Viscosity solutions of Hamilton-Jacobi equations - summer course at Universidade Federal Fluminense, Niterói

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Introduction

These lecture notes are an abridged version of the book written for a course in the 27th Brazilian Mathematics Colloquium, and correspond to the lectures given at the Universidade Federal Fluminense, Niterói, Brazil in 2011.

This course covered basic notions in viscosity solutions and its applications to deterministic optimal control and differential games. This books is partially based on a course on Calculus of Variations and Partial Differential Equations that I have taught over the years at the Mathematics Department of Instituto Superior Técnico. I would like to thank my students: Tiago Alcaria, Patrícia Engrácia, Sílvia Guerra, Igor Kravchenko, Anabela Pelicano, Ana Rita Pires, Verónica Quítalo, Lucian Radu, Joana Santos, Ana Santos, and Vitor Saraiva, which took my courses and suggested me several corrections and improvements. Also my post-docs Andrey Byriuk, Filippo Cagnetti, and Milena Chermisi, and my colleagues Pedro Girão, Cláudia Nunes Philipart, and António Serra have suggested numerous improvements on the original text. I would like to thank Artur Lopes that challenged me to present the proposal of the original course at IMPA. I would like to thank Max Souza for the invitation to give these lectures to a very interested and engaging audience.

The structure of this text is the following: we start with a survey of classical mechanics and classical calculus of variations. Then we present the basic tools in classical optimal control. We continue with a discussion of viscosity solutions for the terminal value problem. We follow with a brief discussion of zero sum differential games. We ended the course with an introduction to games based upon the author's joint work with Joana Mohr and Rafael Souza from the Universidade Federal do Rio Grande do Sul.

For additional material, the reader should consult the bibliographical references. In each chapter we have a section on bibliographical notes that lists the main references on the material of that chapter.

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Classical calculus of variations

This chapter is dedicated to the study of classical mechanics and calculus of variations. We start by discussing the minimum action principle, Euler-Lagrange equations and some applications to Classical Mechanics. In section 2 we establish further necessary conditions for minimizers. The following section is dedicated to the Hamiltonian formalism. We end the chapter with some bibliographical notes.

1. Euler-Lagrange Equations

In classical mechanics, the trajectories $\mathbf{x} : [0, T] \to \mathbb{R}^n$ of a mechanical system are determined by a variational principle called the minimal action principle. This principle asserts that the trajectories are minimizers (or at least critical points) of an integral functional. In this section we study this problem and discuss several examples.

Consider a mechanical system on $\mathbb{R}^n$ with kinetic energy $K(x, v)$ and potential energy $U(x, v)$. We define the Lagrangian, $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ to be difference between the kinetic energy K and potential energy U of the system, that is, $L = K - U$. The variational formulation of classical mechanics asserts that trajectories of this mechanical system minimize (or are at least critical points) of the action functional

$$S[\mathbf{x}] = \int_0^T L(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt,$$

under fixed boundary conditions. More precisely, a C^1 trajectory $\mathbf{x} : [0, T] \to \mathbb{R}^n$ is a minimizer S under fixed boundary conditions if for any C^1 trajectory $\mathbf{y} : [0, T] \to \mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{y}(0)$ and $\mathbf{x}(T) = \mathbf{y}(T)$ we have

$$S[\mathbf{x}] \leq S[\mathbf{y}].$$

In particular, for any C^1 function $\varphi : [0, T] \to \mathbb{R}^n$ with compact support in $(0, T)$, and any $\epsilon \in \mathbb{R}$ we have

$$i(\epsilon) = S[\mathbf{x} + \epsilon\varphi] \geq S[\mathbf{x}] = i(0).$$

Thus $i(\epsilon)$ has a minimum at $\epsilon = 0$. So, if i is differentiable, $i'(0) = 0$. A trajectory $\mathbf{x}$ is a *critical point* of S, if for any C^1 function $\varphi : [0, T] \to \mathbb{R}^n$ with compact support in $(0, T)$ we have

$$i'(0) = \frac{d}{d\epsilon} S[\mathbf{x} + \epsilon\varphi]\Big|_{\epsilon=0} = 0.$$

The critical points of the action which are of class C^2 are solutions to an ordinary differential equation, the *Euler-Lagrange equation*, that we derive in what follows. Any minimizer of the action functional satisfies further necessary conditions which will be discussed in section 2.

Theorem 1 (Euler-Lagrange equation). *Let $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Suppose that $\mathbf{x} : [0, T] \to \mathbb{R}^n$ is a C^2 critical point of the action S under fixed boundary conditions $\mathbf{x}(0)$ and $\mathbf{x}(T)$. Then*

$$\frac{d}{dt} D_v L(\mathbf{x}, \dot{\mathbf{x}}) - D_x L(\mathbf{x}, \dot{\mathbf{x}}) = 0. \tag{1}$$

Proof. Let $\mathbf{x}$ be as in the statement. Then for any $\varphi : [0, T] \to \mathbb{R}^n$ with compact support on $(0, T)$, the function

$$i(\epsilon) = S[\mathbf{x} + \epsilon\varphi]$$

has a minimum at $\epsilon = 0$. Thus

$$i'(0) = 0,$$

that is,

$$\int_0^T D_x L(\mathbf{x}, \dot{\mathbf{x}})\varphi + D_v L(\mathbf{x}, \dot{\mathbf{x}})\dot{\varphi} = 0.$$

Integrating by parts, we conclude that

$$\int_0^T \left[\frac{d}{dt} D_v L(\mathbf{x}, \dot{\mathbf{x}}) - D_x L(\mathbf{x}, \dot{\mathbf{x}})\right] \varphi = 0,$$

for all $\varphi : [0, T] \to \mathbb{R}^n$ with compact support in $(0, T)$. This implies (1) and ends the proof of the theorem. ☐

Example 1. In classical mechanics, the kinetic energy K of a particle with mass m with trajectory $\mathbf{x}(t)$ is:

$$K = m\frac{|\dot{\mathbf{x}}|^2}{2}.$$

Suppose that the potential energy $U(x)$ depends only on the position x. Assume also that U is smooth. Then the Lagrangian for this mechanical system is then

$$L = K - U.$$

and the corresponding Euler-Lagrange equation is

$$m\ddot{\mathbf{x}} = -U'(\mathbf{x}),$$

which is the Newton's law. ◀

EXERCISE 1. *Let $P \in \mathbb{R}^n$, and consider the Lagrangian $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $L(x, v) = g(x)|v|^2 + P \cdot v - U(x)$, where g and U are C^2 functions. Determine the Euler-Lagrange equation and show that it does not depend on P.*

EXERCISE 2. *Suppose we form a surface of revolution by connecting a point (x_0, y_0) with a point (x_1, y_1) by a curve $(x, y(x))$, $x \in [0, 1]$, and then revolving it around the y axis. The area of this surface is*

$$\int_{x_0}^{x_1} x\sqrt{1+\dot{y}^2}dx.$$

Compute the Euler-Lagrange equation and study its solutions.

To understand the behavior of the Euler-Lagrange equation it is sometimes useful to change coordinates. The following proposition shows how this is achieved:

PROPOSITION 2. *Let $\mathbf{x} : [0, T] \to \mathbb{R}^n$ be a critical point of the action*

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}})dt.$$

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 diffeomorphism and $\hat{L}$ given by

$$\hat{L}(y, w) = L(g(y), Dg(y)w).$$

Then $\mathbf{y} = g^{-1} \circ \mathbf{x}$ is a critical point of

$$\int_0^T \hat{L}(\mathbf{y}, \dot{\mathbf{y}})dt.$$

PROOF. This is a simple computation and is left as an exercise to the reader. □

Before proceeding, we will discuss some applications of variational methods to classical mechanics. As mentioned before, the trajectories of a mechanical system with kinetic energy K and potential energy U are critical points of the action corresponding to the Lagrangian $L = K - U$. In the following examples we use this variational principle to study the motion of a particle in a central field, and the planar two body problem.

EXAMPLE 2 (Central field motion). Consider the Lagrangian of a particle in the plane subjected to a radial potential field.

$$L(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) = \frac{\dot{\mathbf{x}}^2 + \dot{\mathbf{y}}^2}{2} - U(\sqrt{\mathbf{x}^2 + \mathbf{y}^2}).$$

Consider polar coordinates, (r, θ), that is $(x, y) = (r\cos\theta, r\sin\theta) = g(r, \theta)$, We can change coordinates (see proposition 2) and obtain the Lagragian in these new coordinates

$$\hat{L}(\mathbf{r}, \theta, \dot{\mathbf{r}}, \dot{\theta}) = \frac{\mathbf{r}^2\dot{\theta}^2 + \dot{\mathbf{r}}^2}{2} - U(\mathbf{r}).$$

Then the Euler-Lagrange equations can be written as

$$\frac{d}{dt}\mathbf{r}^2\dot{\theta} = 0 \qquad \frac{d}{dt}\dot{\mathbf{r}} = -U'(\mathbf{r}) + \mathbf{r}\dot{\theta}^2.$$

The first equation implies that $\mathbf{r}^2\dot{\theta} \equiv \eta$ is conserved. Therefore, $\mathbf{r}\dot{\theta}^2 = \frac{\eta^2}{\mathbf{r}^3}$. Multiplying the second equation by $\dot{\mathbf{r}}$ we get

$$\frac{d}{dt}\left[\frac{\dot{\mathbf{r}}^2}{2} + U(\mathbf{r}) + \frac{\eta^2}{2\mathbf{r}^2}\right] = 0.$$

Consequently

$$E_\eta = \frac{\dot{\mathbf{r}}^2}{2} + U(\mathbf{r}) + \frac{\eta^2}{2\mathbf{r}^2}$$

is a conserved quantity. Thus, we can solve for $\dot{\mathbf{r}}$ as a function of $\mathbf{r}$ (given the values of the conserved quantities E_η and η) and so obtain a first-order differential equation for the trajectories. ◀

Example 3 (Planar two-body problem). Consider now the problem of two point bodies in the plane, with trajectories $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$. Suppose that the interaction potential energy U depends only on the distance $\sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{y}_1 - \mathbf{y}_2)^2}$ between them. We will show how to reduce this problem to the one of a single body under a radial field.

The Lagrangian of this system is

$$L = m_1\frac{\dot{\mathbf{x}}_1^2 + \dot{\mathbf{y}}_1^2}{2} + m_2\frac{\dot{\mathbf{x}}_2^2 + \dot{\mathbf{y}}_2^2}{2} - U(\sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{y}_1 - \mathbf{y}_2)^2}).$$

Consider new coordinates (X, Y, x, y), where (X, Y) is the center of mass

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \qquad Y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2},$$

and (x, y) the relative position of the two bodies

$$x = x_1 - x_2, \qquad y = y_1 - y_2.$$

In these new coordinates the Lagrangian, using proposition 2, is

$$\hat{L} = \hat{L}_1(\dot{\mathbf{X}}, \dot{\mathbf{Y}}) + \hat{L}_2(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}).$$

Therefore, the equations for the variables $\mathbf{X}$ and $\mathbf{Y}$ are decoupled from the ones for $\mathbf{x}, \mathbf{y}$. Elementary computations show that

$$\frac{d^2}{dt^2}\mathbf{X} = \frac{d^2}{dt^2}\mathbf{Y} = 0.$$

Thus $\mathbf{X}(t) = X_0 + V_X t$ and $\mathbf{Y}(t) = Y_0 + V_Y t$, for suitable constants X_0, Y_0, V_X and V_Y.

