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On Generalized and Viscosity Solutions of Nonlinear Elliptic Equations

David Hartenstine, Klaus Schmitt
Department of Mathematics, University of Utah,
155 South 1400 East, Salt Lake City, Utah 84112–0090
e-mail: hartenst@math.utah.edu, schmitt@math.utah.edu

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Abstract

There are many notions of solutions of nonlinear elliptic partial differential equations. This paper is concerned with solutions which are obtained as suprema (or infima) of so-called subfunctions (superfunctions) or viscosity subsolutions (viscosity supersolutions). The paper also explores the relationship of these (generalized) solutions of differential inequalities and provides a relevant example for which existence questions have been studied using these concepts.

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1 Introduction

A seminal paper concerning the existence of harmonic functions satisfying given Dirichlet boundary conditions appeared in 1923. In it, [33], Perron used the notions of sub- and superharmonic functions to define sub- and superfunctions for the given boundary value problem and then obtained the existence of a harmonic function, which satisfies the boundary data in a certain generalized sense, as the supremum of all subfunctions. The process (since called the Perron process) is in some sense a constructive one and has been abstracted in many different ways. It was Beckenbach who introduced the concept of generalized convex functions, [5], which then gave rise to the concepts of sub- and superfunctions of several variables, [6]; these concepts were subsequently used by Jackson, [22], [23], to study boundary value problems for certain quasilinear elliptic equations. This approach was refined for the case of ordinary differential equations in [15], [4] and others, and has been summarized in detail in [24]. It turned out that such functions, when smooth enough, were solutions of differential inequalities, cf. [32], [15], [4], [35], which then provided the basis of the so-called sub- and supersolution method (lower- and uppersolution method) (in a classical and weak sense) for semilinear and quasilinear elliptic partial differential equations; this allowed for the creation of a multitude of existence results for many different types of boundary value problems for second order ordinary and semi- and quasilinear elliptic partial differential equations. The origin of such results likely lies in the work by Scorza-Dragoni, [36], (who subsequently extended and refined his work further), the work of Nagumo, [30], [31], several papers by Akô, including [1], [2], and many others, e.g. [3], [11], [10], [14], [17], [27], [28], [29], [34]. The area of viscosity solutions of nonlinear partial differential equations has enjoyed tremendous activity during the past two and a half decades and has become a standard approach in the study of such equations. This is largely because of the broad applicability of the methods to nonlinear elliptic equations, problems of Hamilton - Jacobi - Bellman type (and hence for problems of stochastic control and differential games) etc., see [13]. Since viscosity solutions are, in general, obtained as suprema of collections of viscosity subsolutions, the area is reminiscent of the abstract approach of Beckenbach [5] and Beckenbach and Jackson [6]. In this paper we shall discuss such a relationship at least for some specific and special class of problems and hence complement some of the work of Ishii [18] and [19]. We will show, in Section 3, that for this class of elliptic problems, the subfunctions of Beckenbach and Jackson are equivalent to viscosity subsolutions. This relationship is explored further in the somewhat more delicate case of the k-Hessian equations in Section 5.2. We also describe in some detail the axiomatic approach of Beckenbach and Jackson (Section 4), and demonstrate how this approach applies to Dirichlet problems for the k-Hessian equations in Section 5.3. The plan of this paper is as follows. In Section 2, we introduce some basic terminology and notation, define subfunctions and viscosity solutions, and review some of the relevant theory for these topics. Next we show the equivalence of viscosity subsolutions and subfunctions. We then turn to the abstract Perron method of Beckenbach and Jackson in Section 4. We conclude with a section devoted to k-Hessian equations and their connection to the ideas of the previous sections.

2 Preliminaries

In this section, we introduce the class of partial differential equations we shall consider, define viscosity solutions and discuss the part of the theory relevant for the Perron method. We also introduce the subfunctions of Beckenbach and Jackson.

2.1 A class of differential equations

The class of partial differential equations we shall consider is the following: Let $\Omega \subset \mathbb{R}^N$ be an open connected set and $\mathcal{S}^N$ be the set of $N \times N$ symmetric matrices. Let

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$$

be a continuous mapping. Associated with F will be the differential equation

$$F(x, u, Du, D^2u) = 0, \tag{2.1}$$

where Du is the gradient and D^2u is the Hessian matrix of second derivatives of a function

$$u : \Omega \to \mathbb{R}.$$

In the set $\mathcal{S}^N$ the following is a partial order:

$$X \leq Y \Longleftrightarrow Y - X \text{ is positive semidefinite.}$$

It will be assumed throughout that F has the following monotonicity property (usually called *degenerate ellipticity*) either for all X and Y in $\mathcal{S}^N$ or in a specific subset (see Section 5).

$$F(x, r, p, X) \geq F(x, r, p, Y), \text{ whenever } X \leq Y. \tag{2.2}$$

We remark that in the absence of degenerate ellipticity, classical solutions may fail to be viscosity solutions, as one can easily see.

