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VISCOSITY SOLUTIONS OF NONLINEAR SECOND ORDER ELLIPTIC PDES INVOLVING NONLOCAL OPERATORS

Dedicated to Professor Hiroki Tanabe in commemoration of his sixtieth birthday

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1. Introduction

This paper deals with viscosity solutions of nonlinear degenerate elliptic partial differential equations (PDEs) involving nonlocal operators.

To begin with, we show model problems. Let $\Omega\subset\boldsymbol{R}^N$ be a bounded domain.

Model I. (*Integro-differential equation with obstacle*)

$$(1.1)\qquad \begin{cases} \max\{Lu-f, u-\varphi\}=0 & \text{in } \Omega\,, \\ u(x)=\int_\Omega u(y)\,Q(dy,x) & \text{for } x\in\partial\Omega\,, \end{cases}$$

where L is an integro-differential operator of the form:

$$Lu(x)=-\sum_{i=1}^N g_i(x)\,u_{x_i}(x)+\alpha(x)\,u(x)+\lambda(x)\int_\Omega (u(x)-u(y))\,Q(dy,x)\,,$$

and $Q(\cdot,x)$ is a probability measure in Ω for $x\in\bar{\Omega}$.

Model II. (*Second order elliptic PDE with implicit obstacle*)

$$(1.2)\qquad \begin{cases} \max\{Lu-f, u-Mu\}=0 & \text{in } \Omega\,, \\ \max\{u-g, u-Mu\}=0 & \text{on } \partial\Omega\,, \end{cases}$$

where L denotes the following linear (possibly degenerate) second order elliptic operator:

$$Lu(x)=-\sum_{i,j=1}^N a_{ij}(x)\,u_{x_ix_j}(x)+\sum_{i=1}^N b_i(x)\,u_{x_i}(x)+c(x)\,u(x)$$

and Mu is a nonlocal term defined by

$$Mu(x)=\inf\{k(\xi)+u(x+\xi)\,|\,\xi\in(\boldsymbol{R}^+)^N,\ x+\xi\in\bar{\Omega}\}\,.$$

Model I is derived from the optimal stopping problem for piecewise-deterministic (PD) processes. S.M. Lenhart-Y.C. Liao [8] discussed the optimal stopping for PD processes and characterized a $W^{1,\infty}$ solution of (1.1) as the minimal cost function of it. S.M. Lenhart [6] proved the uniqueness and existence of viscosity solutions of (1.1). Her existence result was obtained by an iterative approximation scheme. S.M. Lenhart-the second author [9] showed that Perron's method can be used for general integro-differential equations including (1.1). In the case $\Omega=\boldsymbol{R}^N$, the similar results were obtained in [8] and [6].

Model II is the dynamic programming equation arising in the impulse control problem for diffusion processes. See A. Bensoussan-J.L. Lions [3] for more backgrounds. In Perthame [11] the existence and uniqueness of viscosity solutions of (1.2) were obtained and the solution was represented as the optimal cost function for the associated impulse control problem. His existence result was also based upon an iterative approximation scheme. The first author [5] extended the result of [11] to the case where the principal part of (1.2) is a degenerate elliptic operator. When $\Omega=\boldsymbol{R}^N$, G. Barles [1] and B. Perthame [10] treated the impulse control problem and G. Barles [2] discussed the existence and uniqueness of viscosity solutions of (1.2) in a general first order operator case.

As explained above, Models I and II have been considered as separate problems. In this paper we shall unify these two models from the view point of viscosity solution. More precisely, we shall get the comparison principle and existence of viscosity solutions for the boundary value problems of the general form:

$$
\begin{cases} F(x, u, Du, D^2 u, u-Mu) = 0 & \text{in } \Omega\,, \\ B(x, u, u-Mu) = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.3}
$$

where Du, D^2u are, respectively, the gradient and Hessian matrix of u and M is a nonlocal operator line in Models I and II.

This paper is organized as follows. In Section 2 we state our assumptions and recall the definitions of viscosity solutions. In Section 3 we establish the comparison principle and existence of viscosity solutions for the problem (1.3). As to the fundamental arguments, see M.G. Crandall-H. Ishii-P.L. Lions [4]. Our methods are essentially based upon them. Section 4 is devoted to treating the case $\Omega=\boldsymbol{R}^N$. In Section 5 we mention Models I and II precisely.

Finally we refer the reader to S.M. Lenhart [7] which discussed the uniqueness and existence of viscosity solutions of nonlinear PDEs involving the operators in Models I and II.

2. Assumptions and Definitions

In this section we state our assumptions and recall the notion of viscosity solutions. We set $\Gamma=\Omega\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N\times\boldsymbol{R}$ and $\Sigma=\partial\Omega\times\boldsymbol{R}\times\boldsymbol{R}$, where $\boldsymbol{S}^N$ denotes the set of all $N\times N$ real symmetric matrices. For a topological space T, we denote by $USC(T)$, $LSC(T)$ and $C(T)$, respectively, the set of all real valued upper semi-, lower semi- and continuous functions defined on T. We make the following assumptions on F.

(D) $\Omega\subset\boldsymbol{R}^N$ is a bounded domain.

(F.1) $F\in C(\Gamma)$ and satisfies the degenerate ellipticity condition, that is,

$$F(x,r,p,X+Y,m)\leqq F(x,r,p,X,m)$$

for all $(x,r,p,X,m)\in\Gamma$ and $Y\in\boldsymbol{S}^N$ such that $Y\geqq O$.

(F.2) There exists $\omega_1\in C(\boldsymbol{R}^+)$ with $\omega_1(0)=0$, such that if X, $Y\in\boldsymbol{S}^N$, $\alpha>1$ and

$$-3\alpha\begin{pmatrix} I & O\\ O & I\end{pmatrix}\leqq\begin{pmatrix} X & O\\ O & -Y\end{pmatrix}\leqq 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix},$$

then

$$\begin{aligned}&F(y,r,\alpha(x-y),Y,m)-F(x,r,\alpha(x-y),X,m)\\ &\leqq\omega_1(\alpha|x-y|^2+|x-y|)\end{aligned}$$

for all $x,y\in\Omega$, $r,m\in\boldsymbol{R}$.

(F.3) There exists $\omega_2\in C(\boldsymbol{R}^+)$ with $\omega_2(0)=0$ such that

$$|F(x,r,p,X,m)-F(x,r,q,Y,m)|\leqq\omega_2(|p-q|+||X-Y||)$$

for all $(x,r,p,X,m)\in\Gamma$, $q\in\boldsymbol{R}^N$ and $Y\in\boldsymbol{S}^N$, where $||X||$ is the operator norm of $X\in\boldsymbol{S}^N$ as a self-adjoint operator.

