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ON SOLVING CERTAIN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY ACCRETIVE OPERATOR METHODS

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ABSTRACT

We use similar functional analytic methods to solve (a) a fully nonlinear second order elliptic equation, (b) a Hamilton–Jacobi equation, and (c) a functional/partial differential equation from plasma physics. The technique in each case is to approximate by the solutions of simpler problems, and then to pass to limits using a modification of G. Minty's device to the space L^∞.

1. Introduction

In this paper, a continuation of Evans [7], we solve three nonlinear problems: a second order fully nonlinear elliptic equation, a first order Hamilton–Jacobi equation, and a second order functional equation from plasma physics. In each case we solve a sequence of approximate problems, make estimates, and then pass to limits by means of a modification to L^∞ of G. Minty's Hilbert space method. This technique arose in our study of the Bellman p.d.e. of dynamic programming (see the end of this section for references). The theme of this paper is that these same procedures can be extended to cover several other quite different nonlinear problems.

Only in the first application, solving a non-quasilinear elliptic equation, are our results new. However, for the Hamilton–Jacobi equation our method is simpler than those earlier techniques using stochastic differential game theory; and for the plasma problem we do not rely on the quasi-variational formulation. In both situations our techniques should perhaps be extendable to include wider classes of similar problems. In any case, the point here is not so much the particular theorems, as it is the common plan of attack. We therefore defer the

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detailed statements and proofs to later sections and instead present here an informal discussion of the method.

Each of our problems can be written in form

$$\text{(P)} \qquad A(u) = f,$$

where A is some nonlinear differential operator, f is a given function, and the unknown is the function u. We attempt to find and then solve a sequence of simpler, approximate problems

$$\text{(P)}_\varepsilon \qquad A_\varepsilon(u_\varepsilon) = f, \qquad \varepsilon > 0,$$

where the A_ε are "nice" operators which converge in some sense to A. Next we try to obtain estimates independent of ε on the approximate solutions u_ε, their derivatives, etc. Assuming success so far we now send ε to zero and attempt to show the u_ε converge to a function u, which — we hope — solves (P). The major difficulty in this entire plan is that usually the nonlinearity is such that only relatively poor uniform estimates can be obtained for the u_ε: in practice it is often not too hard to show that the u_ε (or a subsequence) converge to some limit u, but it *is* difficult to prove u solves problem (P).

A method of G. Minty (cf. [18]) circumvents this problem for certain operators A satisfying a monotonicity condition; i.e.,

$$\text{(M)} \qquad 0 \leqq (u - \hat{u}, A(u) - A(\hat{u}))$$

for all $u, \hat{u} \in D(A)$, the domain of A ((,) denotes the inner product in L^2). Choose any fixed $\phi \in D(A)$. Then if u_ε solves $(\text{P})_\varepsilon$ and if the operators A_ε are also monotone, we have

$$\begin{aligned} 0 &\leqq (u_\varepsilon - \phi, A_\varepsilon(u_\varepsilon) - A_\varepsilon(\phi)) \\ &= (u_\varepsilon - \phi, f - A_\varepsilon(\phi)) \qquad \text{by } (\text{P}_\varepsilon). \end{aligned}$$

Suppose now that $A_\varepsilon(\phi) \to A(\phi)$, $u_\varepsilon \to u$ as $\varepsilon \searrow 0$; then

$$0 \leqq (u - \phi, f - A(\phi)).$$

(Notice that the operators A_ε pass to limits applied to a fixed "test function" ϕ.) Now set $\phi = u - \lambda\psi$ for $\psi \in D(A)$, $\lambda > 0$, and substitute into the above to find

$$0 \leqq (\psi, f - A(u - \lambda\psi)).$$

Let $\lambda \searrow 0$; if $A(u - \lambda\psi) \to A(u)$ (as is the case for many nonlinear differential operators), we have

$$0 \leqq (\psi, f - A(u)) \qquad \text{for all } \psi \in D(A).$$

This implies

$$A(u)=f$$

whenever $D(A)$ is dense in L^2.

The preceding argument has several variants and extensions, and applies to many nonlinear partial differential equations: see Lions [16, chap. 2], Browder [4], etc. Unfortunately the examples we have in mind fail to satisfy the monotonicity condition (M). Instead we observe that, formally at least, in each case a kind of maximum principle holds, namely

$$(A(u)-A(\hat{u}))(x_0)\geqq 0$$

at any point x_0 where $u-\hat{u}$ attains the value $\|u-\hat{u}\|_{L^\infty}$. Therefore

$$0\leqq[u-\hat{u}, A(u)-A(\hat{u})]_+ \tag{A}$$

for all $u, \hat{u}\in D(A)$, where

$$[f,g]_+\equiv \sup_{\substack{x_0\\ |f(x_0)|=\|f\|_{L^\infty}}} g(x_0)\operatorname{sgn} f(x_0)$$

is a kind of "partial inner product" on L^∞ (or C). An operator satisfying (A) is called *accretive* in L^∞: see the appendix (§8) or Crandall [6], Evans [8], Barbu [1].

The main idea of this paper (and of [7]) is to modify Minty's L^2 argument to L^∞, with the bracket $[\ ,\ ]_+$ playing the role of a true inner product. The technical difficulties we must still face are these:

(a) we must discover approximations A_ε which are nice enough that (P_ε) is solvable, and which are themselves accretive in L^∞,

(b) we must obtain estimates on the u_ε,

(c) we must pass to limits as $\varepsilon\searrow 0$.

Difficulty (c) is the worst. We note in particular that $[\ ,\ ]_+$ is only upper semicontinuous with respect to uniform convergence in each argument (it behaves badly with respect to weak convergence) and that $D(A)$ is *not* dense in L^∞ for the applications mentioned above. Nevertheless we will be able to mimic Minty's argument at least to the point of asserting

$$0\leqq[u-\phi, f-A(\phi)]_+$$

for all sufficiently nice "test functions" ϕ. Then we show that for a.e. x_0, there exist smooth functions ϕ^n such that $u-\phi^n$ attains its norm precisely at x_0, and $A(\phi^n)(x_0)\rightarrow A(u)(x_0)$. Hence the definition of $[\ ,\ ]_+$ implies

$$f(x_0) \geqq A(u)(x_0).$$

The reverse inequality follows from a similar argument, whence

$$A(u) = f \qquad \text{a.e.}$$

Concrete realizations of this scheme involve of course considerable technical difficulties, and we delay our discussion of these until later. For the reader's convenience we include an appendix (§8) of the key facts about accretive operator theory; various technical lemmas ensuring that the class of test functions ϕ is large enough are collected in the second appendix (§9).

We close by noting that another application of our convergence method appears in Evans–Friedman [9] and P. L. Lions [17]; here we follow the general scheme outlined above to solve the Bellman p.d.e. of dynamic programming (cf. Fleming–Rishel [10])

$$\max_{1 \leqq k \leqq m} \{L^k u\} = f,$$

when the L^k are second order linear elliptic operators.

Part 1. A Nonlinear Elliptic Equation

2. Statement of the problem; preliminaries

For the rest of the paper Ω will denote a bounded, smooth domain in $\mathbf{R}^n$ ($n \geqq 2$). Let us assume that $F : \mathbf{R}^{n^2} \to \mathbf{R}$ satisfies these hypotheses:

(F1) F is continuously differentiable, with bounded gradient,

(F2) for each $x \in \mathbf{R}^{n^2}$ the matrix $((\partial F(x)/\partial x_{ij}))$ is nonnegative definite,

and

(F3) $\lim_{|x|\to\infty} |F(x)|/|x| = 0$.

