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ON DERIVATIVES OF VISCOSITY SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We prove that partial derivatives of viscosity solutions of elliptic fully nonlinear equations are viscosity solutions of linear elliptic equations.

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1. Introduction

Let

$$F(D^2u) = 0 \tag{1}$$

be a fully nonlinear second-order elliptic equation defined in a domain of $\mathbb{R}^n$. Here D^2u denotes the Hessian of the function u. We assume that F is a smooth function defined on the set $S^2(\mathbb{R}^n)$, the space of $n \times n$ symmetric matrices. Recall also that (1) is called uniformly elliptic if there exists a constant $C = C(F) \geqslant 1$ (called an *ellipticity constant*) such that

$$\frac{1}{C}|\xi|^2 \leqslant F_{u_{ij}}\xi_i\xi_j \leqslant C|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Here, u_{ij} denotes the partial derivative $\partial^2 u/\partial x_i \partial x_j$. A function u is called a *classical* solution of (1) if $u \in C^2(\Omega)$ and u satisfies (1). Actually, any classical solution of (1) is a smooth ($C^{\alpha+3}$) solution, provided that F is a smooth ($C^{1+\alpha}$) function of its arguments.

Let u_1, u_2 be two classical solutions of the equation (1). Then the difference $v = u_1 - u_2$ is a solution of a linear uniformly elliptic equation

$$Lv = \sum a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} = 0, \tag{2}$$

where

$$a_{ij} = F_{u_{ij}}(\theta D^2 u_1 + (1-\theta)D^2 u_2), \tag{3}$$

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and $0 < \theta < 1$. Equality (3) follows from a multidimensional version of Rolle's theorem. The coefficients $a_{ij}(x)$ satisfy the inequalities

$$C^{-1}|\xi|^2 \leqslant \sum a_{ij}\xi_i\xi_j \leqslant C|\xi|^2,$$

where $C > 0$ is an ellipticity constant.

A function v is called a *classical* solution of equation (2) with measurable coefficients if $v \in C^2$ and v satisfies (2) almost everywhere.

In oder to get a solution to the Dirichlet problem for each of the equation (1) or (2) the notion of classical solutions has to be extended. Such extension, known as *viscosity* (weak) solutions can be done for equations (1) and (2) in different ways.

For the fully nonlinear equation (1) the set of the viscosity solution can be defined as an intersection of C-closures of the sets of super and subsolutions.

For the linear operator (2) one can define a continuous strong Markov process $x(\cdot)$ such that for small $h > 0$, $x(t+h) - x(t)$ behaves as a Gaussian process with mean zero and covariance $a(x(t))$.

The main goal of this paper is to show that the Rolle's relation (2), (3) holds in a weak sense, i.e., if u_1, u_2 are viscosity solutions of (1) then v is a viscosity solution of (2).

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2. Viscosity Solutions to Linear and Nonlinear Elliptic Equations

We recall first the formal definitions of viscosity solutions to the equation (1) and (2).

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let L be a linear uniformly elliptic operator (2) defined in Ω with the ellipticity constant C.

We consider a Dirichlet problem in Ω:

$$\begin{cases} Lv = 0 & \text{in } \Omega \\ v = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4}$$

Due to the result of Jensen [J2] two following definitions of the viscosity solutions to the Dirichlet problem (4) are equivalent.

Definition 1. Function v is a viscosity solution of (1) if

$$v = \lim v_k,$$

where

$$Lv_k = \sum a_{ij}^k(x)\frac{\partial^2 v_k}{\partial x_i \partial x_j} = 0,$$

a_{ij}^k are continuous and $a_{ij}^k \to a_{ij}$ in $L_1(\Omega)$.

Definition 2. Function v is a viscosity solution of (1) if

(i) $$\lim_{\epsilon\to+0} \sup \epsilon^{-n} \int_{|x-y|<\epsilon} \Big[\sum a_{ij}(y)\Big(\frac{\partial^2\phi}{\partial x_i\partial x_j} + \eta\delta_{ij}\Big)\Big]^+ dy > 0$$

for all $\eta > 0$, whenever $x \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$0 = (v - \phi)(x) \geqslant (v - \phi)(y)$$

for all $y \in \Omega$, and if

(ii) $$\lim_{\epsilon \to +0} \sup \epsilon^{-n} \int_{|x-y|<\epsilon} \Big[\sum a_{ij}(y) \Big(\frac{\partial^2 \phi}{\partial x_i \partial x_j} - \eta \delta_{ij} \Big) \Big]^- dy > 0$$

for all $\eta > 0$, whenever $x \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$0 = (v - \phi)(x) \leqslant (v - \phi)(y)$$

for all $y \in \Omega$, where $[t]^+$ denote $\max\{0, t\}$, and $[t]^-$ denote $\max\{0, -t\}$.