(F.4) For each $0<\mu\leqq 1$, there exist functions $\sigma_1(x,\mu)$, $\sigma_2(x,\mu)\in C(\overline{\Omega}\times(0,1])$, $\beta\in C(\Omega)$ and constants $\alpha_1>0$, $\alpha_2\geqq 0$ satisfying

$$\begin{aligned}&F(x,r,p,X,m)-F_\mu(x,s,p,X,n)\\ &\quad\leqq\max\{\alpha_1(r-s)+\sigma_1(x,1-\mu),\\ &\qquad\alpha_2(r-s+\sigma_2(x,1-\mu))+\beta(x)(m-n)\},\\ &||\sigma_1(\cdot,\mu)||_{C(\overline{\Omega})},\ ||\sigma_2(\cdot,\mu)||_{C(\overline{\Omega})}\to 0\quad(\mu\to 0),\\ &\beta(x)>0\quad\text{for}\quad x\in\Omega\end{aligned}$$

for all $(x,r,p,X,m)\in\Gamma$ and $s,n\in\boldsymbol{R}$ such that $r\leqq s$, where

$$F_\mu(x,r,p,X,m)=\mu F\left(x,\frac{r}{\mu},\frac{p}{\mu},\frac{X}{\mu},\frac{m}{\mu}\right).$$

REMARK 2.1. The assumption (F.4) includes the monotonicity of F. Taking $\mu=1$, we have

$$F(x,r,p,X,m)-F(x,s,p,X,n)\leqq\max\{\alpha_1(r-s),\ \alpha_2(r-s)+\beta(x)(m-n)\}.$$

Then we see that F is nondecreasing in m and that

F is strictly increasing in r if $\alpha_2>0$ (e.g., Medel I (1.1)),
F is nondecreasing in r if $\alpha_2=0$ (e.g., Model II (1.2)).

Next we mention the assumptions on the nonlocal operator M. Let u, v be bounded functions on $\bar{\Omega}$.

(M.1) $M: USC(\bar{\Omega}) \to USC(\bar{\Omega})$ and $M: LSC(\bar{\Omega}) \to LSC(\bar{\Omega})$.
(M.2) $u(z)-v(z) \geqq Mu(z)-Mv(z)$ for all $z\in\bar{\Omega}$ such that $u(z)-v(z)= \sup_{\bar{\Omega}}(u-v)$.
(M.3) For each $0<\mu\leqq 1$, there exists $k_\mu\geqq 0$ satisfying

$$k_\mu \to 0 \quad (\mu \to 0),$$
$$M((1-\mu)u)(x) \geqq (1-\mu)Mu(x)+k_\mu .$$

We make the following assumptions on B.

(B.1) $B\in C(\Sigma)$.
(B.2) For each $0<\mu\leqq 1$, there exist function $\sigma_3(x, \mu)\in C(\partial\Omega\times(0, 1])$ and constants $\gamma_1\geqq 0$, $\gamma_2\geqq 0$ such that $\gamma_1+\gamma_2>0$, satisfying

$$B(x, r, m)-B_\mu(x, s, n)$$
$$\leqq \gamma_1 \max\{(r-s)+\sigma_3(x, 1-\mu), m-n\}+\gamma_2(m-n),$$
$$\|\sigma_3(\cdot, \mu)\|_{C(\partial\Omega)} \to 0 \quad (\mu \to 0),$$

for all $(x, r, m)\in\Sigma$ and $s, n\in\boldsymbol{R}$ such that $r\leqq s$ and $m\leqq n$, where

$$B_\mu(x, r, m) = \mu B\left(x, \frac{r}{\mu}, \frac{m}{\mu}\right).$$

REMARK 2.2. As in Remark 2.1, B is monotone with respect to r, m, respectively.

We conclude this section by recalling the notion of viscosity solutions. We prepare some notations. For the function $u: \bar{\Omega}\to\boldsymbol{R}$, we define the upper semicontinuous (u.s.c) envelope u^* and lower semicontinuous (l.s.c.) envelope u_* of u by

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid y\in\bar{\Omega},\ |y-x|<r\},\ u_*(x) = -(-u)^*(x).$$

We observe easily that $u_*\leqq u\leqq u^*$ on $\bar{\Omega}$ and that u^* and u_* are, respectively, u.s.c., l.s.c. on $\bar{\Omega}$ with values in $\boldsymbol{R}\cup\{\pm\infty\}$.

DEFINITION 2.1. *Let u be a bounded function defined on $\bar{\Omega}$.*
(1) *We call u a viscosity subsolution of* (1.3) *provided that for each $\varphi\in C^2(\Omega)$, if $u^*-\varphi$ has a local maximum at $x_0\in\Omega$, then*

$$F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0), u^*(x_0)-Mu^*(x_0)) \leqq 0 .$$

(2) *We call u a viscosity supersolution of* (1.3) *provided that for each* $\varphi \in C^2(\Omega)$, *if* $u_* - \varphi$ *has a local minimum at* $x_0 \in \Omega$, *then*

$$F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0), u_*(x_0) - Mu_*(x_0)) \geqq 0 .$$

(3) *We call u a viscosity solution of* (1.3) *if u is a viscosity sub- and supersolution of* (1.3).

In the following we surpress the term "viscosity" since we are concerned mainly with viscosity sub-, super- and solutions.

To prove the comparison principle, it will be convenient for us to have at hand certain alternative definitions. For the function $u: \Omega \rightarrow \boldsymbol{R}$, $J^{2,+}u(x)$ denotes the super 2-jet at $x \in \Omega$:

$$J^{2,+}u(x) = \{(p, X) \in \boldsymbol{R}^N \times \boldsymbol{S}^N \mid u(y) \leqq u(x) + \langle p, y-x \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle + o(|y-x|^2) \quad \text{as } \Omega \ni y \rightarrow x\} ,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidian inner product in $\boldsymbol{R}^N$. We denote by $\bar{J}^{2,+}u(x)$ the "graph closure" of $J^{2,+}u(x)$:

$$\bar{J}^{2,+}u(x) = \{(p, X) \in \boldsymbol{R}^N \times \boldsymbol{S}^N \mid \exists (x_n, p_n, X_n) \in \Omega \times \boldsymbol{R}^N \times \boldsymbol{S}^N \text{ such that } (p_n, X_n) \in J^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X) \quad \text{as} \quad n \rightarrow +\infty\} ,$$

We define the sub 2-jet $J^{2,-}u(x)$ of u at x and its closure $\bar{J}^{2,-}u(x)$ by similar way. It is easily seen that the following propositions hold.