Here and afterwards the components of $x \in \mathbf{R}^{n^2}$ are denoted by x_{ij} ($1 \leqq i, j \leqq n$); i.e.,

$$x = (x_{11}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, x_{31}, \cdots, x_{nn}).$$

Consider now the elliptic partial differential equation

$$\text{(E)} \qquad \begin{cases} \Delta u(x) + F(D^2 u(x)) = f(x) & \text{a.e. } x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where

$$F(D^2 u) \equiv F(u_{x_1 x_1}, \cdots, u_{x_i x_j}, \cdots, u_{x_n x_n}),$$

$f \in C(\bar{\Omega})$ is given and u is the unknown. Hypothesis (F2) is an ellipticity assumption, whereas (F3) ensures that F grows less rapidly than linearly at infinity (cf. Remark 3.3).

Our existence result is this:

THEOREM 2.1. *Assume that F satisfies* (F1)–(F3). *Then for each $f \in C(\bar{\Omega})$ there exists a unique $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (for all $1 \leqq p < \infty$) solving* (E).

This proposition extends theorem 2 in [7], a similar assertion for the case that F depends only on the pure second order derivatives of u. The new idea here is a more sophisticated approximation scheme to take care of mixed partial derivatives. Note that in light of (F3) *a priori* $W^{2,p}$ estimates are easy: the entire difficulty is therefore the devising of a procedure to exploit these estimates (see Remark 3.2).

Our method depends upon a "quasilinearization" representation of F:

LEMMA 2.2. *Suppose that* (F1) *holds. Then*

$$F(x) = \max_{y \in \mathbf{R}^{n^2}} \min_{z \in \mathbf{R}^{n^2}} \left\{ \left[\int_0^1 \frac{\partial F}{\partial x_{ij}}((1-\lambda)y + \lambda z)) d\lambda \right] (x_{ij} - y_{ij}) + F(y) \right\} \tag{2.1}$$

for each $x \in \mathbf{R}^{n^2}$. (Here we employ the implicit summation convention.)

PROOF. Fix $x \in \mathbf{R}^{n^2}$.

Now for any given $y \in \mathbf{R}^{n^2}$, set $z = x$ in the expression within the brackets $\{\ \}$. Then

$$\begin{aligned} \min_z \{\ \} &\leqq \left\{ \left[\int_0^1 \frac{\partial F}{\partial x_{ij}}((1-\lambda)y + \lambda x) d\lambda \right] (x_{ij} - y_{ij}) + F(y) \right\} \\ &= \int_0^1 \frac{d}{d\lambda} F((1-\lambda)y + \lambda x) d\lambda + F(y) \\ &= F(x); \end{aligned}$$

therefore

$$\max_y \min_z \{\ \} \leqq F(x).$$

On the other hand, for $y = x$ and any z,

$$\{\ \} = \left[\int_0^1 \frac{\partial F}{\partial x_{ij}}((1-\lambda)x + \lambda z) d\lambda \right] (x_{ij} - x_{ij}) + F(x) = F(x);$$

so that

$$\max_y \min_z \{\ \} \geqq F(x). \qquad \blacksquare$$

Now define

$$\bar{F}(x) \equiv -F(-x), \qquad x \in \mathbf{R}^{n^2}.$$

$\bar{F}$ satisfies (F1)–(F3), and (E) may be rewritten to read

$$\text{(E)} \qquad \begin{cases} -\Delta u(x) + \bar{F}(-D^2 u(x)) = -f(x) & \text{a.e. } x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{cases}$$

Define also for each $(y, z) \in \mathbf{R}^{n^2} \times \mathbf{R}^{n^2}$,

$$\text{(2.2)} \qquad \begin{aligned} a_{ij}(y, z) &\equiv \int_0^1 \frac{\partial \bar{F}}{\partial x_{ij}}((1-\lambda)y + \lambda z)\,d\lambda \\ &= \int_0^1 \frac{\partial F}{\partial x_{ij}}(-(1-\lambda)y - \lambda z)\,d\lambda \end{aligned}$$

and

$$\text{(2.3)} \qquad \begin{aligned} f(y, z) &\equiv \bar{F}(y) - \left[\int_0^1 \frac{\partial \bar{F}}{\partial x_{ij}}((1-\lambda)y + \lambda z)\,d\lambda\right] y_{ij} \\ &= -F(-y) - a_{ij}(y, z) y_{ij}\,; \end{aligned}$$

by Lemma 2.2 (applied to $\bar{F}$ instead of F)

$$\text{(2.4)} \quad \bar{F}(-u_{x_1x_1}, \cdots, -u_{x_ix_j}, \cdots, -u_{x_nx_n}) = \max_y \min_z \{-a_{ij}(y, z) u_{x_ix_j} + f(y, z)\}.$$

Hypothesis (F2) implies

$$\text{(2.5)} \qquad 0 \leqq a_{ij}(y, z)\xi_i\xi_j \qquad \text{for each } \xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n.$$

According now to (2.4) and (2.5) $\bar{F}$ is the max-min of affine elliptic operators. This representation is at the heart of the proof of Theorem 2.1 in the next section. The idea is to replace each operator $-a_{ij}(y, z)u_{x_ix_j}$ by its Yosida approximation. The resulting approximations are Lipschitz and yet are still accretive with respect to the supremum norm topology. As such we can both obtain estimates and then eventually pass to limits.

3. Proof of Theorem 2.1

The proof consists of four parts:
(a) approximation,
(b) first passage to limits,
(c) second passage to limits,
(d) uniqueness.

(a) *Approximation*

Let $0 < \eta < \frac{1}{2}$ be a small fixed number (to be selected later). We define for each $(y, z) \in \mathbf{R}^{n^2} \times \mathbf{R}^{n^2}$ the uniformly elliptic operator

$$A^{y,z}u \equiv -\eta \Delta u - a_{ij}(y, z)u_{x_i x_j}, \tag{3.1}$$

for

$$u \in D(A^{y,z}) \equiv \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)(1 \leqq p < \infty) \mid A^{y,z}u \in C(\bar{\Omega})\}.$$

According to standard elliptic theory the operator $A^{y,z}$ is m-accretive in $C(\bar{\Omega})$ (see the appendix (§8) for definitions of technical terms below).

Fix $\lambda > 0$ and let $A_\lambda^{y,z}$ denote the λth Yosida approximation of $A^{y,z}$; according to (8.6) and (8.7) each $A_\lambda^{y,z}$ is an everywhere defined, Lipschitz, accretive operator on $C(\bar{\Omega})$.

Next choose $M > 1$ and select some smooth function $\beta = \beta_M$ such that

$$\begin{aligned} &\beta(x) = x, && |x| \leqq M - 1, \\ &\beta(x) = M, && |x| \geqq M, \\ &0 \leqq \beta' \leqq 1. \end{aligned} \tag{3.2}$$

We intend eventually to send M to infinity (in part (c) below); but for the moment it is fixed, and so we suppress any reference to M in the notation.