Though from Definition 1 follows the existence of the viscosity solution to the Dirichlet problem (4), the principal question on the uniqueness of the viscosity solution remains open. For a general uniformly elliptic operator (2) we showed [N2] that the viscosity solutions to the Dirichlet problem (4) are not unique, (see also some extension of the result in [S1]). However under certain restriction on the coefficients of operator L number of the uniqueness results are known, see, e.g., the survey [K].

We will need the following proposition [N1].

Proposition 1. *Let u be a viscosity solution of the equation* (2)*. Then for almost every point $y \in \Omega$ there exists a second order polynomial $p_y(x)$ such that $u(x) - p_y(x) = o(|x - y|^2)$ and $Lp_y = 0$.*

Now we consider the Dirichlet problem for the fully nonlinear equation

$$\begin{cases} F(D^2 u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

Definition 3. A continuous function u in Ω is a viscosity subsolution (resp. viscosity supersolution) of (1) in Ω when the following condition holds: if $x_0 \in \Omega$, $f \in C^2(\Omega)$ and $u - f$ has a local maximum at x_0 then

$$F(D^2 f(x_0)) \geqslant 0$$

(resp. if $u - f$ has a local minimum at x_0 then $F(D^2 f(x_0)) \leqslant 0$).

We say that u is a viscosity solution of (1) when it is subsolution and supersolution.

The existence and the uniqueness of the viscosity solution to Dirichlet problem (5) was shown by Crandall, Lions, Evans and Jensen, see [CC].

Following Ishii [I], one can also define viscosity solutions of (5) using Perron's method, see also [CIL].

The existence of nonclassical viscosity solutions to fully nonlinear elliptic equations was shown in [NV].

Denote by U^+ the set of C^2-supersolutions of the problem (5): $u^+ \in U^+$ if $u^+ \in C^2(\Omega)$ and $F(D^2 u^+) \leqslant 0$, and $u^+ \geqslant \varphi$ on $\partial\Omega$. Correspondingly U^- be the set of C^2-subsolutions of the problem (5): $u^- \in U^-$ if $u^- \in C^2(\Omega)$ and $F(D^2u^-) \geqslant 0$, and $u^- \leqslant \varphi$ on $\partial\Omega$.

The following important result is due to M. Safonov [S2].

Theorem 1. *Let u be a viscosity solution of (5). Then there are sequences $u_n^+ \in U^+$, $u_n^- \in U^-$ such that*

$$u = \lim_{n\to\infty} u_n^+ = \lim_{n\to\infty} u_n^-.$$

Proof. Let u be a continuous function in Ω and let $H \subset\subset \Omega$ be an open set. Define, for $\epsilon > 0$, the upper ϵ-envelope of u:

$$u^\epsilon(x_0) = \sup_{x\in H}\{u(x) + \epsilon - |x - x_0|^2/\epsilon\}.$$

According to a result of Jensen [J1] (see also [CC, Section 5.1]), if u is a viscosity subsolution of $F(D^2u) = 0$, then the upper ϵ-envelopes u^ϵ are also viscosity subsolutios of $F(D^2u) = 0$. Moreover, $u^\epsilon \to u$ as $\epsilon \to 0$, uniformly on compact subsets, and u^ϵ are $C^{1,1}$ from below. In particular, they have second differential almost everywhere and $F(D^2u^\epsilon) \geqslant 0$ a.e.

Let η^δ be a smooth nonnegative function with the support in B_δ and the total integral 1. Consider the standard mollifiers $u^{\epsilon,\delta} = u^\epsilon * \eta^\delta$, which are smooth and satisfy

$$D^2u^{\epsilon,\delta} \to D^2u^\epsilon$$

almost everywhere, as $\delta \to 0$. Since the functions $u^{\epsilon,\delta} + C|x|^2$ are convex, with $C = C(\epsilon)$, we have

$$0 \leqslant f^{\epsilon,\delta} := (F(D^2(u^{\epsilon,\delta})))_- \leqslant C = C(\epsilon),$$

and $f^{\epsilon,\delta} \to 0$ a.e. as $\delta \to 0$.

