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On the Equivalence of Two Notions of Weak Solutions, Viscosity Solutions and Distribution Solutions

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Introduction

We shall be mainly concerned with the linear, degenerate elliptic, partial differential equation

$$\mathscr{L}u = f \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open subset of $\boldsymbol{R}^N$ and $\mathscr{L}$ is the operator defined by

$$\mathscr{L}u(x) = -\sum_{i,j=1}^{N} a_{ij}(x)u_{x_ix_j}(x) + \sum_{i=1}^{N} b_i(x)u_{x_i}(x) + c(x)u(x).$$

Throughout this paper we assume that the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ and $f(x)$ are real and that the matrices $a(x) \equiv (a_{ij}(x))$ are symmetric and nonnegative definite and

$$a_{ij} \in C^{1,1}(\Omega),\ b_i \in C^{0,1}(\Omega),\ c, f \in C(\Omega)\ \forall i, j = 1,\dots, N.$$

It is known that under these assumptions the square root $\sigma \equiv a^{1/2}$ of a is in $C^{0,1}(\Omega)$. E.g., see [10] for a proof of this fact.

We consider weak solutions of (1.1) in the class of continuous functions. Subsolutions in the distribution sense are defined as follows. A function $u \in C(\Omega)$ is a distribution subsolution of (1.1) if

$$\int_{\Omega} (u\mathscr{L}^*\varphi - f\varphi)\,dx \le 0 \tag{1.2}$$

for any $\varphi \in \mathscr{D}_+(\Omega) \equiv \{\varphi \in C_0^\infty(\Omega) \mid \varphi \ge 0\}$, where $\mathscr{L}^*$ is the formal adjoint operator of $\mathscr{L}$, i.e.,

$$\mathscr{L}^*\varphi = -\sum_{i,j=1}^{N} (a_{ij}\varphi)_{x_ix_j} - \sum_{i=1}^{N} (b_i\varphi)_{x_i} + c\varphi \qquad \forall \varphi \in C^2(\Omega).$$

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Likewise, a distribution supersolution is defined to be a continuous function u which satisfies (1.2) with $\geq$ replacing $\leq$. We shall indicate that u is a distribution subsolution (respectively, a distribution supersolution) by writing

$$\mathscr{L}u \leq f \quad \text{in } \mathscr{D}'(\Omega) \qquad (\text{respectively, } \mathscr{L}u \geq f \quad \text{in } \mathscr{D}'(\Omega)).$$

A distribution solution of (1.1) is a function which is both a distribution subsolution and a distribution supersolution of (1.1). Equivalently, $u \in C(\Omega)$ is a distribution solution of (1.1) if

$$\int_\Omega (u\mathscr{L}^*\varphi - f\varphi)\,dx = 0 \qquad \forall\varphi \in C_0^\infty(\Omega).$$

This is indicated by writing $\mathscr{L}u = f$ in $\mathscr{D}'(\Omega)$.

For our exposition it is convenient to consider the general second-order, degenerate elliptic partial differential equation

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{1.3}$$

Here $F: \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \times \boldsymbol{S}^N \to \boldsymbol{R}$ is a continuous function, where $\boldsymbol{S}^N$ denotes the set of real $N \times N$ symmetric matrices, and Du and D^2u denote the gradient $(u_{x_1}, \dots, u_{x_N})$ and the Hessian matrix $(u_{x_ix_j})$. The precise meaning of "degenerate ellipticity" is this. The function F or equation (1.3) is degenerate elliptic if $F(x, r, p, X) \leq F(x, r, p, Y)$ provided $X \geq Y$, i.e., $X - Y$ is nonnegative definite.

A function $u \in C(\Omega)$ is a viscosity subsolution of (1.3) if $F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$ whenever $\varphi \in C^2(\Omega)$, $x \in \Omega$ and $(u - \varphi)(x) = \sup_\Omega(u - \varphi)$. Similarly, $u \in C(\Omega)$ is a viscosity supersolution of (1.3) if $F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0$ whenever $\varphi \in C^2(\Omega)$, $x \in \Omega$ and $(u - \varphi)(x) = \inf_\Omega(u - \varphi)$. $u \in C(\Omega)$ is a viscosity solution of (1.3) if it is both a viscosity subsolution and a viscosity supersolution of (1.3). When convenient, we shall indicate that u is a viscosity subsolution (respectively, a viscosity supersolution, or a viscosity solution) of (1.3) by writing

$$F(x, u, Du, D^2u) \leq 0 \text{ (respectively, } \geq 0, \text{ or } = 0) \text{ in } \Omega \text{ in the viscosity sense.}$$

We set

$$F_{\mathscr{L}}(x, r, p, X) = -\operatorname{tr} a(x)X + \langle b(x), p\rangle + c(x)r - f(x).$$

Now (1.1) reads $F_{\mathscr{L}}(x, u, Du, D^2u) = 0$ in Ω. Since $a(x) \geq 0$, it is seen that $F_{\mathscr{L}}$ is degenerate elliptic. Subsolutions, supersolutions and solutions of (1.1) in the viscosity sense are defined with $F_{\mathscr{L}}$.

The definitions of distribution solutions and viscosity solutions are based on the integration by parts and on the maximum principle, respectively. The maximum principle here means that if $v \in C^2(\Omega)$ attains its maximum at $x \in \Omega$, then $Dv(x) = 0$ and $D^2v(x) \leq 0$.

The question we address here is if these two notions of weak solutions of (1.1) are equivalent. An affirmative answer has ben given in [8] by P.-L. Lions. The arguments there are largely based on probabilistic techniques to deduce the answer. We will give here another approach based on purely PDE and viscosity solutions methods to obtain a similar conclusion.

Theorem 1 *If* $u \in C(\Omega)$ *is a viscosity subsolution of* (1.1), *then it is a distribution subsolution of* (1.1).

Theorem 2 *Assume that* $\sigma \in C^1(\Omega)$. *If* $u \in C(\Omega)$ *is a distribution subsolution of* (1.1), *then it is also a viscosity subsolution of* (1.1).

Our results are slightly better in the sense that the regularity requirements on a is less than those in [8]. In deed, it is assumed in [8] that σ is in $C^{1,1}(\Omega)$.

The paper is organized as follows. In Section 1 we explain an observation concerning the sup-convolution of viscosity solutions. Section 2 is devoted to the proof of Theorem 1. In Section 3 we collect solvability and regularity results (Theorems 4 and 5) of solutions of (1.1) which are needed in the proof of Theorem 2. Theorem 2 is proved in Section 4. Theorems 4 and 5 are proved in Section 5.

§1 Approximation of viscosity solutions

It is well known that the sup-convolutions and inf-convolutions yield good approximations of viscosity subsolutions and supersolutions, respectively. We give here an additional remark concerning these approximations.

Throughout this section, for simplicity of presentation we assume that Ω is bounded and only consider those solutions u which are bounded, uniformly continuous, i.e., $u \in BUC(\Omega)$. For a function $u \in BUC(\Omega)$ and $\varepsilon > 0$ the sup-convolution is defined by

$$u^{\varepsilon}(x) = \sup_{y \in \Omega} \left(u(y) - \frac{1}{2\varepsilon} |x - y|^2 \right).$$

We shall write $\Omega_{\varepsilon} = \{x \in \Omega \mid \operatorname{dist}(x, \Omega^c) > \varepsilon\}$.

To formulate the result, we introduce some conditions on F.