Finally define the nonlinear operator

$$B^\lambda(u) \equiv \beta\left(\max_{\substack{y \\ |y| \leqq \lambda^{-1}}} \min_z \left(A_\lambda^{y,z}(u) + f(y, z)\right)\right). \tag{3.3}$$

Since

$$|f(y, z)| \leqq \frac{C}{\lambda} \qquad \text{for } |y| \leqq \frac{1}{\lambda}, \quad \text{any } z,$$

B_λ is defined on all of $C(\bar{\Omega})$. Furthermore B_λ is Lipschitz (since each $A_\lambda^{y,z}$ is Lipschitz with the same constant $2/\lambda$ by (8.7)); and according to Lemma 8.2 B_λ is accretive on $C(\bar{\Omega})$.

Hence the Perturbation Lemma 8.1 (applied to $A = -(1-\eta)\Delta$ and $B = B^\lambda$) from the appendix implies the existence of a unique $u_\lambda \in W^{2,p} \cap W_0^{1,p}$ solving

$$\lambda u_\lambda - (1-\eta)\Delta u_\lambda + B^\lambda(u_\lambda) = -f \qquad \text{in } \Omega. \tag{3.4}$$

Since $|B^\lambda| \leqq \sup|\beta| \leqq M$ by (3.2) and (3.3), we have

$$\|u_\lambda\|_{W^{2,p}(\Omega)} \leqq C(p, M) \qquad \text{for each } 1 \leqq p < \infty; \tag{3.5}$$

the constant depends on p and M, but not on λ.

(b) *First Passage to Limits*

Owing to (3.5) there exists a sequence $\lambda_j \searrow 0$ and a function $u \in W^{2,p} \cap W_0^{1,p}$ such that

$$
\begin{aligned}
&u_{\lambda_j} \rightharpoonup u && \text{weakly in } W^{2,p}\ (1 \leqq p < \infty),\\
&Du_{\lambda_j} \to Du && \text{uniformly on } \bar{\Omega}, \qquad (3.6)\\
&u_{\lambda_j} \to u && \text{uniformly on } \bar{\Omega}.
\end{aligned}
$$

Consider now some given $\phi \in C_0^\infty(\Omega)$. We claim

$$B^\lambda(\phi) \to \beta(-\eta\Delta\phi + \bar{F}(-D^2\phi)) \tag{3.7}$$

uniformly on Ω as $\lambda \searrow 0$. Indeed, for any y, z

$$
\begin{aligned}
\|A_\lambda^{y,z}\phi - A^{y,z}\phi\|_{C(\bar{\Omega})} &= \|J_\lambda^{y,z}A^{y,z}\phi - A^{y,z}\phi\|_{C(\bar{\Omega})} \quad (J_\lambda^{y,z} = (I + \lambda A^{y,z})^{-1})\\
&\leqq \lambda\,\|(A^{y,z})^2\phi\|_{C(\bar{\Omega})} \qquad \text{by } (8.8) \qquad (3.8)\\
&\leqq \lambda C\|\phi\|_{C^4} \leqq \lambda C_1,
\end{aligned}
$$

and the constant C_1 does not depend on y or z. Furthermore for $\lambda^{-1} \geqq \sup|D^2\phi|$,

$$\max_{|y| \leqq \lambda^{-1}} \min_z \{-a_{ij}(y,z)\phi_{x_ix_j} + f(y,z)\} = \bar{F}(-D^2\phi); \tag{3.9}$$

to see this note that

$$\max_{|y| \leqq \lambda^{-1}} \min_z \{\quad\} \leqq \bar{F}(-D^2\phi),$$

by (2.4), whereas substituting $y = (\phi_{x_1x_1}, \cdots, \phi_{x_nx_n})$ yields equality. Therefore for λ small enough

$$\left|\max_{|y| \leqq \lambda^{-1}} \min_z \{A_\lambda^{y,z}\phi + f(y,z)\} - (-\eta\Delta\phi + \bar{F}(-D^2\phi))\right|$$

$$\leqq \max_{|y| \leqq \lambda^{-1}} \min_z |A_\lambda^{y,z}\phi - A^{y,z}\phi| \leqq \lambda C_1 \qquad \text{on } \Omega, \text{ by } (3.8).$$

Since β is continuous, this proves (3.7).

Now according to the accretiveness of $-(1-\eta)\Delta + B^\lambda$ we have

$$0 \leqq [u_\lambda - \phi, -(1-\eta)\Delta u_\lambda + B^\lambda(u_\lambda) - (-(1-\eta)\Delta\phi + B^\lambda(\phi))]_+$$

for any $\phi \in C_0^\infty(\Omega)$; (3.4) then implies

$$0 \leqq [u_\lambda - \phi, -\lambda u_\lambda - f + (1-\eta)\Delta\phi - B^\lambda(\phi)]_+.$$

Let $\lambda = \lambda_j \searrow 0$. By (3.6), (3.7), and the upper-semicontinuity of $[\ \ ,\ \ ]_+$ with respect to uniform convergence,

$$0 \leqq [u - \phi, -f + (1-\eta)\Delta\phi - \beta(-\eta\Delta\phi + \bar{F}(-D^2\phi))]_+ \tag{3.10}$$

for all $\phi \in C_0^\infty(\Omega)$.

By Lemma 9.1 for a.e. $x_0 \in \Omega$, there exists a sequence $\phi^k \in C_0^\infty(\Omega)$ such that

(i) $\phi^k(x_0) - u(x_0) = \|\phi^k - u\|_{C(\bar{\Omega})} > |\phi^k(x) - u(x)|$ for all $x \in \bar{\Omega}$, $x \neq x_0$,

and

(ii) $D^2\phi^k(x_0) \to D^2u(x_0)$, $D\phi^k(x_0) \to Du(x_0)$, $\phi^k(x_0) \to \phi(x_0)$.

Fix such a point x_0. By the characterization (8.14) of $[\ \ ,\ \ ]_+$ and by (i), we have

$$-f(x_0) + (1-\eta)\Delta\phi^k(x_0) - \beta(-\eta\Delta\phi^k(x_0) + \bar{F}(-D^2\phi^k(x_0)) \leqq 0.$$

Let $k \to \infty$ and use (ii) to find

$$-f(x_0) \leqq -(1-\eta)\Delta u(x_0) + \beta(-\eta\Delta u(x_0) + \bar{F}(-D^2u(x_0)) \text{ for a.e. } x_0 \in \Omega. \tag{3.11}$$

In the same way we can find for a.e. x_0 a sequence $\psi^k \in C_0^\infty(\Omega)$ satisfying (ii) and

(i)′ $u(x_0) - \psi^k(x_0) \equiv \|u - \psi^k\|_{C(\bar{\Omega})} > |u(x) - \psi^k(x)|$ for all $x \in \bar{\Omega}$, $x \neq x_0$.

Substituting $\phi = \psi^k$ in (3.10) and then letting $k \to \infty$ yields the reverse inequality to (3.11). Therefore

$$-(1-\eta)\Delta u + \beta(-\eta\Delta u + \bar{F}(-D^2u)) = -f \qquad \text{a.e.} \tag{3.12}$$

(c) *Second Passage to Limits*

Next we remove the β from (3.12). Recall (3.2) and now denote by u^M the solution of (3.12) (for $\beta = \beta_M$) constructed above.

