of K dividing $\mathfrak{n}$.

(ii) The $\mathbb{Z}[G]$-submodule U' of $\mathbb{Q}[G]$ generated by $\rho'_{\mathfrak{n}}$ where $\mathfrak{n}$ runs through all integral ideals of $\mathcal{O}_K$ is called *Sinnott's module.*

(iii) Define U'_0 to be the kernel of multiplication by $s(G)$ in U'.

REMARK 3.2. (i) The notation U' and $\rho'_{\mathfrak{n}}$ is adopted from [CK19]. In the second part of the present article, we use a modification of Sinnott's module which will be denoted by U.

(ii) Note that for $\mathfrak{n} \nmid \mathfrak{m}$, we have $L_{\mathfrak{n}} = L_{\gcd(\mathfrak{n},\mathfrak{m})}$, hence $\rho'_{\mathfrak{n}} = \rho'_{\gcd(\mathfrak{n},\mathfrak{m})}$. Therefore, it suffices to consider the elements $\rho'_{\mathfrak{n}}$ with $\mathfrak{n} \mid \mathfrak{m}$.

(iii) If $\mathfrak{n} \neq (1)$, we have $\rho'_{\mathfrak{n}} \in U'_0$. As in the imaginary quadratic case (cf. [O03]), the component of U' generated by $\rho'_{(1)}$ intersected with U'_0 is generated by

$$\rho'_{(1)}(1-\sigma), \qquad \sigma \in G.$$

If $\sigma, \sigma' \in G$ are lifts of the same element $\tau \in \mathrm{Gal}(L_{(1)}/K)$, then

$$\rho'_{(1)}(1-\sigma) = \rho'_{(1)}(1-\sigma'),$$

hence it suffices to consider the elements

$$\rho'_{(1)}(1-\widetilde{\tau}), \qquad \tau \in \mathrm{Gal}(L_{(1)}/K),$$

where $\widetilde{\tau} \in G$ is an arbitrary lift of τ.

Now recall the convention introduced in Section 2.5 and consider the logarithmic map

$$l_L : L^\times \to \mathbb{Q}[G], \qquad x \mapsto \sum_{\sigma \in G} \mathrm{ord}_\infty(x^\sigma)\sigma^{-1},$$

and the element

$$\omega := h w_\infty \sum_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} L(\overline{\chi}, 0) e_\chi.$$

Also define

$$l_L^* := (1-e_G) l_L.$$

PROPOSITION 3.3 (cf. [O03, Prop. 6]). *Let* $\mathfrak{n} \neq 1$ *be such that* $\mathfrak{n} \mid \mathfrak{m}$ *and let* $\tau \in \mathrm{Gal}(L_{(1)}/K)$. *Then*

$$l_L^*(\varphi_{L,\mathfrak{n}}) = \omega \rho'_{\mathfrak{n}}\ , \qquad l_L^*\left(\frac{\partial_L(1)}{\partial_L(\tau)}\right) = \omega \rho'_{(1)}(1-\widetilde{\tau}),$$

where $\widetilde{\tau} \in G$ *is any lift of* τ.

Proof. It suffices to prove the equations on the χ-component for each non-trivial character $\chi \in \widehat{G}$. This follows from Proposition 2.14 by a direct computation. ∎

COROLLARY 3.4. *We have* $l_L^*(P_L) = \omega \cdot U'_0$.

Proof. This follows directly from Remark 3.2. ∎

4. An index formula. We briefly recall the definition of Sinnott's index (see [O03, §4]). Let V be a finite-dimensional vector space over $L = \mathbb{Q}$ or $\mathbb{R}$. A subgroup X of V is a *lattice* if $\mathrm{rk}_{\mathbb{Z}}(X) = \dim_L(V)$ and $L \cdot X = V$. If A and B are lattices of V and if γ is an automorphism of V such that $\gamma(A) = B$, then we define

$$[A : B] := |\det(\gamma)|.$$

If $B \subseteq A$, then $[A : B]$ is the usual group index. Now we can prove

PROPOSITION 4.1 (cf. [O03, Prop. 7]). *We have*

$$[U_0' : l_L^*(P_L)] = \left(\frac{hw_\infty}{d_\infty}\right)^{[L:K]-1} \cdot \frac{w_K h_L R_L}{w_L h}.$$

Proof. Using Proposition 2.10(iv) and

$$L'_{\{\infty\}}(\chi, 0) = \log(N\infty) \cdot L(\chi, 0) = d_\infty \log(q) \cdot L(\chi, 0),$$

we obtain

$$\zeta^*_{L,S_\infty}(0) = \zeta^*_{K,\{\infty\}}(0) \cdot \prod_{\chi \neq 1} L'_{\{\infty\}}(\chi, 0) = -\frac{h}{w_K}(d_\infty \log(q))^{[L:K]-1} \prod_{\chi \neq 1} L(\chi, 0).$$

Together with the analytic class number formula in 2.12, this yields

$$\prod_{\chi \neq 1} L(\chi, 0) = \frac{w_K h_L R_L}{w_L h d_\infty^{[L:K]-1}}.$$

Therefore, we obtain

$$\begin{aligned} |\det(\omega)| = \prod_{\chi \neq 1} \chi(\omega) &= (hw_\infty)^{[L:K]-1} \prod_{\chi \neq 1} L(\overline{\chi}, 0) \\ &= \left(\frac{hw_\infty}{d_\infty}\right)^{[L:K]-1} \cdot \frac{w_K h_L R_L}{w_L h}, \end{aligned}$$

where the first equality follows from [S80, Lemma 1.2(b)]. Since this is non-zero, we find that multiplication by ω is an automorphism of $V = \mathbb{Q} \cdot U_0'$. By Corollary 3.4 we have $l_L^*(P_L) = \omega U_0'$ and hence the desired Sinnott index exists and is given by

$$[U_0' : l_L^*(P_L)] = [U_0' : \omega U_0'] = |\det(\omega)| = \left(\frac{hw_\infty}{d_\infty}\right)^{[L:K]-1} \cdot \frac{w_K h_L R_L}{w_L h}. \ \blacksquare$$

Let $\mathfrak{p} \mid \mathfrak{m}$ be a prime ideal of K. The norm relation of Proposition 2.9 implies that $x_{\mathfrak{p}}^{w_\infty / w_K [H:L_{(1)}]} \in P_L$, where $x_{\mathfrak{p}}$ is a generator of $\mathfrak{p}^h$.

DEFINITION 4.2. Let Q_L be the subgroup of P_L generated by $\mu(L)$, Δ_L and the elements $x_{\mathfrak{p}}^{w_\infty / w_K [H:L_{(1)}]}$ for all $\mathfrak{p} \mid \mathfrak{m}$.

Now we can state

Proposition 4.3 (cf. [O03, Prop. 8]). *We have*

$$[l_L^*(P_L) : l_L(C_L)] = \frac{\prod_{\mathfrak{p}}[L \cap H_{\mathfrak{p}^\infty} : L_{(1)}]}{[P_L^{w_L} \cap K : Q_L^{w_L} \cap K]}$$

where $\mathfrak{p}$ *runs through all maximal ideals of* $\mathcal{O}_K$.

