Fall term 2022

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Selected Topics in Complex Geometry

Sheet 06

Exercise 1. Let $f : X \to Y$ be a holomorphic map between compact Kähler manifolds of complex dimension n and m. Show that the pushforward maps

$$f_* : H^k(X) \to H^{2(m-n)+k}(Y)$$

defined in the lecture are maps of Hodge structures.

Exercise 2. Let Γ, Λ be free abelian groups of finite ranks e, f which carry Hodge structures $H^{\bullet,\bullet}$, $I^{\bullet,\bullet}$ of weights k, l. Construct Hodge structures on $\Gamma \otimes \Lambda$, $\Gamma \oplus \Lambda$, $\Gamma^\vee$ and $Hom_{\mathbb{Z}}(\Gamma, \Lambda)$ such that the natural maps

$$\begin{aligned} \Gamma \to \Gamma \oplus \Lambda \to \Gamma, \\ \Lambda \to \Gamma \oplus \Lambda \to \Lambda, \\ \Gamma^\vee \otimes \Lambda \xrightarrow{\sim} Hom_{\mathbb{Z}}(\Gamma, \Lambda) \end{aligned}$$

induce homomorphisms of Hodge structures.

Exercise 3. Let Γ, Λ be as in the previous exercise.

1. Show that for any map of abelian groups $f : \Gamma \to \Lambda$ which underlies a homomorphism of Hodge structures, its kernel and image are sub Hodge structures of H, resp. I. Is this still true one merely has a map $g : \Gamma \otimes \mathbb{C} \to \Lambda \otimes \mathbb{C}$ compatible with the grading?
2. Let $A = \bigoplus A_i$ be a cohomology algebra with Hodge structure as in the lecture. Show that for any $a, b \in \mathbb{Z}$ the image of the multiplication map

   $$A_{2a} \otimes A_{2b+1} \to A_{2(a+b)+1}.$$

   has to have even rank. [This generalizes the fact that the ranks of the odd degree pieces have to be even.]

Hand-in: Via Email or in person to Jonas Stelzig until We, 08.06., 14:00.