Since $|\beta_M(x)| \leqq |x|$ for all x, we have

$$\begin{aligned}(1-\eta)\|u^M\|_{W^{2,p}} &\leqq C\|(1-\eta)\Delta u^M\|_{L^p} \\ &\leqq C\|f\|_{L^p} + C\|\beta(\ \ )\|_{L^p} \\ &\leqq C + C\|\eta\Delta u^M\|_{L^p} + C\|\bar{F}(-D^2u^M)\|_{L^p} \\ &\leqq C + \eta C\|u^M\|_{W^{2,p}} + \varepsilon\|u^M\|_{W^{2,p}} + C(\varepsilon),\end{aligned}$$

for any $\varepsilon > 0$, by hypothesis (F3). For fixed $\eta, \varepsilon > 0$ small enough, we obtain the bounds

$$\|u^M\|_{W^{2,p}} \leqq C(p) \qquad (1 \leqq p < \infty), \tag{3.13}$$

the constant depending on p, but not on M.

Estimate (3.13) implies the existence of a subsequence (also denoted "u^M") and a function u such that

$$u^M \rightharpoonup u \quad \text{weakly in } W^{2,p} \quad (1 \leqq p < \infty),$$

$$Du^M \to Du \quad \text{uniformly on } \bar{\Omega},$$

$$u^M \to u \quad \text{uniformly on } \bar{\Omega}.$$

Furthermore for each $\beta = \beta_M$ the differential operator on the left hand side of (3.12) is accretive in $L^\infty(\Omega)$: see Lemma 8.2 and the maximum principle of Bony [3]. Hence for each $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} 0 &\leqq [u_M - \phi, -(1-\eta)\Delta u^M + \beta_M(-\eta\Delta u^M + \bar{F}(-D^2u^M)) \\ &\qquad -(-(1-\eta)\Delta\phi + \beta_M(-\eta\Delta\phi + \bar{F}(-D^2\phi))]_+ \\ &= [u_M - \phi, -f + (1-\eta)\Delta\phi - \beta_M(-\eta\Delta\phi + \bar{F}(-D^2\phi))]_+. \end{aligned}$$

Let $M \to \infty$; since $\beta_M(x) \to x$ for all x and ϕ is smooth we have

$$0 \leqq [u - \phi, -f + \Delta\phi - \bar{F}(-D^2\phi)]_+$$

for all $\phi \in C_0^\infty(\Omega)$. Exactly as in section (b) above this implies

$$\text{(E)} \qquad \begin{cases} \Delta u + F(D^2u) = f & \text{a.e. in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

(d) *Uniqueness*

Assume now that $u, \hat{u} \in W^{2,p}(\Omega)$ $(1 \leqq p < \infty)$ are two solutions of (E). Define for a.e. x,

$$\bar{a}_{ij}(x) = \int_0^1 \frac{\partial F}{\partial x_{ij}}(\lambda D^2\hat{u}(x) + (1-\lambda)D^2u(x))d\lambda.$$

Then

$$\Delta v + \bar{a}_{ij}v_{x_ix_j} = 0 \quad \text{a.e.} \quad \text{for } v \equiv u - \hat{u} \in W^{2,p} \cap W_0^{1,p}.$$

By the maximum principle of Bony [3] $v \equiv 0$. ■

REMARK 3.1. Various extensions of Theorem 2.1 — to allow for example lower order nonlinearities — are clearly possible. We do not pursue this, except to note that the hypothesized ((F1)) bound on the gradient of F, or even its existence, is not essential and can be removed by approximating F with smooth functions, uniformly on compact sets. ■

REMARK 3.2. The rather elaborate scheme (3.3) and (3.4) is the only approximation the author could devise to obtain well behaved, *accretive* operators converging formally to $-\Delta u + \bar{F}(-D^2u)$. It is not very hard to invent various other plausible procedures (e.g. replace the derivatives in F by difference quotients, mollify each derivative in F, try successive guesses, add on $\varepsilon\Delta^2 u$, etc.); but for none of these are the approximation operators accretive in $C(\bar{\Omega})$. This failing, we do not know how to justify the limiting procedure with just $W^{2,p}$ estimates, implying only weak and not a.e. convergence of the second derivatives. ■

REMARK 3.3. Hypothesis (F3) can be replaced with a somewhat weaker assumption as follows. Select $n < p_0 < \infty$ and suppose that

$$\text{(F3)}' \qquad \lim_{|x|\to\infty} \frac{|F(x)|}{|x|} \leqq \varepsilon(p_0),$$

where $\varepsilon(p_0)$ is an appropriate small positive constant, which depends only on p_0 and Ω. Under (F1), (F2), and (F3)′ the proof of Theorem 3.1 still works, except that in section (c) above we obtain estimates only in the space $W^{2,p_0}(\Omega)$. ■

Part II. The Hamilton–Jacobi Equation

4. Statement of the problem; preliminaries

This and the next section study the solvability of the Cauchy problem for the Hamilton–Jacobi equation:

$$\text{(HJ)} \quad \begin{cases} u_t(x,t) + H(x,t,u(x,t),Du(x,t)) = 0 & \text{a.e. } (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^n. \end{cases}$$

In this problem $T > 0$, $H : \mathbf{R}^n \times [0,T] \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ and $u_0 : \mathbf{R}^n \to \mathbf{R}$ are given; and the unknown is u. $Du \equiv (u_{x_1}, \cdots, u_{x_n})$.

There is of course a wide literature concerning (HJ); see, for example, Benton [2] for a readable discussion of various techniques and for references. Here we intend only to demonstrate the applicability of our convergence methods to justifying the "vanishing viscosity" method (and hence impose convenient, but excessive, restrictions on H). Following Kruzkov [15], Fleming [10], and Friedman [13], we consider for each $\varepsilon > 0$ the approximate problem

$$\text{(HJ)}_\varepsilon \qquad \begin{cases} u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + H(x,t,u^\varepsilon,Du^\varepsilon) = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u^\varepsilon(x,0) = u_0(x), \quad x \in \mathbf{R}^n, \end{cases}$$

and will prove that a subsequence of the u^ε converges as $\varepsilon \to 0$ to a solution of (HJ). In [10] and [13] this follows from rather complicated considerations involving stochastic differential game theory. Tamburro [24] and Burch [5] directly pass to limits (for a time independent version of (HJ)), but under the assumption that H is convex with respect to Du, in which case there is an extra estimate on the second derivatives of the u_ε (implying a.e. convergence of the Du^ε). (The papers [24] and [5] use accretive operator methods to construct a semigroup solution of (HJ).)

We will assume

(H1) $\qquad H$ is continuously differentiable

and

there exists a constant M such that

(H2) $$|F| + |F_x| + |F_t| + |F_u| + |F_p| \leqq M$$

for all $(x, t, u, p) \in \mathbf{R}^n \times [0, T] \times \mathbf{R} \times \mathbf{R}^n$.

THEOREM 4.1. *Suppose that H satisfies conditions* (H1) *and* (H2), *and that* $u_0 : \mathbf{R}^n \to \mathbf{R}$ *is Lipschitz, bounded. Then there is a sequence* $\varepsilon_j \searrow 0$ *such that*

$$\lim_{\varepsilon_j \searrow 0} u^{\varepsilon_j}(x, t) \equiv u(x, t)$$

exists, uniformly on compact sets of $\mathbf{R}^n \times [0, T]$. *Furthermore u is Lipschitz and is a solution of* (HJ).