2.2 Semicontinuous envelopes

For a function

$$u : \Omega \to \mathbb{R}$$

one defines the upper semicontinuous envelope u^* by

$$u^*(x) = \limsup_{y \to x} u(y)$$

and the lower semicontinuous envelope u_* by

$$u_*(x) = \liminf_{y \to x} u(y).$$

2.3 Viscosity solutions

A function

$$u : \Omega \to \mathbb{R}$$

is called a *viscosity subsolution* of (2.1) provided that $u^* : \Omega \to \mathbb{R}$ and for $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$u^*(x) - \phi(x) = \max_{\Omega}(u^* - \phi),$$

it follows that

$$F(x, u^*(x), D\phi(x), D^2\phi(x)) \leq 0.$$

Similarly, a function

$$u : \Omega \to \mathbb{R}$$

is called a *viscosity supersolution* of (2.1) provided that $u_* : \Omega \to \mathbb{R}$ and for $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$u_*(x) - \phi(x) = \min_{\Omega}(u_* - \phi),$$

it follows that

$$F(x, u_*(x), D\phi(x), D^2\phi(x)) \geq 0.$$

A function

$$u : \Omega \to \mathbb{R}$$

is called a *viscosity solution* of (2.1) if it is both a viscosity sub- and a viscosity supersolution. We note that the class of test functions employed in the preceding definitions can be restricted to quadratic polynomials (see for example [9]). Solutions of differential inequalities in the viscosity sense and classical solutions are related in the following way. If

$$u : \Omega \to \mathbb{R}$$

is of class C^2 and is a viscosity subsolution, then it must satisfy

$$F(x, u(x), Du(x), D^2u(x)) \leq 0.$$

In other words, if u is a C^2 viscosity subsolution, then it is a classical subsolution. Conversely, it follows from the degenerate ellipticity, (2.2), that classical subsolutions are viscosity subsolutions. Furthermore if u is a viscosity subsolution and has first and second order superdifferentials at a point x, i.e.

$$0 \leq u(x) - u(y) + p \cdot (x - y) + \frac{1}{2}(x - y)^T X(x - y) + o(|x - y|^2)$$

for some $p \in \mathbb{R}^N$, some $X \in \mathcal{S}^N$, and all y near x, then

$$F(x, u(x), p, X) \leq 0.$$

For the above assertions, see [18]. When a comparison principle for viscosity sub- and supersolutions holds, the Perron method can be used to demonstrate the existence and uniqueness of viscosity solutions to Dirichlet problems. More precisely, let us consider the following assumption.

Assumption 2.1 (Comparison Principle) *Suppose* u *is a viscosity solution and* v *is a viscosity supersolution for* (2.1) *in a domain* D*. If* $u^* \leq v_*$ *on* ∂D*, then* $u^* \leq v_*$ *in* D*.*

With this assumption, the following theorem holds.

Theorem 2.2 (Proposition II.1, [20]) *Let $g \in C(\partial\Omega)$, and suppose that Assumption 2.1 holds. If there exist a viscosity subsolution $u \in C(\overline{\Omega})$ and a viscosity supersolution $v \in C(\overline{\Omega})$ of (2.1) satisfying $u = v = g$ on $\partial\Omega$, then there exists a unique viscosity solution $W \in C(\overline{\Omega})$ to (2.1) with $W = g$ on $\partial\Omega$. Furthermore, if*

$$S = \{w : w \text{ is a viscosity subsolution of (2.1)},\ w = g \text{ on } \partial\Omega\},$$

then

$$W(x) = \sup_{w \in S} w(x).$$

Thus the existence and uniqueness of a viscosity solution to the Dirichlet problem can be reduced to demonstrating Assumption 2.1 and finding a continuous viscosity subsolution and a continuous viscosity supersolution both of which attain the boundary data. To demonstrate this comparison principle, it is often assumed that F is nondecreasing in r for each (x, p, X). This assumption should be compared to the case of linear second-order elliptic equations. Conditions on F which permit the establishment of Assumption 2.1 are given in [20], [13], [26], and [37], among others. For a particular example, we quote the following result of Ishii ([18]).

Theorem 2.3 *Let Ω be a bounded domain, and let $G = G(r, p, X)$ be non-decreasing in r for all (p, X), continuous and degenerate elliptic. Define the mapping F by:*

$$F[u] = u + G(u, Du, D^2u).$$

Let u be a viscosity subsolution to $F[w] = 0$ and v be a viscosity supersolution to $F[w] = 0$. Suppose also that $u^* \le v_*$ on $\partial\Omega$. Then $u^* \le v_*$ in Ω.

Assumption 2.1 implies that viscosity solutions (and hence also classical solutions) to the Dirichlet problem are unique. Therefore, this comparison principle cannot hold for any equation that admits multiple solutions to the Dirichlet problem.

2.4 Subfunctions, superfunctions and generalized solutions

An upper semicontinuous function

$$u : \Omega \to \mathbb{R}$$

is called a *subfunction* relative to (2.1) provided that if B is any ball with $\overline{B} \subset \Omega$, and $\phi \in C^2(B) \cap C(\overline{B})$ is a solution of (2.1) with

$$u \le \phi \text{ on } \partial B,$$

then

$$u \le \phi \text{ in } \overline{B}.$$

A lower semicontinuous function

$$u : \Omega \to \mathbb{R}$$

is called a *superfunction* relative to (2.1) provided that if B is any ball with $\overline{B} \subset \Omega$, and $\phi \in C^2(B) \cap C(\overline{B})$ is a solution of (2.1) with

$$u \geq \phi \text{ on } \partial B,$$

then

$$u \geq \phi \text{ in } \overline{B}.$$

We now define a *generalized solution* to (2.1) as a function that is both a subfunction and a superfunction in Ω. When a comparison principle of the type in Postulate 4.2 below holds for classical solutions of (2.1), classical solutions are generalized solutions. In Section 3 we compare these subfunctions and viscosity subsolutions, and demonstrate their equivalence under some assumptions on the equation (2.1). We remark that the theory can be built on the notions of local subfunctions and superfunctions, for which the necessary comparison properties are required to hold only on suitably small domains. The same is also true in the viscosity case, see for example the definition in [9].