Proof. We can use the proof of [O03, Prop. 8] here. The essential inputs

(i) $\ker(l_L) \cap \mathcal{O}_L^\times = \mu(L)$,
(ii) $l_L(C_L) = l_L^*(C_L)$

also hold in the function field case. ■

Now we can state the desired index formula (cf. Theorem A):

Theorem 4.4 (cf. [O03, Thm. 1]). *Set* $d(L) := [P_L^{w_L} \cap K : Q_L^{w_L} \cap K]$. *Then*

$$[\mathcal{O}_L^\times : C_L] = \frac{(hw_\infty)^{[L:K]-1} w_K h_L}{w_L h} \frac{\prod_{\mathfrak{p}}[L \cap H_{\mathfrak{p}^\infty} : L_{(1)}]}{[L : L_{(1)}]} \frac{[\mathbb{Z}[G] : U']}{d(L)}.$$

Proof. Let $R = \mathbb{Z}[G]$ and R_0 be the kernel of multiplication with $s(G)$ in R. Since $\ker(l_L) \cap \mathcal{O}_L^\times = \mu(L)$ we get

$$\begin{aligned}
[\mathcal{O}_L^\times : C_L] &= [l_L(\mathcal{O}_L^\times) : l_L(C_L)] = [l_L(\mathcal{O}_L^\times) : R_0][R_0 : l_L(C_L)] \\
&= \frac{[R_0 : U_0']}{[R_0 : l_L(\mathcal{O}_L^\times)]}[U_0' : l_L(C_L)] \\
&= \frac{[R_0 : U_0']}{[R_0 : l_L(\mathcal{O}_L^\times)]}[U_0' : l_L^*(P_L)][l_L^*(P_L) : l_L(C_L)].
\end{aligned}$$

Note that all the indices above are defined, since each of the $\mathbb{Z}$-modules has the same rank. By definition of Sinnott's index, one can easily show that

$$[R_0 : l_L(\mathcal{O}_L^\times)] = |\det(A)|,$$

where A is the matrix with entries

$$(\mathrm{ord}_w(u_i))_{\substack{w \in S_\infty(L)\setminus\{w_0\} \\ i \in \{1,\ldots,[L:K]-1\}}},$$

where w_0 is an arbitrary place in $S_\infty(L)$ and the units $u_1, \ldots, u_{[L:K]-1} \in \mathcal{O}_L^\times$ project to a basis of $\mathcal{O}_L^\times/\mu(L)$. By the definition of the regulator, we hence get

$$R_L = |\det(-d_\infty A)| = d_\infty^{[L:K]-1}|\det(A)|,$$

so

$$[R_0 : l_L(\mathcal{O}_L^\times)] = \frac{R_L}{d_\infty^{[L:K]-1}}.$$

As in [O03] we find that

$$[R_0 : U'_0] = \frac{[R : U']}{[L : L_{(1)}]}.$$

Using these computations and the results of Propositions 4.1 and 4.3 we obtain

$$[\mathcal{O}_L^\times : C_L] = \frac{(hw_\infty)^{[L:K]-1} w_K h_L}{w_L h} \frac{\prod_{\mathfrak{p}} [L \cap H_{\mathfrak{p}^\infty} : L_{(1)}]}{[L : L_{(1)}]} \frac{[R : U']}{d(L)}. \ \blacksquare$$

We state some results on $[R : U']$ similar to [O03, §6, §7]:

PROPOSITION 4.5.

(i) *The index* $[R : U']$ *is an integer divisible only by primes dividing* $[L : L_{(1)}]$. *Moreover, if* $\mathrm{Gal}(L/L_{(1)})$ *is the direct product of its inertia groups or if at most two primes ramify in* L/K, *then* $[R : U'] = 1$.

(ii) *If* G *is cyclic, then* $[R : U'] = 1$.

(iii) *If* $L = H_{\mathfrak{m}}$ *for some integral ideal* $\mathfrak{m} = \prod_{i=1}^{s} \mathfrak{p}_i^{e_i}$ *for some* $s \geq 3$ *and if* h *is coprime to* w_K, *we have*

$$[R : U'] = w_K^{e(2^{s-2}-1)},$$

where e *is the index of the subgroup generated by the classes of* $\mathfrak{p}_i$ *in* $\mathrm{Cl}(K)$.

Proof. (i) is [O03, Prop. 16]; (ii) is [S80, Thm. 5.3]; and (iii) is [O03, Prop. 18].

Note that the arguments are only based on the group structure of G and hence can also be applied in the case of function fields. ■

REMARK 4.6. There is a list of cases in [O03, Remark 2] in which the author gets $d(L) = 1$. With similar methods we can show that if one of the following conditions holds, we have $d(L) = 1$:

(i) $L \subseteq H$,

(ii) $H \subseteq L$,

(iii) $[H : L_{(1)}]$ and $[L : L_{(1)}]$ are coprime,

(iv) $\mathrm{Gal}(L/L_{(1)})$ is cyclic,

(v) $\mathrm{Gal}(L/L_{(1)})$ is the direct product of its inertia subgroups,

(vi) at most two primes ramify in L/K.

REMARK 4.7. (i) In [O92] H. Oukhaba defines a group $\mathcal{E}_L$ of elliptic units in an unramified extension L/K. He also shows that the elements of $\mathcal{E}_L^{w_K w_\infty h}$ are of the form

$$\prod_{\tau \in G} \left(\frac{\partial_L(1)\partial_L(\tau\sigma^{-1})}{\partial_L(\sigma^{-1})\partial_L(\tau)} \right)^{w_K m_\tau}$$

for $\sigma \in G$ and certain rational numbers $m_\tau \in \mathbb{Q}$ (cf. [O92, Prop. 3.6]). He also derives an index formula in this case:

$$[\mathcal{O}_L^\times : \mathcal{E}_L] = \frac{h_L}{[H:L]}.$$

In this case, our index formula yields

$$[\mathcal{O}_L^\times : C_L] = (hw_\infty)^{[L:K]-1}\frac{w_K h_L}{w_L h}.$$

From the above description, we find that $\mathcal{E}_L^{w_\infty h} \subseteq C_L$ and so

$$[C_L : \mathcal{E}_L^{w_\infty h}] = h\frac{w_L}{w_K}.$$

(ii) In [Y97b] L. Yin defines a group $\overline{C}$ of extended cyclotomic units in the ray class fields $K_\mathfrak{m}$. The ramified elliptic units in the present article are in fact norms of Yin's cyclotomic units. However our construction of the unramified units is quite different from the one in [Y97b]. Nevertheless, Yin also computes an index formula

$$[\mathcal{O}_{H_\mathfrak{m}}^\times : (\overline{C} \cap \mathcal{O}_{H_\mathfrak{m}}^\times)] = w_K^a h_{H_\mathfrak{m}},$$

where $a = 0$ if $s \leq 2$ and $a = e(2^{s-2}-1) - (s-2)$ if $s \geq 3$. Note that there is the additional assumption $(h, w_K) = 1$ in the case $s \geq 3$. With these assumptions, from our index formula we get

$$[\mathcal{O}_{H_\mathfrak{m}}^\times : C_{H_\mathfrak{m}}] = (hw_\infty)^{[H_\mathfrak{m}:K]-1}\frac{w_K h_{H_\mathfrak{m}}}{w_\infty h} w_K^{-(s-1)}[R:U'].$$

With Proposition 4.5, this yields

$$[\mathcal{O}_{H_\mathfrak{m}}^\times : C_{H_\mathfrak{m}}] = (hw_\infty)^{[H_\mathfrak{m}:K]-2} w_K^a h_{H_\mathfrak{m}}.$$

5. A non-trivial root of an elliptic unit.

With this definition of elliptic units we can prove an analogue of the main result of [CK19] in the case of global function fields.

5.1. Preliminaries.

We use the notation of Section 1.1 with the following additional assumptions:

- p is an odd prime such that $p \nmid q \cdot (q-1) \cdot h$.
- L is a real cyclic extension of K of degree p^k for some positive integer k.
- We change notation to $\Gamma := \mathrm{Gal}(L/K)$. Let σ be a generator of Γ.

Remark 5.1. Note that the assumption on L and $p \nmid h$ are exactly the same as in [CK19]. The assumption $p \nmid (q-1) = w_K$ is also implied by the hypotheses stated there. The only new assumption is $p \nmid q$, i.e. we suppose that p is prime to the characteristic of K, which is a natural hypothesis when dealing with function fields.