REMARK 4.2. Examples show that the solution of (HJ) need not be unique: see Benton [2]. According to the stochastic differential game method of Fleming [10], in fact $\lim_{\varepsilon \searrow 0} u^\varepsilon = u$; we do not know a direct proof of this. ∎

We continue by quoting Theorem 2.1 of Friedman [13], which implies the solvability of $(\mathrm{HJ})_\varepsilon$:

LEMMA 4.3. *Assume* (H1) *and* (H2). *Then for each* $0 < \varepsilon < 1$, *there exists a unique bounded function* $u^\varepsilon \in C^{2+\alpha, 1+\alpha/2}_{\mathrm{loc}}(\mathbf{R}^n \times (0, T))$ $(0 < \alpha < 1)$ *solving* $(\mathrm{HJ})_\varepsilon$.

Furthermore there exists a constant K, independent of $\varepsilon > 0$, *such that*

$$|u^\varepsilon(x, t)| \leqq K$$
(4.1) $$|Du^\varepsilon(x, t)| \leqq K$$
$$|u^\varepsilon_t(x, t)| \leqq K$$

for all $(x, t) \in \mathbf{R}^n \times (0, T)$.

5. Proof of Theorem 4.1

According to (4.1) there exists a subsequence $\varepsilon_j \searrow 0$ and a function $u \in C^{0,1}(\mathbf{R}^n \times [0, T])$ such that

$$u^{\varepsilon_j} \to u \qquad \text{uniformly on compact sets,}$$

$$Du^{\varepsilon_j} \rightharpoonup Du \qquad \text{weak * in } L^\infty \text{, on compact sets,}$$

$$u_t^{\varepsilon_j} \rightharpoonup u_t \qquad \text{weak * in } L^\infty \text{, on compact sets.}$$

Lemma 9.2 implies that, for a.e. $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$, u is differentiable at (x_0, t_0) and there exists a C^1 function ϕ, with compact support $\bar{\Omega}$, satisfying

(i) $\phi(x_0, t_0) - u(x_0, t_0) = \|\phi - u\|_{C(\bar{\Omega})} > |\phi(x, t) - u(x, t)|$ for all $(x, t) \in \bar{\Omega}$, $(x, t) \neq (x_0, t_0)$.

(ii) $\phi(x_0, t_0) \geqq 4K$ (K is the constant in (4.1)).

Choose $\{\psi^k\} \subset C_0^\infty(\mathbf{R} \times (0, T))$ so that

$$\psi^k \to \phi, \quad D\psi^k \to D\phi, \quad \psi_t^k \to \phi_t \text{ uniformly on } \mathbf{R}^n \times [0, T]. \tag{5.1}$$

Now (i) and (ii) imply that $\phi - u$ attains its L^∞-norm over all of $\mathbf{R}^n \times [0, T]$ at (x_0, t_0); and so, for K large enough and ε small enough,

$$\psi^k - u^\varepsilon \text{ must attain its } L^\infty\text{-norm over } \mathbf{R}^n \times [0, T] \text{ in } \Omega. \tag{5.2}$$

Hence by (8.16) and the maximum principle

$$0 \leqq [\psi^k - u^\varepsilon, \psi_t^k - \varepsilon\Delta\psi^k + H(x, t, u^\varepsilon, D\psi^k) - (u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + H(x, t, u^\varepsilon, Du^\varepsilon))]_+$$
$$= [\psi^k - u^\varepsilon, \psi_t^k - \varepsilon\Delta\psi^k + H(x, t, u^\varepsilon, D\psi^k)]_+$$

by $(\mathrm{HJ})_\varepsilon$. Now according to (5.2) and (8.16), the behavior of $\psi^k - u^\varepsilon$ off $\bar{\Omega}$ does not affect the bracket $[\ ,\ ]_+$, whereas $u^\varepsilon \to u$ uniformly on $\bar{\Omega}$.

Since $[\ ,\ ]_+$ is upper semicontinuous, we therefore may conclude, after sending $\varepsilon \to 0$,

$$0 \leqq [\psi^k - u, \psi_t^k + H(x, t, u, D\psi^k)]_+ \qquad (k = 1, 2, \cdots).$$

Now let $k \to \infty$ and recall (5.1):

$$0 \leqq [\phi - u, \phi_t + H(x, t, u, D\phi)]_+;$$

by (i) and the characterization (8.16) we find

$$u_t(x_0, t_0) + H(x_0, t_0, u(x_0, t_0), Du(x_0, t_0)) \geqq 0, \tag{5.3}$$

since $D\phi = Du$, $\phi_t = u_t$ at (x_0, t_0) (where $\phi - u$ attains its maximum). In a similar way we can find a C^1 function ϕ so that $u-\phi$ attains its maximum at (x_0, t_0). Then after approximating by C_0^∞ functions ψ^k as above and passing to limits, we conclude

$$u_t(x_0, t_0) + H(x_0, t_0, u(x_0, t_0), Du(x_0, t_0)) \leqq 0. \tag{5.4}$$

Since (5.3) and (5.4) hold for a.e. (x_0, t_0), u solves (HJ). ■

REMARK 5.1. We note that in contrast to the difficulties discussed in Remark 3.2 that the "vanishing viscosity" approach to (HJ) has the important property that the approximations still satisfy the maximum principle and, more precisely, still correspond to accretive operators. ■

Part III. A Functional/Differential Equation

6. Statement of the problem; preliminaries

J. Mossino in [19], [20], and [21] has studied this nonlinear functional p.d.e., a simplified form of the Grad–Mercier equations in plasma physics ([14]):

$$\text{(P)} \qquad \begin{cases} -\Delta u(x) + \bar\beta(u)(x) \ni f(x) & \text{a.e. } x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

Here Ω is a bounded smooth domain in $\mathbf{R}^n$, $f \in C(\bar\Omega)$ is given, and

$$\bar\beta(u)(x) \equiv [\operatorname{meas}\{y \mid u(y) < u(x)\}, \operatorname{meas}\{y \mid u(y) \leqq u(x)\}] \tag{6.1}$$

is a multivalued functional defined on $C(\bar\Omega)$; the unknown is u.

Mossino solved (P) in [20] by means of a certain quasi-variational formulation of the problem. As a further and simpler example of accretive operator methods we present here a new existence proof:

THEOREM 6.1. *For each* $f \in C(\bar\Omega)$ *there exists* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(1 \leqq p < \infty)$ *solving* (P).