Proposition 2.2. *Let u be a bounded function on* $\bar{\Omega}$. *Then u is a subsolution of* (1.3) *if and only if*

$$F(x, u^*(x), p, X, u^*(x) - Mu^*(x)) \leqq 0$$

for all $x \in \Omega$, *and* $(p, X) \in J^{2,+}u^*(x)$.

Proposition 2.3. *Let u be a bounded function on* $\bar{\Omega}$. *Suppose that F is continuous on* Γ *and nondecreasing in the variable m and that M satisfies* (M.1). *Then u is a subsolution of* (1.3) *if and only if*

$$F(x, u^*(x), p, X, u^*(x) - Mu^*(x)) \leqq 0$$

for all $x \in \Omega$, $(p, X) \in \bar{J}^{2,+}u^*(x)$.

As to the supersolutions of (1.3), we can get the equivalent propositions similar to the above ones. (See [4, Section 2].)

3. Uniqueness and existence of solutions

In this section we establish the comparison principle and existence of solutions of (1.3).

Theorem 3.1. *Assume* (D), (F.1)-(F.4), (M.1)-(M.3) *and* (B.1)-(B.2) *hold. Moreover assume either* (3.1) *or* (3.2) *holds*:

(3.1) *When* $\alpha_2=0$ *in* (F.4), $k_\mu>0$ $(0<\mu<1)$ *holds in* (M.3).

(3.2) *When* $\alpha_2>0$ *in* (F.4), *if the maximum of the function* $u-v$ *is attained only on* $\partial\Omega$, *then we have in* (M.2)

$$u(z)-v(z)>Mu(z)-Mv(z)$$

for all $z\in\partial\Omega$ *such that* $u(z)-v(z)=\sup_{\bar{\Omega}}(u-v)$.

Let u *and* v *be, respectively, a subsolution and a supersolution of* (1.3). *If* u, v *satisfy*

$$B(x, u^*, u^*-Mu^*)\leqq 0 \quad \text{and} \quad B(x, v_*, v_*-Mv_*)\geqq 0 \quad \text{on } \partial\Omega\,, \tag{3.3}$$

then $u^*\leqq v_*$ *on* $\bar{\Omega}$.

Proof. We may assume $u\in USC(\bar{\Omega})$ and $v\in LSC(\bar{\Omega})$ because, if otherwise, we replace u and v with u^* and v_*, respectively. We suppose $\sup_{\bar{\Omega}}(u-v)=2\theta>0$ and shall get a contradiction.

For each $m\in\boldsymbol{N}$, we set $u_m=(1-1/m)\,u$. Since $(p, X)\in \bar{J}^{2,+}u_m(x)$ implies $(m'p, m'X)\in \bar{J}^{2,+}u(x)$, where $m'=m/(m-1)$, and u is a subsolution of (1.3), we have

$$F(x, u(x), m'p, m'X, u(x)-Mu(x))\leqq 0\,.$$

By (M.3) we obtain

$$\begin{aligned} u(x)-Mu(x) &= m'(u_m(x)-(1-m^{-1})\,Mu(x)) \\ &\geqq m'(u_m(x)-Mu_m(x)+k_{1/m})\,. \end{aligned}$$

Thus from the monotonicity of F and the definition of $F_{m'}$ we get

$$F_{1/m'}(x, u_m(x), p, X, u_m(x)-Mu_m(x)+k_{1/m})\leqq 0\,. \tag{3.4}$$

Similarly, using (M.3) and (B.2) we obtain

$$B_{1/m'}(x, u_m, u_m-Mu_m+k_{1/m})\leqq 0 \quad \text{on } \partial\Omega\,. \tag{3.5}$$

Now, let $z\in\bar{\Omega}$ be a maximum point of the function u_m-v on $\bar{\Omega}$. Since $u_m(z)-v(z)\to 2\theta$ as $m\to+\infty$, we have $u_m(z)-v(z)\geqq\theta$ for sufficiently large $m\in\boldsymbol{N}$. Consider the case $z\in\partial\Omega$ and $(u_m-v)(x)<(u_m-v)(z)$ for $x\in\Omega$. In this case we obtain by (3.3), (3.5) and (B.2),

$$\begin{aligned} 0\leqq\, & B(z, v(z), v(z)-Mv(z)) \\ & -B_{1/m'}(z, u_m(z), u_m(z)-Mu_m(z)+k_{1/m}) \end{aligned} \tag{3.6}$$

$$\leqq \gamma_1 \max\{v(z)-u_m(z)+\sigma_3(z, m^{-1}),\ v(z)-Mv(z)-u_m(z)+Mu_m(z)-k_{1/m}\} +\gamma_2(v(z)-Mv(z)-u_m(z)+Mu_m(z)-k_{1/m})\,.$$

If (3.1) holds, then by using (M.2) we get

$$0\leqq \gamma_1 \max\{-\theta+\sigma_3(z, m^{-1}), -k_{1/m}\} -\gamma_2 k_{1/m}<0$$

for sufficiently large $m\in \boldsymbol{N}$. Hence we have a contradiction. If (3.2) holds, then we get

$$\gamma_1 \max\{-\theta+\sigma_3(z, m^{-1}), v(z)-Mv(z)-u_m(z)+Mu_m(z)\} +\gamma_2(v(z)-Mv(z)-u_m(z)+Mu_m(z))<0\,.$$

Combining this with (3.6), we obtain a contradiction. Therefore we can see that there exists a maximum point $z\in\Omega$ of u_m-v. Then, for any $\alpha>1$, we define the function $\Phi(x, y)$ on $\bar{\Omega}\times\bar{\Omega}$ by

$$\Phi(x, y) = u_m(x)-v(y)-\frac{\alpha}{2}\,|x-y|^2-|y-z|^4$$

and let $(\bar{x}, \bar{y})\in\bar{\Omega}\times\bar{\Omega}$ be a maximum point of $\Phi(x, y)$. By the usual calculation we obtain the behaviors of $\bar{x}$, $\bar{y}$, $u_m(\bar{x})$, $v(\bar{y})$ as $\alpha\to+\infty$:

$$\bar{x}, \bar{y} \to z,\ u_m(\bar{x}) \to u_m(z),\ v(\bar{y}) \to v(z),\ \alpha|\bar{x}-\bar{y}|^2 \to 0\,.$$

Then we apply the maximum principle for semicontinuous functions to obtain X, $Y\in \boldsymbol{S}^N$ satisfying

$$(\alpha(\bar{x}-\bar{y}), X)\in \bar{J}^{2,+} u_m(\bar{x}),\ (\alpha(\bar{x}-\bar{y}), Y)\in \bar{J}^{2,-}(v(\bar{y})+|\bar{y}-z|^4)\,,$$
$$-3\alpha\begin{pmatrix} I & O \\ O & I\end{pmatrix}\leqq\begin{pmatrix} X & O \\ O & -Y\end{pmatrix}\leqq 3\alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}.$$