Let $v^{\epsilon,\delta}$ be a classical solution of the Dirichlet problem with the concave minimal Pucci operator [CC, p. 17] and Lipschitz right side $f^{\epsilon,\delta}$:

$$\begin{cases} \mathcal{M}^-(D^2v^{\epsilon,\delta}) = f^{\epsilon,\delta} & \text{in } \Omega \\ v^{\epsilon,\delta} = 0 & \text{on } \partial\Omega \end{cases}$$

By the Alexandrov–Bakelman–Pucci estimates, [CC], $0 \geqslant v^{\epsilon,\delta} \to 0$ as $\delta \to 0$, uniformly on Ω. Then one can choose small positive ϵ, δ and c_ϵ such that the function $w^{\epsilon,\delta} := u^{\epsilon,\delta} + v^{\epsilon,\delta} - c_\epsilon < u$, and it can be made arbitrarily close to u. Finally

$$F(D^2(w^{\epsilon,\delta}) \geqslant F(D^2(u^{\epsilon,\delta})) + \mathcal{M}^-(D^2v^{\epsilon,\delta}) = F(D^2u^{\epsilon,\delta})) + (F(D^2(u^{\epsilon,\delta})))_- \geqslant 0.$$

Thus we proved that a viscosity solution can be uniformly approximated from below by classical subsolutions. The "upper" approximation is quite similar. The theorem is proved. □

We will need the following propositions, see [A].

Proposition 2. *Let $B \subset R^n$ be a unit ball. Let $u \in W^{2,n}(B)$ be such that*

$$Lu \geqslant -1 \quad \textit{in } B,$$

where L is the uniformly elliptic operator (2) *and*

$$u|_{\partial B} \leqslant 0.$$

Let U be the convex envelope of the graph of u^+, and

$$\nu \colon U \to S^n$$

be the Gauss normal map. Let ds be the element of the surface area of U and Jds be the Jacobian of the map ν. Then

$$J \leqslant C_1,$$

where a positive constant C_1 depends on the ellipticity constant of the operator L. Moreover the support of the function J is on the set of coincidence of the graphs of function u^+ and J.

As an immediate corollary of Proposition 2 we have the following

Proposition 3. *Let $B \subset R^n$ be a unit ball. Let $u_n \in W^{2,n}(B)$, $n = 1, 2, \dots$ be such that*

$$L_n u_n \geqslant -1 \quad \text{in } B,$$

where L_n is a sequence of uniformly elliptic operators (2) *with a joint constant of ellipticity. Assume that the sequence converges in $C(B)$, $u_n \to u$, and*

$$u|_{\partial B} \leqslant 0.$$

Let U be the convex envelope of the graph of u^+, and

$$\nu \colon U \to S^n$$

be the Gauss normal map. Let ds be the element of the surface area of U and Jds be the Jacobian of the map ν. Then

$$J \leqslant C_1,$$

where a positive constant C_1 depends on the ellipticity constant of the operators L_n.

Two following propositions are due to Trudinger [T2], [T1].

Proposition 4. *Let u be a viscosity solution of the fully nonlinear equation* (1). *Then for almost every point $y \in \Omega$ there exists a second order polynomial $p_y(x)$ such that $u(x) - p_y(x) = o(|x-y|^2)$ and $F(D^2 p_y) = 0$.*

Proposition 5. *Let u be a viscosity solution of the fully nonlinear equation* (1). *Then $u \in C^{1,\delta}$, $\delta > 0$.*

Theorem 2. *Let u_1, u_2 be viscosity solutions of equation* (1), *$v = u_1 - u_2$. Then v is a viscosity solution of the equation* (2) *where coefficients a_{ij} satisfy equality* (3).

Proof of Theorem 2. Since by Proposition 5 the functions u_1, u_2 have almost everywhere approximative second differentials which satisfy the equation (1) it follows that function v has almost everywhere an approximative second differential which satisfies the equation (2) with the coefficients satisfying (3).