(A1) For each $R > 0$ there is a function $\omega_{1R} \in C([0, \infty))$ satisfying $\omega_{1R}(0) = 0$ such that if $-R \leq r \leq s \leq R$, then $F(x, r, p, X) \leq F(x, s, p, X) + \omega_{1R}(s - r)$.

(A2) For each $R > 0$ there is a function $\omega_{2R} \in C([0, \infty))$ satisfying $\omega_{2R}(0)$ such that if $|r| \leq R$ and if $\alpha > 1$ and $X, Y \in \boldsymbol{S}^N$ satisfy

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y, r, \alpha(x-y), -Y) \le F(x, r, \alpha(x-y), X) + \omega_{2R}(\alpha|x-y|^2 + 1/\alpha).$$

We note that if F satisfies (A2), then F is degenerate elliptic. Note also that $F_{\mathscr{L}}$ satisfies (A1) and (A2) provided σ and b are Lipschitz continuous and c and f are uniformly continuous on Ω. See for these [2].

Theorem 3 *Let* (A1) *and* (A2) *hold. Let* $u \in BUC(\Omega)$ *be a viscosity subsolution of* (1.3). *Then for each* $\varepsilon > 0$ *there is* $\delta_0 > 0$ *such that for* $0 < \delta < \delta_0$,

$$F(x, u^\delta, Du^\delta, D^2u^\delta) \le \varepsilon \qquad \textit{in } \Omega_\varepsilon \textit{ in the viscosity sense.}$$

Remark The constant δ_0 can be chosen so that it depends on u only through $\sup_\Omega |u|$ and its modulus of continuity.

Proof. We choose $M > 0$ so that $M \ge 2\sup_\Omega |u|$, and a nondecreasing function $\omega \in C([0, \infty))$ satisfying $\omega(0) = 0$ so that

$$\sup\{u(x) - u(y) \mid x, y \in \Omega, |x-y| \le r\} \le \omega(r) \qquad \forall r \ge 0,$$

and $\max\{\omega_{1M}, \omega_{2M}\} \le \omega$, where ω_{1M} and ω_{2M} are from (A1) and (A2) with $R = M$, respectively.

Let $\delta > 0$. It is obvious that $u \le u^\delta$ on Ω. Therefore, it is easily seen that if $\gamma = (2\delta M)^{1/2}$ and $x \in \Omega_\gamma$, then $B(x, \gamma) \subset \Omega$ and

$$u^\delta(x) = \max\left\{u(y) - \frac{1}{2\delta}|x-y|^2 \mid y \in B(x, \gamma)\right\}.$$

For each $x \in \Omega_\gamma$ we fix $y(x, \delta) \in B(x, \gamma)$ so that

$$u^\delta(x) = u(y(x, \delta)) - \frac{1}{2\delta}|x - y(x, \delta)|^2.$$

We observe that from the inequality $u \le u^\delta$ on Ω that

$$\frac{1}{2\delta}|x - y(x, \delta)|^2 \le u(y(x, \delta)) - u(x) \le \omega(|x - y(x, \delta)|) \le \omega(\gamma).$$

We recall that if $x \in \Omega_\gamma$ and $(p, X) \in J^{2,+}u^\delta(x)$, then $y(x, \delta) = x + \delta p$. See [2] for this, the definitions of semijets $J^{2,\pm}u$, $\bar{J}^{2,\pm}u$ and relevant facts. Now, fix $x \in \Omega_\gamma$ and $(p, X) \in J^{2,+}u^\delta(x)$. We set

$$v(z) = \langle p, z - x\rangle + \frac{1}{2}\langle X(z-x), z - x\rangle \qquad \forall z \in \boldsymbol{R}^N,$$

and $w(y, z) = u(y) - v(z)$ for $y \in \Omega$, $z \in \boldsymbol{R}^N$. We observe that

$$w(y, z) - \frac{1}{2\delta}|y - z|^2 \leq u^\delta(z) - v(z) \leq u^\delta(x) + o(|z - x|^2)$$

$$= w(y(x, \delta), x) - \frac{1}{2\delta}|y(x, \delta) - x|^2 + o(|z - x|^2) \qquad \text{as } z \longrightarrow x,$$

i.e.,

$$\left(\frac{1}{\delta}(y\,(x, \delta) - x), \frac{1}{\delta}(x - y(x, \delta)), \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right)$$

$$= \left(p, -p, \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right) \in J^{2,+} w(y(x, \delta), x).$$

By the maximum principle for semicontinuous functions (see [2]), we see that there are $Y, Z \in \boldsymbol{S}^N$ such that

$$-\frac{3}{\delta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \leq \frac{3}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

$$(p, Y) \in \bar{J}^{2,+} u(y(x, \delta)), \qquad (p, -Z) \in \bar{J}^{2,-} v(x).$$

The last inclusion implies that $-Z \leq D^2 v(x) = X$. Since u is a viscosity subsolution of (1.3), we have

$$F(y(x, \delta), u(y(x, \delta)), p, Y) \leq 0.$$

To proceed, we assume that $\delta < 1$. Assumption (A2) now yields

$$F\left(x, u(y(x, \delta)), \frac{1}{\delta}(y(x, \delta) - x), -Z\right)$$

$$\leq F(y(x, \delta), u(y(x, \delta)), \frac{1}{\delta}(y(x, \delta) - x), Y) + \omega\left(\frac{1}{\delta}|y(x, \delta) - x|^2 + \delta\right).$$

Consequently,

$$\begin{aligned} 0 &\geq F(x, u(y(x, \delta)), p, -Z) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u(y(x, \delta)), p, X) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u^\delta(x) + (1/2\delta)|y(x, \delta) - x|^2, p, X) - \omega(2\omega(\gamma) + \delta) \\ &\geq F(x, u^\delta(x), p, X) - \omega(\omega(\gamma)) - \omega(2\omega(\gamma) + \gamma). \end{aligned}$$

Thus

$$F(x, u^\delta(x), p, X) \leq 2\omega(2\omega(\gamma) + \delta) \qquad \text{in } \Omega_\gamma$$

in the viscosity sense. Noting that $\gamma \equiv (2\gamma M)^{1/2} \to 0$ and $2\omega(2\omega(\gamma) + \delta) \to 0$ as $\delta \downarrow 0$, we finish the proof. ■

§2 Proof of Theorem 1

Theorem 3 and the following lemma will be key observations in our proof of Theorem 1. We denote by $\mathcal{M}(\Omega)$ and by $\mathcal{D}'(\Omega)$ the spaces of Radon measures on Ω and of distributions on Ω, respectively. Recall that we may identify $\mathcal{M}(\Omega)$ with the dual space $C_0(\Omega)'$ of $C_0(\Omega)$.

Lemma 1 (A. D. Aleksandrov) *Let $u \in C(\boldsymbol{R}^N)$ be semiconvex. Then there are matrices $U = (u_{ij})_{1 \le i,j \le N}$ with $u_{ij} \in L^1_{loc}(\boldsymbol{R}^N)$ and $V = (v_{ij})_{1 \le i,j \le N}$ with $v_{ij} \in \mathcal{M}(\boldsymbol{R}^N)$ such that*

$$D^2 u = U + V \text{ in } \mathcal{D}'(\Omega), \quad V \ge 0 \text{ in } \mathcal{M}(\boldsymbol{R}^N),$$

$$(Du(x), D^2u(x)) \in J^2 u(x) \text{ a.e. in } \boldsymbol{R}^N,$$

where $J^2 u(x) = J^{2,+} u(x) \cap J^{2,-} u(x)$. Moreover, the measures v_{ij} are singular with respect to the Lebesgue measure.

For a proof of this lemma we refer the reader to [5].