Since

$$L_2 = \frac{m_1 m_2}{m_1 + m_2} \frac{\dot{\mathbf{x}}^2 + \dot{\mathbf{y}}^2}{2} - U(\sqrt{\mathbf{x}^2 + \mathbf{y}^2}),$$

the problem now is reduced to the previous example. ◀

Exercise 3 (Two body problem). *Consider a system of two point bodies in* $\mathbb{R}^3$ *with masses* m_1 *and* m_2*, whose relative location is given by the vector* $\mathbf{r} \in \mathbb{R}^3$*. Assume that the interaction depends only on the distance between the bodies. Show that by choosing appropriate coordinates, the motion can be reduced to the one of a single point particle with mass* $M = \frac{m_1 m_2}{m_1 + m_2}$ *under a radial potential. Show, by proving that* $\mathbf{r} \times \dot{\mathbf{r}}$ *is conserved, that the orbit of a particle under a radial field lies in a fixed plane for all times.*

Exercise 4. *Let* $\mathbf{x} : [0, T] \to \mathbb{R}^n$ *be a solution to the Euler-Lagrange equation associated to a* C^2 *Lagrangian* $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*. Show that*

$$E(t) = -L(\mathbf{x}, \dot{\mathbf{x}}) + \dot{\mathbf{x}} \cdot D_v L(\mathbf{x}, \dot{\mathbf{x}})$$

is constant in time. For mechanical systems this is simply the conservation of energy. Occasionally, the identity $\frac{d}{dt} E(t) = 0$ *is also called the Beltrami identity.*

Exercise 5. *Consider a system of* n *point bodies of mass* m_i*, and positions* $\mathbf{r}_i \in \mathbb{R}^3$*,* $1 \leq i \leq n$*. Suppose the kinetic energy is* $T = \sum_i \frac{m_i}{2} |\dot{\mathbf{r}}|^2$ *and the potential energy is* $U = -\sum_{i, j \neq i} \frac{m_i m_j}{2|\mathbf{r}_i - \mathbf{r}_j|}$*. Let* $I = \sum_i m_i |\mathbf{r}_i|^2$*. Show that*

$$\frac{d^2}{dt^2} I = 4T + 2U,$$

which is strictly positive if the energy $T + U$ *is positive. What implications does this identity have for the stability of planetary systems?*

Exercise 6 (Jacobi metric). *Let* $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a* C^2 *Lagrangian. Let* $\mathbf{x} : [0, T] \to \mathbb{R}^n$ *be a solution to the corresponding Euler-Lagrange*

$$\frac{d}{dt} D_v L - D_x L = 0, \tag{2}$$

for the Lagrangian

$$L(x, v) = \frac{|v|^2}{2} - V(x).$$

Let $E(t) = \frac{|\dot{\mathbf{x}}(t)|^2}{2} + V(\mathbf{x}(t))$*.*

1. *Show that* $\dot{E} = 0$*.*

2. *Let* $E_0 = E(0)$*. Show that* $\mathbf{x}$ *is a solution to the Euler-Lagrange equation*

$$\frac{d}{dt} D_v L_J - D_x L_J = 0 \tag{3}$$

associated to $L_J = \sqrt{E_0 - V(\mathbf{x})}|\dot{\mathbf{x}}|$.

3. *Show that any reparametrization in time of* $\mathbf{x}$ *is also a solution to* (3) *and observe that the functional*

$$\int_0^T \sqrt{E_0 - V(\mathbf{x})}|\dot{\mathbf{x}}|$$

represents the lenght of the path between $\mathbf{x}(0)$ *and* $\mathbf{x}(T)$ *using the Jacobi metric* $g = \sqrt{E_0 - V(x)}$.

4. *Show that the solutions to the Euler-Lagrange* (3) *when reparametrized in time in such a way that the energy of the reparametrized trajectory is* E_0 *satisfy* (2).

Exercise 7 (Braquistochrone problem). *Let* (x_1, y_1) *be a point in a (vertical) plane. Show that the curve* $\mathbf{y} = \mathbf{u}(x)$ *that connects* $(0,0)$ *to* (x_1, y_1) *in such a way that a particle with unit mass moving under the influence a unit gravity field reaches* (x_1, y_1) *in the minimum amount of time minimizes*

$$\int_0^{x_1} \sqrt{\frac{1 + \dot{\mathbf{u}}^2}{-2\mathbf{u}}} dx.$$

Hint: *use the fact that the sum of kinetic and potential energy is constant.*

Determine the Euler-Lagrange equation and study its solutions, using exercise 4.

Exercise 8. *Consider a second-order variational problem:*

$$\min_{\mathbf{x}} \int_0^T L(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) \tag{4}$$

where the minimum is taken over all trajectories $\mathbf{x} : [0,T] \to \mathbb{R}^n$ *with fixed boundary data* $\mathbf{x}(0), \mathbf{x}(T), \dot{\mathbf{x}}(0), \dot{\mathbf{x}}(T)$*. Determine the Euler-Lagrange equation corresponding to .*

2. Further necessary conditions

A classical strategy in the study of variational problems consists in establishing necessary conditions for minimizers. If there exists a minimizer and if the necessary conditions have a unique solution, then this solution has to be the unique minimizer and thus the problem is solved. In addition to Euler-Lagrange equations, several other necessary conditions can be derived. In this section we discuss boundary conditions which arise, for instance when the end-points are not fixed, and second-order conditions.

2.1. Boundary conditions. In certain problems, the boundary conditions, such as end point values are not prescribed a-priori. In this case, it is possible to prove that the minimizers satisfy certain boundary conditions automatically. These are called natural boundary conditions.

EXAMPLE 4. Consider the problem of minimizing the integral

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}})dt, \tag{5}$$

over all C^2 curves $\mathbf{x} : [0, T] \to \mathbb{R}^n$. Note that the boundary values for the trajectory $\mathbf{x}$ at $t = 0, T$ are not prescribed a-priori.

Let $\mathbf{x}$ be a minimizer of (5) (with free endpoints). Then for all $\varphi : [0, T] \to \mathbb{R}^n$, not necessarily compactly supported,

$$\int_0^T D_x L(\mathbf{x}, \dot{\mathbf{x}})\varphi + D_v L(\mathbf{x}, \dot{\mathbf{x}})\dot{\varphi} dt = 0.$$

Integrating by parts and using the fact that $\mathbf{x}$ is a solution to the Euler-Lagrange equation, we conclude that

$$D_v L(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = D_v L(\mathbf{x}(T), \dot{\mathbf{x}}(T)) = 0.$$

◀

EXERCISE 9. *Consider the problem of minimizing the integral*

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}})dt,$$

over all C^2 curves $\mathbf{x} : [0, T] \to \mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{x}(T)$. Deduce that

$$D_v L(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = D_v L(\mathbf{x}(T), \dot{\mathbf{x}}(T)).$$

Use the previous identity to show that any periodic (smooth) minimizer is in fact a periodic solutions to the Euler-Lagrange equations.

EXERCISE 10. *Consider the problem of minimizing*

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}})dt + \psi(\mathbf{x}(T)),$$

with $\mathbf{x}(0)$ fixed and $\mathbf{x}(T)$ free. Derive a boundary condition at $t = T$ for the minimizers.

EXERCISE 11 (Free boundary).

Consider the problem of minimizing

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}}),$$

over all terminal times T and all C^2 curves $\mathbf{x} : [0,T] \to \mathbb{R}^n$. Show that $\mathbf{x}$ is a solution to the Euler-Lagrange equation and that

$$\begin{aligned} &L(\mathbf{x}(T), \dot{\mathbf{x}}(T)) = 0,\\ &D_x L(\mathbf{x}(T), \dot{\mathbf{x}}(T))\dot{\mathbf{x}}(T) + D_v L(\mathbf{x}(T), \dot{\mathbf{x}}(T))\ddot{\mathbf{x}}(T) \geq 0,\\ &D_v L(\mathbf{x}(T), \dot{\mathbf{x}}(T)) = 0. \end{aligned}$$

Let $q \in \mathbb{R}$ and $L : \mathbb{R}^2 \to \mathbb{R}$ given by

$$L(x, v) = \frac{(v-q)^2}{2} + \frac{x^2}{2} - 1$$

If possible, determine T and $\mathbf{x} : [0,T] \to \mathbb{R}$ that are (local) minimizers of

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}}) ds,$$

with $\mathbf{x}(0) = 0$.

2.2. Second-order conditions. If $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function which has a minimum at a point x_0 then $f'(x_0) = 0$ and $f''(x_0) \geq 0$. For the minimal action problem, the analog of the vanishing of the first derivative is the Euler-Lagrange equation. We will now consider the analog to the second derivative being non-negative.

The next theorem concerns second-order conditions for minimizers:

THEOREM 3 (Jacobi's test). *Let $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 Lagrangian. Let $\mathbf{x} : [0,T] \to \mathbb{R}^n$ be a C^1 minimizer of the action under fixed boundary conditions. Then, for each $\eta : [0,T] \to \mathbb{R}^n$, with compact support in $(0,T)$, we have*

$$\int_0^T \frac{1}{2}\eta^T D^2_{xx} L(\mathbf{x}, \dot{\mathbf{x}})\eta + \eta^T D^2_{xv} L(\mathbf{x}, \dot{\mathbf{x}})\dot{\eta} + \frac{1}{2}\dot{\eta}^T D^2_{vv} L(\mathbf{x}, \dot{\mathbf{x}})\dot{\eta} \geq 0. \tag{6}$$

PROOF. If $\mathbf{x}$ is a minimizer, the function $\epsilon \mapsto I[\mathbf{x} + \epsilon\eta]$ has a minimum at $\epsilon = 0$. By computing $\frac{d^2}{d\epsilon^2} I[\mathbf{x} + \epsilon\eta]$ at $\epsilon = 0$ we obtain (6). ☐

A corollary of the previous theorem is Lagrange's test that we state next:

COROLLARY 4 (Lagrange's test). *Let $L(x, v) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 Lagrangian. Suppose $\mathbf{x} : [0,T] \to \mathbb{R}^n$ is a C^1 minimizer of the action under fixed boundary conditions. Then*

$$D^2_{vv} L(\mathbf{x}, \dot{\mathbf{x}}) \geq 0.$$

PROOF. Use Theorem 3 with $\eta = \epsilon \xi(t) \sin \frac{t}{\epsilon}$, for $\xi : [0, T] \to \mathbb{R}^n$, with compact support in $(0, T)$, and let $\epsilon \to 0$. $\square$

EXERCISE 12. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous Lagrangian and let $\mathbf{x} : [0, T] \to \mathbb{R}^n$ be a continuous piecewise C^1 trajectory. Show that for each $\delta > 0$ there exists a trajectory $\mathbf{y}_\delta : [0, T] \to \mathbb{R}^n$ of class C^1 such that*

$$\left| \int_0^T L(\mathbf{x}, \dot{\mathbf{x}}) - \int_0^T L(\mathbf{y}_\delta, \dot{\mathbf{y}}_\delta) \right| < \delta.$$

As a corollary, show that the value of the infimum of the action over piecewise C^1 trajectories is the same as the infimum over trajectories globally C^1. Note, however, that a minimizer may not be C^1.