3 Subfunctions and viscosity subsolutions

We now show that under some natural assumptions, the notions of subfunctions and upper semicontinuous viscosity subsolutions are equivalent. The same argument shows that lower semicontinuous viscosity supersolutions are superfunctions, and hence that continuous viscosity solutions and generalized solutions coincide for the class of equations considered here. We assume first that Assumption 2.1 holds. This trivially guarantees that upper semicontinuous viscosity subsolutions are subfunctions by the following argument. If u is an upper semicontinuous viscosity subsolution in Ω and v is a classical solution in B such that $u \leq v$ on ∂B, for some ball B compactly contained in Ω, then by Assumption 2.1, $u \leq v$ in B. In other words, u is a subfunction. Under the following additional assumption, subfunctions are viscosity subsolutions.

Assumption 3.1 (Local Solvability) *There exists $\delta > 0$ such that the problem*

$$\begin{cases} F(x, u, Du, D^2u) &= 0 \quad \text{in } B \\ \qquad\qquad u &= \phi \quad \text{on } \partial B \end{cases}$$

has a solution $u \in C^2(B) \cap C(\overline{B})$ for any ball B of radius smaller than δ and any $\phi \in C(\partial B)$.

Proposition 3.2 *If Assumptions 2.1 and 3.1 hold, a subfunction is a viscosity subsolution. Hence, under these assumptions, subfunctions and upper semicontinuous viscosity subsolutions are equivalent.*

Proof. Suppose that u is a subfunction but is not a viscosity subsolution. Then there exist $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$u(x) - \phi(x) = \max_{\Omega}(u - \phi),$$

and

$$F(x, u(x), D\phi(x), D^2\phi(x)) > 0. \tag{3.1}$$

We note that (3.1) remains valid for $\phi + c$ for any constant c, and thus we may assume that $u(x) = \phi(x)$ and (3.1) becomes

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) > 0. \tag{3.2}$$

By the continuity of F and ϕ and its derivatives, (3.2) holds in the ball $B = B_r(x)$, where r is smaller than the number δ of Assumption 3.1. Let v satisfy

$$\begin{cases} F(x, v, Dv, D^2v) &= 0 \quad \text{in } B \\ \qquad\qquad v &= \phi \quad \text{on } \partial B. \end{cases} \tag{3.3}$$

Because v is a classical solution, it is a viscosity subsolution, so we conclude from Assumption 2.1 and (3.2) that

$$v(y) \leq \phi(y), \ y \in \overline{B}.$$

Since u is a subfunction and $v = \phi$ on ∂B, it follows that

$$u(y) \leq v(y) \leq \phi(y), \ y \in \overline{B}.$$

Therefore, since $u(x) = \phi(x)$, $v - \phi$ has a local maximum at x, and we must have that

$$u(x) = v(x) = \phi(x), \ Dv(x) = D\phi(x), \text{ and } \ D^2v(x) \leq D^2\phi(x).$$

Hence by (3.3) and degenerate ellipticity, (2.2):

$$\begin{aligned} 0 = F(x, v(x), Dv(x), D^2v(x)) &= F(x, \phi(x), D\phi(x), D^2v(x)) \\ &\geq F(x, \phi(x), D\phi(x), D^2\phi(x)), \end{aligned}$$

contradicting (3.2).

Remark 3.3 By restricting the class of test functions for viscosity solutions to quadratic polynomials, we can prove Proposition 3.2 provided Assumption 3.1 holds only for $\phi \in C^\infty(\partial B)$.

4 Axiomatic approach to boundary value problems

In this section, we summarize the abstract Perron method used by Beckenbach and Jackson as found in [23] and [6]. This method concerns families of functions that satisfy a list of postulates, which describe the essential elements of the Perron argument. We remark that Jackson [23] applied this approach to the Dirichlet problem for the minimal surface equation for nonconvex planar domains that satisfy an exterior sphere condition, provided the boundary data satisfy some technical conditions. We also show that this method produces a generalized solution to the Dirichlet problem when a modification (relevant for the k-Hessian equations considered in Section 5) to the axiomatic structure is made. Let $\mathcal{F}$ be a family of functions satisfying the following postulates.

Postulate 4.1 *For any ball B with $\overline{B} \subset \Omega$ and any $g \in C(\partial B)$, there is a unique $u \in \mathcal{F} \cap C(\overline{B})$ such that $u = g$ on ∂B.*

Postulate 4.2 *For any ball B with $\overline{B} \subset \Omega$, let g_1, $g_2 \in C(\partial B)$ be such that $g_1 - g_2 \leq M$ on ∂B, with $M \geq 0$. Let u_1 and u_2 be those elements of $\mathcal{F}$ corresponding to g_1, g_2 and B which exist by Postulate 4.1. Then $u_1 - u_2 \leq M$ in B. Furthermore, if there exists $x \in \partial B$ such that $g_1(x) - g_2(x) < M$, then $u_1 - u_2 < M$ in B.*

Postulate 4.3 *For any ball B with $\overline{B} \subset \Omega$, if $\{g_n\} \subset C(\partial B)$ is a uniformly bounded sequence, the corresponding sequence of functions $\{u_n\} \subset \mathcal{F}$ is an equicontinuous family in B.*

Before stating the fourth postulate, we need to introduce some terminology. In this abstract setting, an upper semicontinuous function $u : D \to \mathbb{R}$ is called a *subfunction in D* if for any ball $B \subset D \subset \Omega$ and $w \in C(\partial B)$ for which $u \leq w$ on ∂B, we also have that u is less than or equal to that element of $\mathcal{F}$ that is continuous on $\overline{B}$ with boundary values w. Superfunctions are defined similarly. Compare this definition with that given in Section 2.4. Note that by (the weak part of) Postulate 4.2, elements of $\mathcal{F}$ are both sub- and superfunctions on their domains of definition. Conversely, if a function is a subfunction and a superfunction, then it is in $\mathcal{F}$. Suppose D is a bounded domain with $\overline{D} \subset \Omega$ and that g is bounded on ∂D. Then the function $s \in C(\overline{D})$ is said to be a *subsolution (with respect to g, $\mathcal{F}$ and D)* if $s \leq g$ on ∂D and s is a subfunction in D. The function $S \in C(\overline{D})$ is called a *supersolution* if $S \geq g$ on ∂D and S is a superfunction in D.