Note that since $p \nmid h$, we have

$$L \cap H = K,$$

and we assume that there are exactly $s \geq 2$ primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of K which ramify in L. Now we introduce some more notation:

- $I := \{1, \ldots, s\}$.
- $x_j := x_{\mathfrak{p}_j}$ is a generator of $\mathfrak{p}_j^h$.
- $\mathfrak{P}_j$ is a fixed prime ideal of L over $\mathfrak{p}_j$.
- For any abelian extension M/K let $D_j(M) := D_{\mathfrak{p}_j} \subseteq \mathrm{Gal}(M/K)$ be the decomposition group of $\mathfrak{p}_j$ and $T_j(M) := T_{\mathfrak{p}_j} \subseteq D_j(M)$ be the inertia group of $\mathfrak{p}_j$.
- $t_j := |T_j(L)|$ is the ramification index of $\mathfrak{P}_j$ over $\mathfrak{p}_j$.
- $n_j := [G : D_j(L)]$.

Then it follows that $t_j n_j \mid p^k$ and

$$\mathfrak{p}_j \mathcal{O}_L = \prod_{i=0}^{n_j - 1} \mathfrak{P}_j^{t_j \sigma^i}.$$

Since $p \nmid q$, this implies that the extension L/K is tamely ramified and hence its conductor is square-free. Therefore the conductor is given by $\mathfrak{m} := \mathfrak{m}_I := \prod_{j \in I} \mathfrak{p}_j$.

5.2. The distinguished subfields F_j. For any subset $\emptyset \neq J \subseteq I$ we set $\mathfrak{m}_J := \prod_{j \in J} \mathfrak{p}_j$. With our previous observation we find that $L \subseteq H_{\mathfrak{m}_I}$.

Lemma 5.2. *We have $L \subseteq \prod_{j \in I} H_{\mathfrak{p}_j}$.*

Proof. By class field theory, we have a canonical isomorphism

$$\mathrm{Gal}(H_{\mathfrak{m}}/H) \cong (\mathcal{O}_K/\mathfrak{m})^\times / \mathrm{im}(\mu(K))$$

(see e.g. [H85, eq. (3.1)]). With the Chinese Remainder Theorem, we get

$$\Big[H_{\mathfrak{m}} : \prod_{j \in I} H_{\mathfrak{p}_j}\Big] = \frac{[H_{\mathfrak{m}} : H]}{[\prod_{j \in I} H_{\mathfrak{p}_j} : H]} = \frac{|(\mathcal{O}_K/\mathfrak{m})^\times| / w_K}{\prod_{j \in I} [H_{\mathfrak{p}_j} : H]}$$
$$= \frac{\prod_{j \in I} |(\mathcal{O}_K/\mathfrak{p}_j)^\times|}{w_K \prod_{j \in I} |(\mathcal{O}_K/\mathfrak{p}_j)^\times| / w_K} = w_K^{s-1}.$$

The second equality follows, since we obtain $H_{\mathfrak{p}_j} \cap \prod_{i=1}^{j-1} H_{\mathfrak{p}_i} = H$ for any $2 \leq j \leq s$ by considering the ramification of $\mathfrak{p}_j$. Since $p \nmid w_K$, we get $L \subseteq \prod_{j \in I} H_{\mathfrak{p}_j}$. ∎

Using the canonical isomorphism of the proof above, we obtain

$$\mathrm{Gal}(H_{\mathfrak{p}_j}/H) \cong (\mathcal{O}_K/\mathfrak{p}_j)^\times / \mathrm{im}(\mu(K)),$$

which is a cyclic group. Since $t_j \mid [L_{\mathfrak{p}_j} : K] \mid [H_{\mathfrak{p}_j} : K]$ and $p \nmid h$, it follows that $t_j \mid [H_{\mathfrak{p}_j} : H]$. Using $p \nmid h$ and [CK19, Lemma 2.1] we can define F_j to be the unique subfield of $H_{\mathfrak{p}_j}$ such that $[F_j : K] = t_j$. Then $F_j \cap H = K$ and F_j/K is totally ramified at $\mathfrak{p}_j$ and unramified everywhere else.

From now on, we will write $H_J := H_{\mathfrak{m}_J}$ for each $\emptyset \neq J \subseteq I$ and

$$F_J := \prod_{j \in J} F_j \subseteq H_J.$$

Note that the conductor of F_J is $\mathfrak{m}_J$. The definition of F_I implies that the Galois group $\mathrm{Gal}(F_I/F_{I\setminus\{j\}}) = T_j(F_I)$ is the inertia subgroup of a prime of F_I above $\mathfrak{p}_j$, in particular for each $j \in I$ we have $|\mathrm{Gal}(F_I/F_{I\setminus\{j\}})| = t_j$.

LEMMA 5.3. *For any two subsets $\emptyset \neq J_1 \subseteq J_2 \subseteq I$, we have $F_{J_1} = F_{J_2} \cap H_{J_1}$. Moreover, $F_I \cap H = K$.*

Proof. The inclusion $F_{J_1} \subseteq F_{J_2} \cap H_{J_1}$ is clear. For the other inclusion, we use induction on $n = |J_2 \setminus J_1|$. The case $n = 0$, i.e. $J_1 = J_2$, is clear. If $n \geq 1$, we fix an index $j \in J_2 \setminus J_1$ and we see that

$$F_{J_2} \cap H_{J_1} \subseteq F_{J_2} \cap H_{J_2\setminus\{j\}} \subseteq F_{J_2\setminus\{j\}}$$

by the induction hypothesis. But we clearly also have $F_{J_2} \cap H_{J_1} \subseteq H_{J_1}$, hence

$$F_{J_2} \cap H_{J_1} \subseteq F_{J_2\setminus\{j\}} \cap H_{J_1} \subseteq F_{J_1}$$

by the induction hypothesis.

The second assertion follows, since $[F_I : K]$ is a p-power and $p \nmid h$. ∎

PROPOSITION 5.4 (cf. [CK19, Prop. 2.2]). *We have $F_j H_{I\setminus\{j\}} = L H_{I\setminus\{j\}}$ for each $j \in I$. The Galois group*

$$G = \mathrm{Gal}(F_I/K) = \prod_{j \in I} \mathrm{Gal}(F_I/F_{I\setminus\{j\}})$$

is the direct product of its inertia subgroups. Moreover $L \subseteq F_I$.

Proof. We can take the proof of [CK19, Prop. 2.2] here, since there are no changes necessary. ∎

COROLLARY 5.5 (cf. [CK19, Cor. 2.3]).

(i) *For each $j \in I$ we have*

$$T_j(L) = \mathrm{Gal}(L/L \cap F_{I\setminus\{j\}}) = \langle \sigma^{p^k/t_j} \rangle.$$

Moreover, $F_{I\setminus\{j\}} L = F_I$ and $[L \cap F_{I\setminus\{j\}} : K] = p^k/t_j$.

(ii) *F_I/L is an unramified abelian extension.*

(iii) *There exists a $j_0 \in I$ such that $t_{j_0} = p^k$, and hence $G = \mathrm{Gal}(F_I/K)$ has exponent p^k.*

5.3. The elliptic units. Since $F_I \cap H = K$ by Lemma 5.3, there are no unramified elliptic units and we define

$$\eta_J := N_{H_J/F_J}(\alpha_{\mathfrak{m}_J}) = \varphi_{F_I,\mathfrak{m}_J}^{1/h} \in \mathcal{O}_{F_J};$$

cf. Remark 2.6. Let $\sigma_j \in G = \mathrm{Gal}(F_I/K)$ be the lift of the Frobenius of $\mathfrak{p}_j$ uniquely defined by $\sigma_j|_{F_{I\setminus\{j\}}} = (\mathfrak{p}_j, F_{I\setminus\{j\}}/K)$ and $\sigma_j|_{F_j} = 1$. Then we can state

LEMMA 5.6 (cf. [CK19, Lemma 3.1]). *For any $j \in I$ we have*

$$D_j(L) = \langle \sigma^{n_j} \rangle = \langle \sigma_j|_L, \sigma^{p^k/t_j} \rangle.$$

LEMMA 5.7 (cf. [CK19, Lemma 3.2]). *We have $\mu(F_I) = \mu(K)$.*

Proof. For $\zeta \in \mu(F_I)$, the extension $K(\zeta)/K$ is a constant field extension. Since all constant field extensions are unramified, we obtain

$$K(\zeta) \subseteq F_I \cap H = K,$$

so $\zeta \in \mu(K)$. ∎

Proposition 2.9 implies that for each $J \subseteq I$ and each $j \in J$,

$$N_{F_J/F_{J\setminus\{j\}}}(\eta_J) = \begin{cases} \eta_{J\setminus\{j\}}^{1-\sigma_j^{-1}}, & J \setminus \{j\} \neq \emptyset, \\ \zeta x_j^{w_\infty/w_K}, & J \setminus \{j\} = \emptyset, \end{cases} \tag{5.1}$$

for some $\zeta \in \mu(K)$.

