Fall term 2022

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Selected Topics in Complex Geometry

Sheet 11

Exercise 1. (A rational cohomology torus which is not a torus) Let $\rho : C \to E_1$ be a (ramified) double cover of Riemann surfaces where C has genus ≥ 2 and E_1 is an elliptic curve. Let τ be the corresponding involution on C. Let $E_2' \to E_2$ be a degree 2 unramified double cover of elliptic curves (e.g. the quotient of multiplication by 2) and let σ be the corresponding involution on E_2'. Define $S := C \times E_2')(\tau \times \sigma)$. Show

1. S is a smooth complex surface.
2. σ acts trivially on the cohomology of E_2'.
3. The rational cohomology ring of S is isomorphic to that of $E_1 \times E_2$.
4. There is a map $S \to E_1 \times E_2$ and this can be identified with the Albanese map.
5. S is not a torus.

Exercise 2. (An Albanese torus for general complex manifolds) Let X be a compact complex manifold. Find a complex torus $Alb(X)$ and a holomorphic map $\alpha_X : X \to Alb(X)$ s.t. every analytic map from X to a complex torus T factors uniquely through α_X. (Hint: Consider the space of closed holomorphic one-forms).

Exercise 3. Show that on any compact complex manifold of dimension n, if a holomorphic n-form ω is ∂-exact $\omega = \partial\eta$, it is zero. Deduce that on compact complex surfaces, every holomorphic one-form is closed.

Hand-in: Via Email or in person to Jonas Stelzig until We, 12.06., 14:00.