SOME PROPERTIES OF VISCOSITY SOLUTIONS
OF HAMILTON-JACOBI EQUATIONS

BY

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ABSTRACT. Recently M. G. Crandall and P. L. Lions introduced the notion of "viscosity solutions" of scalar nonlinear first order partial differential equations. Viscosity solutions need not be differentiable anywhere and thus are not sensitive to the classical problem of the crossing of characteristics. The value of this concept is established by the fact that very general existence, uniqueness and continuous dependence results hold for viscosity solutions of many problems arising in fields of application. The notion of a "viscosity solution" admits several equivalent formulations. Here we look more closely at two of these equivalent criteria and exhibit their virtues by both proving several new facts and reproving various known results in a simpler manner. Moreover, by forsaking technical generality we hereby provide a more congenial introduction to this subject than the original paper.

Introduction. Recently two of the authors (see [2,3]) defined a class of generalized solutions of nonlinear scalar partial differential equations of the form

$$(0.1) \qquad F(y, u(y), Du(y)) = 0 \quad \text{for } y \in \mathcal{O},$$

where $\mathcal{O}$ is an open set in $\mathbf{R}^m$, $F: \mathcal{O} \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ is continuous and $Du = (\partial^u/\partial_{y_1}, \ldots, \partial^u/\partial_{y_m})$ denotes the gradient of u. These generalized solutions—called *viscosity solutions* in [2]—need not be differentiable anywhere, as the only regularity required in the definition is continuity. M. G. Crandall and P. L. Lions [2] utilized this new concept to establish uniqueness, stability, and certain existence theorems for a wide class of equations of the form (0.1). In addition, P. L. Lions in [8] has extended these techniques to obtain further and more general existence results.

Our goal here is first to look more closely at two alternative definitions of solutions of (0.1), each of which was proved equivalent to the "viscosity" notion in [2], and second to present some new properties of these solutions. Although these alternative definitions were mentioned in [2], they were not used there. Here we emphasize that they are more appealing in some respects and more convenient for certain purposes than the one taken as basic in [2] (see, e.g., Evans [6], which stimulated the current work). In particular, choosing appropriately each time one of

these equivalent notions, we can simplify the proofs of several results given in [2]. Furthermore, the concept of viscosity solutions is closely related to some previous work by L. C. Evans [5]. We should also point out that this paper is essentially self-contained and makes easier reading than [2], as we forsake generality here.

Let us first formulate the definition of viscosity solutions in the form we think the most appealing (even if not always the most convenient to use). We begin by recalling that a function u from $\mathcal{O}$ into $\mathbf{R}$ is said to be differentiable at $y_0 \in \mathcal{O}$, and that $Du(y_0) = p_0 \in \mathbf{R}^m$, if we have

$$(0.2) \qquad u(y) = u(y_0) + p_0 \cdot (y - y_0) + o(|y - y_0|).$$

Here $a \cdot b$ is the Euclidean scalar product of a and b, and $g(y) = o(|y - y_0|)$ means $\lim_{y \to y_0} g(y)|y - y_0|^{-1} = 0$. Obviously, (0.2) is the conjunction of the two relations

$$(0.3) \qquad \limsup_{y \to y_0} (u(y) - u(y_0) - p_0 \cdot (y - y_0))|y - y_0|^{-1} \leqslant 0$$

and

$$(0.4) \qquad \liminf_{y \to y_0} (u(y) - u(y_0) - p_0 \cdot (y - y_0))|y - y_0|^{-1} \geqslant 0.$$

It is well known that if u is continuous, it may fail to be differentiable at every $y_0 \in \mathcal{O}$. Nevertheless, there are—as we will see below—many choices of $(y_0, p_0) \in \mathcal{O} \times \mathbf{R}^m$ for which (0.3) or (0.4) holds. It will thus be convenient to give the following definition.

DEFINITION. Let u be a function from $\mathcal{O}$ into $\mathbf{R}$ and let $y_0 \in \mathcal{O}$. Then the *superdifferential* of u at y_0 is the set, denoted by $D^+ u(y_0)$, of $p_0 \in \mathbf{R}^m$ such that (0.3) holds. Similarly, the *subdifferential* of u at y_0 is the set, denoted by $D^- u(y_0)$, of $p_0 \in \mathbf{R}^m$ such that (0.4) holds.

In general, $D^\pm u(y_0)$ are closed and convex sets. There is an obvious relation between our "subdifferential" and the notion used in convex analysis. We have also learned that the subdifferentials used here were previously employed by E. DeGiorgi, A. Marino and M. Tosques in another context in [4]. We may now define the concept of viscosity solution of (0.1).

DEFINITION 1. A *viscosity solution* of

$$(0.1) \qquad F(y, u(y), Du(y)) = 0 \quad \text{in } \mathcal{O}$$

is a function $u \in C(\mathcal{O})$ satisfying

$$(0.5) \qquad F(y, u(y), p) \leqslant 0 \qquad \forall y \in \mathcal{O}, \forall p \in D^+ u(y),$$

and

$$(0.6) \qquad F(y, u(y), p) \geqslant 0 \qquad \forall y \in \mathcal{O}, \forall p \in D^- u(y).$$

In a similar way, $u \in C(\mathcal{O})$ is said to be a viscosity subsolution (resp. supersolution) of (0.1) if (0.5) (resp. (0.6)) holds.

We will reprove in §1 that this is equivalent to the following notion of solution of (0.1).

DEFINITION 2. $u \in C(\mathcal{O})$ is a *viscosity solution* of (0.1) provided for all $\phi \in C^1(\mathcal{O})$,

$$(0.7) \qquad \text{if } u - \phi \text{ attains a local maximum at } y_0 \in \mathcal{O}, \text{ then}$$

$$F(y_0, u(y_0), D\phi(y_0)) \leqslant 0$$

and

(0.8)   if $u - \phi$ attains a local minimum at $y_0 \in \mathcal{O}$, then

$$F(y_0, u(y_0), D\phi(y_0)) \geq 0.$$

Moreover, each of these notions is equivalent to the one introduced in [2] (see §1).

In §1 we also establish various properties of viscosity solutions, such as, for example, the consistency with classical solution. §2 is devoted to a uniqueness result concerning equations of the type

(0.9) $$u + H(Du) = f, \qquad x \in \mathbf{R}^n,$$

while in §4 we give a general uniqueness result for the initial value problem

(0.10) $$\begin{cases} u_t + H(Du) = 0, & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n. \end{cases}$$

Here Du denotes the gradient with respect to $x (\in \mathbf{R}^n)$. Of course (0.10) is a special case of (0.1) with $\mathcal{O} = \mathbf{R}^n \times (0, +\infty)$, $y = (x, t)$, $m = n + 1$.

In §3 we show that the well-known vanishing viscosity method yields viscosity solutions; and finally in §5 we consider the relations between viscosity solutions of (0.10) and nonlinear semigroup theory.

Let us recall that many of the results presented here have been already proved in [2], but the proofs herein seems to be simpler (essentially due to our freedom in choosing among the equivalent definitions of viscosity solution). Since the main point here is simplicity, we will not consider more general Hamiltonians H than in (0.9) or (0.10). Technical generality (e.g., $H(x, t, u, Du)$ in (0.10)) is available in [2].