Proof of Theorem 1. Because of the local property of the assertion, we may assume that Ω is bounded and that $a \in C^{1,1}(\bar{\Omega})$, $b \in C^{0,1}(\bar{\Omega})$, $c, f \in C(\bar{\Omega})$, $\sigma \in C^{0,1}(\bar{\Omega})$ and $u \in C(\bar{\Omega})$. This guarantees that $F_{\mathcal{L}}$ satisfies (A1) and (A2).

Now, fix $\varphi \in \mathcal{D}_+(\Omega)$. Choose $\varepsilon > 0$ so that $\operatorname{supp} \varphi \subset \Omega_\varepsilon$. By virtue of Theorem 3, there is $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then

(2.1) $$F_{\mathcal{L}}(x, u^\delta, Du^\delta, D^2u^\delta) \le \varepsilon \text{ in } \Omega_\varepsilon \text{ in the viscosity sense.}$$

Fix $\delta \in (0, \delta_0)$. By Lemma 1 we find $U_\delta = (u^\delta_{ij})$ with $u^\delta_{ij} \in L^1_{loc}(\boldsymbol{R}^N)$ and $V_\delta = (v^\delta_{ij})$ with $v^\delta_{ij} \in \mathcal{M}(\boldsymbol{R}^N)$ such that

$$D^2 u^\delta = U_\delta + V_\delta \text{ in } \mathcal{D}'(\boldsymbol{R}^N), \quad V_\delta \ge 0 \text{ in } \mathcal{M}(\boldsymbol{R}^N),$$

$$(Du^\delta(x), U_\delta(x)) \in J^2 u^\delta(x) \text{ a.e.}$$

The last inclusion and (2.1) yield

$$F_{\mathcal{L}}(x, u^\delta(x), Du^\delta(x), U_\delta(x)) \le \varepsilon \qquad \text{a.e. in } \Omega_\varepsilon,$$

and multiplying this by φ and integrating over Ω yield

(2.2) $$\int_\Omega (F_{\mathcal{L}}(x, u^\delta(x), Du^\delta(x), U_\delta(x)) - \varepsilon)\varphi(x)dx \le 0.$$

Now we observe that

$$\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}\varphi dv_{ij}^{\delta}(x) = \sum_{i,j,k=1}^{N} \int_{\Omega} (\sigma_{ik}\varphi^{1/2})(\sigma_{jk}\varphi^{1/2}) dv_{ij}^{\delta}(x) \geq 0,$$

and that if we identify $\mathscr{M}(\boldsymbol{R}^N)$ with $C_0(\boldsymbol{R}^N)' \subset \mathscr{D}'(\boldsymbol{R}^N)$, then

$$\begin{aligned}\sum_{i,j=1}^{N} &\left\{\int_{\Omega} a_{ij}\varphi dv_{ij}^{\delta}(x) + \int_{\Omega} a_{ij}\varphi u_{ij}^{\delta} dx\right\} \\ &= \sum_{i,j=1}^{N} \langle u_{ij}^{\delta} + v_{ij}^{\delta}, a_{ij}\varphi\rangle = \sum_{i,j=1}^{N} \langle u_{x_i x_j}^{\delta}, a_{ij}\varphi\rangle \\ &= \sum_{i,j=1}^{N} \langle u^{\delta}, (a_{ij}\varphi)_{x_i x_j}\rangle = \sum_{i,j=1}^{N} \int_{\Omega} u^{\delta}(a_{ij}\varphi)_{x_i x_j} dx.\end{aligned}$$

Here $\langle g, \psi\rangle$ denotes the duality pairing between $g \in \mathscr{D}'(\boldsymbol{R}^N)$ and $\psi \in C_0^{\infty}(\boldsymbol{R}^N)$ and we may assume by approximation that the a_{ij} are C^{∞}. Combining these, we have

$$\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}\varphi u_{ij}^{\delta} dx \leq \sum_{i,j=1}^{N} \int_{\Omega} u^{\delta}(a_{ij}\varphi)_{x_i x_j} dx.$$

Therefore, from (2.2) we obtain

$$\int_{\Omega} (u^{\delta}\mathscr{L}^*\varphi - f\varphi - \varepsilon\varphi) dx \leq 0.$$

Noting that $u^{\delta}(x) \to u(x)$ uniformly in Ω as $\varepsilon \downarrow 0$ and passing to the limit as $\varepsilon \downarrow 0$, we conclude that

$$\int_{\Omega} (u\mathscr{L}^*\varphi - f\varphi) dx \leq 0.$$

This completes the proof. ■

§3 Solvability of (1.1)

In this section we treat the case when $\Omega = \boldsymbol{R}^N$, and consider the solvability of (1.1). The results here are more or less known.

Concernng the regularity of a we do not assume that $a \in W^{2,\infty}(\boldsymbol{R}^N)$ except in the assertion (ii) of Theorem 5, and instead we only assume that $\sigma \in W^{1,\infty}(\boldsymbol{R}^N)$.

We define

$$c_0 = \inf_{\boldsymbol{R}^N} c, \ \lambda_0 = \sup_{x \neq y} \left\{ \frac{\operatorname{tr}(\sigma(x) - \sigma(y))^2 - \langle b(x) - b(y), x - y \rangle}{|x - y|^2} \right\}.$$

We note that λ_0 may be negative.

Theorem 4 *Assume that* $c_0 > 0$ *and* $c, f \in BUC(\boldsymbol{R}^N)$. *Then there is a unique viscosity solution* $u \in BUC(\boldsymbol{R}^N)$ *of* (1.1) *and moreover,*

$$\|u\|_{L^\infty} \leq \frac{1}{c_0} \|f\|_{L^\infty}. \tag{3.1}$$

Theorem 5 *Assume that* $c_0 \geq 0$, *and let* $u \in BUC(\boldsymbol{R}^N)$ *be a viscosity solution of* (1.1). *Then*: (i) *if* $c_0 > \lambda_0$ *and* $c, f \in W^{1,\infty}(\boldsymbol{R}^N)$, *then* $u \in W^{1,\infty}(\boldsymbol{R}^N)$ *and*

$$\|Du\|_{L^\infty} \leq \frac{1}{c_0 - \lambda_0} (\|Df\|_{L^\infty} + \|Dc\|_{L^\infty} \|u\|_{L^\infty}). \tag{3.2}$$

(ii) *if* $c_0 > \lambda_1 \equiv \max\{\lambda_0, 2\lambda_0\}$ *and* $\sigma, b, c, f \in W^{2,\infty}(\boldsymbol{R}^N)$, *then* $u \in W^{2,\infty}(\boldsymbol{R}^N)$ *and*

$$\|D^2 u\|_{L^\infty} \leq C(\|D^2 \sigma\|_{L^\infty} + 1), \tag{3.3}$$

where

$$C = M(\lambda_1, 1/(c_0 - \lambda_1), \|D\sigma\|_{L^\infty}, \|D^2 b\|_{L^\infty}, \|Df\|_{W^{1,\infty}}, \|c\|_{W^{2,\infty}}, \|u\|_{W^{1,\infty}})$$

for some continuous function M *on* $\boldsymbol{R}^7$.

