LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Fall term 2022

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Selected Topics in Complex Geometry

Sheet 02

Exercise 1. Given two complex vector bundles E_1, E_2 with connections ∇_1 and ∇_2, construct natural connections on their direct sum and tensor product and homomorphism bundle. Compute the curvatures of the new connections in terms of the curvatures of ∇_1 and ∇_2.

Exercise 2. Let (E, h) be a hermitian vector bundle which is a direct sum of subbundles $E = E_1 \oplus E_2$. Denote by h_i the induced metrics on E_i. Let ∇ be a connection on E. Define the second fundamental forms b_i of E_i to be $b_i := \mathrm{pr}_{E/E_i} \circ \nabla$.

1. Show that b_i is linear over functions.
2. Denoting by ∇_1 the induced connection on E_1, show that $\nabla_1^2 = \mathrm{pr}_{E_1} \circ \nabla^2 - b_2 \circ b_1$.
3. Assume ∇ to be compatible with h. Show that

$$h_1(s, b_2(t)) + h_2(b_1(s), t) = h_1(b_2(s), t) + h_2(s_2, b_1(t)) = 0,$$

where s, t are local sections of E_1, resp. E_2.

Exercise 3. Let X be a compact complex manifold. A holomorphic line bundle is called negative if there is a representative $\omega \in c_1(L)$ s.t. $-\omega$ is a Kähler form. Show that Kodaira vanishing is equivalent to the following statement: For a negative line bundle L, the cohomology groups $H^q(X, \Omega_X^p \otimes L)$ vanish whenever $p + q < n = \dim X$.

Exercise 4. (Kodaira vanishing for Riemann surfaces) Give a direct proof of the Kodaira vanishing theorem when $\dim_{\mathbb{C}} X = 1$. Use the equivalent formulation given by the previous exercise.

Hand-in: Per Email or in person to Jonas Stelzig until Wed, 11.04., 18:00.