Postulate 4.4 *If D is any bounded domain with $\overline{D} \subset \Omega$, and g is bounded on ∂D, then there exists a subsolution and a supersolution with respect to $\mathcal{F}$, g and D.*

We now link this axiomatic framework to the more concrete setting of partial differential equations. The family $\mathcal{F}$ represents the collection of (local) solutions of a certain partial differential equation in the larger domain Ω. In other words, $f \in \mathcal{F}$ if it satisfies the partial differential equation on some subdomain of Ω. Then Postulate 4.1 concerns the unique solvability of the Dirichlet problem on balls, possibly in a weak or generalized sense. Postulate 4.2 is a comparison principle. Postulate 4.3 is a compactness result, and Postulate 4.4 deals with the existence of subsolutions and supersolutions. The following theorem should be compared with Theorem 2.2.

Theorem 4.5 ([6], [23]) *Let Ω be bounded and $g \in C(\partial\Omega)$. Then if $\mathcal{F}$ satisfies Postulates 4.1, 4.2, 4.3 and 4.4, and there exist a subsolution s and a supersolution S, with $s = S = g$ on $\partial\Omega$, then there exists $u \in C(\bar{\Omega})$ that solves the Dirichlet problem in the sense that $u = g$ on $\partial\Omega$ and on any ball B compactly contained in Ω, u agrees with that element of $\mathcal{F}$ that is equal to u on ∂B. Furthermore,*

$$u(x) = \sup\{v(x) : \ v \ \ \textit{a subsolution}\} = \inf\{w(x) : \ w \ \ \textit{a supersolution}\}.$$

We remark that if g is only assumed to be bounded or if no subsolutions or supersolutions attain the boundary data, there is still the notion of a generalized solution to the Dirichlet problem, obtained in precisely the same way, although then the supremum of the subsolutions need not coincide with the infimum of the supersolutions. See Theorem 4.8 below. The proof of these theorems relies on "lifting" a subsolution over a ball B compactly contained in Ω to produce a new subsolution. To show that this lift is also a subsolution, Beckenbach and Jackson in [6] use the strong comparison principle of Postulate 4.2. However, if the strong comparison principle does not hold or is not easily established, the same result can be obtained if the following weak comparison principle holds in general domains and not just in balls.

Postulate 4.6 *Suppose $u \in C(\bar{D})$ is a subfunction and $v \in C(\bar{D})$ is a superfunction in D, and that $u \leq v$ on ∂D. Then $u \leq v$ in D.*

Postulate 4.6 is a consequence of Postulates 4.1 and 4.2 as the following simple argument shows. Suppose Postulate 4.6 does not hold. Then there exist a subfunction $u \in C(\bar{D})$, a superfunction $v \in C(\bar{D})$, and a point $x \in D$ such that $u \leq v$ on ∂D and $u(x) > v(x)$. Let $M = \max_D u - v > 0$. Let E be the set of points in D where $u - v = M$. Then E is nonempty, closed and does not intersect ∂D. Let $x_0 \in E$ be a closest point to ∂D, and let B be any ball compactly contained in D centered at x_0. Then $(u - v)|_{\partial B} \leq M$ and this inequality is strict at some point on ∂B (otherwise there is a point in E closer to ∂D than x_0). Invoking Postulate 4.1, let $U \in C(\bar{B})$ be the unique element of $\mathcal{F}$ with boundary values u and let V be the member of $\mathcal{F}$ in $C(\bar{B})$ with boundary values v. Then we have that $u \leq U$ and $v \geq V$ in B. By the strong comparison principle of Postulate 4.2, $U - V < M$ in B. This implies that $u - v < M$ in B. In particular this is true at x_0, but this contradicts $x_0 \in E$. Postulate 4.6 is the subfunction version of Assumption 2.1 for viscosity solutions.

Theorem 4.7 *Let u be a continuous subfunction in Ω, and let B be a ball compactly contained in Ω. Suppose that Postulates 4.1 and 4.6 hold; suppose that the weak part of Postulate 4.2 also holds. Define the lift of u over B by*

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \setminus B \\ U(x) & x \in \overline{B} \end{cases}$$

where $U \in \mathcal{F} \cap C(\bar{B})$ satisfies $U|_{\partial B} = u$. Then $\tilde{u}$ is a subfunction in Ω.

Proof. Since u and U agree on ∂B, $\tilde{u}$ is continuous in Ω. Also, since u is a subfunction that agrees with U on ∂B, we have that $U \geq u$ in B and hence $\tilde{u} \geq u$ in Ω. Since u is a subfunction in $\Omega \setminus B$ and U is a subfunction in B, we only need to verify that $\tilde{u}$ satisfies the defining condition for a subfunction on balls $\tilde{B}$ that intersect both B and $\Omega \setminus B$. Suppose $w \in C(\tilde{B})$ is such that $w \geq \tilde{u}$ on $\partial\tilde{B}$. We let w also denote the element of $\mathcal{F}$ that is continuous on the closure of $\tilde{B}$ with boundary values w. On $\partial\tilde{B}$, $w \geq u$, and u is a subfunction, so $w \geq u$ in $\tilde{B}$, and hence $w \geq \tilde{u}$ in the portion of $\tilde{B}$ that lies outside of B. We then have that $w \geq U$ on $\partial(B \cap \tilde{B})$, and therefore by Postulate 4.6, $w \geq U$ in $(B \cap \tilde{B})$, and $\tilde{u}$ is a subfunction.