In analogy to [CK19], we use the following definition of elliptic units:

DEFINITION 5.8.

- The *group of elliptic numbers* $\mathcal{P}_{F_I}$ of F_I is defined to be the $\mathbb{Z}[G]$-submodule of $F_I^\times$ generated by $\mu(K)$ and by η_J for all $\emptyset \neq J \subseteq I$.
- The *group of elliptic units* $\mathcal{C}_{F_I}$ of F_I is defined as $\mathcal{C}_{F_I} := \mathcal{P}_{F_I} \cap \mathcal{O}_{F_I}^\times$.
- The *group of elliptic numbers* $\mathcal{P}_L$ of L is the $\mathbb{Z}[\Gamma]$-submodule of $L^\times$ generated by $\mu(K)$ and $N_{F_J/F_J\cap L}(\eta_J)$ for all $\emptyset \neq J \subseteq I$.
- The *group of elliptic units* $\mathcal{C}_L$ of L is defined as $\mathcal{C}_L := \mathcal{P}_L \cap \mathcal{O}_L^\times$.

Since $F_I \cap H = K = L \cap H$, one can check that these elliptic units are related to the units of Definition 2.8 by

$$C_{F_I} = \mathcal{C}_{F_I}^h \cdot \mu(K), \quad C_L = \mathcal{C}_L^h \cdot \mu(K).$$

This fact and Theorem 4.4 imply the next lemma. We first need the following

NOTATION 5.9. Let $\widetilde{L}$ be the maximal subfield of L containing K such that $\widetilde{L}/K$ is ramified in at most one prime ideal of K.

Note that since Γ is cyclic and of prime power order, the field $\widetilde{L}$ is unique.

LEMMA 5.10 (cf. [CK19, Lemma 3.4]).

(i) *We have*

$$[\mathcal{O}_{F_I}^\times : \mathcal{C}_{F_I}] = w_\infty^{[F_I:K]-1} \frac{h_{F_I}}{h}, \quad [\mathcal{O}_L^\times : \mathcal{C}_L] = w_\infty^{[L:K]-1} \frac{h_L}{h[L:\widetilde{L}]}.$$

(ii) *For $\beta \in \mathcal{P}_{F_I}$ we have $\beta \in \mathcal{C}_{F_I}$ if and only if $N_{F_I/K}(\beta) \in \mu(K)$.*

Sketch of proof. For more details and part (ii) see [CK19, Lemma 3.4]. First, we deduce from the above observation that $[\mathcal{C}_{F_I} : C_{F_I}] = h^{[F_I:K]-1}$ and $[\mathcal{C}_L : C_L] = h^{[L:K]-1}$. Moreover, it follows from Propositions 5.4, 4.5(ii) (resp. (i)) and Remark 4.6(iv) (resp. (v)) that the last quotient in 4.4 is equal to 1 for L (resp. F_I). We also obtain $w_{F_I} = w_L = w_K$ by Lemma 5.7, $L_{(1)} = K$, $F_I \cap H_{\mathfrak{p}^\infty} = F_j$ for $\mathfrak{p} = \mathfrak{p}_j$ and $\prod_{j=1}^{s}[F_j : K] = [F_I : K]$, which yields the first equation. For the second equation, we consider the definition of $\widetilde{L}$. By Corollary 5.5(iii) we know that there is a prime $\mathfrak{p}_i$ which is totally ramified in L. Therefore, $\widetilde{L}$ is the maximal subfield of L which is unramified at every prime except $\mathfrak{p}_i$, hence $\widetilde{L} = L \cap H_{\mathfrak{p}_i^\infty}$. Since for $\mathfrak{p} \neq \mathfrak{p}_i$ the extension $L \cap H_{\mathfrak{p}^\infty}$ is unramified at $\mathfrak{p}_i$ and $\mathfrak{p}_i$ is totally ramified in L, we find that $L \cap H_{\mathfrak{p}^\infty} = K$ for $\mathfrak{p} \neq \mathfrak{p}_i$, and hence $\prod_{\mathfrak{p}}[L \cap H_{\mathfrak{p}^\infty} : K] = [\widetilde{L} : K]$. ∎

Now we use a modification of Sinnott's module, defined in [GK14]. This module U is a $\mathbb{Z}[G]$-submodule of $\mathbb{Q}[G] \oplus \mathbb{Z}^s$ generated over $\mathbb{Z}[G]$ by certain elements ρ_J, $J \subseteq I$. Each $\mathbb{Z}$ summand is endowed with the trivial G-action and has a standard basis element denoted by e_j.

Define

$$\Psi : \mathcal{P}_{F_I} \to U, \qquad \eta_J \mapsto \rho_{I \setminus J},$$

for $\emptyset \neq J \subseteq I$ and $\Psi(\mu(K)) = 0$.

LEMMA 5.11 (cf. [CK19, Lemma 3.5]). *Ψ is a well-defined $\mathbb{Z}[G]$-module homomorphism satisfying* $\ker(\Psi) = \mu(K)$ *and* $U = \Psi(\mathcal{P}_{F_I}) \oplus (s(G)\mathbb{Z})$.

We call

$$\eta := N_{F_I/L}(\eta_I)$$

the *top generator* of both $\mathcal{P}_L$ and $\mathcal{C}_L$. Set $B := \mathrm{Gal}(F_I/L) \subseteq \mathrm{Gal}(F_I/K) = G$. Then $\Gamma = \langle \sigma \rangle \cong G/B$.

LEMMA 5.12 (cf. [CK19, Lemma 4.1]). *An elliptic number $\beta \in \mathcal{P}_{F_I}$ belongs to L if and only if $\Psi(\beta)$ is fixed by B, i.e. $\Psi(\mathcal{P}_{F_I})^B = \Psi(\mathcal{P}_{F_I} \cap L)$.*

Recall that n_i is the index of the decomposition group of the ideal $\mathfrak{P}_i \subseteq L$ in Γ. Without loss of generality we can assume

$$n_1 \leq \cdots \leq n_s$$

and set $n = n_s = \max\{n_i \mid i \in I\}$. Since $p \mid t_s$, we have $n \mid p^{k-1}$ and by Corollary 5.5(iii) we get $t_1 = p^k$ and hence $n_1 = 1$. Let L' be the unique subfield of L containing K such that $[L' : K] = n$. Note that $\langle \sigma^n \rangle = \mathrm{Gal}(L/L')$ and that $\mathfrak{p}_s$ splits completely in L'/K. Now we can restate Theorem B:

THEOREM 5.13 (cf. [CK19, Thm. 4.2]). *There is a unique $\alpha \in L$ such that $N_{L/L'}(\alpha) = 1$ and $\eta = \alpha^y$, where $y = \prod_{i=2}^{s-1}(1-\sigma^{n_i})$. This α is an elliptic unit of F_I, so that $\alpha \in \mathcal{C}_{F_I} \cap L$. Moreover, there is $\gamma \in L^\times$ such that $\alpha = \gamma^{1-\sigma^n}$.*

Proof. We have proven all ingredients which are used in the proof of [CK19, Thm. 4.2], hence we obtain the same result for function fields. ∎

6. Enlarging the group $\mathcal{C}_L$ of elliptic units of L. We label the subfields of L containing K by

$$K = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k = L,$$

hence we obtain $[L_i : K] = p^i$. Moreover, we define

$$M_i := \{j \in I \mid t_j > p^{k-i}\}.$$

Since we have already seen that $n_1 = 1$, we deduce from the definition of M_i that

$$1 \in M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = I.$$

For $j \in M_i$ we get $p^i > p^k/t_j$ and with Corollary 5.5(i) we can see that $\mathfrak{p}_j$ ramifies in L_i. On the other hand, if $\mathfrak{p}_j$ ramifies in L_i, this implies that $t_j > [L : L_i] = p^{k-i}$. This shows that the conductor of L_i is equal to $\mathfrak{m}_{M_i}$ and so $L_i \subseteq F_{M_i}$ by Proposition 5.4 applied to L_i. Define

$$\eta_i := N_{F_{M_i}/L_i}(\eta_{M_i})$$

for $i = 1, \ldots, k$. Then $\eta_k = \eta \in L$ is the top generator of $\mathcal{C}_L$.