EXERCISE 13 (Weierstrass test). *Let $\mathbf{x} : [0, T] \to \mathbb{R}^n$ be a C^1 minimum of the action corresponding to a Lagrangian L. Let $v, w \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$ be such that $\lambda v + (1 - \lambda) w = 0$. Show that*

$$\lambda L(\mathbf{x}, \dot{\mathbf{x}} + v) + (1 - \lambda) L(\mathbf{x}, \dot{\mathbf{x}} + w) \geq L(\mathbf{x}, \dot{\mathbf{x}}).$$

Hint: *To prove the inequality at a point t_0, choose η such that*

$$\dot{\eta}(t) = \begin{cases} v & \text{if} \quad t_0 \leq t \leq t + \lambda\epsilon \\ w & \text{if} \quad t + \lambda\epsilon < t \leq t_0 + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and consider $I[\mathbf{x} + \eta]$, as $\epsilon \to 0$.

3. Hamiltonian dynamics

In this section we introduce the Hamiltonian formalism of Classical Mechanics. We start by discussing the main properties of the Legendre transform. Then we derive Hamilton's equations. Afterwards we discuss briefly the classical theory of canonical transformations. The section ends with a discussion of additional variational principles.

3.1. Legendre transform. Before we proceed, we need to discuss the Legendre transform of convex functions. The Legendre transform is used to define the Hamiltonian of a mechanical system and it plays an essential role in many problems in calculus of variations. Additionally, it illustrates many of the tools associated with convexity.

Let $L(v) : \mathbb{R}^n \to \mathbb{R}$ be a convex function, satisfying the following superlinear growth condition:

$$\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty.$$

The *Legendre transform* L^* of L is

$$L^*(p) = \sup_{v\in\mathbb{R}^n} \left[-v \cdot p - L(v)\right].$$

This is the usual definition of Legendre transform in optimal control, see [**FS06**] or [**BCD97**]. However, it differs by a sign from the Legendre transform traditionally used in classical mechanics:

$$L^\sharp(p) = \sup_{v\in\mathbb{R}^n} \left[v \cdot p - L(v)\right],$$

as it is defined, for instance, in [**AKN97**] or [**Eva98**]. They are related by the elementary identity

$$L^*(p) = L^\sharp(-p).$$

We will frequently denote $L^*(p)$ by $H(p)$. The Legendre transform of H is denoted by H^* and is

$$H^*(v) = \sup_{p\in\mathbb{R}^n} \left[-p \cdot v - H(p)\right].$$

In classical mechanics, the Lagrangian L can depend also on a position coordinate $x \in \mathbb{R}^n$, $L(x, v)$, but for purposes of the Legendre transform x is taken as a fixed parameter. In this case we write also $H(p, x) = L^*(p, x)$.

PROPOSITION 5. *Let $L(x, v)$ be a C^2 function, which for each x fixed is strictly convex and superlinear in v. Let $H = L^*$. Then*

1. *$H(p, x)$ is convex in p;*
2. *$H^* = L$;*
3. *for each x*
$$\lim_{|p|\to\infty} \frac{H(p, x)}{|p|} = \infty;$$
4. *let v^* be defined by $p = -D_v L(x, v^*)$, then*
$$H(p, x) = -v^* \cdot p - L(x, v^*);$$
5. *in a similar way, let p^* be given by $v = -D_p H(p^*, x)$, then*
$$L(x, v) = -v \cdot p^* - H(p^*, x);$$
6. *if $p = -D_v L(x, v)$ or $v = -D_p H(p, x)$, then*
$$D_x L(x, v) = -D_x H(p, x).$$

PROOF. The first statement follows from the fact that the supremum of convex functions is a convex function. To prove the second point, observe that

$$\begin{aligned} H^*(x,w) &= \sup_p \left[-w \cdot p - H(p,x)\right] \\ &= \sup_p \inf_v \left[(v-w)\cdot p + L(x,v)\right]. \end{aligned}$$

For $v = w$ we conclude that

$$H^*(x,w) \leq L(x,w).$$

The opposite inequality is obtained by observing, since L is convex in v, that for each $w \in \mathbb{R}^n$ there exists $s \in \mathbb{R}^n$ such that

$$L(x,v) \geq L(x,w) + s \cdot (v-w).$$

Therefore,

$$H^*(x,w) \geq \sup_p \inf_v \left[(p+s)\cdot(v-w) + L(x,w)\right] \geq L(x,w),$$

by letting $p = -s$.

To prove the third point observe that

$$\frac{H(p,x)}{|p|} \geq \lambda - \frac{L(x, -\lambda\frac{p}{|p|})}{|p|},$$

by choosing $v = -\lambda\frac{p}{|p|}$. Thus, we conclude

$$\liminf_{|p|\to\infty} \frac{H(p,x)}{|p|} \geq \lambda.$$

Since λ is arbitrary, we have

$$\liminf_{|p|\to\infty} \frac{H(p,x)}{|p|} = \infty.$$

To establish the fourth point, note that for fixed p the function

$$v \mapsto v \cdot p + L(x,v)$$

is differentiable and strictly convex. Consequently, its minimum, which exists by coercivity and is unique by the strict convexity, is achieved for

$$-p - D_v L(x,v) = 0.$$

Note also that v as function of p is a differentiable function by the inverse function theorem.

The proof of the fifth point is similar.

Finally, to prove the last item, observe that for

$$p(x,v) = -D_v L(x,v),$$

we have

$$H(p(x,v),x) = -v \cdot p(x,v) - L(x,v).$$

Differentiating this last equation with respect to x and using

$$v = -D_p H(p(x,v),x),$$

we obtain

$$D_x H = -D_x L.$$

□

EXERCISE 14. *Compute the Legendre transform of the following functions:*

1.
$$L(x,v) = \frac{1}{2}a_{ij}(x)v_i v_j + h_i(x)v_i - U(x),$$
where a_{ij} *is a positive definite matrix and* $h(x)$ *an arbitrary vector field.*

2.
$$L(x,v) = \sqrt{a_{ij}(x)v_i v_j},$$
where a_{ij} *is a positive definite matrix.*

3.
$$L(x,v) = \frac{1}{2}|v|^\lambda - U(x),$$
with $\lambda > 1$.

EXERCISE 15. *By allowing the Lagrangian and its Legendre transform to assume the values* $\pm\infty$ *comute the Legendre transforms of*

1. *for* $\omega \in \mathbb{R}^n$
$$L(v) = \begin{cases} 0 & \text{if} \quad v = \omega \\ +\infty & \text{otherwise.} \end{cases}$$

2. *for* $\omega \in \mathbb{R}^n$ *set*
$$L(v) = \omega \cdot v.$$

3. *for* $R > 0$
$$L(v) = \begin{cases} 0 & \text{if} \quad |v| \le R \\ +\infty & \text{otherwise.} \end{cases}$$

3.2. Hamiltonian formalism. To motivate the Hamiltonian formalism, we consider the following alternative problem. Rather than looking for curves $\mathbf{x} : [0,T] \to \mathbb{R}^n$, which minimize the action

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}}) dt$$

we can consider extended curves $(\mathbf{x}, \mathbf{v}) : [0,T] \to \mathbb{R}^{2n}$ which minimize the action

$$\int_0^T L(\mathbf{x}, \mathbf{v}) dt \tag{7}$$

and that satisfy the additional constraint $\dot{\mathbf{x}} = \mathbf{v}$. Obviously, this problem is equivalent to the original one, however it motivates the introduction of a Lagrange multiplier $\mathbf{p}$ in order to enforce the constraint. Therefore, we will look for critical points of

$$\int_0^T L(\mathbf{x}, \mathbf{v}) + \mathbf{p} \cdot (\mathbf{v} - \dot{\mathbf{x}}) dt. \tag{8}$$

PROPOSITION 6. *Let $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth Lagrangian. Let $(\mathbf{x}, \mathbf{v}) : [0,T] \to \mathbb{R}^{2n}$ be a critical point of (7) under fixed boundary conditions and under the constraint $\dot{\mathbf{x}} = \mathbf{v}$ (the choice of $\mathbf{p}$ is irrelevant since the corresponding term always vanishes). Let*

$$\mathbf{p} = -D_v L(\mathbf{x}, \mathbf{v}).$$

Then the curve $(\mathbf{x}, \mathbf{v}, \mathbf{p})$ is a critical point of (8) under fixed boundary conditions. Additionally, any critical point $(\mathbf{x}, \mathbf{v}, \mathbf{p})$ of (8) *satisfies*

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \mathbf{p} = -D_v L(\mathbf{x}, \mathbf{v}) \\ \dot{\mathbf{p}} = D_x L(\mathbf{x}, \mathbf{v}). \end{cases}$$

In particular, $\mathbf{x}$ is a critical point of (7). *Furthermore, the Euler-Lagrange equation can be rewritten as*

$$\dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}) \qquad \dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x}).$$

PROOF. Let ϕ, ψ and η be $C^2([0,T], \mathbb{R}^n)$ with compact support in $(0,T)$. Then, at $\epsilon = 0$

$$\begin{aligned} &\frac{d}{d\epsilon} \int_0^T L(\mathbf{x} + \epsilon\phi, \mathbf{v} + \epsilon\psi) + (\mathbf{p} + \epsilon\eta) \cdot (\mathbf{v} - \dot{\mathbf{x}}) + \epsilon(\mathbf{p} + \epsilon\eta) \cdot (\psi - \dot{\phi}) \\ &\qquad = \int_0^T D_x L(\mathbf{x}, \dot{\mathbf{x}})\phi + D_v L\psi + \mathbf{p} \cdot (\psi - \dot{\phi}) + \eta \cdot (\mathbf{v} - \dot{\mathbf{x}}) \\ &\qquad = \int_0^T \left[D_x L(\mathbf{x}, \dot{\mathbf{x}}) + \dot{\mathbf{p}} \right] \phi = 0. \end{aligned}$$

If $p = -D_v L(x, v)$, then v maximizes

$$-p \cdot v - L(x, v).$$

Let

$$H(p, x) = \max_v \left[-p \cdot v - L(x, v)\right].$$

By proposition 5 we have

$$D_x H(p, x) = -D_x L(x, v)$$

whenever

$$p = -D_v L(x, v).$$

Additionally, we also have

$$v = -D_p H(p, x).$$

Therefore, the Euler-Lagrange equation can be rewritten as

$$\dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}) \qquad \dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x}).$$

These are the *Hamilton equations.* □

EXERCISE 16. *Suppose $H(p, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^1 function. Show that the energy, which coincides with H, is conserved by the Hamiltonian flow since*

$$\frac{d}{dt} H(\mathbf{p}, \mathbf{x}) = 0.$$

4. Bibliographical notes

There is a very large literature on the topics of this chapter. The main references we have used were [**Arn95**] and [**AKN97**]. Two classical physics books on this subject are [**Gol80**] and [**LL76**]. On the more geometrical perspective, the reader may want to look at [**dC92**] (see also [**dC79**]) and [**Oli02**]. Additional material on classical calculus of variations can be found in [**Dac09**] and the classical book [**Bol61**]. A very good reference in Portuguese is [**Lop06**].

3

Classical optimal control

In this chapter we consider deterministic optimal control problems and its connection with Hamilton-Jacobi equations. We start the discussion, in the next section, with the set up of the problem. Then we present some elementary properties and examples. The dynamic programming principle and Pontryagin maximum principles are discussed in sections 3 and 5, respectively. The Pontryagin maximum principle is the analog of the Euler-Lagrange equation for optimal control problems. Then, in section 6 we will show that if the value function V is differentiable, it satisfies the Hamilton-Jacobi partial differential equation

$$-V_t + H(D_x V, x, t) = 0,$$

in which $H(p, x)$, the Hamiltonian, is the (generalized) Legendre transform of the Lagrangian L

$$H(p, x, t) = \sup_{v \in U} -p \cdot f(x, v) - L(x, v, t). \tag{9}$$

We end this chapter with a verification theorem, section 7 that establishes that a sufficiently smooth solution to the Hamilton-Jacobi equation is the value function.