1. Main properties of viscosity solutions.

1.1. Equivalent definitions.

THEOREM 1.1. *Let $u \in C(\mathcal{O})$. Then the following are equivalent*:

  (i) *u satisfies (0.5) and (0.6)*,

  (ii) *u satisfies (0.7) and (0.8) for all $\phi \in C^1(\mathcal{O})$*,

  (iii) *u satisfies, for all $\phi \in C_c^1(\mathcal{O})$,[3] $\phi \geq 0$ and $k \in \mathbf{R}$,*

(1.1) $$\begin{cases} \textit{if } \max_{\mathcal{O}} \phi(u - k) > 0, \textit{ then for some point } y_0 \textit{ at which } \phi(u - k) \\ \\ \textit{attains its maximum, we have } F\left(y_0, u(y_0), -\dfrac{D\phi(y_0)}{\phi(y_0)}(u(y_0) - k)\right) \leq 0 \end{cases}$$

and

(1.2) $$\begin{cases} \textit{if } \min_{\mathcal{O}} \phi(u - k) < 0, \textit{ then for some point } y_0 \textit{ at which } \phi(u - k) \\ \\ \textit{attains its minimum, we have } F\left(y_0, u(y_0), -\dfrac{D\phi(y_0)}{\phi(y_0)}(u(y_0) - k)\right) \geq 0. \end{cases}$$

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[3]$C_c^1(\mathcal{O}) = \{\phi \in C^1(\mathcal{O})$, the support of ϕ is a compact subset of $\mathcal{O}\}$.

REMARK 1.1. It is easy to prove that in (ii) or (iii) we may replace $\phi \in C^1(\mathcal{O})$ by $\phi \in C_c^\infty(\mathcal{O})$. (See the proof of Theorem 3.1.)

REMARK 1.2. Property (iii) is the original definition of viscosity solutions given in [**2**] (where the equivalence of (i), (ii) and (iii) is also proved).

REMARK 1.3. A more precise statement than Theorem 1.1 asserts the equivalence of: (i) u satisfies (0.5), (ii) u satisfies (0.7) and (iii) u satisfies (1.1). When u satisfies one of these equivalent conditions, it is called a viscosity subsolution of (0, 1). An analogous statement holds for supersolutions.

REMARK 1.4. Let us finally mention that we could define the notion of viscosity subsolutions for upper semicontinuous functions and most of the results of [**2**] would still remain valid.

We will prove (i) ⇔ (ii). For the equivalence with (iii), see [**2**]. That (i) ⇔ (ii) is an immediate consequence of the following

PROPOSITION 1.1. *Let* $u \in C(\mathcal{O})$, $y_0 \in \mathcal{O}$, $p \in \mathbf{R}^m$: *Then the following are equivalent*:
(i) $p \in D^+ u(y_0)$ (*resp.* $p \in D^- u(y_0)$) *and*
(ii) *there exists* $\phi \in C^1(\mathcal{O})$ *such that* $u - \phi$ *has a local maximum* (*resp. minimum*) *at* y_0 *and* $D\phi(y_0) = p$.

PROOF. The key fact is the following form [**2**, Lemma I.4] of a result of [**5**]:

LEMMA 1.1. *Let* $\eta \in C(\mathcal{O})$ *be differentiable at* $y_0 \in \mathcal{O}$. *Then there exists* $\Psi_\pm \in C^1(\mathcal{O})$ *such that* $D\Psi_+(y_0) = D\Psi_-(y_0) = D\eta(y_0)$ *and* $\eta - \Psi_+$ ($\eta - \Psi_-$) *has a strict local maximum* (*resp. minimum*) *value of zero at* y_0.

Accepting this lemma, we prove Proposition 1.1. Assume $p \in D^+ u(y_0)$. Set $\eta(y) = \{u(y) - u(y_0) - p \cdot (y - y_0)\}^+$, where r^+ denotes $\max(r, 0)$. By assumption, η is differentiable at y_0 and $D\eta(y_0) = 0$. Let $\Psi_+ \in C^1(\mathcal{O})$ be as in Lemma 1.1. Then near y_0,

$$\{u(y) - [u(y_0) + p \cdot (y - y_0)]\}^+ - \Psi_+(y) \leqslant 0;$$

so if $\phi(y) = u(y_0) + p \cdot (y - y_0) + \Psi_+(y)$, $u - \phi$ has a local maximum at y_0 and $D\phi(y_0) = p$. Thus (i) implies (ii). It is clear that (ii) implies (i), since if $u - \phi$ has a maximum at y_0, then near y_0 we have

$$u(y) \leqslant u(y_0) - \phi(y_0) + \phi(y) \leqslant u(y_0) + D\phi(y_0) \cdot (y - y_0) + o(|y - y_0|).$$

This implies $D\phi(y_0) \in D^+ u(y_0)$. (We have dealt with D^+ and local maxima; the situation for D^- and local minima is entirely the same.) □

1.2. *Elementary properties of viscosity solutions.* We begin with a simple result which establishes the consistency of the notions of viscosity and classical solutions.

THEOREM 1.2. (i) *Let* $u \in C^1(\mathcal{O})$ *be a classical solution of* (0.1), *that is*

$$F(y, u(y), Du(y)) = 0 \quad in \, \mathcal{O}, u \in C^1(\mathcal{O}).$$

Then u *is a viscosity solution.*
(ii) *Let* u *be a viscosity solution of* (0.1) *which is differentiable at some* $y_0 \in \mathcal{O}$. *Then*

$$F(y_0, u(y_0), Du(y_0)) = 0.$$

REMARK 1.5. This result is proved in [**2**, Corollary I.6]. (See also [**5**, pp. 237–238].) Analogous statements hold for viscosity sub- and supersolutions. Obviously, we deduce from (ii) above that if u is a viscosity solution of (0.1) and if $u \in W_{\text{loc}}^{1,p}(\mathcal{O})$ for $p > m$, then

$$F(y, u, Du) = 0 \quad \text{a.e. in } \mathcal{O}.$$

The proof of Theorem 1.2 is trivial, since if u is C^1, then at every point $y \in \mathcal{O}$, the super- and the subdifferentials coincide and $D^+ u(y) = D^- u(y) = \{Du(y)\}$. In the same way, if u is a viscosity solution of (0.1) and is differentiable at $y \in \mathcal{O}$, then $D^+ u(y_0) = D^- u(y_0) = \{Du(y_0)\}$. $\quad \square$

We next reprove Theorem I.10 of [**2**], which characterizes piecewise C^1 viscosity solutions of $F = 0$. Let $\mathcal{O}$ be divided in two open subsets $\mathcal{O}_+$ and $\mathcal{O}_-$ by a C^1 surface Γ: $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_- \cup \Gamma$. The unit normal to Γ at $y_0 \in \Gamma$ is denoted by $n(y_0)$ and is taken to point into $\mathcal{O}^+$.