Now we fix $j \in \{1, \ldots, s\}$ and let $L_i = L^{T_j}$, hence the index i is determined by $t_j = p^{k-i}$. By Lemma 5.6 we get

$$\langle \sigma^{n_j} \rangle / \langle \sigma^{p^k/t_j} \rangle = \langle \sigma_j|_L, \sigma^{p^k/t_j} \rangle / \langle \sigma^{p^k/t_j} \rangle.$$

This quotient group can be interpreted as the restriction to L_i, since $\sigma^{p^k/t_j} = \sigma^{p^i}$ generates $\mathrm{Gal}(L/L_i)$. Hence we can find a smallest positive integer c_j such that $\sigma^{-c_j n_j}|_{L_i} = \sigma_j|_{L_i}$. Moreover, we see that $\mathfrak{p}_j$ splits completely in L_i/K if and only if $n_j = p^k/t_j$, in which case we get in particular $c_j = 1$ since σ^{n_j} is already an element of the inertia group of $\mathfrak{p}_j$ of L/K. If $\mathfrak{p}_j$ does not split completely in L_i/K, we find that $n_j < p^k/t_j$ and hence $\langle \sigma^{n_j}|_{L_i} \rangle = \langle \sigma_j|_{L_i} \rangle$. In each case, we find that $p \nmid c_j$ and hence $1 - \sigma^{c_j n_j}$ and $1 - \sigma^{n_j}$ are associated in $\mathbb{Z}[\Gamma]$.

Now let $i \in \{1, \ldots, k\}$ be such that $|M_i| > 1$. We apply Theorem 5.13 to the extension L_i/K and obtain an elliptic unit $\alpha_i \in \mathcal{C}_{F_{M_i}} \cap L_i$ and a number $\gamma_i \in L_i^\times$ such that

(i) $\eta_i = \alpha_i^{y_i}$, where $y_i = \prod_{j \in M_i,\, 1 < j < \max M_i} (1 - \sigma^{c_j n_j})$,
(ii) $\alpha_i = \gamma_i^{z_i}$, where $z_i = 1 - \sigma^{c_{\max M_i} n_{\max M_i}}$.

Note that the new c_j factors can be obtained since $1 - \sigma^{n_j}$ and $1 - \sigma^{c_j n_j}$ are associated. In particular we find for $|M_i| = 2$ that $y_i = 1$ and $\alpha_i = \eta_i$ since the product is empty. For $i \in \{1, \ldots, k\}$ with $|M_i| = 1$ we set $\gamma_i = \eta_i$ and $\alpha_i = \eta_i^{1-\sigma}$.

Definition 6.1. The $\mathbb{Z}[\Gamma]$-submodule $\overline{\mathcal{C}_L}$ of $\mathcal{O}_L^\times$ generated by $\mu(K)$ and $\alpha_1, \ldots, \alpha_k$ is called the *extended group of elliptic units*.

Theorem 6.2 (cf. [CK19, Thm. 5.2]). *The group of elliptic units $\mathcal{C}_L$ of L is a subgroup of $\overline{\mathcal{C}_L}$ of index $[\overline{\mathcal{C}_L} : \mathcal{C}_L] = p^\nu$, where*

$$\nu = \sum_{j=1}^{k} \sum_{\substack{i \in M_j \\ 1 < i < \max M_j}} n_i.$$

Moreover, setting $\varphi_L := (\prod_{i=1}^s t_i^{n_i}) \cdot \prod_{j=1}^k p^{-n_{\max M_j}}$, we get

$$p^\nu = \varphi_L \cdot [L : \widetilde{L}]^{-1} \quad \textit{and} \quad [\mathcal{O}_L^\times : \overline{\mathcal{C}_L}] = w_\infty^{p^k - 1} \cdot \frac{h_L}{h} \cdot \varphi_L^{-1}.$$

Proof. We use the same proof as in [GK15, Thm. 3.1]. Note that we need the factors c_j appearing in the definition of the α_i here. ∎

Remark 6.3. If $p \nmid w_\infty$, we obtain $\varphi_L \mid h_L$. As in [CK19, Remark 5.3], this divisibility statement is really stronger than $[F_I : L] \mid h_L$ (which we obtain since F_I/L is unramified). Indeed, by [GK15, Prop. 3.4], $[F_I : L] \mid \varphi_L$ and we obtain equality if and only if $n_1 = \cdots = n_{s-1} = 1$.

7. Semispecial numbers. We use the same notation as before and fix m which is a power of p such that $p^{ks} \mid m$. We know that for a prime ideal $\mathfrak{q}$ of K we have

$$\mathrm{Gal}(H_\mathfrak{q}/H) \cong (\mathcal{O}_K/\mathfrak{q})^\times/\mathrm{im}(\mu(K))$$

via Artin's reciprocity map. In particular, $\mathrm{Gal}(H_\mathfrak{q}/H)$ is cyclic. This enables us to state

Definition 7.1. For a prime ideal $\mathfrak{q}$ of K such that $|\mathcal{O}_K/\mathfrak{q}| \equiv 1 \bmod m$, we define $K[\mathfrak{q}]$ to be the (unique) subfield of $H_\mathfrak{q}$ containing K such that $[K[\mathfrak{q}] : K] = m$. For a finite field extension M/K, we define $M[\mathfrak{q}] := MK[\mathfrak{q}]$.

Note that since $|\mathcal{O}_K/\mathfrak{q}| \equiv 1 \bmod m$ and $p \nmid |\mu(K)|$, we know that the order of $\mathrm{Gal}(H_\mathfrak{q}/H)$ is divisible by m. Hence we get the existence and uniqueness of $K[\mathfrak{q}]$ from the fact that $p \nmid h$ and from [CK19, Lemma 2.1]. Since $K[\mathfrak{q}]$ is contained in $H_\mathfrak{q}$, it is only ramified at $\mathfrak{q}$. Moreover, since $p \nmid h$, we find that $H \cap K[\mathfrak{q}] = K$ and hence it is totally ramified at $\mathfrak{q}$. Finally, since $p \nmid |\mathcal{O}_K/\mathfrak{q}|$, we deduce that this ramification is tame.

Definition 7.2. Let $\mathcal{Q}_m$ be the set of all prime ideals $\mathfrak{q}$ of K such that

(i) $|\mathcal{O}_K/\mathfrak{q}| \equiv 1 + m \bmod m^2$,
(ii) $\mathfrak{q}$ splits completely in L,
(iii) for each $j = 1, \ldots, s$, the class of x_j is an mth power in $(\mathcal{O}_K/\mathfrak{q})^\times$.