(As to the above arguments, see M.G. Crandall-H. Ishii-P.L. Lions [4, Section 3].) Furthermore, we get

$$(\alpha(\bar{x}-\bar{y})-p,\ Y-Z)\in \bar{J}^{2,-} v(\bar{y})\,,$$

where $p=4|\bar{y}-z|^2(\bar{y}-z)$ and $Z=4|\bar{y}-z|^2 I+8(\bar{y}-z)\otimes(\bar{y}-z)$. Thus using (3.4) and the fact that v is a supersolution of (1.3), we have

$$F_{1/m'}(\bar{x}, u_m(\bar{x}), \alpha(\bar{x}-\bar{y}), X, u_m(\bar{x})-Mu_m(\bar{x})+k_{1/m})\leqq 0\,,$$
$$F(\bar{y}, v(\bar{y}), \alpha(\bar{x}-\bar{y})-p, Y-Z, v(\bar{y})-Mv(\bar{y}))\geqq 0\,.$$

Remarking that $u_m(\bar{x})>v(\bar{y})$, we observe from the above inequalities, (F.2) and (F.3) that

$$
\begin{aligned}
0 \leqq & F(\bar{y}, v(\bar{y}), \alpha(\bar{x}-\bar{y})-p, Y-Z, v(\bar{y})-Mv(\bar{y})) \\
& -F_{1/m'}(\bar{x}, u_m(\bar{x}), \alpha(\bar{x}-\bar{y}), X, u_m(\bar{x})-Mu_m(\bar{x})+k_{1/m}) \\
& \leqq \omega_1(\alpha|\bar{x}-\bar{y}|^2+|\bar{x}-\bar{y}|)+\omega_2(|p|+\|Z\|) \\
& +\max\{\alpha_1(v(\bar{y})-u_m(\bar{x}))+\sigma_1(\bar{x}, m^{-1}), \alpha_2(v(\bar{y})+u_m(\bar{x})+\sigma_2(\bar{x}, m^{-1})) \\
& +\beta(\bar{x})(v(\bar{y})-Mv(\bar{y})-u_m(\bar{x})+Mu_m(\bar{x})-k_{1/m})\} .
\end{aligned}
$$

Noting that

$$
\begin{aligned}
& \lim_{\alpha\to+\infty} \omega_1(\alpha|\bar{x}-\bar{y}|^2+|\bar{x}-\bar{y}|) = \lim_{\alpha\to+\infty} \omega_2(|p|+\|Z\|) = 0 , \\
& \limsup_{\alpha\to+\infty} (v(\bar{y})-Mv(\bar{y})-u_m(\bar{x})+Mu_m(\bar{x})) \\
& \quad \leqq v(z)-Mv(z)-u_m(z)+Mu_m(z) ,
\end{aligned}
$$

and letting $\alpha\to+\infty$, we have

$$
\begin{aligned}
(3.7) \qquad 0 \leqq \max\{ & -\alpha_1 \theta+\sigma_1(z, m^{-1}), \alpha_2(-\theta+\sigma_2(z, m^{-1})) \\
& +\beta(z)(v(z)-Mv(z)-u_m(z)+Mu_m(z)-k_{1/m})\} .
\end{aligned}
$$

If $\alpha_2=0$, by (3.1) and $\beta(z)>0$, we get

$$
\beta(z)(v(z)-Mv(z)-u_m(z)+Mu_m(z)-k_{1/m})<0 \quad \text{for large} \quad m\in \boldsymbol{N} .
$$

Thus taking $m\in \boldsymbol{N}$ sufficiently large in (3.7), we obtain a contradiction. In the case $\alpha_2>0$, we also have a contradiction. Therefore the proof is complete. ■

The existence result is stated as follows. It is proved by Perron's method.

Theorem 3.2. *Let Ω, F, M, and B as in Theorem* 3.1. *In addition, assume* (M.4) *M is monotone, that is, if $u\leqq v$ on $\bar{\Omega}$, then $Mu\leqq Mv$ on $\bar{\Omega}$.*

Assume there exist a u.s.c. subsolution $\underline{u}$ and a l.s.c. supersolution $\bar{u}$ of (1.3) *satisfying*

$$
(3.8) \qquad B(x, \underline{u}, \underline{u}-M\underline{u})\leqq 0 \text{ and } B(x, \bar{u}, \bar{u}-M\bar{u})\geqq 0 \text{ on } \partial\Omega .
$$

Then there exists a solution u of (1.3) *satisfying $B(x, u, u-Mu^*)\leqq 0$ on $\partial\Omega$.*

Proof. Let $\mathcal{S}$ and u be defined by

$$
\begin{aligned}
\mathcal{S} = & \{v\text{: subsolution of (1.3)} | \\
& B(x, v, v-Mv^*)\leqq 0 \text{ on } \partial\Omega \text{ and } v\leqq \bar{u} \text{ on } \bar{\Omega}\} \ (\neq\emptyset) , \\
u(x) = & \sup\{v(x) | v\in\mathcal{S}\} \ (x\in\bar{\Omega}) .
\end{aligned}
$$

We note by the definition of u and Theorem 3.1 that $\underline{u}\leqq u\leqq \bar{u}$ on $\bar{\Omega}$. In order to obtain the assertion we shall prove the following properties hold:

(3.9) $u\in\mathcal{S}$,

(3.10) If $v\in\mathcal{S}$ is not a supersolution of (1.3), there exist a function $w\in\mathcal{S}$ and a point $y\in\Omega$ such that $w(y)>v(y)$.

First we show the property (3.9). Fix $x\in\Omega$ and $(p, X)\in J^{2,+}u^*(x)$. The definition of u.s.c. envelope and the function u imply that there exist $(x_n, u_n)\in \Omega\times\mathcal{S}$ such that

$$(x_n, u_n^*(x_n)) \to (x, u^*(x)) \quad \text{as} \quad n\to+\infty .$$

Hence we can find by [4, Proposition 4.3] $\hat{x}_n\in\Omega$ and $(p_n, X_n)\in J^{2,+}u_n^*(\hat{x}_n)$ satisfying

$$(\hat{x}_n, u_n^*(\hat{x}_n), p_n, X_n) \to (x, u^*(x), p, X) \quad \text{as} \quad n\to+\infty .$$

Since $u_n\in\mathcal{S}$, we have

$$F(\hat{x}_n, u_n^*(\hat{x}_n), p_n, X_n, u_n^*(\hat{x}_n)-Mu_n^*(\hat{x}_n))\leqq 0 .$$

