Fall term 2022

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Selected Topics in Complex Geometry

Sheet 03

Exercise 1. A holomorphic line bundle E with hermitian metric h is called (Griffiths) positive if the curvature Ω of the Chern connection satisfies

$$h(\Omega(s), s)(v, \bar{v}) > 0$$

for all local sections s of E and v of $T^{1,0}X$. Analogously, one defines notion of semi-positivity ($\geq$ instead of $>$) and (semi-)negativity. Show:

1. A Griffiths positive line bundle (L, h) is a positive line bundle.
2. A hermitian holomorphic vector bundle (E, h) is (semi-)positive if and only if the dual bundle $E^\vee$ with induced metric is (semi-)negative.
3. If two hermitian holomorphic vector bundles (E_1, h_1) and (E_2, h_2) are (semi-)positive, so are $E_1 \oplus E_2$ and $E_1 \otimes E_2$ with induced metric.

Exercise 2. Let X be a compact complex manifold. Show that the image of the injection

$$\mathcal{C}l(X) \longrightarrow Pic(X)$$

is given by those line bundles admitting a nonzero meromorphic section.

Exercise 3. Construct a complex 2-dimensional torus $X = \mathbb{C}^2/\Gamma$ s.t. $H^{1,1}(X, \mathbb{Z}) = 0$. (Alternatively, show that this holds for a generic choice of Γ). Deduce that on X the map $\mathcal{C}l(X) \to Pic(X)$ is far from being an isomorphism. In particular, X cannot be projective.

Exercise 4. Compute the cohomology groups $H^k(\mathbb{C}\mathrm{P}^n, \mathcal{O}(m))$ for all $n, k \geq 0$, $m \in \mathbb{Z}$. You may use that global sections are given by homogeneous polynomials, i.e. $H^0(\mathbb{C}\mathrm{P}^n, \mathcal{O}(m)) = \mathbb{C}[z_0, ..., z_n]_m$.

Hand-in: Via Email or in person to Jonas Stelzig until Mo, 16.05., 14:00.