1. Optimal Control

A typical problem in optimal control, whose study we begin now is the terminal value optimal control problem. For that let the control space be a closed convex subset U of $\mathbb{R}^m$. A control on an interval $I \subset \mathbb{R}$ is a measurable function $\mathbf{u} : I \to U$. Let $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be a continuous function, Lipschitz in x. For each control $\mathbf{u}$ we can consider the controlled dynamics

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}). \tag{10}$$

We could of course consider control laws depending on time without any problem whatsoever. We consider integral solutions of (10), i.e., $\mathbf{x}$ is a solution of (10) with initial condition $\mathbf{x}(t) = x_0$ if

$$\mathbf{x}(T) = x_0 + \int_t^T f(\mathbf{x}(s), \mathbf{u}(s)) ds,$$

for all $T > t$. It is well known from ODE theory that, at least locally in time, equation (10) admits a unique integral solution, for any (bounded) control $\mathbf{u}$.

We are given a running cost $L : \mathbb{R}^n \times U \to \mathbb{R}$ and a terminal cost $\psi : \mathbb{R}^n \to \mathbb{R}$. To avoid technical problems we assume that both L and ψ are bounded below and, without loss of generality (by adding suitable constants), we suppose for definiteness, $L, \psi \geq 0$. Furthermore, we require $\psi \in L^\infty$ and L to satisfy the following bound: there exists $u_0 \in U$ and a constant C such that

$$L(x, u_0) \leq C. \tag{11}$$

Given a terminal time T, the terminal value optimal control problem consists in determining the optimal trajectories $\mathbf{x}(\cdot)$ which minimize

$$J[\mathbf{u}; x, t] = \int_t^T L(\mathbf{x}, \mathbf{u}) ds + \psi(\mathbf{x}(t_1)),$$

among all bounded controls $\mathbf{u}(\cdot) : [t, t_1] \to \mathbb{R}^n$ and all solutions $\mathbf{x}$ of (10) satisfying the initial condition $\mathbf{x}(t) = x$.

The value function V is

$$V(x, t) = \inf J[\mathbf{u}; x, t] \tag{12}$$

in which the infimum is taken over all controls on $[t, T]$.

An important example is the "calculus of variations setting", where, $f(x, u) = u$, and the optimal trajectories $\mathbf{x}(\cdot)$, as we have shown, are solutions to the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial v}(\mathbf{x}, \dot{\mathbf{x}}) - \frac{\partial L}{\partial x}(\mathbf{x}, \dot{\mathbf{x}}) = 0.$$

Furthermore, $\mathbf{p} = -D_v L(\mathbf{x}, \dot{\mathbf{x}})$ is a solution of Hamilton's equations:

$$\dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x}), \quad \dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}).$$

In the next chapter we will consider this problem under the light of optimal control and generalize the previous results.

In section 4, before considering the "calculus of variations setting", we study a simpler but important situation, the bounded control case. In this the control space U is a compact convex set.

Furthermore, in that section we suppose additionally that $L(x, u)$ is a bounded continuous function, convex in u. We assume further that the function $f(x, u)$ satisfies the following Lipschitz condition

$$|f(x, u) - f(y, u)| \leq C|x - y|.$$

To establish existence of optimal solutions we simplify even more by assuming that $f(x,u)$ has the form

$$f(x,u) = A(x)u + B(x), \tag{13}$$

where A and B are Lipschitz continuous functions.

2. Elementary properties

In this section we establish some elementary properties of the terminal value problem.

PROPOSITION 7. *The value function V satisfies the following inequalities*

$$-\|\psi\|_\infty \leq V \leq c_1|T-t| + \|\psi\|_\infty.$$

PROOF. The first inequality follows from $L \geq 0$. To obtain the second inequality it is enough to observe that

$$V \leq J(x,t;0) \leq c_1|T-t| + \|\psi\|_\infty.$$

□

EXAMPLE 5 (Lax-Hopf formula). Suppose that $L(x,v) \equiv L(v)$, L convex in v and coercive. Assume further that $f(x,v) = v$. By Jensen's inequality

$$\frac{1}{T-t}\int_t^T L(\dot{\mathbf{x}}(s)) \geq L\left(\frac{1}{T-t}\int_t^T \dot{\mathbf{x}}(s)\right) = L\left(\frac{y-x}{T-t}\right),$$

where $y = \mathbf{x}(T)$. Therefore, to solve the terminal value optimal control problem, it is enough to consider constant controls of the form $\mathbf{u}(s) = \frac{y-x}{T-t}$. Thus

$$V(x,t) = \inf_{y\in\mathbb{R}^n}\left[(T-t)L\left(\frac{y-x}{T-t}\right) + \psi(y)\right],$$

and, consequently, the infimum is a minimum. Thus Lax-Hopf formula gives an explicit solution to the optimal control problem. ◀

EXERCISE 17. *Let Q and A be $n\times n$ constant, positive definite, matrices. Let $L(v) = \frac{1}{2}v^TQv$ and $\psi(y) = \frac{1}{2}y^TAy$. Use Lax-Hopf formula to determine $V(x,t)$.*

PROPOSITION 8. *Let $\psi_1(x)$ and $\psi_2(x)$ be continuous functions such that*

$$\psi_1 \leq \psi_2.$$

Let $V_1(x,t)$ and $V_2(x,t)$ be the corresponding value functions. Then

$$V_1(x,t) \leq V_2(x,t).$$

PROOF. Fix $\epsilon > 0$. Then there exists an almost optimal control $\mathbf{u}^\epsilon$ and corresponding trajectory $\mathbf{x}^\epsilon$ such that

$$V_2(x,t) > \int_t^T L(\mathbf{x}^\epsilon(s), \mathbf{u}^\epsilon(s), s)ds + \psi_2(\mathbf{x}^\epsilon(T)) - \epsilon.$$

Clearly

$$V_1(x,t) \leq \int_t^T L(\mathbf{x}^\epsilon(s), \mathbf{u}^\epsilon(s), s)ds + \psi_1(\mathbf{x}^\epsilon(T)),$$

and therefore

$$V_1(x,t) - V_2(x,t) \leq \psi_1(\mathbf{x}^\epsilon(t_1)) - \psi_2(\mathbf{x}^\epsilon(t_1)) + \epsilon \leq \epsilon.$$

Since ϵ is arbitrary, this ends the proof. □

An important corollary is the continuity of the value function on the terminal value, with respect to the L^∞ norm.

COROLLARY 9. *Let $\psi_1(x)$ and $\psi_2(x)$ be continuous functions and $V_1(x,t)$ and $V_2(x,t)$ the corresponding value functions. Then*

$$\sup_x |V_1(x,t) - V_2(x,t)| \leq \sup_x |\psi_1(x) - \psi_2(x)|.$$

PROOF. Note that

$$\psi_1 \leq \tilde{\psi}_2 \equiv \psi_2 + \sup_y |\psi_1(y) - \psi_2(y)|.$$

Let $\tilde{V}_2$ be the value function corresponding to $\tilde{\psi}_2$. Clearly,

$$\tilde{V}_2 = V_2 + \sup_y |\psi_1(y) - \psi_2(y)|.$$

By the previous proposition,

$$V_1 - \tilde{V}_2 \leq 0,$$

which implies

$$V_1 - V_2 \leq \sup_y |\psi_1(y) - \psi_2(y)|.$$

By reverting the roles of V_1 and V_2 we obtain the other inequality. □

3. Dynamic programming principle

The *dynamic programming principle*, that we prove in the next theorem, is simply a semigroup property that the evolution of the value function satisfies.

THEOREM 10 (Dynamic programming principle). *Suppose that* $t \leq t' \leq T$. *Then*

$$V(x,t) = \inf_{\mathbf{u}} \left[\int_t^{t'} L(\mathbf{x}(s), \mathbf{u}(s), s) ds + V(y, t') \right], \tag{14}$$

where $\mathbf{x}(t) = x$ *and* $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$.

PROOF. Denote by $\tilde{V}(x,t)$ the right hand side of (14). For fixed $\epsilon > 0$, let $\mathbf{u}^\epsilon$ be an almost optimal control for $V(x,t)$. Let $\mathbf{x}^\epsilon(s)$ be the corresponding trajectory trajectory, i.e., assume that

$$J(x, t; \mathbf{u}^\epsilon) \leq V(x,t) + \epsilon.$$

We claim that $\tilde{V}(x,t) \leq V(x,t) + \epsilon$. To check this, let $\mathbf{x}(\cdot) = \mathbf{x}^\epsilon(\cdot)$ and $y = \mathbf{x}^\epsilon(t')$. Then

$$\tilde{V}(x,t) \leq \int_t^{t'} L(\mathbf{x}^\epsilon(s), \mathbf{u}^\epsilon(s), s) ds + V(y, t').$$

Additionally,

$$V(y, t') \leq J(y, t'; \mathbf{u}^\epsilon).$$

Therefore

$$\tilde{V}(x,t) \leq J(x, t; \mathbf{u}^\epsilon) \leq V(x,t) + \epsilon,$$

and, since ϵ is arbitrary, $\tilde{V}(x,t) \leq V(x,t)$.

To prove the opposite inequality, we will proceed by contradiction. Therefore, if $\tilde{V}(x,t) < V(x,t)$, we could choose $\epsilon > 0$ and a control $\mathbf{u}^\sharp$ such that

$$\int_t^{t'} L(\mathbf{x}^\sharp(s), \mathbf{u}^\sharp(s), s) ds + V(y, t') < V(x,t) - \epsilon,$$

where $\dot{\mathbf{x}}^\sharp = f(\mathbf{x}^\sharp, \mathbf{u}^\sharp)$, $\mathbf{x}^\sharp(t) = x$, and $y = \mathbf{x}^\sharp(t')$. Choose $\mathbf{u}^\flat$ such that

$$J(y, t'; \mathbf{u}^\flat) \leq V(y, t') + \frac{\epsilon}{2}$$

Define $\mathbf{u}^\star$ as

$$\begin{cases} \mathbf{u}^\star(s) = \mathbf{u}^\sharp(s) \text{ for } s < t' \\ \mathbf{u}^\star(s) = \mathbf{u}^\flat(s) \text{ for } t' < s. \end{cases}$$

So, we would have

$$\begin{aligned} V(x,t) - \epsilon &> \int_t^{t'} L(\mathbf{x}^\sharp(s), \mathbf{u}^\sharp(s), s) ds + V(y, t') \geq \\ &\geq \int_t^{t'} L(\mathbf{x}^\sharp(s), \mathbf{u}^\sharp(s), s) ds + J(y, t'; \mathbf{u}^\flat) - \frac{\epsilon}{2} = \\ &= J(x, t; \mathbf{u}^\star) - \frac{\epsilon}{2} \geq V(x,t) - \frac{\epsilon}{2}, \end{aligned}$$

which is a contradiction. □

4. Optimal controls - bounded control space

We now give a proof of the existence of optimal controls for bounded control space. The unbounded case will be addressed in §3.