It follows from the definition of u and (M.4) that $Mu_n^*\leqq Mu^*$ on $\bar{\Omega}$. Moreover using the monotonicity of F with respect to m, we obtain

$$F(\hat{x}_n, u_n^*(\hat{x}_n), p_n, X_n, u_n^*(\hat{x}_n)-Mu^*(\hat{x}_n))\leqq 0 . \tag{3.11}$$

Since $Mu^*\in USC(\bar{\Omega})$ by (M.1) and $\{Mu^*(x_n)\}$ is bounded, we can find a subsequence $\{Mu^*(x_{n_k})\}$ satisfying

$$\lim_{k\to+\infty} Mu^*(x_{n_k}) = \limsup_{n\to+\infty} Mu^*(x_n)\leqq Mu^*(x) .$$

Hence substituting $n=n_k$ in (3.11) and letting to $k\to+\infty$, we obtain

$$F(x, u^*(x), p, X, u^*(x)-Mu^*(x))\leqq 0 .$$

Let $x\in\partial\Omega$ be fixed. Choose $\{u_n\}\subset\mathcal{S}$ such that $u_n(x)\to u(x)$ as $n\to+\infty$. Using $Mu_n^*\leqq Mu^*$, we have $B(x, u_n(x), u_n(x)-Mu^*(x))\leqq 0$. Sending $n\to+\infty$, we have $B(x, u(x), u(x)-Mu^*(x))\leqq 0$. Hence we get $u\in\mathcal{S}$.

Next, we prove the property (3.10). Suppose $v\in\mathcal{S}$ is not a supersolution of (1.3). Then there exist $z\in\Omega$ and $(p, X)\in J^{2,-}v_*(z)$ satisfying

$$F(z, v_*(z), p, X, v_*(z)-Mv_*(z))<0 .$$

We claim $v_*(z)<\bar{u}(z)$. If $v_*(z)=\bar{u}(z)$, we get $(p, X)\in J^{2,-}\bar{u}(z)$. Noting $Mv_*\leqq M\bar{u}$ on $\bar{\Omega}$, we have

$$F(z, v_*(z), p, X, v_*(z)-Mv_*(z))\geqq F(z, \bar{u}(z), p, X, \bar{u}(z)-M\bar{u}(z))\geqq 0 ,$$

because $\bar{u}$ is a l.s.c. supersolution of (1.3). This is a contradiction. Thus we obtain the claim. Since Mv_* is l.s.c. on $\bar{\Omega}$, there exists a function $\psi\in C(\bar{\Omega})$ such that $\psi\leqq Mv_*$ on $\bar{\Omega}$ and $\psi(z)=Mv_*(z)$. Thus we have

$$F(z, v_*(z), p, X, v_*(z)-\psi(z))<0 .$$

We set

$$v_{\delta,\gamma}(x) = v_*(z)+\delta+\langle p, x-z\rangle+\frac{1}{2}\langle X(x-z), x-z\rangle-\gamma|x-z|^2 .$$

Then by the same method as in [4, Section 4] we obtain, for small $r, \delta, \gamma>0$, that $v_{\delta,\gamma}(x)<u_*(x)$ on $B_r(z)$ and $v_{\delta,\gamma}$ is a classical solution of

$$F(x, u, Du, D^2 u, u-\psi)\leqq 0 \quad \text{in} \quad B_r(z) , \tag{3.12}$$

where $B_r(z)=\{x\in \boldsymbol{R}^N: |x-z|<r\}$. We note that $v_{\delta,\gamma}$ is also a subsolution of (3.12) since F is degenerate elliptic. It is easily seen that, taking $\delta=r^2\gamma/8$, $v(x)>v_{\delta,\gamma}(x)$ when $r/2\leqq|x-z|\leqq r$. Hence we define the function w by

$$w(x) = \begin{cases} \max\{v(x), v_{\delta,\gamma}(x)\} & \text{for} \quad x\in B(z, r) , \\ v(x) & \text{otherwise.} \end{cases}$$

and then we see $Mv_*\leqq Mv^*\leqq Mw^*$ on $\bar{\Omega}$. Therefore we get

$$B(x, w, w-Mw^*)\leqq 0 \quad \text{on} \quad \partial\Omega$$

since $w=v$ on $\partial\Omega$ and $v\in\mathcal{S}$. By the similar proof to that of the property (3.9) we obtain $w\in\mathcal{S}$. Since $v_{\delta,\gamma}(z)=v_*(z)+\delta$, there exists a point $y\in\Omega$ such that $w(y)=v_{\delta,\gamma}(y)>v(y)$. ■

As to the boundary condition, we know only $B(x, u, u-Mu^*)\leqq 0$ on $\partial\Omega$ in the above proof. By the following corollary, we can show that u is a unique solution of (1.3) and satisfies the boundary codnition.

Corollary 3.3. *Let the assumpitons in Theorem* 3.2 *hold. Let u be a solution of* (1.3) *constructed in Theorem* 3.2. *If u satisfies both $B(x, u^*, u^*-Mu^*)\leqq 0$ and $B(x, u_*, u_*-Mu_*)\geqq 0$ on $\partial\Omega$, then u is a unique solution of* (1.3). *Moreover u is continuous on $\bar{\Omega}$ and satisfies the boundary condition.*

Proof. Let v be any solution of (1.3) satisfying $B(x, v^*, v^*-Mv^*)\leqq 0$ and $B(x, v_*, v_*-Mv_*)\geqq 0$ on $\partial\Omega$. Theorem 3.1 implies $u\equiv v\in C(\bar{\Omega})$. It is easily seen that u satisfies the boundary condition. ■

4. The case $\Omega=\boldsymbol{R}^N$

We devote this section to establish the comparison principle and existence of solutions of the following problems:

$$F(x, u, Du, D^2 u, u-Mu) = 0 \quad \text{on} \quad \boldsymbol{R}^N , \tag{4.1}$$

where F is continuous on $\boldsymbol{R}^N\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N\times\boldsymbol{R}$.

Let u and v be bounded functions on $\boldsymbol{R}^N$ and $\Gamma=\boldsymbol{R}^N\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N\times\boldsymbol{R}$.