THEOREM 1.3. *Let $u \in C(\mathcal{O})$ and $u = u_+$ in $\mathcal{O}_+ \cup \Gamma$, $u = u_-$ in $\mathcal{O}_- \cup \Gamma$ where u_+, u_- are of class C^1 in $\mathcal{O}_+ \cup \Gamma$, and $\mathcal{O}_- \cup \Gamma$. Then u is a viscosity solution of (0.1) if and only if the following conditions hold*:

(a) *u_+ and u_- are classical solutions of $F = 0$ in $\mathcal{O}_+$ and $\mathcal{O}_-$ respectively, and*

(b) *if $y_0 \in \Gamma$, $Ty_0 = \{\tau \in \mathbf{R}^m : n(y_0) \cdot \tau = 0\}$ is the tangent space to Γ at y_0 and P_T is the orthogonal projection of $\mathbf{R}^m$ onto Ty_0, then*

(1.3) $\quad \begin{cases} \text{if } a = Du_+(y_0) \cdot n(y_0) \leq Du_-(y_0) \cdot n(y_0) = b, \text{ we have} \\ F(y_0, u(y_0), P_T Du_\pm(y_0) + \xi n(y_0)) \leq 0 \quad \text{for } a \leq \xi \leq b \end{cases}$

and

(1.4) $\quad \begin{cases} \text{if } a = Du_+(y_0) \cdot n(y_0) \geq Du_-(y_0) \cdot n(y_0) = b, \text{ we have} \\ F(y_0, u(y_0), P_T Du_\pm(y_0) + \xi n(y_0)) \geq 0 \quad \text{for } b \leq \xi \leq a. \end{cases}$

PROOF. First note that the assumptions imply that $P_T Du_+(y_0) = P_T Du_-(y_0)$ for $y_0 \in \Gamma$; hence $P_T Du_\pm(y_0)$ is unambiguous. Now, by assumption, if $y \in \mathcal{O}_+ \cup \mathcal{O}_-$, then $D^+ u(y) = D^- u(y) = \{Du(y)\}$; therefore (a) is equivalent to the satisfaction of (0.5) and (0.6) for $y \in \mathcal{O} \setminus \Gamma$.

We are now going to prove that (b) is equivalent to (0.5) and (0.6) for $y \in \Gamma$. In order to do so, we just need to compute $D^+ u(y_0)$, $D^- u(y_0)$ for $y_0 \in \Gamma$. By assumption we have

(1.5) $\quad u(y) = u(y_0) + P_T Du_+(y_0) \cdot (y - y_0) + (n(y_0) \cdot Du_+(y_0))$
$$\cdot (n(y_0) \cdot (y - y_0)) + o(|y - y_0|) \quad \text{if } y \in \mathcal{O}_+ \cup \Gamma;$$

and

(1.6) $\quad u(y) = u(y_0) + P_T Du_-(y_0) \cdot (y - y_0) + (n(y_0) \cdot Du_-(y_0))$
$$\cdot (n(y_0) \cdot (y - y_0)) + o(|y - y_0|) \quad \text{if } y \in \mathcal{O}_- \cup \Gamma.$$

On the other hand, $p \in D^+ u(y_0)$ if and only if
(1.7)

$$u(y) \leq u(y_0) + p \cdot (y - y_0) + o(|y - y_0|)$$
$$= u(y_0) + P_T p \cdot (y - y_0) + (n(y_0) \cdot p)(n(y_0) \cdot (y - y_0)) + o(|y - y_0|).$$

Recalling that $P_T Du_+(y_0) = P_T Du_-(y_0)$ and that Γ is of class C^1, one deduces immediately that $p \in D^+ u(y_0)$ if and only if

(1.8) $P_T p = P_T Du_-(y_0)$ and $n(y_0) \cdot Du_+(y_0) \leq n(y_0) \cdot p \leq n(y_0) \cdot Du_-(y_0).$

Thus
(1.9)

$$D^+ u(y_0) = \{ P_T Du_\pm(y_0) + \xi n(y_0) : n(y_0) \cdot Du_+(y_0) \leq \xi \leq n(y_0) \cdot Du_-(y_0) \}$$

(and if $n(y_0) \cdot Du_+(y_0) > n(y_0) \cdot Du_-(y_0)$, $D^+ u(y_0)$ is empty). In the same way, we find
(1.10)

$$D^- u(y_0) = \{ P_T Du_-(y_0) + \xi n(y_0) : n(y_0) \cdot Du_-(y_0) \leq \xi \leq n(y_0) \cdot Du_+(y_0) \}.$$

It is then straightforward to complete the proof of Theorem 1.3. $\square$

REMARK 1.6. It is worth noting that due to the use of Definition 1 of viscosity solutions, the above proof of Theorem 1.3 is more direct than in [2]. This phenomenon is further illustrated by two easy results (the first in Corollary I.8 and the second in Theorem I.2 of [2]).

PROPOSITION 1.2. *Let u be a viscosity solution of* (0.1) *and let* $\Phi \in C^1(\mathbf{R})$ *satisfy* $\Phi'(t) > 0$ *in* $\mathbf{R}$, $\Phi(\mathbf{R}) = \mathbf{R}$. *Then* $\Phi(u)$ *is a viscosity solution of*

$$F(x, \Psi(v), \Psi'(v)Dv) = 0,$$

where Ψ *denotes* Φ^{-1}.

PROOF. This is immediate from the observations

$$D^+ \Phi(u)(y_0) = \Phi'(u(y_0))D^+ u(y_0), \quad D^- \Phi(u)(y_0) = \Phi'(u(y_0))D^- u(y_0). \quad \square$$

PROPOSITION 1.3. (a) *Let u, v be viscosity subsolutions (resp. supersolutions) of* (0.1). *Then* $w = \max(u, v)$ *(resp. $w = \min(u, v)$) is a viscosity subsolution (resp. supersolution) of* (0.1).

(b) *Let* $(u_n)_{n \geq 1}$ *be viscosity subsolutions (resp. supersolutions) of* (0.1). *If* $w = \sup_{n \geq 1} u_n$ *(resp., $\inf_{n \geq 1} u_n$) $\in C(\mathcal{O})$, then w is a viscosity subsolution (resp. supersolution) of* (0.1).

PROOF. The proof of (a) is an immediate consequence of the remark

$$D^+ \{ \max(u, v) \}(y_0) \subset D^+ u(y_0) \cup D^+ v(y_0)$$

(resp. $D^- \{ \min(u, v) \}(y_0) \subset D^- u(y_0) \cup D^- v(y_0)$). And (b) is a consequence of (a) and of the following stability result. Observe that $w_m = \sup_{m \geq n \geq 1} u_n$ converges uniformly on compact sets of $\mathcal{O}$ to w, because of Dini's lemma. $\square$

THEOREM 1.4. *Let $F_n(y, t, p)$ be a sequence of continuous functions such that $F_n(y, t, p)$ converges uniformly on compact subsets of* $\mathcal{O} \times \mathbf{R} \times \mathbf{R}^m$ *to some function*

$F(y, t, p)$. Let u_n be a viscosity solution of $F_n(y, u_n, Du_n) = 0$ in $\mathcal{O}$. We assume that u_n converges uniformly on compact subsets of $\mathcal{O}$ to some u. Then u is a viscosity solution of $F(y, u, Du) = 0$.