Theorems 4 and 5 have been proved in [6], [7], [8], [3] and [4]. See also [9]. The condition that $c_0 > \lambda_1$ in the assertion (ii) of Theorem 5 is slightly sharper than that used in [9]. Theorem 4 and the assertion (i) of Theorem 5 are valid for Hamilton-Jacobi-Bellman-Isaacs equations under similar assumptions. Half of the assertion (ii) of Theorem 5, the estimate on solutions u

$$\langle D^2 u \xi, \xi \rangle \leq C(\|D^2 \sigma\|_{L^\infty} + 1) \qquad \forall \xi \in \boldsymbol{R}^N \text{ with } |\xi| \leq 1$$

(in the viscosity sense or equivalenltly in the distribution sense) is valid for Hamilton-Jacobi-Bellman equation under similar assumtions. This assertion requires convexity of equations. Indeed, [6], [7] and [8] treat Hamilton-Jacobi-Bellman equations and techniques there are largely based on stochastic optimal control theory, and [3] treat Hamilton-Jacobi-Bellman-Isaacs equations.

The proof of these theorems will be postponed until Section 5.

§4 Proof of Theorem 2

We may assume that $c = 0$; otherwise we regard the original $f - cu$ as f in (1.1). Let $u \in C(\Omega)$ satisfy

$$\mathscr{L}u \le f \qquad \text{in } \mathscr{D}'(\Omega).$$

Suppose that u does not satisfy

$$\mathscr{L}u \le f \qquad \text{in } \Omega \text{ in the viscosity sense.}$$

We shall show that this yields a contradiction.

By this supposition we find $z \in \Omega$, $r > 0$ and $\varphi \in C^2(\Omega)$ such that

$$\begin{cases} \mathscr{L}\varphi(x) \ge f(x) + 2r & \forall x \in B(z, r), \\ u(z) = \varphi(z), & \\ u(x) \le \varphi(x) - |x - z|^4 & \forall x \in B(z, r). \end{cases}$$

Of course, we assume here that $B(z, r) \subset \Omega$. Set $U = B(z, r)^\circ$. By continuity, there is $\delta > 0$ such that for any $\varepsilon \in [0, \delta]$, if we define $\varphi_\varepsilon \in C^2(U)$ by $\varphi_\varepsilon(x) = \varphi(x) - \varepsilon$, then $\mathscr{L}\varphi_\varepsilon(x) \ge f(x) + r$ for $\forall x \in U$.

We assume that $\delta^{1/4} < r$, so that $B(z, \delta^{1/4}) \subset U$. Let $0 < \varepsilon \le \delta$, and we set $w_\varepsilon(x) = u(x) - \varphi_\varepsilon(x)$ for $x \in \bar{U}$. Then $w_\varepsilon \in C(\bar{U})$, $\max_{\bar{U}} w_\varepsilon = \varepsilon$, $w_\varepsilon(x) \le 0$ for $\forall x \in \bar{U} \setminus B(z, \varepsilon^{1/4})$ and $\mathscr{L}w_\varepsilon \le -r$ in $\mathscr{D}'(U)$.

Fix $\zeta \in C_0^\infty(U)$ so that $0 \le \zeta \le 1$ in U and $\zeta(x) = 1$ for $\forall x \in B(x, \varepsilon^{1/4})$. Define the operator $\mathscr{L}_\zeta$ by

$$\mathscr{L}_\zeta \psi = \zeta^2 \mathscr{L}\psi = -\operatorname{tr}(\zeta^2 a D^2 \psi) + \langle \zeta^2 b, D\psi \rangle.$$

Then,

$$\mathscr{L}_\zeta w_\varepsilon \le -r\zeta^2 \qquad \text{in } \mathscr{D}'(U).$$

Let $\lambda > 0$ be a constant to be fixed later on. We let $\varepsilon = \{\delta, r/\lambda\}$, so that $\lambda w_\varepsilon \le r\zeta^2$ in U and moreover,

$$\lambda w_\varepsilon + \mathscr{L}_\zeta w_\varepsilon \le 0 \qquad \text{in } \mathscr{D}'(U).$$

Thus

$$\langle w_\varepsilon, \lambda v + \mathscr{L}_\zeta^* v \rangle \le 0 \;\; \forall v \in W^{2,\infty}(U) \text{ with } v \ge 0. \tag{4.1}$$

We put

$$\tilde{\sigma}_{ij} = \zeta \sigma_{ij}, \quad \tilde{b}_i = [\zeta^2 b_i + \sum_{j=1}^N (\zeta^2 a_{ij})_{x_j}],$$

$$\tilde{c} = -\sum_{i,j=1}^{N} (\zeta^2 a_{ij})_{x_i x_j} - \sum_{i=1}^{N} (\zeta^2 b_i)_{x_i}.$$

We extend these functions to $\boldsymbol{R}^N$ by assuming their values to be zero outside of U, and set $\tilde{\sigma} = (\tilde{\sigma}_{ij})_{1 \leq i,j \leq N}$, $\tilde{a} = (\tilde{\sigma})^2$ and $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_N)$. Now we may regard $\mathscr{L}_\zeta^*$ as an operator defined for functions on $\boldsymbol{R}^N$, i.e.,

$$\mathscr{L}_\zeta^* \psi = -\operatorname{tr}(\tilde{a} D^2 \psi) + \langle \tilde{b}, D\psi \rangle + \tilde{c}\psi \qquad \text{for } \psi \in C^2(\boldsymbol{R}^N).$$

Note that $\tilde{\sigma}_{ij} \in C^1(\boldsymbol{R}^N)$, $\tilde{b}_i \in W^{1,\infty}(\boldsymbol{R}^N)$ and $\tilde{c} \in L^\infty(\boldsymbol{R}^N)$. By using standard mollification techniques, we find C_0^∞ functions σ_{ij}^δ, b_i^δ, c^δ, with $\delta \in (0, 1)$ and $1 \leq i, j \leq N$, such that

$$\|\sigma_{ij}^\delta\|_{W^{1,\infty}} \leq \|\tilde{\sigma}_{ij}\|_{W^{1,\infty}}, \ \|D^2 \sigma_{ij}^\delta\|_{L^\infty} \leq \frac{1}{\delta} \|D\tilde{\sigma}_{ij}\|_{L^\infty},$$

$$\|b_i^\delta\|_{W^{1,\infty}} \leq \|\tilde{b}_i\|_{W^{1,\infty}}, \ \|c^\delta\|_{L^\infty} \leq \|\tilde{c}\|_{L^\infty},$$

and as $\delta \downarrow 0$,

$$\text{(4.2)} \qquad \begin{cases} \|\sigma_{ij}^\delta - \tilde{\sigma}_{ij}\|_{L^1} = o(\delta), \\ \|b_i^\delta - \tilde{b}_i\|_{L^1} \longrightarrow 0, \ \|c^\delta - \tilde{c}\|_{L^1} \longrightarrow 0. \end{cases}$$

We may moreover assume that the σ_{ij}^δ, b_i^δ and c^δ vanish outside of a compact subset of U.