(F.2)′ There exist $\omega_3 \in C(\boldsymbol{R}^+)$ with $\omega_3(0)=0$ and functions $\sigma_3(\mu)$ $(0<\mu\leqq 1)$, $\beta\in C(\boldsymbol{R}^N)$ and constants $\alpha_3>0$, $\alpha_4\geqq 0$ satisfying

$$\begin{aligned}
&F(y, r, \alpha(x-y)+p, Y+Z, m)-F_\mu(x, s, \alpha(x-y), X, n)\\
&\quad\leqq \max\{\omega_3(\alpha|x-y|^2+|x-y|+|p|+\|Z\|)\\
&\qquad +\alpha_3(r-s)+\sigma_3(1-\mu),\\
&\qquad \alpha_4(\omega_3(\alpha|x-y|^2+|x-y|+|p|+\|Z\|)\\
&\qquad +r-s+\sigma_3(1-\mu))+\beta(x)(m-n)\},\\
&\sigma_3(\mu)\to 0 \quad (\mu\to 0),\\
&\beta(x)>0 \quad \text{for} \quad x\in\boldsymbol{R}^N \quad \text{and} \quad \|\beta\|_{C(\boldsymbol{R}^N)}<+\infty
\end{aligned}$$

for all $x, y\in\boldsymbol{R}^N$, $r, s\in\boldsymbol{R}$, $p\in\boldsymbol{R}^N$, $X, Y, Z\in S^N$, $m, n\in\boldsymbol{R}$ and $\alpha>1$ such that $r\leqq s$ and

$$-3\alpha\begin{pmatrix} I & O\\ O & I\end{pmatrix}\leqq\begin{pmatrix} X & O\\ O & -Y\end{pmatrix}\leqq 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix},$$

where F_μ is defined in (F.4).

(M.2)′ For any $\eta>0$, and $z\in\boldsymbol{R}^N$ such that $\sup_{\boldsymbol{R}^N}(u-v)-\eta\leqq u(z)-v(z)$,

$$u(z)-v(z)+\eta\geqq Mu(z)-Mv(z).$$

We give the definition of sub-, super- and solutions of (4.1) as in Definition2.1 with $\Omega=\boldsymbol{R}^N$. Then we have the similar propositions to those in Section 2. Our main results in this section are stated as follows.

Theorem 4.1. *Assume* (F.1), (F.2)′, (M.1), (M.2)′ *and* (M.3) *hold. Moreover assume* (4.2) *holds.*

(4.2) *If* $\alpha_4=0$ *in* (F.2)′, *then it holds* $k_\mu>0$ $(0<\mu<1)$ *in* (M.3).

Let u *and* v *be, respectively, a subsolution and a supersolution of* (4.1). *Then* $u^*\leqq v_*$ *on* $\boldsymbol{R}^N$.

Theorem 4.2. *Let* F *and* M *be as in Theorem* 4.1. *Assume* (M.4) *in Theorem* 3.2 *holds with* $\Omega=\boldsymbol{R}^N$. *Assume there exist a u.s.c. subsolution* $\underline{u}$ *and a l.s.c. supersolution* $\overline{u}$ *of* (4.1). *Then there exists a* (*unique*) *solution* u *of* (4.1). *Moreover* $u\in C(\boldsymbol{R}^N)$.

If we admit Theorem 4.1 holds, we can get Theorem 4.2 by Perron's method as in the proof of Theorem 3.2. So we show only Theorem 4.1.

Proof of Theorem 4.1. Since the proof is similar to that of Theorem 3.1, we point out the differences. We may assume $u\in USC(\boldsymbol{R}^N)$ and $v\in LSC(\boldsymbol{R}^N)$. We suppose $\sup_{\boldsymbol{R}^N}(u-v)=2\theta>0$ and get a contradiction.

Let $u_m=(1-1/m)u$. Since u is bounded in $\boldsymbol{R}^N$, we have

$$\sup_{R^N} (u_m - v) \geqq \theta \quad \text{for large} \quad m \in \boldsymbol{N}.$$

Moreover, for any $\eta \in (0, \theta)$, we can find $z \in \boldsymbol{R}^N$ such that

$$0 < \sup_{R^N} (u_m - v) - \eta \leqq u_m(z) - v(z).$$

We remark that the function u_m satisfies

$$F_{1/m'}(x, u_m, p, X, u_m - Mu_m + k_{1/m}) \leqq 0. \tag{4.3}$$

for all $x \in \boldsymbol{R}^N$ and $(p, X) \in \bar{J}^{2,+} u_m(x)$.

For each $\alpha > 1$, Let the function Φ on $\boldsymbol{R}^N \times \boldsymbol{R}^N$ be as in the proof of Theorem 3.1. Then Φ attains its maximum because $\Phi \in USC(\boldsymbol{R}^N \times \boldsymbol{R}^N)$ and

$$\Phi(z, z) > 0 \quad \text{and} \quad \Phi(x, y) \to -\infty \quad (|x| + |y| \to +\infty).$$

Let $(\bar{x}, \bar{y}) \in \boldsymbol{R}^N \times \boldsymbol{R}^N$ be a maximum point of Φ. By $\Phi(z, z) \leqq \Phi(\bar{x}, \bar{y})$ we obtain

$$\frac{\alpha}{2} |\bar{x} - \bar{y}|^2 + |\bar{y} - z|^4 \leqq 2(\|u\| + \|v\|) \quad (\|u\| = \sup_{x \in R^N} |u(x)|),$$

and so $\{\bar{y}\}_{\alpha > 1} \subset \boldsymbol{R}^N$ is bounded and $|\bar{x} - \bar{y}| \to 0$ as $\alpha \to +\infty$. Then we may consider that $\bar{x}$ and $\bar{y}$ converge to some $z_0 \in \boldsymbol{R}^N$ as $\alpha \to +\infty$ by taking subsequence if necessary. By the semicontinuity of u_m and v we get

$$u_m(z) - v(z) \leqq u_m(z_0) - v(z_0), \tag{4.4}$$

$$\limsup_{\alpha \to +\infty} (\alpha |\bar{x} - \bar{y}|^2 + |\bar{y} - z|^4) \leqq \eta. \tag{4.5}$$

Moreover we note that $u_m(\bar{x}) \to u_m(z_0)$ and $v(\bar{y}) \to v(z_0)$ as $\alpha \to +\infty$. As in the proof of Theorem 3.1, using the maximum principle, we have the following inequalities:

$$F_{1/m'}(\bar{x}, u_m(\bar{x}), \alpha(\bar{x} - \bar{y}), X, u_m(\bar{x}) - Mu_m(\bar{x}) + k_{1/m}) \leqq 0,$$
$$F(\bar{y}, v(\bar{y}), \alpha(\bar{x} - \bar{y}) - p, Y - Z, v(\bar{y}) - Mv(\bar{y})) \geqq 0,$$

where $p = 4|\bar{y} - z|^2(\bar{y} - z)$ and $Z = 4|\bar{y} - z|^2 I + 8(\bar{y} - z) \otimes (\bar{y} - z)$. By (F.2)$'$ the same calculation as in the proof of Theorem 3.1 implies

$$\begin{aligned} 0 &\leqq F(\bar{y}, v(\bar{y}), \alpha(\bar{x} - \bar{y}) + p, Y + Z, v(\bar{y}) - Mv(\bar{y})) \\ &\quad - F_{1/m'}(\bar{x}, u_m(\bar{x}), \alpha(\bar{x} - \bar{y}), X, u_m(\bar{x}) - Mu_m(\bar{x}) + k_{1/m}) \\ &\leqq \max\{\omega_3(\alpha|\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{y}| + |p| + \|Z\|) - \alpha_3 \theta + \sigma_3(m^{-1}), \\ &\quad \alpha_4(\omega_3(\alpha|\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{y}| + |p| + \|Z\|) - \theta + \sigma_3(m^{-1})) \\ &\quad + \beta(\bar{x})(v(\bar{y}) - Mv(\bar{y}) - u_m(\bar{x}) + Mu_m(\bar{x}) - k_{1/m})\}. \end{aligned}$$