PROOF. By Theorem 1.1, we need only to consider points of local maximum of $u - \phi$ for $\phi \in C^1(\mathcal{O})$. Let y_0 be such a point and let us prove that

$$F(y_0, u(y_0), D\phi(y_0)) \leq 0.$$

Take $\Psi \in C^1(\mathcal{O})$ such that $0 \leq \Psi < 1$ if $y \neq y_0$ and $\Psi(y_0) = 1$. It is clear that $u - (\phi - \Psi)$ attains a local strict maximum at y_0 and thus for n large enough, there will exist y_n such that $u_n - (\phi - \Psi)$ attains a local maximum at y_n, $y_n \in \mathcal{O}$, and $y_n \to y_0$. By assumption, we have

$$F_n(y_n, u_n(y_n), D\phi(y_n) - D\Psi(y_n)) \leq 0,$$

and we conclude since $u_n(y_n) \to u(y_0)$ and $D\phi(y_n) - D\Psi(y_n) \to D\phi(y_0) - D\Psi(y_0) = D\phi(y_0)$. $\square$

REMARK 1.7. The proof above actually shows that if $u_n \to u$ in $C(\mathcal{O})$, then for all $p \in D^+ u(y_0)$ (resp. $D^- u(y_0)$) there exist $y_n \in \mathcal{O}$, $p_n \in D^+ u_n(y_n)$ (resp $D^- u_n(y_n)$) such that $y_n \to y_0, p_n \to p$. In other words,

$$D^+ u(y_0) \subset \limsup_{\substack{n \to \infty \\ y \to y_0}} D^+ u_n(y).$$

2. Uniqueness for $u + H(Du) = f$ in $\mathbf{R}^n$. We now consider the uniqueness of viscosity solutions of

$$(2.1) \qquad u + H(Du) = f \quad \text{in } \mathbf{R}^n.$$

Of course, (2.1) is a very special case of (0.1): take $m = n$, $y = x$, $F(y, r, p) = r + H(p) - f(y)$. We will prove the following

THEOREM 2.1. *Let $u, v, f, g, H \in C(\mathbf{R}^n)$. Assume that u, v are bounded and f, g are uniformly continuous on $\mathbf{R}^n$. Assume that u and v are viscosity solutions of, respectively, $u + H(Du) = f, v + H(Dv) = g$ in $\mathbf{R}^n$. Then we have*

$$(2.2) \qquad \sup_{\mathbf{R}^n} (u - v)^+ \leq \sup_{\mathbf{R}^n} (f - g)^+ .$$

REMARK 2.1. By symmetry, we also have $\sup_{\mathbf{R}^n}(v - u)^+ \leq \sup_{\mathbf{R}^n}(g - f)^+$ and thus $\|u - v\| \leq \|f - g\|$, where $\|h\| = \sup_{\mathbf{R}^n} |h(x)|$ is the norm on the space $C_b(\mathbf{R}^n)$ of bounded continuous functions on $\mathbf{R}^n$.

PROOF OF THEOREM 2.1. We begin the proof by assuming the extra conditions

$$(2.3) \qquad \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0.$$

This will keep the ideas clearer; later the full result is established.

We choose a function $\beta \in C^\infty(\mathbf{R}^n)$ with the properties

$$(2.4) \qquad 0 \leq \beta \leq 1, \quad \beta(0) = 1, \quad \beta(x) = 0 \quad \text{if } |x| > 1.$$

Let $M = \max(\|u\|, \|v\|)$, $\varepsilon > 0$, and let $\Phi: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be given by

$$(2.5) \qquad \Phi(x, y) = u(x) - v(y) + 3M\beta_\varepsilon(x - y),$$

where

(2.6) $$\beta_\varepsilon(z) = \beta(z/\varepsilon) \quad \text{for } z \in \mathbf{R}^n.$$

By (2.3)–(2.6) and the choice of β_ε, $\limsup_{|x|+|y|\to\infty} \Phi(x, y) \leqslant 3M$. (In fact, off the support of $\beta_\varepsilon(x - y)$, $\Phi \leqslant 2M$, while if $|x| + |y| \to \infty$ on this support, then $|x|$ and $|y| \to \infty$.) We may assume that $u(\bar{x}) > v(\bar{x})$ for some $\bar{x}$, since (2.2) clearly holds if this is not the case. But then

$$\Phi(\bar{x}, \bar{x}) = u(\bar{x}) - v(\bar{x}) + 3M\beta_\varepsilon(0) > 3M.$$

Therefore Φ assumes its maximum value at some point $(x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}^n$; i.e.

(2.7) $$\Phi(x_0, y_0) = u(x_0) - v(y_0) + 3M\beta_\varepsilon(x_0 - y_0)$$
$$\geqslant u(x) - v(y) + 3M\beta_\varepsilon(x - y) \quad \text{for all } (x, y) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Moreover, $|x_0 - y_0| \leqslant \varepsilon$ since $\beta_\varepsilon(x_0 - y_0) > 0$ by the above.

Now x_0 is a maximum point of $x \mapsto u(x) - (v(y_0) - 3M\beta_\varepsilon(x - y))$ and thus, by assumption,

(2.8) $$u(x_0) + H(-3M(D\beta_\varepsilon)(x_0 - y_0)) \leqslant f(x_0).$$

Similarly, y_0 is a minimum point of $y \mapsto v(y) - (u(x_0) + 3M\beta_\varepsilon(x_0 - y))$ and so

(2.9) $$v(y_0) + H(-3M(D\beta_\varepsilon)(x_0 - y_0)) \geqslant g(y_0).$$

Together, (2.8) and (2.9) yield

(2.10) $$u(x_0) - v(y_0) \leqslant f(x_0) - g(y_0).$$

For $x \in \mathbf{R}^n$,

$$u(x) - v(x) + 3M = \Phi(x, x) \leqslant \Phi(x_0, y_0) \leqslant u(x_0) - v(y_0) + 3M;$$

so, by (2.10),

$$\sup_{\mathbf{R}^n} (u(x) - v(x))^+ \leqslant (u(x_0) - v(y_0))^+ \leqslant (f(x_0) - g(y_0))^+$$

$$\leqslant \sup_{\mathbf{R}^n} (f - g)^+ + |g(x_0) - g(y_0)|$$

$$\leqslant \sup_{\mathbf{R}^n} (f - g)^+ + \omega_g(\varepsilon),$$

where $\omega_g(\cdot)$ is the modulus of continuity of g. Now (2.2) follows upon our letting $\varepsilon \to 0$.

The extra assumption (2.3) was used to guarantee that Φ had a global maximum point (x_0, y_0). To treat the general case, choose $\delta > 0$ and then (x_1, y_1) so that

(2.11) $$\Phi(x_1, y_1) \geqslant \sup_{\mathbf{R}^n \times \mathbf{R}^n} \Phi(x, y) - \delta.$$

Now choose $\zeta \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that

(2.12) $$\begin{cases} 0 \leqslant \zeta \leqslant 1, \zeta(x_1, y_1) = 1, \zeta(x, y) = 0 \quad \text{if } |x - x_1|^2 + |y - y_1|^2 > 1, \\ |D\zeta| \leqslant 2 \quad \text{in } \mathbf{R}^n \times \mathbf{R}^n. \end{cases}$$

Finally, set
(2.13)

$$\Psi(x, y) = \Phi(x, y) + 2\delta\zeta(x, y) = u(x) - v(y) + 3M\beta_\varepsilon(x - y) + 2\delta\zeta(x, y).$$

We claim that Ψ has a global maximum point $(x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}^n$. Indeed, from (2.11) one deduces

$$\Psi(x_1, y_1) = \Phi(x_1, y_1) + 2\delta \geqslant \sup \Phi + \delta,$$

whereas $\limsup_{|x|+|y| \to \infty} \Psi(x, y) \leqslant \sup \Phi$. Moreover, with $\bar{x}, (x_0, y_0)$ as above,

$$u(x_0) - v(y_0) + 3M\beta_\varepsilon(x_0 - y_0) + 2\delta\zeta(x_0, y_0)$$
$$\geqslant u(\bar{x}) - v(\bar{x}) + 3M + 2\delta\zeta(\bar{x}, \bar{x}),$$

hence, $2M + 3M\beta_\varepsilon(x_0 - y_0) + 2\delta > 3M$. We conclude that

$$(2.14) \qquad\qquad |x_0 - y_0| \leqslant \varepsilon \quad \text{if } M > 2\delta.$$