Theorem 4.8 *Suppose Ω is bounded and g is a bounded function on $\partial\Omega$. Suppose $\mathcal{F}$ satisfies Postulates 4.1, 4.3, 4.4, 4.6, and the weak part of Postulate 4.2. Then*

$$u(x) = \sup\{v(x) : v \textit{ subsolution}\}$$

is an (interior) generalized solution to the Dirichlet problem in Ω with boundary values g, i.e., u is continuous in Ω, $u \le g$ on $\partial\Omega$, and in any ball B compactly contained in Ω, u agrees with that element of $\mathcal{F}$ that is equal to u on ∂B. The same is true of the infimum of all supersolutions, except that this function dominates g on $\partial\Omega$.

Proof. We prove the statement for the supremum of the subsolutions. Observe first that u is well-defined and bounded. By Postulate 4.4, there exists a subsolution V and a supersolution W. Therefore, by hypothesis $V \le u \le W$ in Ω. The next step is to show that u is continuous. We use an argument similar to ones in [1] and [34]. Since u is the supremum of continuous functions, it is lower semicontinuous. Denote the set of subsolutions by S_g. Let $\{y_n\}_{n=1}^{\infty} \subset \Omega$ be dense in Ω. For each n, there exists a sequence $\{v_{n,m}\} \subset S_g$ such that

$$\lim_{m\to\infty} v_{n,m}(y_n) = u(y_n).$$

From the definition of subsolution, the maximum of any finite set of subsolutions is also a subsolution. Therefore, we may assume that for each n, $\{v_{n,m}\}$ is a monotonically nondecreasing sequence. ¿From the sequence $\{v_{n,m}\}$, we construct a monotonically nondecreasing sequence $\{u_m\}$ in S_g that will converge to u at each point y_n. Let $u_1 = v_{1,1}$. Let

$$u_2(x) = \max\{u_1(x), v_{1,2}(x), v_{2,2}(x)\}.$$

Note that $u_2 \in S_g$, and that $u_2 \ge u_1$. Having defined u_i, define u_{i+1} by

$$u_{i+1}(x) = \max\{u_i(x), v_{1,i+1}(x), v_{2,i+1}(x), \ldots, v_{i+1,i+1}(x)\}.$$

Then $u_{i+1} \in S_g$, and $u_{i+1} \ge u_i \ge \cdots \ge u_1$. We also have that each u_i is continuous in Ω. Since for any n and any $j \ge n$,

$$u(y_n) \ge u_j(y_n) \ge v_{n,j}(y_n),$$

we have that $\lim_{j\to\infty} u_j(y_n) = u(y_n)$. Now let B be any ball compactly contained in Ω. For each i, let $\tilde{u}_i$ be the lift of u_i over B. By Theorem 4.7, $\tilde{u}_i \in S_g$; we also have that $u_i \le \tilde{u}_i$, and the sequence $\{\tilde{u}_i\}$ is monotonically nondecreasing. Since $u_i \le \tilde{u}_i \le u$ and for each n, $u_i(y_n) \to u(y_n)$, $\tilde{u}_i(y_n) \to u(y_n)$ for all n. The functions u_i are uniformly bounded on ∂B. Therefore, by Postulate 4.3, the family $\{\tilde{u}_i\}$ is equicontinuous in B. Because $\{\tilde{u}_i\} \subset S_g$, the $\tilde{u}_i$ are uniformly bounded. Therefore, a subsequence of $\{\tilde{u}_i\}$ converges uniformly on compact subsets of B to a function $T \in C(B)$, but since the sequence $\{\tilde{u}_i\}$ is monotone, we obtain that $\tilde{u}_i \to T$. In particular, if $y_n \in B$, $\tilde{u}_i(y_n) \to T(y_n)$ and therefore, $T(y_n) = u(y_n)$. By density, for any $x \in B$, there is a sequence $\{z_j\} \subset \{y_n\} \cap B$ such that $z_j \to x$. Then

$$u(x) = \liminf_{j\to\infty} u(z_j) = \lim_{j\to\infty} T(z_j) = T(x).$$

Therefore, u agrees with the continuous function T on B, and hence it is continuous. We now show that u agrees with the unique element $U \in \mathcal{F}$ that is continuous on $\bar{B}$ and equal to u on ∂B, where B is any ball compactly contained in Ω. Postulate 4.1 guarantees the existence of U. The argument is similar to that given in [6]. Let $v \in S_g$. Then $u \geq v$ in Ω. Since v is a subfunction, $v \leq \tilde{v}$ in B, where $\tilde{v}$ is the lift of v over B. By the weak part of Postulate 4.2, $\tilde{v} \leq U$ in B. Therefore we have that $v \leq U$ in B, and taking the supremum over all $v \in S_g$, we get that $u \leq U$ in B. To prove the opposite inequality, let $\epsilon > 0$. Then for any point $x \in B$, there exists a subsolution ϕ such that $\phi(x) > u(x) - \epsilon/2$. By continuity, there exists an $r > 0$ such that $\phi(x) > u(x) - \epsilon$ on $\overline{B_0}$, where $B_0 = B_r(x) \subset B$. By the weak part of Postulate 4.2, we have that $\tilde{\phi} > \tilde{u}_\epsilon$ in B_0, where $\tilde{\phi}$ is the lift of ϕ over B_0 and $\tilde{u}_\epsilon$ is the lift of $u - \epsilon$ over B_0. The strict inequality follows from the weak part of Postulate 4.2 because the strict inequality holds at all points on ∂B_0. By using Postulate 4.2 again, we see that $\tilde{u}_\epsilon \geq \tilde{u} - \epsilon$ in $\overline{B_0}$. By the definition of u, $u \geq \tilde{\phi}$. Therefore, for all $x \in \overline{B_0}$, $u(x) \geq \tilde{\phi}(x) \geq \tilde{u}(x) - \epsilon$ but since $\epsilon > 0$ and x are arbitrary, we get that $u \geq \tilde{u}$ in $\overline{B}$.