In view of Theorems 4 and 5 we set

$$\lambda_0 = \sup \left\{ \frac{\operatorname{tr}(\sigma^\alpha(x) - \sigma^\alpha(y))^2 - \langle b^\beta(x) - b^\beta(y), x - y \rangle}{|x - y|^2} \,\middle|\, x \neq y, \alpha, \beta \in (0, 1) \right\},$$

$$c_0 = \inf \{ c^\gamma(x) \mid x \in \boldsymbol{R}^N, \ 0 < \gamma < 1 \},$$

and fix $\lambda > 0$ so that $\lambda > c_0 + 2 \max\{\lambda_0, 0\}$. Fix $\psi \in C_0^\infty(\boldsymbol{R}^N)$ so that $\operatorname{supp} \psi \subset U$. Theorems 4 and 5 guarantee that for each $\alpha, \beta, \gamma \in (0, 1)$ there is a unique viscosity solution $v = v^{\alpha\beta\gamma} \in BUC(\boldsymbol{R}^N)$ of

$$\lambda v + \mathscr{L}^{\alpha\beta\gamma} v = \psi \qquad \text{in } \boldsymbol{R}^N,$$

where

$$\mathscr{L}^{\alpha\beta\gamma} v(x) = -\operatorname{tr} a^\alpha(x) D^2 v(x) + \langle b^\beta(x), Dv(x) \rangle + c^\gamma(x) v(x).$$

Moreover, for any $\alpha, \beta, \gamma \in (0, 1)$ we have $v^{\alpha\beta\gamma} \in W^{2,\infty}(\boldsymbol{R}^N)$, and

$$\text{(4.3)} \qquad \begin{cases} \|D^2 v^{\alpha\beta\gamma}\|_0 \leq \dfrac{1}{\alpha} C_1(\beta, \gamma), \\ \|D v^{\alpha\beta\gamma}\|_0 \leq C_2(\gamma), \\ \|v^{\alpha\beta\gamma}\|_0 \leq C_3, \end{cases}$$

where $C_1(\beta, \gamma)$, $C_2(\gamma)$ and C_3 are constants independent, respectively, of α, of α and β and of α, β and γ. Since the a^α, b^β and c^γ vanish outside of a compact subset of U, so does the $v^{\alpha\beta\gamma}$, i.e., $v^{\alpha\beta\gamma} \in C_0(U)$. Also, by the maximum principle, $v^{\alpha\beta\gamma} \geq 0$ on $\boldsymbol{R}^N$ for all α, β, $\gamma \in (0, 1)$. Therefore, going back to (4.1), we obtain

$$\begin{aligned}\langle w_\varepsilon, \psi \rangle &= \langle w_\varepsilon, \lambda v^{\alpha\beta\gamma} + \mathscr{L}^{\alpha\beta\gamma} v^{\alpha\beta\gamma} \rangle \\ &= \langle w_\varepsilon, \lambda v^{\alpha\beta\gamma} + \mathscr{L}_\zeta^* v^{\alpha\beta\gamma} \rangle + \langle w_\varepsilon, \mathscr{L}^{\alpha\beta\gamma} v^{\alpha\beta\gamma} - \mathscr{L}_\zeta^* v^{\alpha\beta\gamma} \rangle \\ &\leq \| w_\varepsilon \|_0 \{ \| D^2 v^{\alpha\beta\gamma} \|_0 (\| \sigma^\alpha \|_0 + \| \tilde{\sigma} \|_0) \| \sigma^\alpha - \tilde{\sigma} \|_{L^1} \\ &\quad + \| D v^{\alpha\beta\gamma} \|_0 \| \tilde{b} - b^\beta \|_{L_1} + \| v^{\alpha\beta\gamma} \|_0 \| \tilde{c} - c^\gamma \|_{L^1} \}.\end{aligned}$$

In view of (4.2) and (4.3), sending $\alpha \downarrow 0$, $\beta \downarrow 0$ and $\gamma \downarrow 0$ in this order, we see that $\langle w_\varepsilon, \psi \rangle \leq 0$ and hence $w_\varepsilon \leq 0$ on U. This is a contradiction, which completes the proof. ■

§5 Proof of Theorems 4 and 5

In the spirit of being free from probabilistic techniques, it may be important to prove Theorems 4 and 5 without using results based on probalistic techniques.

It is well known (see, e.g., [8] and [3]) that Theorem 4 is valid. However we give a proof for the reader's convenience.

In what follows we use the notation: For a function $u = (u_{ij})\colon \boldsymbol{R}^N \to \boldsymbol{R}^{m \times n}$ we write

$$\| u \|_0 = \| (\sum_{i=1}^m \sum_{j=1}^n | u_{ij} |^2)^{1/2} \|_{L^\infty}, \quad \| u \|_1 = \| (\sum_{k=1}^N \sum_{i=1}^m \sum_{j=1}^n | u_{ijx_k} |^2)^{1/2} \|_{L^\infty},$$

$$\| u \|_2 = \| (\sum_{k,l=1}^N \sum_{i=1}^m \sum_{j=1}^n | u_{ijx_k x_l} |^2)^{1/2} \|_{L^\infty}.$$

In particular, we have

$$\| u \|_{W^{1,\infty}} = \| u \|_0 + \| u \|_1 \quad \text{and} \quad \| u \|_{W^{2,\infty}} = \| u \|_0 + \| u \|_1 + \| u \|_2.$$

Proof of Theorem 4. Since $c_0 > 0$, the constants $\| f \|_0 / c_0$ and $- \| f \|_0 / c_0$ are a supersolution and a subsolution of (1.1), respectively. By the Perron method, we find a viscosity solution u of (1.1) with

$$-\frac{1}{c_0} \| f \|_0 \leq u \leq \frac{1}{c_0} \| f \|_0 \qquad \text{on } \boldsymbol{R}^N.$$

The fact that $u \in UC(\boldsymbol{R}^N)$ follows from the comparison result for viscosity solutions (see for instance [2] and [3]). ■

Proof of Theorem 5. Assume that $c_0 > \lambda_0$. Let $u \in BUC(\mathbf{R}^N)$ be a viscosity solution of (1.1). Let $\varepsilon > 0$, $\delta > 0$ and

$$L > \frac{1}{c_0 - \lambda_0}(\|c\|_1 \|u\|_0 + \|f\|_1), \tag{5.1}$$

and set

$$\Phi(x, y) = u(x) - u(y) - L|x - y| - \delta|x|^2 - \varepsilon \qquad \text{for } x, y \in \mathbf{R}^N.$$

We will show tht $\Phi \leq 0$ on $\mathbf{R}^N$ for all $\varepsilon, \delta > 0$. To this end, suppose that $\sup_{\mathbf{R}^{2N}} \Phi > 0$ for some $\varepsilon > 0$ and $\delta = \delta_0 > 0$. This will lead a contradiction. Fix $\varepsilon > 0$ and $\delta_0 > 0$ so that $\sup_{\mathbf{R}^{2N}} \Phi > 0$ with this $\varepsilon > 0$ and $\delta = \delta_0$, and $0 < \delta \leq \delta_0$. Note that $\sup_{\mathbf{R}^{2N}} \Phi > 0$. Let $(\hat{x}, \hat{y}) \in \mathbf{R}^N \times \mathbf{R}^N$ be a maximum point of Φ. Writing

$$\psi(x) = |x| \quad \text{and} \quad \varphi(x, y) = L|x - y| \text{ for } x, y \in \mathbf{R}^N,$$

and noting that

$$D\psi(x) = \frac{x}{|x|}, \quad D^2\psi(x) = \frac{I}{|x|} - \frac{x \otimes x}{|x|^3} \leq \frac{I}{|x|},$$

and

$$D^2\varphi(x, y) \leq L\begin{pmatrix} D^2\psi(x-y) & -D^2\psi(x-y) \\ -D^2\psi(x-y) & D^2\psi(x-y) \end{pmatrix} \leq \frac{L}{|x-y|}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

we see by the maximum principle (see [2]) that for each $\theta > 1$ there are $X, Y \in \mathbf{S}^N$ such that

$$\begin{cases} (\hat{p}, X) \in \bar{J}^{2,+}u(\hat{x}) - 2\delta(\hat{x}, I), \quad (\hat{p}, -Y) \in \bar{J}^{2,-}u(\hat{y}), \\ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \dfrac{L\theta}{|\hat{x} - \hat{y}|}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{cases} \tag{5.2}$$

where $\hat{p} = L(\hat{x} - \hat{y})/|\hat{x} - \hat{y}|$. Therefore we have

$$-\operatorname{tr} a(\hat{x})X + \langle b(\hat{x}), \hat{p}\rangle + c(\hat{x})u(\hat{x}) \leq f(\hat{x}) - 2\delta\langle b(\hat{x}), \hat{x}\rangle + 2\delta \operatorname{tr} a(\hat{x})$$

and

$$-\operatorname{tr} a(\hat{y})(-Y) + \langle b(\hat{y}), \hat{p}\rangle + c(\hat{y})u(\hat{y}) \geq f(\hat{y}).$$