Next, using the assumptions on u, v as above, one deduces

$$u(x_0) + H(-3MD\beta_\varepsilon(x_0 - y_0) - 2\delta D_x\zeta(x_0, y_0)) \leqslant f(x_0)$$

and

$$v(y_0) + H(-3MD\beta_\varepsilon(x_0 - y_0) + 2\delta D_y\zeta(x_0, y_0)) \geqslant g(y_0).$$

These imply

$$(2.15) \qquad\qquad u(x_0) - v(y_0) \leqslant f(x_0) - g(y_0) + \omega_{H,r}(8\delta),$$

where $\omega_{H,r}$ denotes the modulus of continuity of H on $\{\xi \in \mathbf{R}^n; |\xi| \leqslant r\}$ and $r = 3M\|D\beta_\varepsilon\| + 8\delta$. Therefore, for $x \in \mathbf{R}^n$, we deduce

$$u(x) - v(x) + 3M \leqslant u(x) - v(x) + 3M + 2\delta\zeta(x, x) = \Psi(x, x)$$
$$\leqslant \Psi(x_0, y_0) \leqslant u(x_0) - v(y_0) + 3M + 2\delta$$
$$\leqslant \sup_{\mathbf{R}^n}(f - g)^+ + \omega_g(\varepsilon) + \omega_{H,r}(8\delta) + 2\delta + 3M.$$

We conclude upon sending $\delta \to 0$ and then $\varepsilon \to 0$. $\square$

REMARK 2.2. It is worth noting that the essential ingredients of the above proofs are, in fact, general results on the semidifferentials of $u, v \in C_b(\mathbf{R}^n)$. Indeed, we actually proved the following in the course of the proof of Theorem 2.1.

PROPOSITION 2.1. Let $u, v \in C(\mathbf{R}^n)$.

(a) If u, v satisfy (2.3) and $\sup_{\mathbf{R}^n}(u - v) > 0$, then for each $\gamma > 0$ there exist x_0, $y_0 \in \mathbf{R}^n$ satisfying

$$(2.16) \qquad \begin{cases} u(x_0) - v(y_0) \geqslant \sup_{\mathbf{R}^n}(u - v), \quad |x_0 - y_0| \leqslant \gamma, \quad \text{and} \\ D^+u(x_0) \cap D^-v(y_0) \quad \text{is not empty.} \end{cases}$$

(b) Let $u, v \in C_b(\mathbf{R}^n)$ and $\sup(u - v) > 0$. Let $M = \max(\|u\|, \|v\|)$. Then there is a constant $K > 0$, independent of u and v, such that for $0 < \gamma, \lambda \leqslant 1$ there are x_0, $y_0 \in \mathbf{R}^N$ satisfying

$$(2.17) \qquad \begin{cases} \text{(i) } |x_0 - y_0| \leqslant \gamma \text{ and } u(x_0) - v(y_0) \geqslant \sup(u - v) - \lambda M, \\ \text{(ii) there exist } p \in D^+u(x_0), q \in D^-v(y_0) \\ \text{satisfying } |p|, |q| \leqslant KM(1/\gamma + \lambda) \text{ and } |p - q| \leqslant K\lambda M. \end{cases}$$

Of course, if $u, v \in C^1(\mathbf{R}^n)$ in case (a) and if x_0 is a maximum point of $h = u - v$, then $x_0 = y_0$ solves (2.16) with $\gamma = 0$. The C^1 version of (2.17) is that if $h \in C^1(\mathbf{R}) \cap C_b(\mathbf{R})$, then one can solve $h(x_0) \geqslant \sup h - \gamma$, $|Dh(x_0)| < \gamma$ for each $\gamma > 0$. The uniqueness result, Theorem 2.1, is an easy consequence of Proposition 2.1. For example, if u, v satisfy (2.3) and are viscosity solutions of $u + H(Du) = f$ and $v + H(Dv) = g$ respectively, we then deduce

$$u(x_0) + H(p) \leqslant f(x_0), \quad v(y_0) + H(p) \geqslant g(y_0)$$

where x_0, y_0 are as in (2.16) and p is the common element of $D^+ u(x_0)$ and $D^- v(y_0)$. Thus

$$\sup_{\mathbf{R}^n} (u - v) \leqslant f(x_0) - g(y_0) \leqslant \sup_{\mathbf{R}^n} (f - g) + \omega_g(\gamma)$$

and we conclude by letting $\gamma \to 0$.

The point of these remarks is that proofs of facts about viscosity solutions can, in fact, be regarded as applications of general results concerning the generalized calculus of the semidifferentials D^+, D^- to problems at hand.

3. Existence of viscosity solutions. To demonstrate the existence of viscosity solutions (and also explain the name of these solutions) we quickly reprove the following

THEOREM 3.1. *Let $\varepsilon > 0$, and let $F_\varepsilon(y, t, p)$ be a family of continuous functions such that $F_\varepsilon(y, t, p)$ converges uniformly on compact subsets of $\mathcal{O} \times \mathbf{R} \times \mathbf{R}^n$ to some function $F(y, t, p)$, as ε goes to 0. Finally, suppose $u^\varepsilon \in C^2(\mathcal{O})$ is a solution of*

$$-\varepsilon \Delta u^\varepsilon + F_\varepsilon(y, u^\varepsilon, Du^\varepsilon) = 0 \quad in \ \mathcal{O},$$

and let us assume that the u^ε converge uniformly on compact subsets of $\mathcal{O}$ to some $u \in C(\mathcal{O})$. Then u is a viscosity solution of (0.1).

REMARK 3.1. This result is proved in [2, Proposition IV.1]. (See also [5, pp. 237–238].)

PROOF OF THEOREM 3.1. Let us check (0.7) first for $\phi \in C^2(\mathcal{O})$. Assuming that $u - \phi$ has a local maximum at $y_0 \in \mathcal{O}$, choose $\zeta \in C^\infty(\mathcal{O})$ such that $0 \leqslant \zeta < 1$ if $y \neq y_0$, $\zeta(y_0) = 1$. Obviously, $u - (\phi - \zeta)$ has a strict local maximum at $y_0 \in \mathcal{O}$ and thus for ε small enough, $u^\varepsilon - (\phi - \zeta)$ has a local maximum at some $y_\varepsilon \in \mathcal{O}$ and $y_\varepsilon \to y_0$ as $\varepsilon \to 0$. But at the point $y = y_\varepsilon$, we have

$$Du^\varepsilon(y_\varepsilon) = D(\phi - \zeta)(y_\varepsilon), \quad \Delta u^\varepsilon(y_\varepsilon) \leqslant \Delta(\phi - \zeta)(y_\varepsilon);$$

therefore

$$F_\varepsilon(y_\varepsilon, u^\varepsilon(y_\varepsilon), D(\phi - \zeta)(y_\varepsilon)) \leqslant \varepsilon \Delta(\phi - \zeta)(y_\varepsilon).$$

