Fall term 2022

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Selected Topics in Complex Geometry

Sheet 04

Exercise 1. Let C be a smooth complex plane curve of degree d, i.e. smooth hypersurface $C \subseteq \mathbb{C}\mathrm{P}^2$ s.t. $\deg(\mathcal{O}(C)) = d$. Show the 'degree-genus-formula'

$$g(C) = \frac{(d-1)(d-2)}{2},$$

where $g(C) = \frac{b_1(C)}{2}$ denotes the genus of C. Deduce that there are values of g such that a complex curve of genus g cannot be holomorphically embedded into $\mathbb{C}\mathrm{P}^2$.

Exercise 2. Let X be a connected complex manifold of dimension n and $\hat{X}$ the blow-up at a point $x \in X$. Show that there is a diffeomorphism

$$\hat{X} \cong X \# \overline{\mathbb{C}\mathrm{P}}^n,$$

where the right summand is $\mathbb{C}\mathrm{P}^n$ with reverse orientation. If you know the Mayer-Vietoris sequence, deduce a formula for the de-Rham or singular cohomology of $\hat{X}$.

Exercise 3. Let $\sigma : B \to \mathbb{C}^n$ be the blow-up of the origin $0 \in \mathbb{C}^n$ with exceptional divisor $E = \sigma^{-1}(0) \cong \mathbb{C}\mathrm{P}^{n-1}$. Denote by $p : B \to \mathbb{C}\mathrm{P}^{n-1}$ the projection induced from the inclusion $B \subseteq \mathbb{C}^n \times \mathbb{C}\mathrm{P}^{n-1}$. Show that $\mathcal{O}(E) \cong p^*\mathcal{O}(-1)$. Deduce that $\mathcal{O}(E)|_E = \mathcal{O}(-1)$.

Exercise 4. Consider the analytic set $Z = V(y^2 - x^3 - x^2) \subseteq \mathbb{C}^2$.

1. Show that Z is not a complex manifold at the origin $0 \in Z$.
2. Let $\sigma : B \to \mathbb{C}^2$ be the blow-up of the origin $0 \in \mathbb{C}^2$. Denote by $\bar{Z} \subseteq B$ the closure of $\sigma^{-1}(Z - \{0\})$. Show that $\bar{Z}$ is complex manifold. Draw a picture of the map from $\bar{Z}$ to Z.

Hand-in: Via Email or in person to Jonas Stelzig until Mo, 23.05., 14:00.