5 Example: k-Hessian equations

We now apply the ideas of the preceding sections to a discrete family of operators that includes both the Laplacian and the Monge-Ampère operator. We begin by defining these operators and discussing their ellipticity and other basic properties. In order to be consistent with the literature concerning these operators, in this section we reverse the inequality in the definition of degenerate ellipticity found in Section 2.1. The k-th elementary symmetric polynomial in N variables is

$$P_k(x_1, \dots, x_N) = \sum_{i_1 < \cdots < i_k \leq N} x_{i_1} \cdots x_{i_k}.$$

The k-Hessian operators S_k, acting on $C^2(\Omega)$, are defined as follows: For $1 \leq k \leq N$, let

$$S_k(D^2u)(x) = P_k(\lambda_1(x), \lambda_2(x), \dots, \lambda_N(x)),$$

where $\lambda_1(x), \dots, \lambda_N(x)$ are the eigenvalues of $D^2u(x)$. Equivalently, $S_k(D^2u)$ is the sum of the $k \times k$ principal minors of the Hessian matrix. When $k = 1$, $S_k(D^2u) = \operatorname{trace}(D^2u) = \Delta u$, and, at the other extreme, $S_N(D^2u) = \det(D^2u)$ is the Monge-Ampère operator. When $k \geq 2$, these operators are fully nonlinear. k-Hessians have been studied extensively, e.g. in [8], [21], [39], [38], [40], [41], [25], [44], [12], [43], [42], [16]. In general, these operators are not elliptic. However, they are elliptic when restricted to a certain subset of $C^2(\Omega)$, which we now introduce. A function $u \in C^2(\Omega)$ is called *(uniformly) k-convex* if $S_j(D^2u) \geq (>)\, 0$ for

$j = 1, \ldots, k$. Note that 1-convex functions are subharmonic, and N-convex functions are convex in the usual sense. Equivalently, a C^2 function is k-convex if all of the eigenvalues of its Hessian matrix lie in the convex cone Γ_k^N in $\mathbb{R}^N$ defined by

$$\Gamma_k^N = \{\lambda \in \mathbb{R}^N : P_j(\lambda) \geq 0,\ j = 1, \ldots, k\}.$$

The set of continuous k-convex functions on Ω will be denoted $\Phi^k(\Omega)$, and $\Phi_2^k(\Omega)$ will stand for $\Phi^k(\Omega) \cap C^2(\Omega)$. As an immediate consequence of the definition, we see that $\Phi_2^k(\Omega) \subset \Phi_2^l(\Omega)$ for $l \leq k$. The natural domains for boundary value problems concerning these operators are the $k-$convex domains. The domain $\Omega \subset \mathbb{R}^N$ is *(uniformly) $k-$convex* if the principal curvatures at all points on $\partial\Omega$ are in $\bar{\Gamma}_k^{N-1}$ (Γ_k^{N-1}). In [8] it is shown that if $u \in \Phi_2^k(\Omega)$, then S_k is degenerate elliptic with respect to u. Thus, in order to work in the elliptic realm, we will restrict our attention to $\Phi^k(\Omega)$. If $u \in \Phi_2^k(\Omega)$, then $S_k(D^2u) \geq 0$, so we will consider boundary value problems for the differential equation:

$$S_k(D^2u) = h(x, u, Du), \tag{5.1}$$

where $h = h(x, r, p)$ is a non-negative function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

5.1 Weak solutions

We now introduce two notions of weak solutions for k-Hessian equations.

5.1.1 Viscosity solutions

Since these operators are not elliptic on all of $C^2(\Omega)$, we need to restrict the class of test functions used in defining viscosity solutions. A function

$$u : \Omega \to \mathbb{R}$$

is called a *viscosity subsolution* of (5.1) if $u^* : \Omega \to \mathbb{R}$, and if $x \in \Omega$ and $\phi \in \Phi_2^k(\Omega)$ are such that

$$u^*(x) - \phi(x) = \max_{\Omega}(u^* - \phi),$$

then

$$S_k(D^2\phi(x)) \geq h(x, u^*(x), D\phi(x)).$$

Viscosity supersolutions (and viscosity solutions) are then defined for these operators in the obvious way. If this modification in the definition is not made, there are classical solutions which are not viscosity solutions. See [12] for an example.