We may conclude since $u^\varepsilon(y_\varepsilon) \to u(y_0)$, $D(\phi - \zeta)(y_\varepsilon) \to D(\phi - \zeta)(y_0) = D\phi(y_0)$, $\varepsilon \Delta(\phi - \zeta)(y_\varepsilon) \to 0$. Next, if $\phi \in C^1(\mathcal{O})$ and if $(u - \phi)$ has a local maximum at $y_0 \in \mathcal{O}$, we again prove (0.7). Take $\phi_n \in C^2(\mathcal{O})$ such that $\phi_n \to \phi$ in $C^1(\mathcal{O})$ and, as before, choose $\zeta \in C^\infty(0)$ such that $0 \leqslant \zeta < 1$ if $y \neq y_0$, $\zeta(y_0) = 1$. For n large enough, $u - (\phi_n - \zeta)$ has a local maximum at some point $y_n \in \mathcal{O}$ and $y_n \to y_0$. By the argument made above, we know

$$F(y_n, u(y_n), D\phi_n(y_n) - D\zeta(y_n)) \leqslant 0.$$

We finish by noting $D\phi_n(y_n) - D\zeta(y_n) \to D\phi(y_0) - D\zeta(y_0) = D\phi(y_0)$. $\square$

As an illustration of the above result, we prove

COROLLARY 3.1. *Assume* $H: \mathbf{R}^n \to \mathbf{R}$ *is continuous,* $\lambda > 0$. *Then for each* $f \in$ BUC$(\mathbf{R}^n) = X$,[4] *there exists a unique viscosity solution of* $u \in X$ *of* $u + \lambda H(Du) = f(x)$ *in* $\mathbf{R}^n$. *Let us denote this solution by* $u = R_\lambda f$. *Then for all* $f, g \in$ BUC$(\mathbf{R}^n)$, *we have*

$$(3.1) \qquad \|(R_\lambda f - R_\lambda g)^+\| \leqslant \|(f - g)^+\|.$$

PROOF. The uniqueness and the estimate (3.1) are immediate consequences of Theorem 2.1. To prove the existence, we consider the approximate problem

$$(3.2) \qquad u^\varepsilon + \lambda H(Du^\varepsilon) - \varepsilon\Delta u^\varepsilon = f^\varepsilon \quad \text{in } \mathbf{R}^n$$

where $H_\varepsilon, f^\varepsilon \in C^\infty(\mathbf{R}^n)$ and $H_\varepsilon \to H$ in $C(\mathbf{R}^n)$ as $\varepsilon \to 0$, $f^\varepsilon \to f$ in $C_b(\mathbf{R}^N)$ as $\varepsilon \to 0$. Routine P.D.E. arguments imply the existence of a unique smooth solution $u^\varepsilon \in$ BUC$(\mathbf{R}^n)$ (see, for example, [7, 3, 8]). In addition if, $v^\varepsilon \in X$ solves $v^\varepsilon + \lambda H(Dv^\varepsilon) - \varepsilon\Delta v^\varepsilon = g^\varepsilon$ in $\mathbf{R}^n$, the maximum principle yields

$$(3.3) \qquad \|u^\varepsilon - v^\varepsilon\| \leqslant \|f^\varepsilon - g^\varepsilon\|.$$

If we take $g^\varepsilon(X) = f^\varepsilon(x + y)$ for some fixed $y \in \mathbf{R}^n$, we obtain

$$\sup_{\mathbf{R}^n} |u^\varepsilon(x) - u^\varepsilon(x + y)| \leqslant \|f^\varepsilon - g^\varepsilon\|;$$

that is,

$$(3.4) \qquad \omega_u \varepsilon(\cdot) \leqslant \omega_f \varepsilon(\cdot).$$

Thus the set $\{u^\varepsilon\}$ is bounded in $C_b(\mathbf{R}^n)$ and equicontinuous, and there exists $\varepsilon_j \to 0$ and $u \in X$ such that $u^{\varepsilon_j} \to_j u$ locally uniformly. Now recall Theorem 3.1. $\square$

REMARK 3.1 For more general existence results, we refer to P. L. Lions [8].

As a final result in this section, we obtain an estimate on a bounded viscosity solution u of

$$(3.5) \qquad u + H(Du) \leqslant f$$

under the assumptions

$$(3.6) \qquad |H(p)| \leqslant L|p|, \qquad p \in \mathbf{R}^n,$$

and

$$(3.7) \qquad |f(x)| \leqslant ae^{-b|x|}, \qquad x \in \mathbf{R}^n,$$

where L, a, b are nonnegative constants. Set

$$(3.8) \qquad v(x) = ce^{-d|x|},$$

where $c, d > 0$. Clearly

$$(3.9) \qquad D^-v(x) = \begin{cases} -cde^{-d|x|}x/|x| & \text{if } x \neq 0, \\ \text{empty} & \text{if } x = 0. \end{cases}$$

---

[4] BUC$(\mathbf{R}^n) = \{u \in C_b(\mathbf{R}^n),\ u \text{ is uniformly continuous on } \mathbf{R}^n\}$.

Thus v is a viscosity supersolution of $v + H(Dv) = f$ provided that

$$(3.10) \qquad ce^{-d|x|} + H\big(-cde^{-d|x|}x/|x|\big) \geqslant f(x) \quad \text{for } x \neq 0.$$

In view of (3.6) and (3.7), the estimate (3.10) is valid provided $ce^{-d|x|} - Lcde^{-d|x|} \geqslant ae^{-b|x|}$ or, equivalently $c(1 - Ld) \geqslant ae^{(d-b)|x|}$. If $d \leqslant b$ and $Ld < 1$ we conclude that $v = a/(1 - Ld)e^{-d|x|}$ is a supersolution of $v + H(Dv) = f$. The proof of (2.2) of Theorem 2.1 used only that u is a subsolution and v is a supersolution; so we conclude that (3.5), (3.6), (3.7) imply

$$(3.11) \qquad u(x) \leqslant \frac{a}{1 - Ld} e^{-d|x|} \quad \text{for } d < b, \, Ld < 1.$$

Compare this with the proof of [6, Lemma 2.2].

4. Uniqueness for $u_t + H(Du) = 0$. This section and the next one concern the Hamilton-Jacobi equation

$$(4.1) \qquad u_t + H(Du) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

with the initial condition

$$(4.2) \qquad u(x,0) = u_0(x), \qquad x \in \mathbf{R}^n.$$

According to our definitions and to Theorem 1.1, $u \in C(\mathbf{R}^n \times (0, T))$ is a viscosity solution of

$$(4.3) \qquad u_t + H(Du) = 0 \quad \text{on } \mathbf{R}^n \times (0, T)$$

provided that, for every $\phi \in C^1(\mathbf{R}^n \times (0, T))$, we have $\phi_t + H(D\phi) \leqslant 0$ (resp. $\geqslant 0$) at local maxima (resp. minima) of $u - \phi$. We will prove

THEOREM 4.1. *Let $0 < T < \infty$ and let $u, v \in \mathrm{BUC}(\mathbf{R}^n \times [0, T])$ be viscosity solutions of* (4.3). *Then*

$$\sup_{\mathbf{R}^n \times [0, T]} (u - v)^+ \leqslant \sup_{\mathbf{R}^n} (u(x,0) - v(x,0))^+.$$

REMARK 4.1. The method of §3 is easily adapted to prove that if $u_0 \in \mathrm{BUC}(\mathbf{R}^n)$, then (4.1), (4.2) has a unique viscosity solution u such that $u \in \mathrm{BUC}(\mathbf{R}^n \times [0, T])$ for every $0 < T < \infty$. This will also follow from §5 and nonlinear semigroup theory.

PROOF OF THEOREM 4.1. We will give the proof in the general case. However, let us first dispose of the following technicality.