Letting $\alpha \to +\infty$, we obtain

$$(4.6) \qquad \begin{aligned} 0 \leqq \max\{ & \omega_3(\eta+4\eta^{3/4}+12\eta^{1/2})-\alpha_3\,\theta+\sigma_3(m^{-1}), \\ & \alpha_4(\omega_3(\eta+4\eta^{3/4}+12\eta^{1/2})-\theta+\sigma_3(m^{-1})) \\ & +\beta(z_0)\,(\eta-k_{1/m})\} \end{aligned}$$

since (M.2)$'$ is satisfied at z_0 from (4.4). We suppose $\alpha_4>0$. Using (F.2)$'$, (M.3) and letting $\eta\to 0$, we get

$$0\leqq\max\{-\alpha_3\,\theta+\sigma_3(m^{-1}),\ \alpha_4(-\theta+\sigma_3(m^{-1}))\}\ .$$

Taking sufficiently large $m\in\boldsymbol{N}$, we obtain a contradiction because of (F.2)$'$. Next we suppose $\alpha_4=0$. Sending $\eta\to 0$ in (4.6), we also have a contradiction for sufficiently large $m\in\boldsymbol{N}$ by (4.2). ■

5. Model problems I and II

In this section we mention Models I and II precisely. In the following we consider only the case $\Omega\subset\boldsymbol{R}^N$ is a bounded domain and assume that $\partial\Omega$ is sufficiently smooth.

First we consider Model I. Then we make the following assumptions on Q and the coefficients of (1.1):

(I.1) For all $v\in L^1(\Omega)$,

$$\left|\int_\Omega v(y)\,Q(dy,x)\right|\leqq C_0\,\|v\|_{L^1(\Omega)}\quad\text{for some}\quad C_0>0\,.$$

(I.2) The function

$$x\to\int_\Omega v(y)\,Q(dy,x)\,,$$

is continuous in the variable $x\in\bar{\Omega}$, uniformly for $v\in L^\infty(\Omega)$.

(I.3) $g_i\in W^{1,\infty}(\Omega)\ (i=1,\cdots,N)$, $\alpha,\lambda,f,\varphi\in C(\bar{\Omega})$.

(I.4) $\alpha\geqq\alpha_0$ on $\bar{\Omega}$ for some $\alpha_0>0$ and $\lambda>0$ in Ω.

(I.5) $g_i(x)\,\nu_i(x)>0$ for $x\in\partial\Omega$, where $\nu(x)=(\nu_1(x),\cdots,\nu_N(x))$ denotes the outward unit normal at $x\in\partial\Omega$.

(I.6) $\displaystyle\int_\Omega\varphi(y)\,Q(dy,x)\leqq\varphi(x)$ for $x\in\partial\Omega$.

Under thsee conditions we see (F.1)-(F.4), (M.1)-(M.3) and (B.1)-(B.2) hold. Thus we can apply Thoerem 3.1 to obtain the comparison principle of solutions of (1.1). Since $\underline{u}\equiv -C$ and $\bar{u}\equiv C$ on $\bar{\Omega}$ are, respectively, a subsolution and a supersolution of (1.1) satisfying

$$\underline{u}(x)\leqq\int_\Omega\underline{u}(y)\,Q(dy,x)\quad\text{and}\quad\bar{u}(x)\geqq\int_\Omega\bar{u}(y)\,Q(dy,x)\quad\text{for all}\quad x\in\partial\Omega$$

for sufficiently large $C>0$, Theorem 3.2 holds for (1.1). We prepare the following lemma to apply Corollary 3.3.

Lemma 5.1. *Assume* (I.1)-(I.6) *hold. Let u be the function defined as in the proof of Theorem* 3.2. *Then u satisfies*

$$u^*(x) \leqq \int_\Omega u^*(y)\, Q(dy, x) \quad \text{and} \quad u_*(x) \geqq \int_\Omega u_*(y)\, Q(dy, x) \quad \text{for all} \quad x \in \partial\Omega .$$

Proof. To prove the first inequality, we suppose that there exists $x_0 \in \partial\Omega$ such that $u^*(x_0) > \int_\Omega u^*(y)\, Q(dy, x_0)$ and shall get a contradiction. As in the proof of Theorem 3.2, we can find $(x_n, u_n) \in \bar{\Omega} \times \mathcal{S}$ satisfying $(x_n, u_n(x_n)) \to (x_0, u^*(x_0))$ as $n \to +\infty$. Noting (I.2), there exists $\delta > 0$ such that

$$u^*(x_0) - \delta \geqq \int_\Omega u^*(y)\, Q(dy, x) \quad \text{for} \quad x \in \overline{B(x_0, \delta) \cap \Omega}.$$

Using the facts that $u_n \leqq u$ on $\bar{\Omega}$ and $u_n \in \mathcal{S}$, we get, for all $n \in \boldsymbol{N}$,

$$\begin{aligned} &u_n \text{ is a subsolution of} \\ &\quad \tilde{L}u - \lambda(u^*(x_0) - \delta) - f = 0 && \text{in} \quad B(x_0, \delta) \cap \Omega , \\ &u_n - (u^*(x_0) - \delta) \leqq 0 && \text{on} \quad \overline{B(x_0, \delta)} \cap \partial\Omega , \\ &u_n \leqq \bar{u} && \text{on} \quad \bar{\Omega} , \end{aligned}$$

where $\tilde{L}u = -\sum_{i=1}^N g_i u_{x_i} + (\alpha + \lambda) u$. It is easily seen that from (I.5) that there exist $\varepsilon_0 \in (0, \delta)$ and $\psi \in C^1(B(x_0, \varepsilon_0) \cap \Omega) \cap C(\overline{B(x_0, \varepsilon_0) \cap \Omega})$ satisfying

$$\begin{aligned} &\tilde{L}\psi - \lambda(u^*(x_0) - \delta) - f \geqq 0 && \text{in} \quad B(x_0, \varepsilon_0) \cap \Omega , \\ &\psi(x_0) = u^*(x_0) - \delta , \\ &\psi - (u^*(x_0) - \delta) \geqq 0 && \text{on} \quad \overline{B(x_0, \varepsilon_0)} \cap \partial\Omega , \\ &\psi \geqq \bar{u} && \text{on} \quad \partial B(x_0, \varepsilon_0) \cap \bar{\Omega} . \end{aligned}$$