LEMMA 11. *Let f is as in* (13) *a linear control law. Then J is weakly lower semicontinuous, with respect to weak-* convergence in L^∞.*

PROOF. Let u_n be a sequence of controls such that $\mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty[t, t_1]$. Then, by using Ascoli-Arzela theorem, we can extract a subsequence of $\mathbf{x}_n(\cdot)$ converging uniformly to $\mathbf{x}(\cdot)$. Furthermore, because the control law (13) is linear we have

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}).$$

We have

$$J(x, t; \mathbf{u}_n) = \int_t^{t_1} \left[L(\mathbf{x}_n(s), \mathbf{u}_n(s), s) - L(\mathbf{x}(s), \mathbf{u}_n(s), s)\right] ds + \\ + \int_t^{t_1} L(\mathbf{x}(s), \mathbf{u}_n(s), s) ds + \psi(\mathbf{x}_n(t_1)).$$

The first term, $\int_t^{t_1} \left[L(\mathbf{x}_n(s), \mathbf{u}_n(s), s) - L(\mathbf{x}(s), \mathbf{u}_n(s), s)\right] ds$, converges to zero. Similarly, $\psi(\mathbf{x}_n(t_1)) \to \psi(\mathbf{x}(t_1))$. Finally, the convexity of L implies

$$L(\mathbf{x}(s), \mathbf{u}_n(s), s) \geq L(\mathbf{x}(s), \mathbf{u}(s), s) + D_v L(\mathbf{x}(s), \mathbf{u}(s), s)(\mathbf{u}_n(s) - \mathbf{u}(s)).$$

Since $\mathbf{u}_n \rightharpoonup \mathbf{u}$,

$$\int_t^{t_1} D_v L(\mathbf{x}(s), \mathbf{u}(s), s)(\mathbf{u}_n(s) - \mathbf{u}(s)) ds \to 0.$$

Hence

$$\liminf J(x, t; \mathbf{u}_n) \geq J(x, t; \mathbf{u}),$$

that is, J is weakly lower semicontinuous. □

Using the previous result we can now state and prove our first existence result.

LEMMA 12. *Suppose the control set U is bounded, closed and convex. There exists a minimizer $\mathbf{u}^*$ of J.*

PROOF. Let $\mathbf{u}_n$ be a minimizing sequence, that is, such that

$$J(x, t; \mathbf{u}_n) \to \inf_{\mathbf{u} \in \mathcal{U}_R} J(x, t; \mathbf{u}).$$

Because this sequence is bounded in L^∞, by Banach-Alaoglu theorem we can extract a sequence $\mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u}^*$. Clearly, we have $\mathbf{u}^* \in U$, by closeness and convexity. We claim now that

$$J(x, t; \mathbf{u}^*) = \inf_{\mathbf{u}} J(x, t; \mathbf{u}).$$

This just follows from the weak lower semicontinuity:

$$\inf_{\mathbf{u}} J(x,t;\mathbf{u}) \le J(x,t;\mathbf{u}^*) \le \liminf J(x,t;\mathbf{u}_n) = \inf_{\mathbf{u}} J(x,t;\mathbf{u}),$$

which ends the proof. □

EXAMPLE 6 (Bang-Bang principle). Consider the case of a bounded closed convex control space U and suppose the Lagrangian L is constant. Suppose $f(x,u) = Au + B$, for suitable constant matrices A and B, and that the terminal value ψ is convex.

In this setting we first observe that the set of all optimal controls is convex. As such it admits an extreme point $\mathbf{u}^*$. We claim that $\mathbf{u}^*$ takes values on ∂U.

To see this, choose a time r and suppose that for some ϵ there is a set of positive measure in $[r, r+\epsilon]$ for which $\mathbf{u}^*$ is in the interior of U. Then there exists an L^∞ function ν supported on this set such that $\int_r^{r+\epsilon} d\nu = 0$, and such that $\mathbf{u}^* \pm \nu$ is an admissible control. By our assumptions it is also an optimal control. It is clear then that $\mathbf{u}^*$ is not an extreme point, which is a contradiction. ◀

5. Pontryagin maximum principle

In this section we assume the control space U is bounded and that we can apply the results of the previous section to establish existence of an optimal control $\mathbf{u}^*$ and corresponding optimal trajectory $\mathbf{x}^*$. We assume also that the terminal data ψ is differentiable.

Let $r \in [t, t_1)$ be a point where $\mathbf{u}^*$ is strongly approximately continuous, i.e.,

$$\varphi(\mathbf{u}^*(r)) = \lim_{\delta\to 0} \frac{1}{\delta} \int_r^{r+\delta} \varphi(\mathbf{u}^*(s))ds,$$

for all continuous functions φ (note that in the limit δ can take both positive and negative values). Note that almost any r is a point of approximate continuity, see [**EG92**]. Denote by Ξ_0 the fundamental solution of

$$\dot{\xi}_0 = D_x f(\mathbf{x}^*, \mathbf{u}^*)\xi_0, \tag{15}$$

with $\Xi_0(r) = I$.

Let $\mathbf{p}^*$ be given by

$$\mathbf{p}^*(r) = D_x\psi(\mathbf{x}_R(t_1))\Xi_0(t_1) + \int_r^{t_1} D_x L(\mathbf{x}^*(s), \mathbf{u}^*(s), s)\Xi_0(s)ds. \tag{16}$$

LEMMA 13 (Pontryagin maximum principle). *Suppose that ψ is differentiable. Let $\mathbf{u}^*$ be an optimal control for initial data (x,t) and $\mathbf{x}^*$ the corresponding optimal trajectory. Then, for almost all $r \in (t,T)$,*

$$\begin{aligned} f(\mathbf{x}^*(r),\mathbf{u}^*(r))\cdot\mathbf{p}^*(r)+L(\mathbf{x}^*(r),\mathbf{u}^*(r),r)& \\ =\min_{v\in U}\left[f(\mathbf{x}^*,v)\cdot\mathbf{p}^*(r)+L(\mathbf{x}^*(r),v,r)\right].& \end{aligned} \tag{17}$$

PROOF. Let $v \in U$. For almost all $r \in (t,T)$ $\mathbf{u}^*$ is strongly approximately continuous (see [**EG92**]). Let r be one of these points. Let $\delta \geq 0$ Define

$$\mathbf{u}_\delta(s)=\begin{cases} v & \text{if} \quad r-\delta<s<r \\ \mathbf{u}^*(s) & \text{otherwise.}\end{cases}$$

Let

$$\mathbf{x}_\delta(s)=\begin{cases} \mathbf{x}^*(s) & \text{if} \quad t<s<r-\delta \\ \mathbf{x}^*(r)+\int_{r-\delta}^s f(\mathbf{x}_\delta^*,v) & \text{if} \quad r-\delta<s<r \\ \mathbf{x}^*(s)+\delta\xi_\delta(s) & \text{if} \quad r-\delta<s<T,\end{cases}$$

where

$$\xi_\delta(r)=\frac{1}{\delta}\int_{r-\delta}^r\left[f(\mathbf{x}_\delta^*(s),v)-f(\mathbf{x}^*(s),\mathbf{u}^*(s))\right]ds,$$

and $\mathbf{y}_\delta \equiv \mathbf{x}^*(s)+\delta\xi_\delta(s)$ solves, for $r<s<t_1$,

$$\dot{\mathbf{y}}_\delta = f(\mathbf{y}_\delta,\mathbf{u}^*).$$

Observe that

$$\xi_0(r)=\lim_{\delta\to 0}\xi_\delta(r+\delta)=f(\mathbf{x}^*(r),v)-f(\mathbf{x}^*(r),\mathbf{u}^*(r)). \tag{18}$$

By a standard ODE result we have that ξ_δ converges, as $\delta \to 0$, to a solution ξ_0 of (15) with initial data given by (18). Thus $\xi_0(s)=\Xi_0(s)\left(f(\mathbf{x}^*(r),v)-f(\mathbf{x}^*(r),\mathbf{u}^*(r))\right)$.

Clearly

$$J(t,x;\mathbf{u}^*)\leq\int_t^T L(\mathbf{x}_\delta(s),\mathbf{u}_\delta(s),s)ds+\psi(\mathbf{x}^*(T)+\delta\xi_\delta).$$

This last inequality implies

$$\begin{aligned} &\frac{1}{\delta}\int_{r-\delta}^r\left[L(\mathbf{x}_\delta(s),v,s)-L(\mathbf{x}^*(s),\mathbf{u}^*(s),s)\right]ds+ \\ &+\frac{1}{\delta}\int_r^T\left[L(\mathbf{x}^*(s)+\delta\xi_\delta,\mathbf{u}^*(s),s)-L(\mathbf{x}^*(s),\mathbf{u}^*(s),s)\right]ds+ \\ &+\frac{1}{\delta}\left[\psi(\mathbf{x}^*(T)+\delta\xi_\delta)-\psi(\mathbf{x}^*(T))\right]\geq 0.\end{aligned}$$

When $\delta \to 0$, the first term converges to

$$L(\mathbf{x}^*(r),v,r)-L(\mathbf{x}^*(r),\mathbf{u}^*(r),r),$$

since $\mathbf{u}^*$ is strongly approximately continuous. The second term converges to

$$\int_r^T D_x L(\mathbf{x}^*(s), \mathbf{u}^*(s), s)\xi_0(s)ds,$$

whereas the third one has the following limit:

$$D_x\psi(\mathbf{x}_R(T)) \cdot \xi_0(T)).$$

This implies that for almost all $r \in (t, T)$,

$$\begin{aligned} &L(\mathbf{x}^*(r), v, r) - L(\mathbf{x}^*(r), \mathbf{u}^*(r), r) \\ &\qquad + \mathbf{p}^*(r) \cdot (f(\mathbf{x}^*(r), v) - f(\mathbf{x}^*(r), \mathbf{u}^*(r))) \geq 0. \end{aligned}$$

Consequently

$$\begin{aligned} &f(\mathbf{x}^*(r), \mathbf{u}^*(r)) \cdot \mathbf{p}^*(r) + L(\mathbf{x}^*(r), \mathbf{u}^*(r), r) \\ &\qquad = \min_{v \in U} \left[f(\mathbf{x}^*(r), v) \cdot \mathbf{p}^*(r) + L(\mathbf{x}_R(r), v, r) \right], \end{aligned}$$

as required. □

6. The Hamilton-Jacobi equation

We now show that if the value function is differentiable then it is a solution to the Hamilton-Jacobi equation.