Now we want to study condition (iii) in some more detail. Let $\mathfrak{q}$ be such that $|\mathcal{O}_K/\mathfrak{q}| \equiv 1 \bmod m$. Since $H \cap K[\mathfrak{q}] = K$, we get $\mathrm{Gal}(H[\mathfrak{q}]/H) \cong \mathrm{Gal}(K[\mathfrak{q}]/K)$ by restriction. The first group is the unique quotient of the cyclic group $\mathrm{Gal}(H_\mathfrak{q}/H)$ of order m, hence it is obtained by factoring out mth powers. Therefore with the Artin reciprocity map and $p \nmid w_K$ we get

$$(\mathcal{O}_K/\mathfrak{q})^\times/m \cong \mathrm{Gal}(H[\mathfrak{q}]/H) \cong \mathrm{Gal}(K[\mathfrak{q}]/K),$$

where the composition map takes the class of $\alpha \in \mathcal{O}_K \setminus \mathfrak{q}$ to $(\alpha\mathcal{O}_K, K[\mathfrak{q}]/K)$. Now the facts that $x_j\mathcal{O}_K = \mathfrak{p}_j^h$ and $p \nmid h$ imply that condition (iii) is equivalent to

$$(\mathfrak{p}_j, K[\mathfrak{q}]/K) = 1 \quad \forall j = 1, \ldots, s.$$

Definition 7.3. A number $\varepsilon \in L^\times$ is called *m-semispecial* if for all but finitely many $\mathfrak{q} \in \mathcal{Q}_m$, there exists a unit $\varepsilon_\mathfrak{q} \in \mathcal{O}_{L[\mathfrak{q}]}^\times$ such that

(i) $N_{L[\mathfrak{q}]/L}(\varepsilon_\mathfrak{q}) = 1$,
(ii) if $\mathfrak{q}_{L[\mathfrak{q}]}$ is the product of all primes of $L[\mathfrak{q}]$ above $\mathfrak{q}$, then ε and $\varepsilon_\mathfrak{q}$ have the same image in $(\mathcal{O}_{L[\mathfrak{q}]}/\mathfrak{q}_{L[\mathfrak{q}]})^\times/(m/p^{k(s-1)})$.

Since each $\mathfrak{q} \in \mathcal{Q}_m$ is totally ramified in $K[\mathfrak{q}]/K$ and splits completely in L/K, we find that $L[\mathfrak{q}]/L$ is totally ramified at each prime above $\mathfrak{q}$ and we obtain $L \cap K[\mathfrak{q}] = K$. Therefore, the two restriction maps

$$\mathrm{Gal}(L[\mathfrak{q}]/L) \to \mathrm{Gal}(K[\mathfrak{q}]/K), \qquad \mathrm{Gal}(L[\mathfrak{q}]/K[\mathfrak{q}]) \to \mathrm{Gal}(L/K)$$

are isomorphisms.

Theorem 7.4 (cf. [CK19, Thm. 6.4]). *The elliptic unit $\alpha \in \mathcal{C}_{F_I} \cap L$ of Theorem* 5.13 *is m-semispecial.*

Proof. Recall that α is a yth root of the top generator η of $\mathcal{C}_L$. We need to show that for almost all $\mathfrak{q} \in \mathcal{Q}_m$, there exists an $\varepsilon_\mathfrak{q}$ satisfying the conditions (i) and (ii) of Definition 7.3. In fact, we can construct such an $\varepsilon_\mathfrak{q}$ for *each* $\mathfrak{q} \in \mathcal{Q}_m$ with the methods of [CK19, Thm. 6.4]. There is only a slight change in the result which implies the congruence relation. Therefore we skip the proof of the theorem here and refer to [CK19] once again. Our version of Proposition 6.6 of that paper is stated below. ∎

Proposition 7.5 (cf. [CK19, Prop. 6.6]). *Let $\mathfrak{q} \in \mathcal{Q}_m$, $Q := |\mathcal{O}_K/\mathfrak{q}|$ and let $\mathfrak{q}_{L[\mathfrak{q}]}$ be the product of all primes of $L[\mathfrak{q}]$ above $\mathfrak{q}$. Then*

$$\hat{\eta}^{Q(1-\sigma)} \equiv \eta^{(1-\sigma)(Q-1)/m} \bmod \mathfrak{q}_{L[\mathfrak{q}]},$$

where η is the top generator of $\mathcal{C}_L$ and $\hat{\eta}$ is the top generator of $\mathcal{C}_{L[\mathfrak{q}]}$.

Proof. Let $x \in \mathcal{O}_K$ be such that $x\mathcal{O}_K = \mathfrak{q}^h$. Let $K_m := K(\zeta_m)$, where ζ_m is a primitive mth root of unity. Then K_m/K is a constant field extension, and hence it is unramified everywhere. Moreover, it is an abelian extension. Now we can define $M := K_m(x^{1/p})$, and since $\mathcal{O}_K^\times = \mu(K)$, $p \nmid |\mu(K)|$ and

K_m contains a primitive pth root of unity, this definition is independent of the choice of the generator x and of its pth root. Then M/K is a Galois extension. We claim that x is not a pth power in K_m. If $x = \alpha^p$, then the valuation of x at $\mathfrak{q}$ would be p-times the valuation of α at $\mathfrak{q}$ since K_m/K is unramified. But $x\mathcal{O}_K = \mathfrak{q}^h$, and since $p \nmid h$, this is a contradiction. Hence the extension M/K_m is cyclic of degree p. For finishing the proof, we need

LEMMA 7.6. *Let $\mathfrak{q} \in \mathcal{Q}_m$ and σ be the unique generator of* $\mathrm{Gal}(L[\mathfrak{q}]/K[\mathfrak{q}])$ *which restricts to the original generator of* $\mathrm{Gal}(L/K)$. *Then there exists a prime $\mathfrak{l}$ of K such that*

(i) $|\mathcal{O}_K/\mathfrak{l}| \equiv 1 \bmod m$,
(ii) $\mathfrak{l}$ *is unramified in* $L[\mathfrak{q}]$ *and* $(\mathfrak{l}, L[\mathfrak{q}]/K) = \sigma^{-1}$,
(iii) $\mathfrak{q}$ *is inert in* $K[\mathfrak{l}]/K$.

Proof. By an explicit analysis of the Galois automorphisms, one checks that K_m/K is an abelian extension whereas M/K_m is not. As $[M : K_m] = p$, there are no intermediate fields and hence K_m/K is the maximal abelian subextension of M. This implies that $M \cap L[\mathfrak{q}] = K_m \cap L[\mathfrak{q}]$, since $L[\mathfrak{q}]/K$ is an abelian extension. Since $K_m \cap L[\mathfrak{q}]$ is unramified and $p \nmid h$, we find $K_m \cap L[\mathfrak{q}] = K$. Then there exists a $\tau \in \mathrm{Gal}(L[\mathfrak{q}] \cdot M/K)$ which restricts to $\sigma^{-1} \in \mathrm{Gal}(L[\mathfrak{q}]/K)$ and to a generator of $\mathrm{Gal}(M/K_m) \subseteq \mathrm{Gal}(M/K)$.

Using a variant of Chebotarev's Density Theorem (cf. [R02, Thm. 9.13B]), we see that there exists a prime $\mathfrak{l}$ such that the Frobenius of $\mathfrak{l}$ is the conjugacy class of τ and $|\mathcal{O}_K/\mathfrak{l}| \equiv 1 \bmod m$. Then conditions (i) and (ii) are satisfied and it remains to prove that $\mathfrak{q}$ is inert in $K[\mathfrak{l}]$.

Since τ acts as the identity on K_m, we find that $\mathfrak{l}$ splits completely in K_m/K. Let $\mathfrak{L}$ be a prime of K_m over $\mathfrak{l}$; then $\mathcal{O}_{K_m}/\mathfrak{L} \cong \mathcal{O}_K/\mathfrak{l}$. Moreover, since

$$\langle \tau|_M \rangle = \mathrm{Gal}(M/K_m) \cong \mathbb{Z}/p\mathbb{Z},$$

$\mathfrak{L}$ must be inert in M. It is easily seen that $\mathcal{O}_M/\mathfrak{L}\mathcal{O}_M \cong (\mathcal{O}_{K_m}/\mathfrak{L})[\xi]$, where ξ is the class of $x^{1/p}$ modulo $\mathfrak{L}\mathcal{O}_M$. If x was a pth power in $(\mathcal{O}_{K_m}/\mathfrak{L})^\times$, this extension would be trivial, hence the inertia degree of $\mathfrak{L}$ would be 1. This is a contradiction, since $\mathfrak{L}$ is inert in M, so we have shown that x cannot be a pth power in $(\mathcal{O}_K/\mathfrak{l})^\times$.