Hence

$$\begin{aligned} &c(\hat{x})(u(\hat{x}) - u(\hat{y})) - \operatorname{tr}(a(\hat{x})X + a(\hat{y})Y) \\ &\quad + \langle b(\hat{x}) - b(\hat{y}), \hat{p}\rangle \leq (c(\hat{y}) - c(\hat{x}))u(\hat{y}) \end{aligned}$$

$$+f(\hat{x})-f(\hat{y})+2\delta(\operatorname{tr} a(\hat{x})-\langle b(\hat{x}), \hat{x}\rangle)$$
$$\leq (\|c\|_1\|u\|_0+\|f\|_1)|\hat{x}-\hat{y}|+2\delta(\operatorname{tr} a(\hat{x})-\langle b(\hat{x}), \hat{x}\rangle).$$

The latter of (5.2) yields

$$\operatorname{tr}(a(\hat{x})X+a(\hat{y})Y)=\operatorname{tr}\left\{(\sigma(\hat{x})\sigma(\hat{y}))\begin{pmatrix} X & 0 \\ 0 & Y\end{pmatrix}\begin{pmatrix}\sigma(\hat{x}) \\ \sigma(\hat{y})\end{pmatrix}\right\}$$
$$\leq \frac{L\theta}{|\hat{x}-\hat{y}|}\operatorname{tr}\left\{(\sigma(\hat{x})\sigma(\hat{y}))\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}\begin{pmatrix}\sigma(\hat{x}) \\ \sigma(\hat{y})\end{pmatrix}\right\}$$
$$\leq \frac{L\theta}{|\hat{x}-\hat{y}|}\operatorname{tr}(\sigma(\hat{x})-\sigma(\hat{y}))^2.$$

Thus, recalling that $\Phi(\hat{x}, \hat{y})>0$, we have

$$c_0 L|\hat{x}-\hat{y}| \leq L|\hat{x}-\hat{y}|\frac{\operatorname{tr}(\sigma(\hat{x})-\sigma(\hat{y}))^2-\langle b(\hat{x})-b(\hat{y}), \hat{x}-\hat{y}\rangle}{|\hat{x}-\hat{y}|^2}$$
$$+L(\theta-1)\frac{\operatorname{tr}(\sigma(\hat{x})-\sigma(\hat{y}))^2}{|\hat{x}-\hat{y}|^2}+(\|c\|_1\|u\|_0+\|f\|_1)|\hat{x}-\hat{y}|$$
$$+2\delta(\|\operatorname{tr} a\|_0+\|b\|_0|\hat{x}|).$$

Since $\theta>1$ is arbitrary, sending $\theta\downarrow 1$, we obtain

$$(c_0-\lambda_0)L|\hat{x}-\hat{y}| \leq (\|c\|_1\|u\|_0+\|f\|_1)|\hat{x}-\hat{y}|+2\delta(\|\operatorname{tr}\sigma\|_0+\|b.\|_0|\hat{x}|).$$

Since $\Phi(\hat{x}, \hat{y})>0$ and $u\in BUC(\boldsymbol{R}^N)$, it follows tht $\delta|\hat{x}|^2\leq 2\|u\|_0$ and also that $\gamma\leq|\hat{x}-\hat{y}|\leq\gamma^{-1}$ for some constant $\gamma>0$ independnet of $\delta>0$. Therefore, passing to the limit as $\delta\downarrow 0$, we see that

$$(c_0-\lambda_0)Lr\leq(\|c\|_1\|u\|_0+\|f\|_1)r$$

for some $r\geq\gamma$, and hence

$$L\leq\frac{1}{c_0-\lambda_0}(\|c\|_1\|u\|_0+\|f\|_1).$$

This contradicts our choice (5.1) of L. Thus we know that $\Phi(x, y)\leq 0$ for all $x, y\in\boldsymbol{R}^N$ and $\varepsilon, \delta>0$, which implies

$$u(x)-u(y)\leq\frac{\|c\|_1\|u\|_0+\|f\|_1}{c_0-\lambda_0}|x-y| \qquad \forall x, y\in\boldsymbol{R}^N,$$

and thus proves the assertion (i).

Next we prove (ii). We begin with preliminary calculations. Let $L>0$, and set

$$\varphi(x, y, z) = L|x-y|^2 + (|x-y|^4 + |x+y-2z|^2)^{1/2}$$
$$\equiv L|x-y|^2 + \varphi_1(x, y, z)$$

for $x, y, z \in \boldsymbol{R}^N$. Let $(x, y, z) \in \boldsymbol{R}^{3N}$ be an arbitrary point with $\varphi_1(x, y, z) \neq 0$. We then have:

(5.3)
$$D\varphi = 2L\begin{pmatrix} x-y \\ y-x \\ 0 \end{pmatrix} + \frac{1}{\varphi_1}\left\{ 2|x-y|^2 \begin{pmatrix} x-y \\ y-x \\ 0 \end{pmatrix} + \begin{pmatrix} x+y-2z \\ x+y-2z \\ -2x-2y+4z \end{pmatrix}\right\}$$

and

(5.4)
$$D^2\varphi = 2L\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1}\left\{ 2|x-y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix}\right.$$
$$\left. + 4\begin{pmatrix} x-y \\ y-x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x-y \\ y-x \\ 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}\right\} - \frac{1}{\varphi_1^2} D\varphi_1 \otimes D\varphi_1$$
$$\leq 2L\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1}\left\{ 2|x-y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix}\right.$$
$$\left. + 4|x-y|^2 \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}\right\}$$
$$\leq \left(2L + \frac{6|x-y|^2}{\varphi_1}\right)\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1}\begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}.$$

Here and later φ_1 denotes its value evaluated at (x, y, z). Now, setting

$$J = \operatorname{tr}(\sigma(x) + \sigma(y) - 2\sigma(z))^2 - \langle b(x) + b(y) - 2b(z), x + y - 2z \rangle,$$
$$\xi = \sigma(x) + \sigma(y) - 2\sigma\left(\frac{x+y}{2}\right), \quad \eta = 2\left[\sigma\left(\frac{x+y}{2}\right) - \sigma(z)\right],$$
$$\alpha = b(x) + b(y) - 2b\left(\frac{x+y}{2}\right), \quad \beta = 2\left[b\left(\frac{x+y}{2}\right) - b(x)\right],$$

and noting that for any $g \in C_b^2(\boldsymbol{R}^N)$,

$$g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) \leq \|D^2 g\|_0 \left|\frac{x+y}{2}\right|^2 \leq \|D^2 g\|_0 |x-y|^2,$$

we calculate that

$$\operatorname{tr} \xi^2 \leq \|\sigma\|_2^2 |x-y|^4, \ \operatorname{tr} \eta^2 \leq \|\sigma\|_1^2 |x+y-2z|^2,$$
$$|\alpha| \leq \|b\|_2 |x-y|^2, \ |\beta| \leq \|b\|_1 |x+y-2z|,$$