5.1.2 Weak solutions defined by approximation

Trudinger, considering (5.1) where $h \in L^p(\Omega)$ is independent of u and Du, defined another concept of weak solution for k-Hessian equations in [39] in terms of continuous k-convex functions. A similar definition was also employed in [40] for equation (5.1) when the right-hand side is replaced by a finite Borel measure. For this further generalization of (5.1), the notion of k-convexity was extended to upper semicontinuous functions. A function $u \in C(\Omega)$ is *k-convex* ($u \in \Phi^k(\Omega)$) if there exists a sequence $\{u_n\} \subset C^2(\Omega)$, such that on any subdomain Ω' with $\overline{\Omega'} \subset \Omega$, u_n converges to u uniformly, and $u_n \in \Phi_2^k(\Omega')$ for all n sufficiently large. Equivalently, $u \in \Phi^k(\Omega)$ if u is a continuous viscosity solution of $S_k(D^2u) \geq 0$. A function $u \in \Phi^k(\Omega)$ is a *weak solution* to (5.1), where $h = h(x) \in L^p(\Omega)$ if there exists an approximating sequence $\{u_n\}$, as described above, such that

$$S_k(D^2u_n) \to h \text{ in } L^1_{loc}(\Omega).$$

When $h \in C(\Omega)$, this notion of weak solution coincides with that of viscosity solution ([41]). We will show, in Section 5.3, that the axiomatic approach described in Section 4 applies to these weak solutions when $h \in L^\infty(\Omega)$.

5.2 Subfunctions and viscosity subsolutions

In this section, we observe that continuous viscosity subsolutions are continuous subfunctions for a certain class of k-Hessian equations. In order for Proposition 3.2 to apply to equation (5.1), we need to know that Assumptions 2.1 and 3.1 hold. The following result of Urbas supplies the necessary comparison principle for continuous viscosity solutions.

Theorem 5.1 (Proposition 2.3, [43]) *Let Ω be bounded, and suppose h is positive, uniformly continuous on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, Lipschitz continuous in r and p, nondecreasing in r, and $|h_p|$ is bounded above. If $u, v \in C(\Omega)$ are respectively a viscosity subsolution and a viscosity supersolution of equation (5.1) and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.*

We turn now to the local solvability of (5.1). To establish existence results in small balls for these problems, we will need to impose more conditions on the function h and require more regularity of the boundary data. This necessity is made clear by the following result of Urbas [43].

Theorem 5.2 *For any integers k and N satisfying $3 \leq k \leq N$ and any positive function $h \in C^\infty(\overline{B_1} \times \mathbb{R} \times \mathbb{R}^N)$, there exists $\epsilon \in (0,1)$ and $u \in C^{0,1}(\overline{B}_\epsilon)$ which is a viscosity solution of $(S_k(D^2u))^{\frac{1}{k}} = h(x, u, Du)$ in B_ϵ such that $u \notin C^{1,\alpha}(B_\epsilon)$ for any α.*

In the situation where viscosity solutions of the Dirichlet problem are unique (e.g. when the comparison principle holds), Theorem 5.2 implies that there may be no classical solutions. In [8] it is proved that the problem

$$\begin{cases} S_k(D^2u) &= h(x) \text{ in } B \\ u &= g \text{ on } \partial B, \end{cases}$$

where $h \in C^\infty(\overline{B})$, $h > 0$, and $g \in C^\infty(\partial B)$, has a unique k-convex solution $u \in C^\infty(\overline{B})$. The strong smoothness requirements for the boundary data do not pose a problem for us here, as we can restrict the class of test functions used for viscosity solutions to the class of k-convex quadratic polynomials (see Remark 3.3). Also, the assumption that $h \in C^\infty(\overline{B})$ can be relaxed to $h \in C^{1,1}(\overline{B})$, in which case the solution will lie in $C^{3,\alpha}(\overline{B})$, see [38] for a statement of this result (and also the corresponding result for ratios of k-Hessian operators). Therefore, when $h = h(x) \in C^{1,1}(\bar{\Omega})$ is positive, continuous subfunctions coincide with continuous viscosity subsolutions of (5.1).

5.3 Application of axiomatic approach

By citing the work of Trudinger and Wang, we show that the modified axiomatic approach of Beckenbach and Jackson, as in Theorem 4.8, applies to weak solutions (as described in Section 5.1.2) of the equation

$$S_k(D^2u) = h(x) \geq 0, \text{ where } h \in L^\infty(\Omega). \tag{5.2}$$

In this way, we can obtain a generalized solution to a Dirichlet problem with bounded boundary data in any bounded domain.

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $g : \partial\Omega \to \mathbb{R}$ be a bounded function. There exists a generalized solution to the problem*

$$\begin{cases} S_k(D^2u) &= h \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{cases} \tag{5.3}$$

Proof. Let $\mathcal{F}$ be the family of weak (local) solutions (as in Section 5.1.2) of (5.2). In other words, $u \in \mathcal{F}$ if u is a weak solution of (5.2) in some domain $D \subset \Omega$. We prove the theorem by showing that $\mathcal{F}$ satisfies the hypotheses of Theorem 4.8.

Postulate 4.1: To demonstrate the local existence of solutions, we apply the following theorem of Trudinger when D is a ball B.

Theorem 5.4 (Theorem 1.1, [39]) *Let $k \geq 2$, D be a uniformly $(k-1)-$convex domain, $h(x) \in L^p(D)$, where $p > N/2k$, and $g \in C(\bar{D})$. Then*

$$\begin{cases} S_k(D^2u) &= h \text{ in } D \\ u &= g \text{ on } \partial D \end{cases}$$

has a unique weak solution $u \in \Phi^k(D) \cap C(\bar{D}) \cap C^\alpha(D)$, where $\alpha < 1$ satisfies $\alpha \leq 2 - N/kp$.

Postulates 4.6 and the weak part of 4.2: We establish these weak comparison results by using the following result of Trudinger and Wang. Note that because equation (5.2) is simpler than (5.1), the hypotheses are simpler than those in Theorem 5.1. Theorem 5.5 in fact applies to the case where h is replaced by a Borel measure in (5.2).