By the standard comparison argument we have

$$u_n \leqq \psi \quad \text{on} \quad \overline{B(x_0, \varepsilon_0) \cap \Omega} \quad \text{for all} \quad n \in \boldsymbol{N} .$$

By the way, there exists $n_0 \in \boldsymbol{N}$ such that $x_n \in \overline{B(x_0, \varepsilon_0) \cap \Omega}$ for all $n > n_0$. Thus we obtain $u_n(x_n) \leqq \psi(x_n)$ for such $n \in \boldsymbol{N}$. Letting $n \to +\infty$, we get

$$u^*(x_0) \leqq \psi(x_0) = u^*(x_0) - \delta ,$$

which is a contradiction.

Next we show the second inequality. As the above argument, we suppose $u_*(x_0) < \int_\Omega u_*(y)\, Q(dy, x_0)$ for some $x_0 \in \partial\Omega$ and shall get a contradiction. We remark $u_*(x_0) < \varphi(x_0)$ holds from (I.6). By (I.2) and (I.3), there exists $\delta > 0$ such that

$$u_*(x_0)+\delta<\min\{\int_\Omega u_*(y)\,Q(dy,x),\,\varphi(x)\} \quad \text{for} \quad x\in B(x_0,\delta)\cap\bar{\Omega}\,.$$

It is easily seen from (I.5) that there exist $\varepsilon_0\in(0,\delta)$ and $\psi\in C^1(B(x_0,\varepsilon_0)\cap\Omega)\cap C(\overline{B(x_0,\varepsilon_0)\cap\Omega})$ satisfying

$$\begin{aligned}
&\max\{\tilde{L}\psi-\tilde{f},\,\psi-\varphi\}\leqq 0 && \text{in} \quad B(x_0,\varepsilon_0)\cap\Omega\,,\\
&\psi(x)\leqq\int_\Omega u_*(y)\,Q(dy,x) && \text{for} \quad x\in\overline{B(x_0,\varepsilon_0)}\cap\partial\Omega\,,\\
&\psi(x_0)=u_*(x_0)+\delta\,, && \\
&\psi\leqq\varphi && \text{on} \quad \overline{B(x_0,\varepsilon_0)\cap\Omega}\,,\\
&\psi<\inf_{\bar{\Omega}} u && \text{on} \quad \partial B(x_0,\varepsilon_0)\cap\bar{\Omega}\,,
\end{aligned}$$

where $\tilde{f}(x)=f(x)-\lambda(x)\int_\Omega u_*(y)\,Q(dy,x)$. We define the function w by

$$w(x)=\begin{cases}\max\{u(x),\,\psi(x)\} & x\in B(x_0,\varepsilon_0)\cap\Omega\,,\\ u(x) & \text{otherwise.}\end{cases}$$

Then it is easily verified that $w(x)\leqq\int_\Omega w^*(y)\,Q(dy,x)$ for $x\in\partial\Omega$. Therefore by similar argument to the proof of (3.10), we get a contradiction. ■

Hence, using Corollary 3.3 we can see that the solution u of (1.1) is unique and satisfies the boundary condition. Besides S.M. Lenhart-the second author [9, Section 2] proved by another method that the solution u of (1.1) satisfies the boundary condition.

REMARK 5.1. We can also treat a second order equation:

$$\max\{\sum_{i,j=1}^N a_{ij}\,u_{x_ix_j}+Lu-f,\,u-\varphi\}=0 \quad \text{in} \quad \Omega\,,$$

where the matrix $(a_{ij}(x))$ satisfies $\langle(a_{ij}(x))\,\xi,\xi\rangle\geqq 0$ for all $x\in\bar{\Omega}$ and $\xi\in\boldsymbol{R}^N$.

Next we mention Model II. We assume the following:

(II.1) There exists $P\colon\bar{\Omega}\times(\boldsymbol{R}^+)^N\to(\boldsymbol{R}^+)^N$ satisfying

$$\begin{aligned}
&x+P(x,\xi)\in\bar{\Omega} \quad \text{for any} \quad x\in\bar{\Omega},\ \xi\in(\boldsymbol{R}^+)^N\,,\\
&P(x,\xi)=\xi \qquad \text{if} \quad x+\xi\in\bar{\Omega}\,,\\
&P(\cdot,\xi) \text{ is continuous on } \bar{\Omega} \text{ for each } \xi\in(\boldsymbol{R}^+)^N\,.
\end{aligned}$$

(II.2) $k\in C((\boldsymbol{R}^+)^N)$, $k(\xi)\geqq k_0$ on $(\boldsymbol{R}^+)^N$ for some $k_0>0$.

(II.3) For the matrix $(a_{ij}(x))$, there exists a nonnegative matrix $(\sigma_{ij}(x))$ such that

$$(a_{ij})={}^t(\sigma_{ij})\,(\sigma_{ij}) \text{ with } \sigma_{ij}\in W^{1,\infty}(\Omega) \quad (i,j=1,\cdots,N)\,,$$

where ${}^tA$ is the transposed matrix of A.

(II.4) $b_i \in W^{1,\infty}(\Omega)$ $(i=1, \cdots, N)$, $c, f, g \in C(\bar{\Omega})$.

(II.5) $b_i(x)\, \nu_i(x) < 0$ on $\{x \in \partial\Omega \mid \sum_{i,j=1}^{N} a_{ij}(x)\, \nu_i(x)\, \nu_j(x) = 0\}$ where $\nu(x) = (\nu_1(x), \cdots, \nu_N(x))$ is the outward unit normal to Ω at $x \in \partial\Omega$.

(II.6) $c \geqq c_0$ on $\bar{\Omega}$ for some $c_0 > 0$.

By the assumptions (II.1), (II.2) it is seen that the operator M maps $USC(\bar{\Omega})$ into itself. (cf. A. Bensoussan-J.L. Lions [3, Chapter 4, Lemma 1.6] or the first author [5, Proposition 2.3].) From (II.1)-(II.6) we can check that (F.1)-(F.4), (M.1)-(M.3) and (B.1)-(B.2) hold. Thus we get the comparison principle of solutions of (1.2) by Theorem 3.1. Moreover, using the barrier argument, we can apply Corollary 3.3 to obtain the existence of a unique solution of (1.2) satisying the boundary condition. (cf. The first author [5, Section 4].)

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