and that

$$\begin{aligned}
J &\leq 4\lambda_0 \left|\frac{x+y}{2} - 2z\right|^2 + \operatorname{tr} \xi^2 + 2 \operatorname{tr} \xi\eta - \langle \alpha, x+y-2z \rangle \\
&\leq \lambda_1 |x+y-2z|^2 + \|\sigma\|_2^2 |x-y|^4 + 2\|\sigma\|_1 \|\sigma\|_2 |x-y|^2 |x+y-2z| \\
&\quad + \|b\|_2 |x-y|^2 |x+y-2z| \\
&\leq \left(\lambda_1 + \frac{c_0 - \lambda_1}{2}\right) |x+y-2z|^2 \\
&\quad + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1}(\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2\right] |x-y|^4 \\
&= \left(\frac{c_0 + \lambda_1}{2}\right) |x+y-2z|^2 + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1}(\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2\right] |x-y|^4.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\left(2L + \frac{6|x-y|^2}{\varphi_1}\right) \operatorname{tr} (\sigma(x) - \sigma(y))^2 \\
&\quad - 2\left(L + \frac{|x-y|^2}{\varphi_1}\right) \langle b(x) - b(y), x-y \rangle + \frac{J}{\varphi_1} \\
&\leq 2\left(L + \frac{|x-y|^2}{\varphi_1}\right) \lambda_0 |x-y|^2 + 4\|\sigma\|_1^2 \frac{|x-y|^4}{\varphi_1} + \frac{c_0 + \lambda_1}{2\varphi_1} |x+y-2z|^2 \\
&\quad + \left[\|\sigma\|_2^2 + \frac{2}{c_0 - \lambda_1}(\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2\right] \frac{|x-y|^4}{\varphi_1} \\
&\leq \left[\lambda_1 L + \lambda_1^+ + 4\|\sigma\|_1^2 + \frac{2}{c_0 - \lambda_1}(\|b\|_2 + \|\sigma\|_1 \|\sigma\|_2)^2\right] |x-y|^2 \\
&\quad + \frac{c_0 + \lambda_1}{2} |x+y-2z|,
\end{aligned}$$

where $\lambda_1^+ = \max\{\lambda_1, 0\}$. We now choose $L > 0$ so that

$$\frac{c_0+\lambda_1}{2}L \geq \lambda_1 L + \lambda_1^+ + 4\|\sigma\|_1^2 + \frac{2}{c_0-\lambda_1}(\|b\|_2 + \|\sigma\|_1\|\sigma\|_2)^2,$$

i.e.,

$$L \geq \frac{2}{c_0-\lambda_1}\left[\lambda_1^+ + 4\|\sigma\|_1^2 + \frac{2}{c_0-\lambda_1}(\|b\|_2 + \|\sigma\|_1\|\sigma\|_2)^2\right].$$

Then we have

(5.5)
$$\left(2L + \frac{6|x-y|^2}{\varphi_1}\right)\operatorname{tr}(\sigma(x)-\sigma(y))^2 - 2\left(L + \frac{|x-y|^2}{\varphi_1}\right)\langle b(x)-b(y), x-y\rangle$$
$$+ \frac{J}{\varphi_1} \leq \frac{c_0+\lambda_1}{2}(L|x-y|^2 + |x+y-2z|) \leq \frac{c_0+\lambda_1}{2}\varphi(x, y, z)$$

for all $x, y, z \in \boldsymbol{R}^N$.

Now, we observe that

$$(5.6)\qquad f(x)+f(y)-2f(z) = f(x)+f(y)-2f\left(\frac{x+y}{2}\right)$$
$$+ 2\left(f\left(\frac{x+y}{2}\right)-f(z)\right) \leq \|f\|_2|x-y|^2 + \|f\|_1|x+y-2z|$$
$$\leq \|Df\|_{W^{1,\infty}}\varphi_1(x, y, z).$$

Noting that for any $g \in C_b^1(\boldsymbol{R}^N)$,

$$|g(x)-g(z)| \leq \left|g(x)-g\left(\frac{x+y}{2}\right)\right| + \left|g\left(\frac{x+y}{2}\right)-g(z)\right|$$
$$\leq \|g\|_1\frac{|x-y|}{2} + \left(2\|g\|_0\|g\|_1\frac{|x+y-2z|}{2}\right)^{1/2}$$
$$\leq \|g\|_{W^{1,\infty}}\varphi_1(x, y, z)^{1/2},$$

we see that

$$|(c(x)-c(z))(u(x)-u(z)) + (c(y)-c(z))(u(y)-u(z))|$$
$$\leq (\|c\|_0 + \|c\|_1)(\|u\|_0 + \|u\|_1)\varphi_1(x, y, z) \leq \|c\|_{W^{1,\infty}}\|u\|_{W^{1,\infty}}\varphi_1(x, y, z)$$

and hence

$$(5.7)\qquad c(x)u(x) + c(y)u(y) - 2c(z)u(z)$$
$$\geq c(z)(u(x)+u(y)-2u(z)) + (c(x)+c(y)-2c(z))u(z)$$

$$+ (c(x) - c(z))(u(x) - u(z)) + (c(y) - c(z))(u(y) - u(z))$$
$$\geq c(z)(u(x) + u(y) - 2u(z)) - \|u\|_0 \|Dc\|_{W^{1,\infty}} \varphi_1(x, y, z)$$
$$- \|c\|_{W^{1,\infty}} \|u\|_{W^{1,\infty}} \varphi_1(x, y, z)$$
$$\geq c(z)(u(x) + u(y) - 2u(z)) - 2\|c\|_{W^{2,\infty}} \|u\|_{W^{1,\infty}} \varphi_1(x, y, z).$$

Now we are ready to go into the proof. We shall show that

$$u(x) + u(y) - 2u(z) \leq \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}) \varphi(x, y, z) \tag{5.8}$$

for all $x, y, z \in \boldsymbol{R}^N$. By linearity, we then have

$$|u(x) + u(y) - 2u(z)| \leq \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}) \varphi(x, y, z)$$

for all $x, y, z \in \boldsymbol{R}^N$, from which follows the assertion (ii) of Theorem 5.

Fix any

$$M > \frac{2}{c_0 - \lambda_1} (\|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}}).$$

For $\varepsilon > 0$ and $\delta > 0$ we set

$$\Phi(x, y, z) = u(x) + u(y) - 2u(z) - M\varphi(x, y, z) - \delta|x|^2 - \varepsilon \qquad \text{for } x, y, z \in \boldsymbol{R}^N.$$

We shall show that $\Phi \leq 0$ on $\boldsymbol{R}^{3N}$ for all $\varepsilon, \delta > 0$. To this end, suppose that $\sup \Phi > 0$ for some $\varepsilon > 0$ and $\delta = \delta_0 > 0$. Fix such $\varepsilon > 0$ and $\delta_0 > 0$, and fix $0 < \delta \leq \delta_0$, so that $\sup \Phi > 0$.