THEOREM 14. *Let V be the value function to the terminal value problem. Suppose V is C^1. Then it solves the Hamilton-Jacobi equation*

$$-V_t + H(D_x V, x) = 0.$$

PROOF. Fix any constant control u^*. Then, by the dynamic programming principle

$$V(x, t) \leq \int_t^{t+h} L(\mathbf{x}(s), u^*) + V(\mathbf{x}(t+h), t+h).$$

By using Taylor's formula, dividing by h we obtain, as $h \to 0$,

$$\begin{aligned} 0 &\leq V_t + L(x, u^*) + f(x, u^*) \cdot D_x V \\ &\leq V_t - H(D_x V, x), \end{aligned}$$

that is

$$-V_t + H(D_x V, x) \leq 0.$$

Suppose now that in fact the previous inequality were strict at a point (x_0, t_0), that is

$$-V_t(x_0, t_0) + H(D_x V(x_0, t_0), x_0) = -\delta < 0.$$

Then in a neighborhood N of (x_0, t_0) we have

$$-V_t(x,t) + H(D_xV(x,t), x) < -\frac{\delta}{2}.$$

Let $\mathbf{u}^*$ be an optimal control, and let τ be the exit time of N of the corresponding trajectory. Then, by Taylor's formula,

$$V(\mathbf{x}(\tau),\tau) - V(x_0,t_0) = \int_{t_0}^{\tau} V_t + f\cdot D_xV$$

and, by the dynamic programming principle

$$V(x_0,t_0) = \int_{t_0}^{\tau} L(\mathbf{x},\mathbf{u}^*)dt + V(\mathbf{x}(\tau),\tau).$$

Thus

$$\begin{aligned} 0 &= \int_{t_0}^{\tau} L(\mathbf{x},\mathbf{u}^*) + V_t + f\cdot D_xV dt \\ &\geq \int_{t_0}^{\tau} V_t - H(D_xV(\mathbf{x},t),\mathbf{x}) \geq \frac{\delta}{2}(\tau - t_0), \end{aligned}$$

which is a contradiction. □

EXERCISE 18. *Let $M(t), N(t)$ be $n\times n$ matrices with time-differentiable coefficients. Suppose that is N invertible. Let D be a $n\times n$ constant matrix. Consider the Lagrangian*

$$L(x,v) = \frac{1}{2}x^TM(t)x + \frac{1}{2}v^TN(t)v$$

and the terminal condition $\psi = \frac{1}{2}x^TDx$. Show that there exists a solution to the Hamilton-Jacobi with terminal condition ψ at $t = T$ (at least for t close to T) of the form

$$V = \frac{1}{2}x^TP(t)x,$$

where $P(t)$ satisfies the Ricatti equation

$$\dot{P} = P^TN^{-1}P - M$$

and $P(T) = D$.

7. Verification theorem

Now we will show that any sufficiently smooth solution to the Hamilton-Jacobi equation is the value function and it can be used to compute an optimal control.

THEOREM 15. *Let $L(x,v)$ be a C^1 Lagrangian, strictly convex in v, and let $f(x,u)$ be a linear control law as in* (13), *and H be the generalized Legendre transform* (9) *of L. Let $\Phi(x,t)$ be a classical solution to the Hamilton-Jacobi equation*

$$-\Phi_t + H(D_x\Phi, x) = 0 \tag{19}$$

on the time interval $[0,T]$*, with terminal data* $\Phi(x,T)=\psi(x)$*. Then, for all* $0\leq t\leq T$*,*

$$\Phi(x,t)=V(x,t),$$

where V *is the value function.*

Proof. Let $\mathbf{u}$ be a control on $[t,T]$ and $\mathbf{x}$ be the corresponding solution to

$$\dot{\mathbf{x}}=f(\mathbf{x},\mathbf{u}),$$

with $\mathbf{x}(t)=x$. Then, using $\Phi(\mathbf{x}(T),T)=\psi(\mathbf{x}(T))$ we have

$$\begin{aligned}\psi(\mathbf{x}(T))-\Phi(\mathbf{x}(t),t)&=\int_t^T\frac{d}{ds}\Phi(\mathbf{x}(s),s)ds\\&=\int_t^T D_x\Phi(\mathbf{x}(s),s)\cdot f(\mathbf{x},\mathbf{u})+\Phi_s(\mathbf{x}(s),s)ds.\end{aligned}$$

Adding $\int_t^T L(\mathbf{x}(s),\mathbf{u}(s))ds+\Phi(\mathbf{x}(t),t)$ to the above equality and taking the infimum over all controls $\mathbf{u}$, we obtain

$$\begin{aligned}&\inf\left(\int_t^T L(\mathbf{x}(s),\mathbf{u}(s))ds+\psi\big(\mathbf{x}(T)\big)\right)\\&\quad=\Phi(\mathbf{x}(t),t)\\&\quad+\inf\left(\int_t^T\Phi_s(\mathbf{x}(s),s)+L(\mathbf{x}(s),\mathbf{u}(s))+D_x\Phi(\mathbf{x}(s),s)\cdot f(\mathbf{x},\mathbf{u})ds\right).\end{aligned}$$

Now recall that for any v,

$$-H(p,x)\leq L(x,v)+p\cdot f(x,v),$$

therefore

$$\begin{aligned}V(x,t)&=\inf\left(\int_t^T L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds+\varphi\big(\mathbf{x}(T)\big)\right)\\&\geq\Phi(\mathbf{x}(t),t)+\inf\left(\int_t^T\big(\Phi_s(\mathbf{x}(s),s)-H(D_x\Phi(\mathbf{x}(s),s),\mathbf{x}(s))\big)ds\right)\\&=\Phi(\mathbf{x}(t),t).\end{aligned}$$

Let $r(x,t)$ be uniquely defined (uniqueness follows from convexity) as

$$r(x,t)\in\operatorname{argmin}_{v\in U}L(x,v)+D_x\Phi(x,t)\cdot f(x,v).\tag{20}$$

A simple argument shows that r is a continuous function.

Now consider the trajectory $\mathbf{x}$ obtained by solving the following differential equation

$$\dot{\mathbf{x}}(s)=f(\mathbf{x},r(\mathbf{x}(s),s)),$$

with initial condition $\mathbf{x}(t) = x$. Note that since the right-hand side is continuous there is a solution, although it may not be unique. Then

$$
\begin{aligned}
V(x,t) &= \inf \left(\int_t^T L(\mathbf{x}(s), \dot{\mathbf{x}}(s)) ds + \psi\big(\mathbf{x}(T)\big) \right) \\
&\leq \Phi(\mathbf{x}(t), t) + \int_t^T \Big(\Phi_s\big(\mathbf{x}(s), s\big) - H\big(D_x\Phi(\mathbf{x}(s), s), \mathbf{x}(s)\big) \Big) ds \\
&= \Phi(\mathbf{x}(t), t),
\end{aligned}
$$

which ends the proof. $\square$

We should observe from the proof that (20) gives an optimal feedback law for the optimal control, provided we can find a solution to the Hamilton-Jacobi equation (19).

8. Bibliographical notes

The main references we have used on optimal control are [**BCD97**], [**FS06**], [**Lio82**], [**Bar94**], and [**Eva98**].

Viscosity solutions

In this chapter we build upon the theory developed previously to study the terminal value problem. In section 2 we give some technical results concerning subdifferentials and semiconcavity. Then, in section 3 we consider the issue of existence of controls and regularity of the value function in the calculus of variations setting. It is well known that first order partial differential equations such as the Hamilton-Jacobi equation may not admit classical solutions. Using the method of characteristics, the next exercise gives an example of non-existence of smooth solutions:

Exercise 19. *Solve, using the method of characteristics, the equation*

$$\begin{cases} u_t + u_x^2 = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \pm x^2. \end{cases}$$

It is therefore necessary to consider weak solutions to the Hamilton-Jacobi equation: viscosity solutions. In section §4 we develop the theory of viscosity solutions for Hamilton-Jacobi equations, and show that the value function is the unique viscosity solution of the Hamilton-Jacobi equation.

1. Viscosity Solutions

A bounded uniformly continuous function V is a *viscosity subsolution (resp. supersolution)* of the Hamilton-Jacobi equation (27) if for any C^1 function ϕ and any interior point $(x,t) \in \operatorname{argmax}\ V - \phi$ (resp. argmin) then

$$-D_t\phi + H(D_x\phi, x, t) \leq 0$$

(resp. ≥ 0) at (x,t). A bounded uniformly continuous function V is a *viscosity solution* of the Hamilton-Jacobi equation if it is both a sub and supersolution.

The value function is a viscosity solution of (27), although it may not be a classical solution. The motivation behind the definition of viscosity solution is the following: if V is differentiable and $(x,t) \in argmaxV - \phi$ (or argmin) then

$D_x V = D_x \phi$ and $D_t V = D_t \phi$, therefore we should have both inequalities. The specific choice of inequalities is related with the following parabolic approximation of the Hamilton-Jacobi equation

$$-D_t u^\epsilon + H(D_x u^\epsilon, x, t) = \epsilon \Delta u^\epsilon. \tag{21}$$

This equation arises naturally in optimal stochastic control. The limit $\epsilon \to 0$ corresponds to the case in which the diffusion coefficient vanishes.

PROPOSITION 16. *Let u^ϵ be a family of solutions of (21) such that, as $\epsilon \to 0$, the sequence $u^\epsilon \to u$ uniformly. Then u is a viscosity solution of (27).*

PROOF. Suppose that $\phi(x,t)$ is a C^2 function such that $u - \phi$ has a strict local maximum at $(\overline{x}, \overline{t})$. We must show that

$$-D_t \phi + H(D_x \phi, \overline{x}, \overline{t}) \leq 0.$$

By hypothesis, $u^\epsilon \to u$ uniformly. Therefore we can find sequences $(\overline{x}_\epsilon, \overline{t}_\epsilon) \to (\overline{x}, \overline{t})$ such that $u^\epsilon - \phi$ has a local maximum at $(\overline{x}_\epsilon, \overline{t}_\epsilon)$. Therefore,

$$Du^\epsilon(\overline{x}_\epsilon, \overline{t}_\epsilon) = D\phi(\overline{x}_\epsilon, \overline{t}_\epsilon)$$

and

$$\Delta u^\epsilon(\overline{x}_\epsilon, \overline{t}_\epsilon) \leq \Delta \phi(\overline{x}_\epsilon, \overline{t}_\epsilon).$$

Consequently,

$$-D_t \phi(\overline{x}_\epsilon, \overline{t}_\epsilon) + H(D_x \phi(\overline{x}_\epsilon, \overline{t}_\epsilon), \overline{x}_\epsilon, \overline{t}_\epsilon) \leq \epsilon \Delta \phi(\overline{x}_\epsilon, \overline{t}_\epsilon).$$

It is therefore enough to take $\epsilon \to 0$ to end the proof. ∎

THEOREM 17. *Assume we are in the bounded control case. The value function is a viscosity solution to*

$$-V_t + H(DV, x) = 0.$$

PROOF. Let $\varphi(x,t)$ be a smooth function and let $(x_0, t_0) \in \operatorname{argmin} V - \varphi$. Without loss of generality we may assume $V(x_0, t_0) = \varphi(x_0, t_0)$, and so $V(x,t) \geq \varphi(x,t)$ for all (x,t).

By the dynamic programming principle, there exists $\mathbf{u}^h$ and $\mathbf{x}^h$ such that

$$\begin{aligned}\varphi(x_0, t_0) = V(x_0, t_0) &\geq \int_{t_0}^{t_0+h} L(\mathbf{x}^h, \mathbf{u}^h) dt + V(\mathbf{x}^h(t_0+h), t_0+h) - h^2 \\ &\geq \int_{t_0}^{t_0+h} L(\mathbf{x}^h, \mathbf{u}^h) dt + \varphi(\mathbf{x}^h(t_0+h), t_0+h) - h^2.\end{aligned}$$

Using Taylor's formula we then conclude that

$$\begin{aligned} 0 &\geq \int_{t_0}^{t_0+h} (L(\mathbf{x},\mathbf{u}) + \varphi_t + fD_x\varphi)\, dt - h^2 \\ &\geq \int_{t_0}^{t_0+h} (\varphi_t - H(D\varphi(\mathbf{x}),\mathbf{x}))\, dt - h^2. \end{aligned}$$

Dividing by h and sending $h \to 0$, we conclude that

$$-\varphi_t(x_0,t_0) + H(D_x\varphi(x_0,t_0),x_0) \geq 0,$$

since $\sup_{t_0\leq s\leq t_0+h} |\mathbf{x}^h(s) - x_0| \to 0$, since we are in the bounded control setting.

To obtain the second inequality, suppose $(x_0,t_0) \in \operatorname{argmax} V - \varphi$. As before, without loss of generality we may assume $V(x_0,t_0) = \varphi(x_0,t_0)$, and so $V(x,t) \leq \varphi(x,t)$ for all (x,t).