REMARK 6.2. Mossino [20] has constructed examples showing nonuniqueness is possible. The solution we construct is maximal. ■

Our idea — following the general plan outlined in §1 — is to approximate (P) by a sequence of simpler problems $(\mathrm{P})_\varepsilon$. To define these first select, for each $\varepsilon > 0$, a smooth function β_ε satisfying

$$\text{(6.1)} \qquad \begin{aligned} &\beta_\varepsilon(x) = 0, \qquad x \leqq -\varepsilon, \\ &\beta_\varepsilon(x) = 1, \qquad x \geqq 0, \\ &\beta_\varepsilon' \geqq 0. \end{aligned}$$

We suppose in addition that

$$\beta_\varepsilon(x) \leqq \beta_{\varepsilon'}(x) \quad \text{for } x \in \mathbf{R}, \quad 0 < \varepsilon < \varepsilon', \tag{6.2}$$

so that

$$\beta_\varepsilon(x) \searrow 0 \qquad x < 0. \tag{6.3}$$

Consider now the approximate problems

$$(\mathrm{P})_\varepsilon \quad \begin{cases} \varepsilon u^\varepsilon(x) - \Delta u^\varepsilon(x) + \int_\Omega \beta_\varepsilon(u_\varepsilon(x) - u_\varepsilon(y))dy = f(x), & x \in \Omega, \\ u_\varepsilon(x) = 0, \quad x \in \partial\Omega. \end{cases}$$

LEMMA 6.3. *For each* $\varepsilon > 0$, *there exists a unique* $u^\varepsilon \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(1 \leqq p < \infty)$ *solving* $(\mathrm{P})_\varepsilon$.

PROOF. We claim first that the operator B_ε defined by

$$B_\varepsilon(u)(x) \equiv \int_\Omega \beta_\varepsilon(u(x) - u(y))dy, \qquad x \in \bar{\Omega}$$

is Lipschitz, everywhere defined, and accretive on $C(\bar{\Omega})$. The first two statements are clear. To prove the last assume that $u, \hat{u} \in C(\bar{\Omega})$, $(u - \hat{u})(x_0) = \| u - \hat{u} \|_{C(\bar{\Omega})}$ for some $x_0 \in \bar{\Omega}$. Then

$$u(x_0) - u(y) \geqq \hat{u}(x_0) - \hat{u}(y), \qquad y \in \bar{\Omega},$$

and so

$$\beta_\varepsilon(u(x_0) - u(y)) \geqq \beta_\varepsilon(\hat{u}(x_0) - \hat{u}(y)) \qquad \text{for all } y \in \bar{\Omega}.$$

Integration over Ω implies

$$(B_\varepsilon(u) - B_\varepsilon(\hat{u}))(x_0) \geqq 0;$$

that is, B_ε is accretive on $C(\bar{\Omega})$ (by (8.12) and (8.14)).

According to the Perturbation Lemma 8.1 (applied to B_ε and $A = -\Delta$), $(\mathrm{P})_\varepsilon$ has a unique solution u_ε. ■

REMARK 6.4. This lemma, as well as the remainder of the theory in §6–7, works for $f \in L^\infty(\Omega)$; but since the characterization of $[\ ,\ ]_+$ (cf. (8.17)) in $L^\infty(\Omega)$ is somewhat more complicated than in $C(\bar{\Omega})$, we do not discuss this extension. The observation that accretive operator methods apply to (P) is due to Ph. Benilan. ■

7. Proof of Theorem 6.1

Since $|B_\varepsilon(u^\varepsilon)| \leqq \text{meas}(\Omega)$, we have the estimate

$$\|u^\varepsilon\|_{W^{2,p}(\Omega)} \leqq C(p), \qquad 1 \leqq p < \infty, \quad \varepsilon > 0;$$

the constant does not depend on ε. Hence there exists a subsequence $\varepsilon_j \searrow 0$ and a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(1 \leqq p < \infty)$ such that

$$\text{(7.1)} \qquad \begin{aligned} u^{\varepsilon_j} &\rightharpoonup u && \text{weakly in } W^{2,p} \quad (1 \leqq p < \infty), \\ Du^{\varepsilon_j} &\to Du && \text{uniformly on } \bar{\Omega}, \\ u^{\varepsilon_j} &\to u && \text{uniformly on } \bar{\Omega}. \end{aligned}$$

We claim u solves (P), and this fact is easier to prove than the corresponding assertions in §3, 5 since the nonlinearity here does not involve the highest order derivatives.

Fix $x \in \bar{\Omega}$ for the moment and consider any point $y \in \bar{\Omega}$ so that

$$\text{(7.2)} \qquad u(x) > u(y).$$

Then

$$u^{\varepsilon_j}(x) > u^{\varepsilon_j}(y)$$

and hence

$$\text{(7.3)} \qquad \beta_{\varepsilon_j}(u^{\varepsilon_j}(x) - u^{\varepsilon_j}(y)) = 1$$

for ε_j small enough. Similarly, suppose

$$\text{(7.4)} \qquad u(x) < u(y);$$

then

$$\text{(7.5)} \qquad \beta_{\varepsilon_j}(u^{\varepsilon_j}(x) - u^{\varepsilon_j}(y)) = 0$$

for ε_j small enough.

In view of (7.3), (7.5), and Fatou's lemma we have

$$\text{(7.6)} \qquad \begin{aligned} \text{meas}\{y \in \bar{\Omega} \mid u(y) < u(x)\} &\leqq \liminf_{\varepsilon_j \to 0} B_{\varepsilon_j}(u^j)(x) \\ &\leqq \limsup_{\varepsilon_j \to 0} B_{\varepsilon_j}(u^{\varepsilon_j})(x) \\ &\leqq \text{meas}\{y \in \bar{\Omega} \mid u(y) \leqq u(x)\} \end{aligned}$$

for each $x \in \bar{\Omega}$.

Next we note from (7.1) that

$$-\Delta u^{\varepsilon_j} \rightharpoonup -\Delta u \quad \text{weakly in } W^{2,p},$$
$$\varepsilon_j u^{\varepsilon_j} \to 0 \quad \text{uniformly on } \bar{\Omega}, \tag{7.7}$$
$$B_{\varepsilon_j}(u^{\varepsilon_j}) \rightharpoonup B \quad \text{weakly in } L^p,$$

for some function B. According to (7.6)

$$\text{meas}\{y \mid u(y) < u(x)\} \leqq B(x) \leqq \text{meas}\{y \mid u(y) \leqq u(x)\}$$

for a.e. $x \in \Omega$; that is,

$$B(x) \in \bar{\beta}(u)(x) \quad \text{a.e.} \tag{7.8}$$

Passing to limits as $\varepsilon_j \to 0$ in $(\mathrm{P})_{\varepsilon_j}$, we find that u solves (P). ■

8. Appendix 1: Basic theory of accretive operators

The book of Barbu [1] and the survey article of Crandall [6] contain proofs and more explanation concerning the results in this appendix.

Let X be a real Banach space with norm $\|\ \|$. A (possibly nonlinear) operator $A : D(A) \subset X \to X$ is *accretive* if

$$\|x - \hat{x}\| \leqq \|x - \hat{x} + \lambda(A(x) - A(\hat{x}))\| \qquad \text{for all } x, \hat{x} \in D(A), \quad \lambda > 0. \tag{8.1}$$

If in addition $R(I + \lambda A) = X$ for some $\lambda > 0$, A is *m-accretive*, in which case $R(I + \lambda A) = X$ for all $\lambda > 0$.