Recall that we get $(\mathcal{O}_K/\mathfrak{l})^\times/m \cong \mathrm{Gal}(K[\mathfrak{l}]/K)$ from Artin's Reciprocity Theorem and $p \nmid w_K$. Since x is not a pth power in $(\mathcal{O}_K/\mathfrak{l})^\times$, it follows that the Frobenius $(x\mathcal{O}_K, K[\mathfrak{l}]/K) = (\mathfrak{q}, K[\mathfrak{l}]/K)^h$ is not a pth power in $\mathrm{Gal}(K[\mathfrak{l}]/K)$. But since $\mathrm{Gal}(K[\mathfrak{l}]/K)$ is cyclic of order m and $p \nmid h$, we deduce that $(\mathfrak{q}, K[\mathfrak{l}]/K)$ generates $\mathrm{Gal}(K[\mathfrak{l}]/K)$ and hence $\mathfrak{q}$ is inert in $K[\mathfrak{l}]$. ∎

Using the prime $\mathfrak{l}$ satisfying the conditions of the previous lemma, we can define the elliptic units

$$\eta_{\mathfrak{l}} := N_{H_{\mathfrak{l}\mathfrak{m}_I}/L[\mathfrak{l}]}(\alpha_{\mathfrak{l}\mathfrak{m}_I}), \qquad \hat{\eta}_{\mathfrak{l}} := N_{H_{\mathfrak{l}\mathfrak{q}\mathfrak{m}_I}/L[\mathfrak{q}\mathfrak{l}]}(\alpha_{\mathfrak{l}\mathfrak{q}\mathfrak{m}_I}),$$

where $L[\mathfrak{q}\mathfrak{l}]$ is the compositum of $L[\mathfrak{q}]$ and $L[\mathfrak{l}]$. Using the norm relation, we find

$$N_{L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{l}]}(\hat{\eta}_{\mathfrak{l}}) = \eta_{\mathfrak{l}}^{1-\sigma_{\mathfrak{q}}^{-1}},$$
$$N_{L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{q}]}(\hat{\eta}_{\mathfrak{l}}) = \hat{\eta}^{1-\sigma_{\mathfrak{l}}^{-1}} = \hat{\eta}^{1-\sigma},$$
$$N_{L[\mathfrak{l}]/L}(\eta_{\mathfrak{l}}) = \eta^{1-\sigma_{\mathfrak{l}}^{-1}} = \eta^{1-\sigma},$$

where $\sigma_{\mathfrak{q}} = (\mathfrak{q}, L[\mathfrak{l}]/K)$ and $\sigma_{\mathfrak{l}} = (\mathfrak{l}, L[\mathfrak{q}]/K) = \sigma^{-1}$ by condition (ii).

Since $\mathfrak{q} \in \mathcal{Q}_m$, $\mathfrak{q}$ splits completely in L/K, and by condition (iii), the primes of L above $\mathfrak{q}$ are inert in $L[\mathfrak{l}]/L$. Then each prime of $L[\mathfrak{q}]$ above $\mathfrak{q}$ must also be inert in $L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{q}]$. Moreover, since each prime above $\mathfrak{q}$ is unramified in $L[\mathfrak{l}]/L$ and totally ramified in $L[\mathfrak{q}]/L$, it is also totally ramified in $L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{l}]$, hence the product of all primes of $L[\mathfrak{q}\mathfrak{l}]$ above $\mathfrak{q}$ is given by $\mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}$. Therefore, we get the following isomorphism of rings:

$$\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}/\mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]} \cong \mathcal{O}_{L[\mathfrak{l}]}/\mathfrak{q}\mathcal{O}_{L[\mathfrak{l}]}.$$

Since $L[\mathfrak{q}]$ and $L[\mathfrak{l}]$ are linearly disjoint over L, and $\mathfrak{q}$ splits completely in L/K, we can extend $\sigma_{\mathfrak{q}} \in \mathrm{Gal}(L[\mathfrak{l}]/K)$ to $L[\mathfrak{q}\mathfrak{l}]$ in such a way that this extension (also denoted by $\sigma_{\mathfrak{q}}$) restricts to the identity on $L[\mathfrak{q}]$. In particular, $\sigma_{\mathfrak{q}}$ generates $\mathrm{Gal}(L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{q}])$.

From the above isomorphism, we see that $\sigma_{\mathfrak{q}}$ acts as raising to the Qth power on $\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}/\mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}$. Moreover, the group $\mathrm{Gal}(L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{l}])$ is the inertia group at $\mathfrak{q}$ and acts trivially on $\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}/\widetilde{\mathfrak{q}}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}$. Therefore, we can express the action of the norms $N_{L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{l}]}$ and $N_{L[\mathfrak{q}\mathfrak{l}]/L[\mathfrak{q}]}$ on $\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}/\mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}$ as raising to the power m respectively to the power $\sum_{i=0}^{m-1} Q^i$. Since $Q \equiv 1 \bmod m$, there exists a positive integer r such that $\sum_{i=0}^{m-1} Q^i = mr$. Combining our results, we get

$$\hat{\eta}^{Q(1-\sigma)} \equiv \hat{\eta}_{\mathfrak{l}}^{Qmr} \equiv \eta_{\mathfrak{l}}^{Qr(1-\sigma_{\mathfrak{q}}^{-1})} \equiv \eta_{\mathfrak{l}}^{r(Q-1)} \equiv (\eta_{\mathfrak{l}}^{mr})^{(Q-1)/m}$$
$$\equiv \eta^{(1-\sigma)(Q-1)/m} \bmod \mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}.$$

Since the natural map $\mathcal{O}_{L[\mathfrak{q}]}/\mathfrak{q}_{L[\mathfrak{q}]} \to \mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}/\mathfrak{q}_{L[\mathfrak{q}]}\mathcal{O}_{L[\mathfrak{q}\mathfrak{l}]}$ is injective, we obtain the assertion of Proposition 7.5. ∎

8. Annihilating the ideal class group. Using the same notation as before, we define

$$\mu_i := n_{\max M_i}.$$

This is always a power of p, and since $M_i \subseteq M_{i+1}$, we get $\mu_i \leq \mu_{i+1}$. We call an index $i \in \{1, \ldots, k-1\}$ a *jump* if $\mu_i < \mu_{i+1}$. Further, we declare 0 and k to be jumps and set $\mu_0 = 0$. Then we get

LEMMA 8.1 (cf. [CK19, Lemma 7.1]). *Let $0 = s_0 < s_1 < \cdots < s_\kappa = k$ be the ordered sequence of all jumps. Then the set*

$$\bigcup_{t=1}^{\kappa}\{\alpha_{s_t}^{\sigma^i} \mid 0 \leq i < p^{s_t} - p^{s_{t-1}}\}$$

is a $\mathbb{Z}$-basis of $\overline{\mathcal{C}_L}/\mu(K)$.

