FULLY NONLINEAR EQUATIONS AND VISCOSITY SOLUTIONS BIBLIOGRAPHY PART 2 FOR THE COURSE: NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ACADEMIC YEAR 2011-2012

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1. Fully nonlinear equations and viscosity solutions

For general background, some textbooks on fully nonlinear equations are Caffarelli-Cabré [4], Gutiérrez [17], Pogorelov [31] and [33] and Chapter 17 of Gilbarg-Trudinger [16]. The first two books also include aspects of the viscosity theory. For the viscosity theory, we have made much use of the two survey papers Crandall-Ishii-Lions [10] and Crandall [8]. The notion of viscosity solutions originates for first order equations in Crandall-Lions [11]. The basic idea to put derivatives on test functions by way of the maximum principle originated in the work of Evans [12] and [13]. The notion of viscosity solutions was extended to second order equations by Lions in [30].

2. Existence of viscosity solutions by the method of Perron

Again, while we have mainly followed the survey papers of Crandall-Ishii-Lions [10] and Crandall [8], much more should be said. The Perron method was first used by Ishii [19] for first order Hamilton-Jacobi equations and then extended by him to second order equations in [20]. The key observation of Ishii was that a "good" uniqueness theory (in particular, suitable maximum or comparison principles) allows for a completely nonlinear adaptation of the method of Perron for existence (see the paper of Ishii-Lions [21] for this discussion and numerous extensions of the Ishii program). The literature is rich with extensions and variants of this method. See for example the work of Bardi and collaborators [1] and [3].

3. Comparison principles, maximum principles and uniqueness

As we have noted, it is the comparison principle that presents the greatest challenge in establishing uniqueness and sets the table for existence. A major breakthrough was the work of Jensen [22] and [23] which freed the theory from its dependence on the convexity (or concavity) in $F(x, r, q, A)$ with respect to $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ and makes use of regularizations by way of the sup and inf convolutions . This was refined in Jensen-Lions-Souganidis [24]. The analytical underpinnings of the comparison principle were further reformulated making use of the technique of doubling variables and penalization and culminates in the *Theorem on Sums* of Crandall and Ishii [9] which in turn had its origins in Ishii [20] and Ishii-Lions [21] and Crandall [8]. For additional progress on maximum and comparison principles for equations with degenerate ellipticity see the papers of Bardi-Da Lio [2] Bardi-Mannucci [3] and Kawohl-Kutev [25], [26] and [27]. On the other hand, for uniformly elliptic equations (and with more regularity) there is older work of Evans [14] and Trudinger [35].

4. Elliptic branches and duality

An important alternative to the viscosity theory begins with the paper of Krylov [29] on the general notion of ellipticity in which the notion of *elliptic branches* associated to fully nonlinear equations is introduced. This concept led Harvey-Lawson [18] to intoduce a suitable notion of *duality* which reformulates the notion of viscosity solutions in a precise topological framework.

5. Regularity of viscosity solutions

We have not had time to touch the important topic of regularity for viscosity solutions. A nice survey in the case $F(D^2u) = f(x)$ is provided in Part I of the lectures by Evans [15] and a more complete picture is given in Chapter 17 of Gilbarg-Trudinger [16]. In particular, important points along this road is the method of Pogorelov [32] , [31], [33], the work of Cheng-Yau [6] and [7], the landmark paper of Caffarelli-Nirenberg-Spruck [5] and the papers of Krylov [28] and Trudinger [34] just to name a few.