For an m-accretive operator A, we define

$$J_\lambda \equiv (I + \lambda A)^{-1} \qquad (\lambda > 0), \tag{8.2}$$

the (nonlinear) *resolvents* of A, and

$$A_\lambda \equiv \frac{I - J_\lambda}{\lambda} \qquad (\lambda > 0), \tag{8.3}$$

the *Yosida approximations* of A. We note here some simple properties of the J_λ and A_λ:

$$\|J_\lambda(x) - J_\lambda(\hat{x})\| \leqq \|x - \hat{x}\|, \qquad \lambda > 0, \quad x, \hat{x} \in X; \tag{8.4}$$

$$J_\lambda(x) \to x \quad \text{as} \quad \lambda \searrow 0, \qquad \text{for } x \in \overline{D(A)}; \tag{8.5}$$

(8.6) A_λ is an everywhere defined, accretive operator on X;

(8.7) $$\|A_\lambda(x) - A_\lambda(\hat{x})\| \leqq \frac{2}{\lambda}\|x - \hat{x}\|, \qquad x, \hat{x} \in X, \quad \lambda > 0;$$

(8.8) $$\|J_\lambda x - x\| \leqq \lambda \|Ax\|, \qquad x \in D(A), \quad \lambda > 0;$$

(8.9) if A is linear, $A_\lambda x = J_\lambda A x, \qquad x \in D(A),\ \lambda > 0.$

For $x, y \in X$, we define the pairings

(8.10) $$[x, y]_+ \equiv \inf_{\lambda > 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda},$$
$$[x, y]_- \equiv \sup_{\lambda < 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

We note at once that

(8.11) $[\ ,\ ]_+ : X \times X \to \mathbf{R}$ is upper semicontinuous

and

(8.12) A is accretive if and only if $0 \leqq [x - \hat{x}, A(x) - A(\hat{x})]_+, \qquad x, \hat{x} \in D(A).$

An operator A is called *strongly* (or *Browder*) *accretive* if

$$0 \leqq [x - \hat{x}, A(x) - A(\hat{x})]_- \qquad x, \hat{x} \in D(A);$$

see Crandall [6] for a simple proof that

(8.13) a continuous, everywhere defined accretive operator is strongly accretive.

To solve various approximate problems in §2–7 we make use of this:

Perturbation Lemma 8.1. *Assume that A is an m-accretive operator on X and that B is accretive, Lipschitz continuous, everywhere defined on X.*
Then $A + B$ is m-accretive on X.
(Here $D(A + B) = D(A)$.)

Proof. By (8.13) B is strongly accretive, and from this it follows that $A + \lambda B$ is accretive for all $\lambda \geqq 0$. Now let $f \in X$ be given. If $R(I + A + \lambda_0 B) = X$ for some $\lambda_0 > 0$, then we can solve

$$x + Ax + \lambda Bx = f$$

for any $\lambda_0 - 1/K < \lambda < \lambda_0 + 1/K$, by rewriting this equation to read

$$x = J_1(A + \lambda_0 B)(f + (\lambda_0 - \lambda)Bx),$$

$(J_1(A+\lambda_0 B)\equiv(I+A+\lambda_0 B)^{-1})$, recalling (8.4), and using the Banach fixed point theorem. Since $R(I+A)=X$ we can start at $\lambda_0=0$ and then proceed in steps of size $1/2K$, say, to prove $R(I+A+B)=X$. ■

We now note that for the case $X=C(\bar{\Omega})$ the brackets $[\ ,\ ]_\pm$ have these characterizations (cf. Sato [22, p. 431] and Sinestrari [23]):

(8.14) $$[f,g]_+=\max_{\substack{x_0\in\bar{\Omega}\\ |f(x_0)|=\|f\|_{C(\bar{\Omega})}}} g(x_0)\cdot\operatorname{sgn} f(x_0),\qquad f\neq 0,$$

(8.15) $$[f,g]_-=\min_{\substack{x_0\in\bar{\Omega}\\ |f(x_0)|=\|f\|_{C(\bar{\Omega})}}} g(x_0)\cdot\operatorname{sgn} f(x_0),\qquad f\neq 0.$$

(8.16) If Ω is unbounded, but $|f|<\|f\|_{C(\bar{\Omega})}$ off some compact set, (8.14) still holds.

If $X=L^\infty(\Omega)$ this analogous, but more complicated representation holds:

(8.17) $$[f,g]_+=\lim_{\varepsilon\searrow 0}\operatorname*{ess\,sup}_{\Omega(f,\varepsilon)} f(x)\cdot\operatorname{sgn} f(x),\qquad f\neq 0,$$

where $\Omega(f,\varepsilon)$ is defined (up to a set of measure zero) by

$$\Omega(f,\varepsilon)\equiv\{x\in\Omega\,|\,|f(x)|>\|f\|_{L^\infty(\Omega)}-\varepsilon\}.$$

In view of (8.14) and (8.15) the class of accretive operators on $C(\bar{\Omega})$ has certain nice properties (cf. [8]):

LEMMA 8.2. (i) *Let A^γ ($\gamma\in\Gamma$) be any collection of strongly accretive operators on $C(\bar{\Omega})$. Then*

$$Au\equiv\sup_{\gamma\in\Gamma}A^\gamma u\qquad\left(\text{resp. } Bu\equiv\inf_{\gamma\in\Gamma}A^\gamma u\right),$$

defined for $D(A)\equiv\{u\in C(\bar{\Omega})\,|\,u\in\bigcap_{\gamma\in\Gamma}D(A^\gamma)$ and $\sup_\gamma A^\gamma u\in C(\bar{\Omega})\}$ (resp. *$D(B)\equiv\{u\in C(\bar{\Omega})\,|\,u\in\bigcap_{\gamma\in\Gamma}D(A^\gamma)$ and $\inf_\gamma A^\gamma u\in C(\bar{\Omega})\}$*), *is strongly accretive on $C(\bar{\Omega})$.*

(ii) *If A is accretive on $C(\bar{\Omega})$ and $\beta:\mathbf{R}\to\mathbf{R}$ is continuous and nondecreasing*

$$Bu\equiv\beta(Au),$$

defined for $u\in D(B)=D(A)$, is accretive on $C(\bar{\Omega})$.

PROOF. (i) Choose $u,\hat{u}\in D(A)$ and let x_0 be any point where

$$(u-\hat{u})(x_0)=\|u-\hat{u}\|_{C(\bar{\Omega})}.$$

Then, since the A^γ are strongly accretive,

$$(A^\gamma(u)-A^\gamma(\hat u))(x_0)\geqq 0, \qquad \gamma\in\Gamma;$$

hence

$$\left(\sup_{\gamma\in\Gamma} A^\gamma(u)-\sup_{\gamma\in\Gamma} A^\gamma(u)\right)(x_0)\geqq 0.$$

Hence A is strongly accretive by (8.12) and (8.15); the proof for B is similar.

(ii) Again select $u,\hat u\in D(B)$. Since A is accretive, there exists some $x_0\in\bar\Omega$ such that

$$(u-\hat u)(x_0)=\|u-\hat u\|_{C(\bar\Omega)}$$

and

$$(A(u)-A(\hat u))(x_0)\geqq 0.$$

Hence

$$(\beta(A(u))-\beta(A(\hat u)))(x_0)\geqq 0,$$

so that B is accretive. ■

9. Appendix 2: Technical lemmas

In each of the convergence proofs in §3, 5 we prove inequalities of the form

$$0\leqq[u-\phi,f-A(\phi)]_+$$

for all sufficiently nice "test" functions ϕ. That this implies $A(u)=f$ follows from these various assertions concerning the richness of the class of such ϕ:

LEMMA 9.1. *Let Ω be a smooth bounded domain of $\mathbf{R}^n$. Assume that $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ for some $p>n$. Then for a.e. $x_0\in\Omega$ there exists a sequence $\{\phi^k\}\subset C_0^\infty(\Omega)$ such that*

(i) $\phi^k(x_0)-u(x_0)=\|\phi^k-u\|_{C(\bar\Omega)}>|\phi^k(x)-u(x)|$ *for all* $x\in\bar\Omega$, $x\neq x_0$,

(ii) $D^2\phi^k(x_0)\to D^2u(x_0)$, $D\phi^k(x_0)\to Du(x_0)$, $\phi^k(x_0)\to u(x_0)$ *as* $k\to\infty$.