Proof. See [GK15, Lemma 5.1]. ∎

With this basis, we obtain our next result:

LEMMA 8.2 (cf. [CK19, Lemma 7.2]). *Let r be the highest jump less than k, i.e. $\mu_r < \mu_{r+1} = n_s$. Assume that $\rho \in \mathbb{Z}[\Gamma]$ is such that $\alpha_k^{\rho} \in \overline{\mathcal{C}_{L_r}}$. Then*

$$(1 - \sigma^{p^r})\rho = 0.$$

Now we need an additional condition on the p-power m. We already know that $(m, q) = 1$, since $p \nmid q$, so $q \in (\mathbb{Z}/m\mathbb{Z})^{\times}$. Let d denote the order of q in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. Then there exists $i \geq 0$ and $b \in \mathbb{Z}$ with $p \nmid b$ such that

$$q^d - 1 = b \cdot p^i m.$$

If we define $m' := p^i m$, we still have $p^{ks} \mid m'$ and d is the order of q modulo m', so we can assume without loss of generality that $i = 0$. Now we can define f to be the order of q in $(\mathbb{Z}/m^2\mathbb{Z})^{\times}$. Then an easy computation shows

LEMMA 8.3. *We have $m \mid f/d$.*

THEOREM 8.4. *Let m be a power of p such that $m \mid f/d$, and $V \subseteq L^{\times}/m$ a finitely generated $\mathbb{Z}_p[\Gamma]$-submodule. Without loss of generality we can choose representatives of generators of V which belong to $\mathcal{O}_L$. Suppose there is a map $z : V \to (\mathbb{Z}/m\mathbb{Z})[\Gamma]$ of $\mathbb{Z}_p[\Gamma]$-modules such that $z(V \cap K^{\times}) = 0$, where $V \cap K^{\times}$ means $V \cap (K^{\times}(L^{\times})^m/(L^{\times})^m)$. Then for any $\mathfrak{c} \in \mathrm{Cl}(\mathcal{O}_L)_p$, there exist infinitely many primes $\mathfrak{Q}$ in L such that:*

(i) *$\mathfrak{q} := \mathfrak{Q} \cap K$ is completely split in L/K,*
(ii) *$[\mathfrak{Q}] = \mathfrak{c}$, where $[\mathfrak{Q}]$ is the projection of the ideal class of $\mathfrak{Q}$ into $\mathrm{Cl}(L)_p$,*
(iii) *$Q := |\mathcal{O}_L/\mathfrak{Q}| \equiv 1 + m \bmod m^2$,*
(iv) *for each $j = 1, \ldots, s$, the class of x_j is an mth power in $(\mathcal{O}_K/\mathfrak{q})^{\times}$,*
(v) *no prime above $\mathfrak{q}$ is contained in the support of the generators of V and there is a $\mathbb{Z}_p[\Gamma]$-linear map $\varphi : (\mathcal{O}_L/\mathfrak{q}\mathcal{O}_L)^{\times}/m \to (\mathbb{Z}/m\mathbb{Z})[\Gamma]$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\quad z \quad} & (\mathbb{Z}/m\mathbb{Z})[\Gamma] \\ \Big\downarrow{\scriptstyle\psi} & \nearrow_{\varphi} & \\ (\mathcal{O}_L/\mathfrak{q}\mathcal{O}_L)^{\times}/m & & \end{array}$$

commutes, where ψ corresponds to the reduction map.

REMARK 8.5. The reduction map ψ is defined on the chosen set of generators: Let $x \in \mathcal{O}_L$ be a representative of such a generator; then $\overline{x}$ is the class of $x \in \mathcal{O}_L/\mathfrak{q}\mathcal{O}_L$. Since no prime above $\mathfrak{q}$ is contained in the support of x, we get $\overline{x} \in (\mathcal{O}_L/\mathfrak{q}\mathcal{O}_L)^\times$. Hence we can set $\psi(x)$ to be the class of $\overline{x}$ in $(\mathcal{O}_L/\mathfrak{q}\mathcal{O}_L)^\times/m$. This yields a well-defined $\mathbb{Z}_p[\Gamma]$-homomorphism.

Proof of Theorem 8.4. The proof is essentially the same as in [GK04, Thm. 17]. We only point out some changes which are necessary in the function field case:

- In the proof of [GK04, Lemma 18(i, ii)], we only get an isomorphism to a subgroup H of $(\mathbb{Z}/m^2\mathbb{Z})^\times$ of order f. Nevertheless, we can choose q as a generator of H and hence prove the vanishing of the coinvariants since $p \nmid q-1$.
- For [GK04, Lemma 18(iii)] we consider the splitting field of ∞ in $L\mathbb{F}_{q^f}$, which is the unique subextension of degree (f, d_∞) (see [R02, Prop. 8.13]). Then the claim follows since $d_\infty \mid h$ and $p \nmid h$, so $p \nmid d_\infty$.
- In the last step of the construction of the element τ which is used for Chebotarev's Density Theorem (cf. [R02, Thm. 9.13A]), we need an element of order m in $\mathrm{Gal}(L\mathbb{F}_{q^f}/L\mathbb{F}_{q^d})$. At this point we need $m \mid f/d$ from Lemma 8.3.
- Condition (iii) uses the fact that ζ_{m^2} is an element of the constant field of $L\mathbb{F}_{q^f}$. ∎

For the desired annihilation result, we need

THEOREM 8.6 (cf. [R87, Thm. (5.1)]). *Let $\mathfrak{q}$ be a prime of K which splits completely in L, set $Q := |\mathcal{O}_K/\mathfrak{q}|$. Let M be a finite extension of L which is abelian over K and such that in M/L, all primes above $\mathfrak{q}$ are totally tamely ramified and no other primes ramify. Write $\mathfrak{q}_M$ for the product of all primes of M above $\mathfrak{q}$ and let $\mathcal{A}$ denote the annihilator in $(\mathbb{Z}/(Q-1)\mathbb{Z})[\Gamma]$ of the cokernel of the reduction map*

$$\{\varepsilon \in \mathcal{O}_M^\times \mid N_{M/L}(\varepsilon) = 1\} \to (\mathcal{O}_M/\mathfrak{q}_M)^\times.$$

Write $w := (Q-1)/[M:L]$. Then $\mathcal{A} \subseteq w(\mathbb{Z}/(Q-1)\mathbb{Z})[\Gamma]$ and for every prime $\mathfrak{Q}$ of L above $\mathfrak{q}$, $w^{-1}\mathcal{A}$ annihilates the ideal class of $\mathfrak{Q}$ in $\mathrm{Cl}(\mathcal{O}_L)/[M:L]$.

Proof. The proof of [R87, Thm. (5.1)] also works for function fields. ∎

The above theorems are the main ingredients for proving

THEOREM 8.7. *Let m be a power of p divisible by p^{ks} such that $m \mid f/d$. Assume that $\varepsilon \in \mathcal{O}_L$ is m-semispecial and let $V \subseteq L^\times/m$ be a finitely generated $\mathbb{Z}[\Gamma]$-module. Suppose that the class of ε belongs to V. Now let $z : V \to (\mathbb{Z}/m\mathbb{Z})[\Gamma]$ be a $\mathbb{Z}[\Gamma]$-linear map such that $z(V \cap K^\times) = 0$. Then $z(\varepsilon)$ annihilates $\mathrm{Cl}(\mathcal{O}_L)_p/(m/p^{k(s-1)})$.*

Proof. See [GK04, Thm. 12]. ∎

The main result of this article is Theorem C:

THEOREM 8.8. *Let r be the highest jump less than k. Then*

$$\mathrm{Ann}_{\mathbb{Z}[\Gamma]}((\mathcal{O}_L^\times/\overline{\mathcal{C}_L})_p) \subseteq \mathrm{Ann}_{\mathbb{Z}[\Gamma]}((1-\sigma^{p^r})\,\mathrm{Cl}(\mathcal{O}_L)_p).$$

Let $J = \{j \in \{1,\dots,s\} \mid n_j = n_s\}$. Then the number r is determined by $p^{k-r} = \max\{t_j \mid j \in J\}$.

Proof. The proof of [CK19, Thm. 7.5] can be used without any changes, as we have proven all the main ingredients in Theorems 8.4, 8.6 and 8.7. ∎

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Abstract (will appear on the journal's web site only)

Let K be a global function field and fix a place ∞ of K. Let L/K be a finite real abelian extension, i.e. a finite, abelian extension such that ∞ splits completely in L. Then we define a group C_L of elliptic units in $\mathcal{O}_L^\times$ analogously to Sinnott's cyclotomic units and compute the index $[\mathcal{O}_L^\times : C_L]$. In the second part of this article, we additionally assume that L is a cyclic extension of prime power degree. Then we can use the methods of Greither and Kučera to take certain roots of these elliptic units and prove a result on the annihilation of the p-part of the class group of L.