Assume by contradiction that v is not a viscosity solution of (2). Then by Definition 2 it follows that either property (i) or (ii) fails to be true. We may assume without loss that (ii) is not satisfied for the function v. That implies the existence of a point $y \in \Omega$, function $\phi \in C^2(\Omega)$, $x\phi(y) = v(y)$ and constant $\delta > 0$ such that for any $\epsilon > 0$ there exists $r > 0$ such that in the ball $B = \{x\colon |x-y| < r\}$ the following inequalities hold $\phi \leqslant v - \delta|x-y|^2$ and $L\phi \geqslant 0$ on $B \setminus E$, where E is a Borel subset of B such that $\operatorname{meas} E < \epsilon \operatorname{meas} B$. Set $\psi = \phi - v + \delta r^2/2$. Let V be the convex envelope of ψ^+. By Proposition 2 there are convergent sequences $u_n^- \to u_1$, $u_n^+ \to u_2$, u_n^-, $u_n^+ \in C^2$ such that u_n^- are subelliptic, $F(D^2 u_n^-) > 0$ and u_n^+ are superelliptic, $F(D^2 u_n^-) < 0$. Set $v_n = u_n^+ - u_n^-$. Then v_n are superelliptic functions for a linear uniformly elliptic operator (2),

$$L_n v_n \leqslant 0.$$

Thus we can apply Proposition 4 to function ψ. Denote by V the convex envelope of the function ψ. Let $J = ads$ be the Jacobian of the Gauss map of the function V. Since the functions ψ_n are subelliptic then by Proposition 4 $a < C$, where constant $C > 0$ depends on C^2-norm of ϕ and the ellipticity constant of the operator F. Since the function v has almost everywhere the second differential satisfying (2) we conclude that $a = 0$ on $B \setminus E$. Thus by the maximum principle of Alexandrov–Bakelman–Pucci [A], [CC] it follows that $\psi < Cr^2\epsilon^{1/n}$, $C > 0$. Since $\psi(y) = \delta r^2/2$ then choosing sufficiently small $\epsilon > 0$ we get a contradiction. The theorem is proved. □

Corollary 1. *Let u be a viscosity solution of the fully nonlinear equation (1). Then any partial derivative $v = u_{x_k}$ is a solution of the uniformly elliptic operator (2), with the coefficients a_{ij} defined almost everywhere by*

$$a_{ij} = F_{u_{ij}}.$$

Proof. We may assume that $k = 1$. Set

$$v_m = m(u(x_1, \dots, x_n) - u(x_1 + 1/m, \dots, x_n)).$$

By Theorem v_m is a viscosity solution of the equation (2) with the coefficients

$$a_{ij}^m = F_{u_{ij}}(\theta D^2 u_1 + (1-\theta) D^2 u_2),$$

$0 < \theta < 1$. For $\epsilon > 0$ we denote

$$E(m, \epsilon) = \{x \in \Omega\colon |D^2 u(x_1, \dots, x_n) - D^2 u(x_1 + 1/m, \dots, x_n)| > \epsilon\}.$$

Since $D^2 u$ is defined almost everywhere and is measurable on Ω then by Lusin's theorem for any $\epsilon > 0$ $\operatorname{meas} E(m, \epsilon) \to 0$ as $m \to \infty$. Thus $a_{ij}^m \to a_{ij}$ in $L_1(\Omega)$ as $m \to \infty$ and from Definition 1 it follows that v is a viscosity solution of (2). The corollary is proved. □

Corollary 2. *Let u be a viscosity solution of the fully nonlinear equation (1). Then function u has almost everywhere the third approximative differential, i.e., for almost every point $y \in \Omega$ there exists a third order polynomial $p_y(x)$ such that $u(x) - p_y(x) = o(|x-y|^3)$.*

Proof. By Corollary 1 the functions u_{x_i}, $i = 1, 2, \dots$, are viscosity solutions of uniformly elliptic equations. Integrating the function u_{x_1} over dx_1 we get by Proposition 1 that the function u has almost everywhere an approximative third differential along the lines parallel to x_1 axis. Consequently integrating functions u_{x_i} over dx_i,

$i = 1, 2, 3, \ldots,$ we get by induction that function u has almost everywhere an approximative third differential along the planes parallel to $x_1 x_2$, along the subspaces parallel to $x_1 x_2 x_3$, etc. The corollary is proved. □

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