This is lemma 2.2 in Evans [7].

LEMMA 9.2. *Assume that Ω is a bounded, smooth domain of $\mathbf{R}^n$ and $u:\bar\Omega\to\mathbf{R}$ is Lipschitz. Then for a.e. $x_0\in\Omega$ and each $M>0$ there exists $\phi\in C_0^1(\Omega)$ satisfying*

(i) $\phi(x_0)-u(x_0)=\|\phi-u\|_{C(\bar\Omega)}>|\phi(x)-u(x)|$ *for all* $x\in\bar\Omega$, $x\neq x_0$,

(ii) $\phi(x_0)\geqq M$.

PROOF. Choose $x_0 \in \Omega$ to be any point where u is differentiable; by Rademacher's theorem (Friedman [12, p. 122]) a.e. x_0 satisfies this. There is no restriction in supposing $x_0 = 0$. Thus

$$|u(x) - u(0) - u_{x_i}(0)x_i| \leqq \rho(|x|), \tag{9.1}$$

where

$$\lim_{\substack{x\to 0\\ x\in\Omega}} \frac{\rho(|x|)}{|x|} = 0. \tag{9.2}$$

Furthermore we may assume $\rho > 0$ for $|x| > 0$ and

$$\rho(|x|) \text{ is a } C^1 \text{ function of } x. \tag{9.3}$$

(For if not, define

$$\rho_1(t) \equiv \sup_{0\leqq s\leqq t} \rho(s) \geqq \rho(t),$$

$$\rho_2(t) \equiv \frac{1}{\log 2}\int_t^{2t} \frac{\rho_1(s)}{s}\, ds \geqq \rho_1(t) \geqq \rho(t),$$

and replace ρ by

$$\rho_3(t) \equiv \frac{1}{\log 2}\int_t^{2t} \frac{\rho_2(s)}{s}\, ds \geqq \rho(t);$$

$\rho_3(|x|)$ is clearly C^1 for $|x| > 0$, ρ_i $(i = 1,2,3)$ satisfies (9.2), and therefore

$$\begin{aligned}\frac{\partial}{\partial x_i}\rho_3(|x|) &= \rho_3'(|x|)\frac{x_i}{|x|}\\ &= \frac{1}{\log 2}\left(\frac{\rho_2(2|x|)}{|x|} - \frac{\rho_2(|x|)}{|x|}\right)\frac{x_i}{|x|} \to 0 = \frac{\partial}{\partial x_i}\rho_3(0), \quad \text{as } |x|\to 0.\Big)\end{aligned}$$

Now choose a C^1 function ψ with compact support in Ω so that

$$\psi(x) = u(0) + u_{x_i}(0)x_i + 1 - 2\rho(|x|)$$

for all x near 0, say $|x| \leqq \varepsilon_0$.

We claim $\psi - u$ has a strict local maximum at 0. Indeed $\psi(0) - u(0) = 1$, whereas for $0 < |x| < \varepsilon_0$,

$$|\psi(x) - u(x)| \leqq 1 - 2\rho(|x|) + |u(x) - u(0) - u_{x_i}(0)x_i| \leqq 1 - \rho(|x|) < 1 \text{ by (9.1).}$$

Finally choose a smooth function ζ satisfying

$$\text{support}\ \zeta \subset B(0, \varepsilon_0), \qquad 0 \leqq \zeta \leqq 1$$

$$1 = \zeta(0) = \max \zeta;$$

then

$$\phi \equiv K\zeta + \psi$$

for large enough K meets the requirements of the lemma. ∎

REFERENCES

1. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, 1976.

2. S. H. Benton, Jr., *The Hamilton Jacobi Equation: A Global Approach*, Academic Press, New York, 1977.

3. J.-M. Bony, *Principe du maximum dans les espaces de Sobolev*, C. R. Acad. Sci. Paris **265** (1967), 333–336.

4. F. Browder, *Variational boundary value problems for quasilinear elliptic equations of arbitrary order*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 31–37.

5. C. Burch, *A semigroup treatment of the Hamilton–Jacobi equation in several space variables*, J. Differential Equations **23** (1977), 107–124.

6. M. G. Crandall, *An introduction to evolution governed by accretive operators*, Proceedings of the International Symposium on Dynamical Systems, Brown University, Providence, R. I., August 1974.

7. L. C. Evans, *A convergence theorem for solutions of nonlinear second order elliptic equations*, Indiana Univ. Math. J. **27** (1978), 875–887.

8. L. C. Evans, *Application of nonlinear semigroup theory to certain partial differential equations*, in *Nonlinear Evolution Equations* (M. G. Crandall, ed.), Academic Press, New York, 1978.

9. L. C. Evans and A. Friedman, *Optimal stochastic switching and the Dirichlet problem for the Bellman equation*, Trans. Amer. Math. Soc. **253** (1979), 365–389.

10. W. H. Fleming, *The Cauchy problem for degenerate parabolic equations*, J. Math. Mech. **13** (1964), 987–1008.

11. W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

12. A. Friedman, *Differential Games*, Wiley-Interscience, New York, 1971.

13. A. Friedman, *The Cauchy problem for first order partial differential equations*, Indiana Univ. Math. J. **23** (1973), 27–40.

14. H. Grad, P. N. Hu and D. C. Stevens, *Proc. Nat. Acad. Sci. U.S.A.* **72** (1975), 3789–3793.

15. S. N. Kruzkov, *First order quasilinear equations in several independent variables*, Math. USSR-Sb. **10** (1970), 217–243.

16. J. L. Lions, *Quelques méthods de résolution des problèmes aux limites non-linéaires*, Dunod, Paris, 1969.

17. P. L. Lions, *Résolution des problèmes de Bellman-Dirichlet: une méthode analytique I*, to appear.

18. G. J. Minty, *Monotone (non-linear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346.

19. J. Mossino, *Sur certaines inéquations quasi-variationnelles apparaissant en physique*, C. R. Acad. Sci. Paris **282** (1976), 187–190.

20. J. Mossino, *Application des inéquations quasi-variationelles a quelques problèmes non linéaires de la physique des plasmas*, Israel J. Math. **30** (1978), 14–50.

21. J. Mossino and J.-P. Zolesio, *Solution variationnelle d'un problème non linéaire de la physique des plasmas*, C. R. Acad. Sci. Paris **285** (1977), 1033–1036.

22. K. Sato, *On the generators of non-negative contraction semi-groups in Banach lattices*, J. Math. Soc. Japan **20** (1968), 423–436.

23. E. Sinestrari, *Accretive differential operators*, Boll. Un. Mat. Ital. **13** (1976), 19–31.

24. M. B. Tamburro, *The evolution operator solution of the Cauchy problem for the Hamilton–Jacobi equation*, Israel J. Math. **26** (1977), 232–264.

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