LEMMA 4.1. *Let $0 < T < \infty$ and let $u \in C(\mathbf{R}^n \times (0, T))$ be a viscosity solution of* (4.3). *If $\phi \in C^1(\mathbf{R}^n \times (0, T])$, then at each local maximum point (resp. minimum point) of $u - \phi$ on $\mathbf{R}^n \times (0, T]$, we have $\phi_t + H(D\phi) \leqslant 0$ (resp. $\geqslant 0$).*

PROOF. The point is that assumptions on $(0, T)$ imply conclusions on $(0, T]$; this is because of the special dependence of the equation on the time derivative. For example, we prove that if (x_0, T) is a local maximum in $\mathbf{R}^n \times (0, T]$ of $u - \phi$ with $\phi \in C^1(\mathbf{R}^n \times (0, T])$, then $\phi_t(x_0, T) + H(D\phi(x_0, T)) \leqslant 0$. As observed many times above, we may assume (x_0, T) is a strict local maximum of $u - \phi$ on $\mathbf{R}^n \times (0, T]$. Then we choose $\mu, r > 0$, $0 < r < T$, so that

$$(4.4) \qquad u(x, t) - \phi(x, t) \leqslant u(x_0, T) - \phi(x_0, T) - \mu$$

for

(4.5) $(x, t) \in K_r = \{(x, t) : |x - x_0| = r$ and

$$0 \leq T - t \leq r \text{ or } |x - x_0| \leq r \text{ and } t = T - r\}.$$

If $\varepsilon > 0$ is small, it is then obvious that $\psi_\varepsilon(x, t) = u(x, t) - \phi(x, t) - \varepsilon(T - t)^{-1}$ has a local maximum point $(x_\varepsilon, t_\varepsilon)$ such that $t_\varepsilon < T$ and $(x_\varepsilon, t_\varepsilon) \to (x_0, T)$ as $\varepsilon \to 0$. By assumption we then have

$$\phi_t(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon}{(T - t_\varepsilon)^2} + H(D\phi(x_\varepsilon, t_\varepsilon)) \leq 0.$$

This implies $\phi_t(x_\varepsilon, t_\varepsilon) + H(D\phi(x_\varepsilon, t_\varepsilon)) \leq 0$ and the result follows upon letting $\varepsilon \to 0$. $\square$

We may now begin the proof of Theorem 4.1, which involves the construction of a rather complicated "test function". Define σ by

(4.6) $$\sup_{\mathbf{R}^n \times [0, T]} (u - v) = \sup_{\mathbf{R}^n} (u_0(x) - v_0(x)) + \sigma,$$

where here and below $u_0 = u(\cdot, 0)$, $v_0 = v(\cdot, 0)$. If $\sigma = 0$, there is nothing to prove, and so we may assume $\sigma > 0$.

Choose $\beta \in C^\infty(\mathbf{R}^n \times \mathbf{R})$ so that

(4.7) $$\begin{cases} 0 \leq \beta \leq 1, \quad \beta(0, 0) = 1 \quad \text{and} \\ \beta(x, t) = 0 \quad \text{if } |x|^2 + t^2 > 1. \end{cases}$$

Set $\beta_\varepsilon(x, t) = \beta(x/\varepsilon, t/\varepsilon)$ and $M = \max(\|u\|, \|v\|)$, where $\|h\|$ denotes the norm of h in $C_b(\mathbf{R}^n \times [0, T])$ (note that since $\sigma > 0$, $M > 0$). Next, let $\lambda > 0$ be fixed and define $\Phi: \mathbf{R}^n \times \mathbf{R}^n \times [0, T] \times [0, T] \to \mathbf{R}$ by

(4.8)

$$\Phi(x, y, t, s) = u(x, t) - v(y, s) - \lambda(t + s) + (5M + 2\lambda T)\beta_\varepsilon(x - y, t - s).$$

If Φ attains its maximum on $\mathbf{R}^{2n} \times [0, T]^2$ at some point, the proof is easily completed, but this need not be so. Therefore we choose $\delta > 0$ and then $(x_0, y_0, t_0, s_0) \in \mathbf{R}^{2n} \times [0, T]^2$ so that

(4.9) $$\Phi(x_0, y_0, t_0, s_0) > \sup \Phi - \delta.$$

The first claim is that if λ, ε and δ are sufficiently small, then for some $\mu > 0$,

(4.10) $$t_0, s_0 \geq \mu,$$

where μ is independent of λ, ε, δ. To see this, first note

(4.11) $$\begin{cases} \Phi(x, y, t, s) \leq 2M \quad \text{if } |x - y|^2 + |t - s|^2 \geq \varepsilon^2, \\ \sup_{\mathbf{R}^{2n} \times [0, T]^2} \Phi \geq \sup_{\mathbf{R}^n} \Phi(x, x, T, T) \geq 3M. \end{cases}$$

Hence, if $\delta < M$, then $|x_0 - y_0|^2 + (t_0 - s_0)^2 < \varepsilon^2$. Therefore for $\delta < M$,

$$
\begin{aligned}
\Phi(x_0, y_0, t_0, s_0) &= u(x_0, t_0) - v(y_0, s_0) - \lambda(t_0 + s_0) \\
&\quad + (5M + 2\lambda T)\beta_\varepsilon(x_0 - y_0, t_0 - s_0) \\
&\leqslant u(x_0, t_0) - v(x_0, t_0) - \lambda(t_0 + s_0) + 5M + 2\lambda T + \omega_v(\varepsilon) \\
&\leqslant u(x_0, 0) - v(x_0, 0) - \lambda(t_0 + s_0) + 5M + 2\lambda T \\
&\quad + \omega_u(t_0) + \omega_v(t_0) + \omega_v(\varepsilon).
\end{aligned}
$$

Since on the other hand we have, in view of (4.6),

$$
\sup \Phi(x, x, t, t) \geqslant \sup(u(x, t) - v(x, t)) + 5M = \sup(u_0 - v_0) + \sigma + 5M,
$$

we finally obtain

$$
2\lambda T + \omega_u(t_0) + \omega_v(t_0) + \omega_v(\varepsilon) \geqslant \sigma - \delta.
$$

Now if $\delta < \sigma/4$, ε is small enough to force $\omega_v(\varepsilon) < \sigma/4$, $2\lambda T < \sigma/4$, and μ is chosen so that $\omega_u(\xi) + \omega_v(\xi) \leqslant \sigma/4$ for $0 < \xi \leqslant \mu$, we conclude $t_0 \geqslant \mu$. Similarly we obtain $s_0 \geqslant \mu$ and (4.10) is proved.