Let $(\hat{x}, \hat{y}, \hat{z}) \in \boldsymbol{R}^{3N}$ be a maximum point of Φ. Set $w(x, y, z) = u(x) - \delta|x|^2 + u(y) - 2u(z)$. Observe that $\varphi_1(\hat{x}, \hat{y}, \hat{z}) \neq 0$. We have

$$M(D\varphi(\hat{x}, \hat{y}, \hat{z}), D^2\varphi(\hat{x}, \hat{y}, \hat{z})) \in J^{2,+} w(\hat{x}, \hat{y}, \hat{z}).$$

By (5.3) and (5.4), we see that if we set

$$p = 2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right) \begin{pmatrix} \hat{x} - \hat{y} \\ \hat{y} - \hat{x} \\ 0 \end{pmatrix} + \frac{M}{\varphi_1} \begin{pmatrix} \hat{x} + \hat{y} - 2\hat{z} \\ \hat{x} + \hat{y} - 2\hat{z} \\ -2\hat{x} - 2\hat{y} + 4\hat{z} \end{pmatrix},$$

and

$$A = 2M\left(L + 3\frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{M}{\varphi_1} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix},$$

then

$$(p, A) \in J^{2,+} w(\hat{x}, \hat{y}, \hat{z}).$$

Here and hereafter φ_1 also denotes its value at $(\hat{x}, \hat{y}, \hat{z})$. Let $\theta > 1$. By the maximum principle for semicontinuous functions, there are $X, Y, Z \in \boldsymbol{S}^N$ such that

$$\left(2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right)(\hat{x} - \hat{y}) + \frac{M}{\varphi_1}(\hat{x} + \hat{y} - 2\hat{z}),\ X\right) \in \bar{J}^{2,+} u(\hat{x}) - 2\delta(\hat{x}, I),$$

$$\left(2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right)(\hat{y} - \hat{x}) + \frac{M}{\varphi_1}(\hat{x} + \hat{y} - 2\hat{z}),\ Y\right) \in \bar{J}^{2,+} u(\hat{y}),$$

$$+\left(\frac{M}{\varphi_1}(-2\hat{x} - 2\hat{y} + 4\hat{z}),\ Z\right) \in -2\bar{J}^{2,-} u(\hat{z}),$$

$$\begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \le \theta M \left\{ 2\left(L + 3\frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\varphi_1} \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right\} \tag{5.9}$$

From the first three we see that

$$-\operatorname{tr} a(\hat{x})(X + I) + \left\langle b(\hat{x}),\ 2\delta\hat{x} + 2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right)(\hat{x} - \hat{y}) + \frac{M}{\varphi_1}(\hat{x} + \hat{y} - 2\hat{z})\right\rangle$$
$$+ c(\hat{x})u(\hat{x}) \le f(\hat{x}),$$

$$-\operatorname{tr} a(\hat{y})Y + \left\langle b(\hat{y}),\ 2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right)(\hat{y} - \hat{x}) + \frac{M}{\varphi_1}(\hat{x} + \hat{y} - 2\hat{z})\right\rangle$$
$$+ c(\hat{y})u(\hat{y}) \le f(\hat{y}),$$

$$-\operatorname{tr} a(\hat{z})\left(-\frac{1}{2}Z\right) + \left\langle b(\hat{z}),\ \frac{M}{\varphi_1}(\hat{x} + \hat{y} - 2\hat{z})\right\rangle + c(\hat{z})u(\hat{z}) \ge f(\hat{z}).$$

From these we have

$$-\operatorname{tr}(a(\hat{x})X + a(\hat{y})Y + a(\hat{z})Z) + 2M\left(L + \frac{|\hat{x} - \hat{y}|^2}{\varphi_1}\right)\langle b(\hat{x}) - b(\hat{y}),\ \hat{x} - \hat{y}\rangle$$
$$+ \frac{M}{\varphi_1}\langle b(\hat{x}) + b(\hat{y}) - 2b(\hat{z}),\ \hat{x} + \hat{y} - 2\hat{z}\rangle$$
$$\le f(\hat{x}) + f(\hat{y}) - 2f(\hat{z}) - (c(\hat{x})u(\hat{x}) + c(\hat{y})u(\hat{y}) - 2c(\hat{z})u(\hat{z}))$$
$$+ 2\delta \operatorname{tr} a(\hat{x}) - 2\delta\langle b(\hat{x}),\ \hat{x}\rangle.$$

From (5.9) we see that

$$\operatorname{tr}(a(\hat{x})X + a(\hat{y})Y + a(\hat{z})Z) = \operatorname{tr}\left\{ (\sigma(\hat{x})\sigma(\hat{y})\sigma(\hat{z})) \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \begin{pmatrix} \sigma(\hat{x}) \\ \sigma(\hat{y}) \\ \sigma(\hat{z}) \end{pmatrix} \right\}$$

$$\leq \theta M\left[2\left(L + 3\frac{|\hat{x}-\hat{y}|^2}{\varphi_1} \right) \operatorname{tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 + \frac{1}{\varphi_1} \operatorname{tr}(\sigma(\hat{x}) + \sigma(\hat{y}) - 2\sigma(\hat{z}))^2 \right].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} 0 \leq \theta M & \left[2\left(L + 3\frac{\hat{x}-\hat{y}|^2}{\varphi_1} \right) \operatorname{tr}(\sigma(\hat{x}) - \sigma(\hat{y}))^2 + \frac{1}{\varphi_1} \operatorname{tr}(\sigma(\hat{x}) + \sigma(\hat{y}) - 2\sigma(\hat{z}))^2 \right] \\ & - 2M\left(L + \frac{|\hat{x}-\hat{y}|^2}{\varphi_1} \right) \langle b(\hat{x}) - b(\hat{y}), \hat{x} - \hat{y} \rangle \\ & - \frac{M}{\varphi_1} \langle b(\hat{x} + b(\hat{y}) - 2b(\hat{z}), \hat{x} + \hat{y} - 2\hat{z} \rangle \\ & + f(\hat{x}) + f(\hat{y}) - 2f(\hat{z}) - (c(\hat{x})u(\hat{x}) + c(\hat{y})u(\hat{y}) - 2c(\hat{z})u(\hat{z})) \\ & + 2\delta \operatorname{tr} a(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle. \end{aligned}$$

Sending $\theta \downarrow 1$ and using (5.5), (5.6) and (5.7), we have

$$\begin{aligned} 0 \leq M & \frac{c_0 + \lambda_1}{.2} \varphi(\hat{x}, \hat{y}, \hat{z}) + \|Df\|_{W^{1,\infty}} \varphi(\hat{x}, \hat{y}, \hat{z}) \\ & - c(\hat{z})(u(\hat{x}) + u(\hat{y}) - 2u(\hat{z})) + 2\|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}} \varphi(\hat{x}, \hat{y}, \hat{z}) \\ & + 2\delta \operatorname{tr} a(\hat{x}) - 2\delta \langle b(\hat{x}), \hat{x} \rangle. \end{aligned}$$

Since $\Phi(\hat{x}, \hat{y}, \hat{z}) > 0$ and $u \in BUC(\boldsymbol{R}^N)$, we have

$$u(\hat{x}) + u(\hat{y}) - 2u(\hat{z}) \geq M\varphi(\hat{x}, \hat{y}, \hat{z}) \qquad \text{and} \quad \gamma \leq \varphi(\hat{x}, \hat{y}, \hat{z}) \leq \gamma^{-1},$$

where γ is a positive constant independent of $\delta > 0$. Hence,

$$\text{(5.10)} \quad 0 \leq \left(-\frac{c_0 - \lambda_1}{2} M + \|Df\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}} \|c\|_{W^{2,\infty}} \right) \varphi(\hat{x}, \hat{y}, \hat{z}) + C\delta^{1/2},$$

where C is a constant independent of δ. Moreover, sending $\delta \downarrow 0$ and (5.10) yield a contradiction. This proves that $\Phi \leq 0$ on $\boldsymbol{R}^{3N}$ for all $\varepsilon, \delta > 0$. It is now easily concluded that (5.8) holds. ■

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