Theorem 5.5 (Theorem 3.1, [40]) *Let u, $v \in C(\overline{D}) \cap \Phi^k(D)$ satisfy*

$$\begin{cases} S_k(D^2u) & \geq & S_k(D^2v) & \text{in } D \\ u & \leq & v & \text{on } \partial D. \end{cases}$$

Then $u \leq v$ in D.

Now let B be a ball with $\bar{B} \subset \Omega$, and let g_1, $g_2 \in C(\partial B)$ satisfy $g_1 - g_2 \leq M$. Let u_i, $i = 1$, 2, be the unique solutions (which exist by Theorem 5.4) of

$$\begin{cases} S_k(D^2u) & = & h & \text{in } B \\ u & = & g_i & \text{on } \partial B. \end{cases}$$

By adding M to u_2 and g_2, we may assume $M = 0$, and by Theorem 5.5, it follows that $u_1 \leq u_2$.

Postulate 4.3: Given a ball B and a uniformly bounded sequence $\{g_n\} \subset C(\partial B)$, we need to demonstrate that the solutions u_n of

$$\begin{cases} S_k(D^2u) & = & h & \text{in } B \\ u & = & g_n & \text{on } \partial B \end{cases}$$

form an equicontinuous family. We use an oscillation estimate and standard arguments. Let $\omega(u, B)$ denote the oscillation of u in B:

$$\omega(u, B) = \sup_B u - \inf_B u.$$

We first observe that a uniform bound on $|g_n|$, say $|g_n| \leq M$, $n = 1, 2, \cdots$, implies an upper bound on $\omega(u_n, B)$. Since $g_n - g_m \leq 2M$ for any n and m, by the weak part of Postulate 4.2 we have $u_n - u_m \leq 2M$ for all n, m. In particular, $u_n(x) - u_1(x) \leq 2M$ for all $x \in B$ and all n, so $\max_B u_n \leq (\max_B u_1) + 2M$ for all n. Similarly, $\min_B u_n \geq (\min_B u_1) - 2M$. Therefore, for all n,

$$\omega(u_n, B) \leq 4M + \omega(u_1, B). \tag{5.4}$$

We now use the following estimate:

Theorem 5.6 (Theorem 4.1, [39]) *Let $u \in \Phi^k(D)$ be a weak solution of (5.1) with $h = h(x) \in L^p_{loc}(D)$ for some $p > N/2k$. Then for any $\alpha < 1$ such that $\alpha \leq 2 - N/kp$, any $B = B_R(y) \subset D$ and any $\sigma \in [0, 1)$,*

$$\omega(u, B_{\sigma R}(y)) \leq C\sigma^\alpha \left\{\omega(u, B) + R^{2-N/kp}||h||_p^{1/k}\right\}, \tag{5.5}$$

where C is a constant depending on k, N, p and α.

Thus, for each n, we may estimate $\omega(u_n, B_{\sigma R}(y))$ by using (5.5). By (5.4), $\omega(u_n, B)$ can be uniformly dominated, and since each u_n is a solution of the same equation (5.2) (and $L^\infty(\Omega) \subset L^p_{loc}(\Omega)$), we get a uniform bound on $\omega(u_n, B_{\sigma R}(y))$. This provides a uniform bound on the α-Hölder coefficient of u_n at y, which then implies the equicontinuity of the family $\{u_n\}$ at y.

Postulate 4.4: It is easy to find subsolutions and supersolutions because h in (5.2) is bounded. Let $w(x) = \text{const} > \sup_{\partial\Omega} g$. Then $S_k(D^2w) = 0$, so if u is any solution of (5.2) in a ball B with $u \leq w$ on ∂B, then $u \leq w$ in B by Theorem 5.5, consequently w is a superfunction and hence a supersolution. Let $v(x) = A|x - x_0|^2 + C$, for constants A and C to be determined. $S_k(D^2v)$ is a constant depending on k, N and A. Choose A so that $S_k(D^2v) \geq ||h||_{L^\infty}$. Then if u is any solution of (5.2) in a ball B with $u \geq v$ on ∂B, then $u \geq v$ in B by Theorem 5.5, so w is a subfunction. Now choose C so that $v \leq g$ on $\partial\Omega$. These choices make v a subsolution. We remark that the boundedness of h was not needed to produce a supersolution. Therefore, all hypotheses of Theorem 4.8 are satisfied and problem (5.3) has a generalized solution. By Theorem 5.3, we get a generalized solution to the problem (5.3) in any bounded domain Ω, where $h \in L^\infty(\Omega)$ is nonnegative. When $g \in C(\partial\Omega)$, a natural question is whether the boundary data are continuously assumed by a generalized solution. The answer is yes if we can find a subsolution and a supersolution which both equal g on $\partial\Omega$. For domains that are not k-convex, this is not trivial, although if the domain is regular for the Laplacian, a solution of Laplace's equation assuming the given boundary data is a supersolution. For problems with dependence on u and its derivatives, the situation is more complicated. The solvability of the Dirichlet problem in the Monge-Ampère case is reduced to the existence of a convex subsolution in [7]. When such a subsolution can be found, the problem has a smooth solution. Similar results hold for the other k-Hessian operators with some conditions on h; see [16]. We conclude this section by mentioning some existence results that do not require finding a subsolution. The problem with zero boundary data and some technical assumptions on h in the p variables was considered by Ivochkina in [21]. Urbas, in [43], proves an existence result for small balls that does not require the structural assumptions on h found in [21] (and needs only positivity and mild smoothness), but requires smoothness and smallness of the boundary data.

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