Let $\mathbf{u}^*$ be a constant control. Then, by the dynamic programming principle

$$\begin{aligned} \varphi(x_0,t_0) = V(x_0,t_0) &\leq \int_{t_0}^{t_0+h} L(\mathbf{x},\mathbf{u}^*)dt + V(\mathbf{x}(t_0+h),t_0+h) \\ &\leq \int_{t_0}^{t_0+h} L(\mathbf{x},\mathbf{u}^*)dt + \varphi(\mathbf{x}(t_0+h),t_0+h). \end{aligned}$$

This then implies, by sending $h \to 0$,

$$0 \leq L(x_0,\mathbf{u}^*) + \varphi_t + fD_x\varphi dt,$$

and so

$$-\varphi_t(x_0,t_0) + H(D_x\varphi(x_0,t_0),x_0) \leq 0.$$

□

2. Sub and superdifferentials

Before proceeding with the general case of unbounded control spaces we will need to discuss some technical results concerning sub-differentials and semiconcavity.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The *superdifferential* $D_x^+\psi(x)$ of ψ at x is the set of vectors $p \in \mathbb{R}^n$ such that

$$\limsup_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p\cdot v}{|v|} \leq 0.$$

Consequently, $p \in D_x^+\psi(x)$ if and only if

$$\psi(x+v) \leq \psi(x) + p\cdot v + o(|v|),$$

as $|v| \to 0$. Similarly, the *subdifferential*, $D_x^-\psi(x)$, of ψ at x is the set of vectors p such that

$$\liminf_{|v|\to 0} \frac{\psi(x+v)-\psi(x)-p\cdot v}{|v|} \geq 0.$$

EXERCISE 20. *Show that if $u : \mathbb{R}^n \to \mathbb{R}$ has a maximum (resp. minimum) at x_0 then $0 \in D^+u(x_0)$ (resp. $D^-u(x_0)$).*

We can regard these sets as one-sided derivatives. In fact, ψ is differentiable then

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

More precisely,

PROPOSITION 18. *If both $D_x^-\psi(x)$ and $D_x^+\psi(x)$ are non-empty then*

$$D_x^-\psi(x) = D_x^+\psi(x) = \{p\},$$

furthermore ψ is differentiable at x with $D_x\psi = p$. Conversely, if ψ is differentiable at x then

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

PROOF. Suppose that $D_x^-\psi(x)$ and $D_x^+\psi(x)$ are both non-empty. Then we claim that these two sets agree and have a single point p. To see this, take $p^- \in D_x^-\psi(x)$ and $p^+ \in D_x^+\psi(x)$. Then

$$\liminf_{|v|\to 0} \frac{\psi(x+v)-\psi(x)-p^-\cdot v}{|v|} \geq 0$$
$$\limsup_{|v|\to 0} \frac{\psi(x+v)-\psi(x)-p^+\cdot v}{|v|} \leq 0.$$

By subtracting these two identities

$$\liminf_{|v|\to 0} \frac{(p^+-p^-)\cdot v}{|v|} \geq 0.$$

In particular, by choosing $v = -\epsilon\frac{p^+-p^-}{|p^--p^+|}$, we obtain

$$-|p^- - p^+| \geq 0,$$

which implies $p^- = p^+ \equiv p$. Consequently

$$\lim_{|v|\to 0} \frac{\psi(x+v)-\psi(x)-p\cdot v}{|v|} = 0,$$

and so $D_x\psi = p$.

To prove the converse it suffices to observe that if ψ is differentiable then

$$\psi(x+v) = \psi(x) + D_x\psi(x)\cdot v + o(|v|).$$

□

EXERCISE 21. *Let ψ be a continuous function. Show that if x_0 is a local maximum of ψ then $0 \in D^+\psi(x_0)$.*

PROPOSITION 19. *Let*
$$\psi : \mathbb{R}^n \to \mathbb{R}$$
be a continuous function. Then, if
$$p \in D_x^+\psi(x_0) \quad (\textit{resp. } p \in D_x^-\psi(x_0)),$$
there exists a C^1 function ϕ such that
$$\psi(x) - \phi(x)$$
has a local strict maximum (resp. minimum) at x_0 and such that
$$p = D_x\phi(x_0).$$
On the other hand, if ϕ is a C^1 function such that
$$\psi(x) - \phi(x)$$
has a local maximum (resp. minimum) at x_0 then
$$D_x\phi(x_0) \in D_x^+\psi(x_0) \quad (\textit{resp. } D_x^-\psi(x_0)).$$

PROOF. By subtracting $p \cdot (x - x_0) + \psi(x_0)$ to ψ, we can assume, without loss of generality, that $\psi(x_0) = 0$ and $p = 0$. By changing coordinates, if necessary, we can also assume that $x_0 = 0$. Because $0 \in D_x^+\psi(0)$ we have
$$\limsup_{|x|\to 0} \frac{\psi(x)}{|x|} \leq 0.$$
Therefore there exists a continuous function $\rho(x)$, with $\rho(0) = 0$, such that
$$\psi(x) \leq |x|\rho(x).$$
More precisely we can choose
$$\rho(x) = \max\{\frac{\psi}{|x|}, 0\}.$$
Let $\eta(r) = \max_{|x|\leq r}\{\rho(x)\}$. Then η is continuous, non decreasing and $\eta(0) = 0$. Let
$$\phi(x) = \int_{|x|}^{2|x|} \eta(r)dr + |x|^2.$$
The function ϕ is C^1 and satisfies $\phi(0) = D_x\phi(0) = 0$. Additionally, if $x \neq 0$,
$$\psi(x) - \phi(x) \leq |x|\rho(x) - \int_{|x|}^{2|x|} \eta(r)dr - |x|^2 < 0.$$
Thus $\psi - \phi$ has a strict local maximum at 0.

To prove the second part of the proposition, suppose that the difference $\psi(x)-\phi(x)$ has a strict local maximum at 0. Without loss of generality, we can assume $\psi(0)-\phi(0)=0$ and $\phi(0)=0$. Then $\psi(x)-\phi(x)\le 0$ or, equivalently,

$$\psi(x)\le p\cdot x+(\phi(x)-p\cdot x).$$

Thus, by setting $p=D_x\phi(0)$, and using the fact that

$$\lim_{|x|\to 0}\frac{\phi(x)-p\cdot x}{|x|}=0,$$

we conclude that $D_x\phi(0)\in D_x^+\psi(0)$. The case of a minimum is similar. □

A continuous function f is *semiconcave* if there exists C such that

$$f(x+y)+f(x-y)-2f(x)\le C|y|^2.$$

Similarly, a function f is *semiconvex* if there exists a constant such that

$$f(x+y)+f(x-y)-2f(x)\ge -C|y|^2.$$

Proposition 20. *The following statements are equivalent:*

1. *f is semiconcave;*
2. *$\tilde{f}(x)=f(x)-\frac{C}{2}|x|^2$ is concave;*
3. *for all λ, $0\le\lambda\le 1$, and any y,z such that $\lambda y+(1-\lambda)z=0$ we have*

$$\lambda f(x+y)+(1-\lambda)f(x+z)-f(x)\le\frac{C}{2}(\lambda|y|^2+(1-\lambda)|z|^2).$$

Additionally, if f is semiconcave, then

a. $D_x^+f(x)\neq\emptyset$;
b. *if $D_x^-f(x)\neq\emptyset$ then f is differentiable at x;*
c. *there exists C such that, for each $p_i\in D_x^+f(x_i)$ $(i=0,1)$,*

$$(x_0-x_1)\cdot(p_0-p_1)\le C|x_0-x_1|^2.$$

Remark. Of course analogous results hold for semiconvex functions.

Proof. Clearly $2\implies 3\implies 1$. Therefore, to prove the equivalence, it is enough to show that $1\implies 2$. Subtracting $C|x|^2$ to f, we may assume $C=0$. Also, by changing coordinates if necessary, it suffices to prove that for all x,y such that $\lambda x+(1-\lambda)y=0$, for some $\lambda\in[0,1]$, we have:

$$\lambda f(x)+(1-\lambda)f(y)-f(0)\le 0. \tag{22}$$

We claim now that the previous equation holds for each $\lambda=\frac{k}{2^j}$, with $0\le k\le 2^j$. Clearly (22) holds for $j=1$. We will proceed by induction on j. Suppose that

(22) if valid for $\lambda = \frac{k}{2^j}$. We will show that it also holds for $\lambda = \frac{k}{2^{j+1}}$. If k is even, we can reduce the fraction and, therefore, we assume that k is odd, $\lambda = \frac{k}{2^{j+1}}$ and $\lambda x + (1-\lambda)y = 0$. Note that

$$0 = \frac{1}{2}\left[\frac{k-1}{2^{j+1}}x + \left(1 - \frac{k-1}{2^{j+1}}\right)y\right] + \frac{1}{2}\left[\frac{k+1}{2^{j+1}}x + \left(1 - \frac{k+1}{2^{j+1}}y\right)\right].$$

consequently,

$$\begin{aligned} f(0) \geq & \frac{1}{2} f\left(\frac{k-1}{2^{j+1}}x + \left(1 - \frac{k-1}{2^{j+1}}\right)y\right) + \\ & + \frac{1}{2} f\left(\frac{k+1}{2^{j+1}}x + \left(1 - \frac{k+1}{2^{j+1}}\right)y\right) \end{aligned}$$

but, since $k-1$ and $k+1$ are even, $\tilde{k}_0 = \frac{k-1}{2}$ and $\tilde{k}_1 = \frac{k+1}{2}$ are integers. Therefore

$$f(0) \geq \frac{1}{2} f\left(\frac{\tilde{k}_0}{2^j}x + \left(1 - \frac{\tilde{k}_0}{2^j}\right)y\right) + \frac{1}{2} f\left(\frac{\tilde{k}_1}{2^j}x + \left(1 - \frac{\tilde{k}_1}{2^j}\right)y\right)$$

But this implies

$$f(0) \geq \frac{\tilde{k}_0 + \tilde{k}_1}{2^{j+1}} f(x) + \left(1 - \frac{\tilde{k}_0 + \tilde{k}_1}{2^{j+1}}\right) f(y).$$

From $\tilde{k}_0 + \tilde{k}_1 = k$ we obtain

$$f(0) \geq \frac{k}{2^{j+1}} f(x) + \left(1 - \frac{k}{2^{j+1}}\right) f(y).$$

Since the function f is continuous and the rationals of the form $\frac{k}{2^j}$ are dense in $\mathbb{R}$, we conclude that

$$f(0) \geq \lambda f(x) + (1-\lambda) f(y),$$

for each real λ, with $0 \leq \lambda \leq 1$.