References

[1] M. Bardi and S. Bottacin. On the Dirichlet problem for nonlinear degenerate elliptic equations and applications to optimal control. *Rend. Sem. Mat. Univ. Pol. Torino*, 56(4):13–39, 1998.
[2] M. Bardi and F. Da Lio. On the strong maximum principle for fully nonlinear degenerate elliptic equations. *Arch. Math.*, 73(4):276–285, 1999.
[3] M. Bardi and P. Mannucci. On the Dirichlet problem for non-totally degenerate fully nonlinear elliptic equations. *Commun. Pure Appl. Anal.*, 73(4):709–731, 2006.
[4] L. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations.*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.
[5] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3):369–402, 1984.
[6] S. Y. Cheng and S. T. Yau. On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure Appl. Math.*, 29(5):495–516, 1976.
[7] S. Y. Cheng and S. T. Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$. *Comm. Pure Appl. Math.*, 30(1):41–68, 1977.
[8] M. G. Crandall. Semidifferentials, quadratic forms and fully nonlinear elliptic equations of second order. *Ann. Inst. H. Poincaré Anal. Nonlinéaire*, 6(6):419–435, 1989.
[9] M. G. Crandall and H. Ishii. The maximum principle for semicontinuous functions. *Differential Integral Equations*, 3(6):1001–1014, 1990.
[10] M. G. Crandall, H. Ishii, and P-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.
[11] M. G. Crandall and P-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1981.
[12] L. C. Evans. A convergence theorem for solutions of nonlinear second-order elliptic equations. *Indiana Univ. Math. J.*, 27(5):875–887, 1978.
[13] L. C. Evans. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel J. Math.*, 36(5):225–247, 1980.
[14] L. C. Evans. Classical solutions of fully nonlinear, convex, second order elliptic equations. *Comm. Pure Appl. Math.*, 25(3):333–363, 1982.

[15] L. C. Evans. Regularity for fully nonlinear elliptic equations and motion by mean curvature. In *Viscosity Solutions and Applications (Montecatini Terme, 1995)*, volume 1660 of *Lecture Notes in Mathematics*, pages 98–133. Academic Press, 1997.

[16] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[17] C. E. Gutiérrez. *The Monge-Ampère equation.*, volume 44 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser Boston, Inc., Boston, 2001.

[18] F. R. Harvey and H. B. Lawson. Dirichlet duality and the nonlinear Dirichlet problem. *Comm. Pure Appl. Math.*, 62(3):396–443, 2007.

[19] H. Ishii. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.*, 55(2):369–384, 1987.

[20] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.*, 42(1):15–45, 1989.

[21] H. Ishii and P-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differential Equations*, 83(1):26–78, 1990.

[22] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rat. Mech. Anal.*, 101(1):1–27, 1988.

[23] R. Jensen. Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations. *Indiana U. Math. J.*, 38(3):629–667, 1989.

[24] R. Jensen, P-L. Lions, and P. E. Souganidis. A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations. *Proc. Amer. Math. Soc.*, 102(4):975–978, 1988.

[25] B. Kawohl and N. Kutev. Strong maximum principle for semicontinuous viscosity solutions of nonlinear partial differential equations. *Arch. Math.*, 70(2):470–478, 1998.

[26] B. Kawohl and N. Kutev. Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations. *Funkcial. Ekvac.*, 43(2):241–253, 2000.

[27] B. Kawohl and N. Kutev. Comparison principle for viscosity solutions of fully nonlinear, degenerate elliptic equations. *Comm. Partial Differential Equations*, 32(7-9):1209–1224, 2007.

[28] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Math. USSR Izv.*, 22(1):67–98, 1984. Translation from the original in Russian (Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 1, 75108).

[29] N. V. Krylov. On the general notion of fully nonlinear second-order elliptic equations. *Trans. Amer. Math. Soc.*, 347(3):857–895, 1995.

[30] P. L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part II: Viscosity solutions and uniqueness. *Comm. Partial Differential Equations*, 8(11):1229–1276, 1983.

[31] A. V. Pogorelov. *Monge-Ampère equations of elliptic type.* P. Noordhoff, Ltd., Groningen, 1964. Translated from the first Russian edition by Leo F. Boron with the assistance of Albert L. Rabenstein and Richard C. Bollinger.

[32] A. V. Pogorelov. The regularity of the generalized solutions of the equation $\det(\partial^2 u/\partial x_i \partial x_j) = \phi(x_1, \ldots, x_n) > 0$. *Soviet Math. Dokl.*, 12:1436–1440, 1971. Translation from the original in Russian (Dokl. Akad. Nauk SSSR 200 (1971), 534-537).

[33] A. V. Pogorelov. *The Minkowski multidimensional problem.* Halsted Press (John Wiley & Sons), New York-Toronto-London, 1978. Translated from the Russian by Vladimir Oliker. Introduction by Louis Nirenberg.

[34] N. S. Trudinger. Fully nonlinear, uniformly elliptic equations under natural structure conditions. *Trans. Amer. Math. Soc.*, 278(2):751–769, 1983.

[35] N. S. Trudinger. Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations. *Rev. Mat. Iberoamericana*, 4(3-4):453–468, 1988.

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