Next select $\zeta \in C^\infty(\mathbf{R}^{2n} \times [0, T]^2)$ satisfying $0 \leqslant \zeta \leqslant 1$, $\zeta(x_0, y_0, t_0, s_0) = 1$, $\zeta = 0$ if $|x - x_0|^2 + |y - y_0|^2 + |t - t_0|^2 + |s - s_0|^2 \geqslant \mu^2/4$. We set $\Psi(x, y, t, s) = \Phi(x, y, t, s) + 2\delta\zeta(x, y, t, s)$. Since $\Psi \leqslant \Phi$ off the support of ζ and since

$$
\Psi(x_0, y_0, t_0, s_0) = \Phi(x_0, y_0, t_0, s_0) + 2\delta > \sup \Phi + \delta,
$$

Ψ attains its maximum at some point (x_1, y_1, t_1, s_1), which lies in the support of ζ. Thus $t_1, s_1 \geqslant \mu/2$. But $(x_1, t_1) \in \mathbf{R}^n \times [0, T]$ is a maximum of $(x, t) \to u(x, t) - v(y_1, s_1) - \lambda(t + s_1) + (5M + 2\lambda T)\beta_\varepsilon(x - y_1, t - s_1) + 2\delta\zeta(x, y_1, t, s_1)$, so that Lemma 4.1 implies

$$
\lambda - (5M + 2\lambda T)\frac{\partial \beta_\varepsilon}{\partial t}(x_1 - y_1, t_1 - s_1) - 2\delta\zeta_t(x_1, y_1, t_1, s_1)
$$
$$
+ H\big(-(5M + 2\lambda T)D_x\beta_\varepsilon(x_1 - y_1, t_1 - s_1) - 2\delta D_x\zeta(x_1, y_1, t_1, s_1)\big) \leqslant 0.
$$

Similarly,

$$
-\lambda + (5M + 2\lambda T)\frac{\partial \beta_\varepsilon}{\partial t}(x_1 - y_1, t_1 - s_1) + 2\delta\zeta_s(x_1, y_1, t_1, s_1)
$$
$$
+ H\big(-(5M + 2\lambda T)D_x\beta_\varepsilon(x_1 - y_1, t_1 - s_1) + 2\delta D_y\zeta(x_1, y_1, t_1, s_1)\big) \geqslant 0.
$$

Combining these two inequalities and letting $\delta \to 0$, we derive $\lambda = 0$, a contradiction. $\square$

REMARK 4.2. The assumption $u, v \in \mathrm{BUC}(\mathbf{R}^n \times [0, T])$ was used in the proof, but minor modifications allow one to weaken this to $u_0, v_0 \in \mathrm{BUC}(\mathbf{R}^n)$, $u(x, t) \to u_0(x)$, $v(x, t) \to v_0(x)$ uniformly on $\mathbf{R}^n$ as $t \to 0$.

REMARK 4.3. Theorem 4.1 is a special case of [2, Theorem V.2]. The proof in [2] is, however, only indicated and involves a more cumbersome comparison. See [2, § V.3] concerning domains of dependence.

5. The semigroup approach. Let $H \in C(\mathbf{R}^n)$, and $X = \mathrm{BUC}(\mathbf{R}^n)$. We now realize the formal expression "$H(Du)$" as a nonlinear operator on X.

DEFINITION. We say that $u \in X$ belongs to $D(A)$ (the *domain* of A) if u is a viscosity solution of $H(Du) = f$ in $\mathbf{R}^n$; for some $f \in X$. We denote by Au the set of all such $f \in X$.

Obviously, $D(A) \supset \{u \in X, Du \in X\}$ (because of the consistency result Theorem 1.2) and so $\overline{D(A)} = X$.

REMARK 5.1. Except in very special cases we do not know whether A is single valued. (See L. C. Evans [6] when H is uniformly continuous on $\mathbf{R}^n$; the case when $H(p) \to \infty$ as $|p| \to \infty$ is easily deduced from the results of M. G. Crandall and P. L. Lions [2].) For simplicity, we write as if A were single valued below.

PROPOSITION 5.1. *A is an m-accretive operator on X.*

REMARK 5.2. See Barbu [1] or L. C. Evans [5, Appendix 1] for definitions.

PROOF OF PROPOSITION 5.1. In view of Corollary 3.1 we have $R(I + \lambda A) = X$ (for $\lambda > 0$) and by (3.1), $\|u - v\| \leq \|u - v + \lambda(Au - Av)\|$ if $\lambda > 0$, $u, v \in D(A)$. $\square$

Proposition 5.1 implies, by the Crandall-Liggett generation theorem (cf. [1]), that A generates a nonlinear semigroup of contractions $(S(t))_{t \geq 0}$ on X and

$$(5.1) \qquad S(t)u_0 = \lim_{\substack{n \to \infty \\ n\lambda \to t}} (I + \lambda A)^{-n} u_0, \qquad u_0 \in X = \overline{D(A)},$$

uniformly for t in bounded subsets of $(0, \infty)$. Furthermore, the mapping $t \mapsto S(t)u_0$ is continuous from $[0, \infty)$ into X.

In general the semigroup generated by a (nonlinear) m-accretive operator A can be regarded as a "mild" solution of the evolution equation

$$\frac{du}{dt} + Au = 0 \quad (t > 0), \qquad u(0) = u_0;$$

see the forthcoming book of P. Benilan, M. G. Crandall and A. Pazy [1]. For the case at hand a stronger interpretation is possible.

PROPOSITION 5.2. *Assume $u_0 \in X$. Then*
$$u(x, t) = (S(t)u_0)(x), \qquad 0 \leq t \leq T, x \in \mathbf{R}^n,$$
is the viscosity solution of (4.1), (4.2).

PROOF. Assume $\phi \in C^1(\mathbf{R}^n \times (0, T))$ and $u - \phi$ attains a local maximum at $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$. We may as well assume (x_0, t_0) is a strict local maximum of $u - \phi$ in view of arguments used above.

For each $\varepsilon > 0$, consider the step function $u^\varepsilon(t)$ solving

$$(5.2) \qquad \begin{cases} \dfrac{1}{\varepsilon}\{u^\varepsilon(t + \varepsilon) - u^\varepsilon(t)\} + Au^\varepsilon(t + \varepsilon) = 0, & t \geq 0, \\ u^\varepsilon(t) = u_0 \quad \text{if } 0 \leq t \leq \varepsilon. \end{cases}$$

We may assume $t_0 \neq k\varepsilon$ for any integer k by appropriate choice of ε. Since $u^\varepsilon(t) \to S(t)u_0$ uniformly on $[0, T]$ in X as $\varepsilon \to 0$, $u^\varepsilon(x, t + \varepsilon) - \phi(x, t)$ has a local maximum at some point $(x^\varepsilon, t^\varepsilon)$, such that $(x^\varepsilon, t^\varepsilon) \in \mathbf{R}^n \times (0, T)$, $x^\varepsilon \to x_0$, $t^\varepsilon \to t_0$ as $\varepsilon \to 0$. Hence,

$$(5.3) \qquad Au^\varepsilon(t^\varepsilon + \varepsilon) - A\phi \geq 0 \quad \text{at } x_\varepsilon,$$

according to the definition of A (note $u^\varepsilon(dot + \varepsilon) \in D(A)$). Also, if ε is small enough, we have

$$\frac{1}{\varepsilon}\left(u^\varepsilon(x^\varepsilon, t^\varepsilon + \varepsilon) - u^\varepsilon(x^\varepsilon, t^\varepsilon)\right) \geqslant \frac{1}{\varepsilon}\left(\phi(x^\varepsilon, t^\varepsilon) - \phi(x^\varepsilon, t^\varepsilon - \varepsilon)\right).$$

Combining this inequality with (5.3) we deduce

$$A\phi(x^\varepsilon, t^\varepsilon) = H(D\phi(x^\varepsilon, t^\varepsilon)) \leqslant \frac{1}{\varepsilon}\left(\phi(x^\varepsilon, t^\varepsilon - \varepsilon) - \phi(x^\varepsilon, t^\varepsilon)\right).$$

If we let $\varepsilon \to 0$, then, since $(x^\varepsilon, t^\varepsilon) \to (x_0, t_0)$, we finally obtain

$$\phi_t(x_0, t_0) + H(D\phi(x_0, t_0)) \leqslant 0.$$

The opposite inequality has an analogous proof should $u - \phi$ attain a local minimum at (x_1, t_1). $\quad\square$

This result corresponds to [**2**, §VI. 3].

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