To prove the second part of the proposition, observe that by proposition 18, $a \implies b$. To check a, i.e., that $D_x^+ f(x) \neq \emptyset$, it is enough to observe that if f is concave then $D_x^+ f(x) \neq \emptyset$. By subtracting $C|x|^2$ to f, we can reduce the problem to concave functions. Finally, if $p_i \in D_x^+ f(x_i)$ $(i = 0, 1)$ then

$$f(x_0) - \frac{C}{2}|x_0|^2 \leq f(x_1) - \frac{C}{2}|x_1|^2 + (p_1 - Cx_1) \cdot (x_0 - x_1),$$

and

$$f(x_1) - \frac{C}{2}|x_1|^2 \leq f(x_0) - \frac{C}{2}|x_0|^2 + (p_0 - Cx_0) \cdot (x_1 - x_0).$$

Therefore,

$$0 \leq (p_1 - p_0) \cdot (x_0 - x_1) + C|x_0 - x_1|^2,$$

and so $(p_0 - p_1) \cdot (x_0 - x_1) \leq C|x_0 - x_1|^2$. ☐

EXERCISE 22. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that if x_0 is a local maximum then $0 \in D^+ f(x_0)$.*

3. Calculus of variations setting

We now consider the calculus of variations setting and prove the existence of optimal controls. The main technical issue is the fact that the control space $U = \mathbb{R}^n$ is unbounded and therefore compactness arguments do not work directly. We will consider the calculus of variations setting, that is $f(x, u) = u$ and we will work under the following assumptions:

$$L(x, v) : \mathbb{R}^{2n} \to \mathbb{R},$$

$x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, is a C^∞ function, strictly convex em v, i.e., $D^2_{vv}L$ is positive definite, and satisfying the coercivity condition

$$\lim_{|v|\to\infty} \frac{L(x, v, t)}{|v|} = \infty,$$

for each (x, t); without loss of generality, we may also assume that $L(x, v, t) \geq 0$, by adding a constant if necessary. We will also assume that

$$L(x, 0, t) \leq c_1, \quad |D_x L| \leq c_2 L + c_3,$$

for suitable constants c_1, c_2 and c_3; finally we assume that there exists a function $C(R)$ such that

$$|D^2_{xx}L| \leq C(R), \quad |D_v L| \leq C(R)$$

whenever $|v| \leq R$. The terminal cost, ψ, is assumed to be a bounded Lipschitz function.

Example 7. Although the conditions on L are quite technical, they are fulfilled by a wide class of Lagrangians, for instance

$$L(x, v) = \frac{1}{2} v^T A(x) v - V(x),$$

where A and V are C^∞,bounded with bounded derivatives, and $A(x)$ is positive definite. ◀

Fortunately, the coercivity of the Lagrangian is enough to establish the existence of a-priori bounds on optimal controls.

Theorem 21. *Let $x \in \mathbb{R}^n$ and $t_0 \leq t \leq t_1$. Suppose that the Lagrangian $L(x, v)$ satisfies:*

A. *L is C^∞, strictly convex in v (i.e., $D^2_{vv}L$ is positive definite), and satisfying the coercivity condition*

$$\lim_{|v|\to\infty} \frac{L(x, v)}{|v|} = \infty,$$

uniformly in (x, t);

B. *L is bounded by below (without loss of generality we assume $L(x,v) \geq 0$);*
C. *L satisfies the inequalities*

$$L(x,0) \leq c_1, \qquad |D_x L| \leq c_2 L + c_3$$

for suitable c_1, c_2, and c_3;
D. *there exist functions $C_0(R), C_1(R) : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|D_v L| \leq C_0(R), \qquad |D^2_{xx} L| \leq C_1(R)$$

whenever $|v| \leq R$.

Then, if ψ is a bounded Lipschitz function,

1. *There exists $\mathbf{u}^* \in L^\infty[t,t_1]$ such that its corresponding trajectory $\mathbf{x}^*$, given by*

$$\dot{\mathbf{x}}^*(s) = \mathbf{u}(s) \qquad \mathbf{x}^*(t) = x,$$

is optimal, that is it satisfies

$$V(x,t) = \int_t^{t_1} L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s))ds + \psi(\mathbf{x}^*(t_1)).$$

2. *There exists C, depending only on L, ψ and $t_1 - t$ but not on x or t such that $|\mathbf{u}(s)| < C$ for $t \leq s \leq t_1$. The optimal trajectory $\mathbf{x}^*(\cdot)$ is a $C^2[t,t_1]$ solution of the Euler-Lagrange equation*

$$\frac{d}{dt} D_v L - D_x L = 0 \tag{23}$$

with initial condition $\mathbf{x}^*(t) = x$.

3. *The adjoint variable $\mathbf{p}$, defined by*

$$\mathbf{p}(t) = -D_v L(\mathbf{x}^*, \dot{\mathbf{x}}^*), \tag{24}$$

satisfies the differential equation

$$\begin{cases} \dot{\mathbf{p}}(s) = D_x H(\mathbf{p}(s), \mathbf{x}^*(s)) \\ \dot{\mathbf{x}}^*(s) = -D_p H(\mathbf{p}(s), \mathbf{x}^*(s)) \end{cases}$$

with terminal condition

$$\mathbf{p}(t_1) \in D_x^- \psi(\mathbf{x}^*(t_1)).$$

Additionally,

$$(\mathbf{p}(s), H(\mathbf{p}(s), \mathbf{x}^*(s))) \in D^- V(\mathbf{x}^*(s), s)$$

for $t < s \leq t_1$.

4. *The value function V is Lipschitz, and so almost everywhere differentiable.*
5. *If $D^2_{vv} L$ is uniformly bounded, then for each $t < t_1$, $V(x,t)$ is semiconcave in x.*

6. *For* $t \leq s < t_1$

$$(\mathbf{p}(s), H(\mathbf{p}(s), \mathbf{x}^*(s))) \in D^+V(\mathbf{x}^*(s), s)$$

and, therefore, $DV(\mathbf{x}^*(s), s)$ *exists for* $t < s < t_1$*.*

Proof. We will divide the proof into several auxiliary lemmas.

For $R > 0$, define $\mathcal{U}_R = \{\mathbf{u} \in \mathcal{U} : \|\mathbf{u}\|_\infty \leq R\}$. From lemma 12 there exists a minimizer $\mathbf{u}_R$ of J in $\mathcal{U}_R$. Then we will show that the minimizer $\mathbf{u}_R$ satisfies uniform estimates in R. Finally, we will let $R \to \infty$.

Let $\mathbf{p}_R$ be the adjoint variable given by the Pontryagin maximum principle. We now will try to estimate the optimal control $\mathbf{u}_R$ uniformly in R, in order to send $R \to \infty$.

Lemma 22. *Suppose* ψ *is differentiable. Then there exists a constant* C*, independent on* R*, such that*

$$|\mathbf{p}_R| \leq C.$$

Proof. Since ψ is Lipschitz and differentiable we have

$$|D_x\psi| \leq \|D_x\psi\|_\infty < \infty.$$

Therefore

$$|\mathbf{p}_R(s)| \leq \int_s^{t_1} |D_x L(\mathbf{x}_R(r), \mathbf{u}_R(r)| dr + \|D_x\psi\|_\infty.$$

Let V_R be the value function for the terminal value problem with the additional constraint of bounded control: $|v| \leq R$. From $|D_x L| \leq c_2 L + c_3$, it follows

$$|\mathbf{p}_R(s)| \leq C(V_R(t, x) + 1),$$

for an appropriate constant C. Proposition 7, shows that there exists a constant C, which does not depend on R, such that $V_R \leq C$. Therefore

$$|\mathbf{p}_R| \leq C.$$

□

As we will see, the uniform estimates for $\mathbf{p}_R$ yield uniform estimates for $\mathbf{u}_R$.

Lemma 23. *Let* ψ *be differentiable. Then there exists* $R_1 > 0$ *such that, for all* R*,*

$$\|\mathbf{u}_R\|_\infty \leq R_1.$$

PROOF. Suppose $|p| \leq C$. Then, for each c_1, the coercivity condition on L implies that there exists R_1 such that, if

$$v \cdot p + L(x, v) \leq c_1$$

then $|v| \leq R_1$. But then,

$$\mathbf{u}_R(s) \cdot \mathbf{p}_R(s) + L(\mathbf{x}_R(s), \mathbf{u}_R(s)) \leq L(\mathbf{x}_R(s), 0) \leq c_1,$$

that is, $\|\mathbf{u}_R\|_\infty \leq R_1$. □

Since $\mathbf{u}_R$ is bounded independently of R, we have

$$V = J(x, t; \mathbf{u}_{R_0}),$$

for $R_0 > R_1$. Let $\mathbf{u}^* = \mathbf{u}_{R_0}$ and $\mathbf{p} = \mathbf{p}_{R_0}$.

LEMMA 24 (Pontryagin maximum principle - II). *If ψ is differentiable, the optimal control $\mathbf{u}^*$ satisfies*

$$\mathbf{u}^* \cdot \mathbf{p} + L(\mathbf{x}^*, \mathbf{u}^*) = \min_v [v \cdot \mathbf{p} + L(\mathbf{x}^*, v)] = -H(\mathbf{p}, \mathbf{x}^*),$$

for almost all s and, therefore,

$$\mathbf{p} = -D_v L(\mathbf{x}^*, \mathbf{u}^*) \qquad \text{and} \qquad \mathbf{u}^* = -D_p H(\mathbf{p}, \mathbf{x}^*),$$

where $H = L^*$. Additionally, $\mathbf{p}$ satisfies the terminal condition

$$\mathbf{p}(t_1) = D_x \psi(\mathbf{x}^*(t_1)).$$

PROOF. Clearly it is enough to choose R sufficiently large. □

LEMMA 25. *Let ψ be differentiable. The minimizing trajectory $\mathbf{x}(\cdot)$ is C^2 and satisfies the Euler-Lagrange equation (23). Furthermore,*

$$\dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}^*) \qquad \dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x}^*).$$

PROOF. By its definition $\mathbf{p}$ is continuous. We know that

$$\dot{\mathbf{x}}^*(s) = -D_p H(\mathbf{p}(s), \mathbf{x}^*(s)),$$

almost everywhere. Since the right hand side of the previous identity is continuous, the identity holds everywhere and, therefore, we conclude that $\mathbf{x}^*$ is C^1. Because $\mathbf{p}$ is given by the integral of a continuous function (16),

$$\mathbf{p}(r) = D_x \psi(\mathbf{x}^*(t_1)) + \int_r^{t_1} D_x L(\mathbf{x}^*(s), \mathbf{u}^*(s)) ds,$$

we conclude that $\mathbf{p}$ is C^1. Additionally,

$$\dot{\mathbf{x}}^* = -D_p H(\mathbf{p}, \mathbf{x}^*)$$

and, therefore, $\dot{\mathbf{x}}^*$ is C^1, which implies that $\mathbf{x}$ is C^2. We have also

$$\mathbf{p} = -D_v L(\mathbf{x}^*, \dot{\mathbf{x}}^*) \qquad \dot{\mathbf{p}} = -D_x L(\mathbf{x}^*, \dot{\mathbf{x}}^*),$$

from which it follows

$$\frac{d}{dt} D_v L(\mathbf{x}^*, \dot{\mathbf{x}}^*) - D_x L(\mathbf{x}^*, \dot{\mathbf{x}}^*) = 0. \tag{25}$$

Thus, since $D_x L(\mathbf{x}^*, \dot{\mathbf{x}}^*) = -D_x H(\mathbf{p}, \mathbf{x}^*)$, we conclude that

$$\dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}^*) \qquad \dot{\mathbf{x}}^* = -D_p H(\mathbf{p}, \mathbf{x}^*),$$

as required. ☐

In the case in which ψ is only Lipschitz and not C^1, we can consider a sequence of C^1 functions, $\psi_n \to \psi$ uniformly, such that

$$\|D_x \psi_n\|_\infty \leq \|D\psi\|_{L^\infty}.$$

for each ψ_n. Let

$$J_n(x, t; \mathbf{u}) = \int_t^{t_1} L(\mathbf{x}_n(s), \mathbf{u}_n(s)) ds + \psi_n(\mathbf{x}_n(t_1)),$$

and $\mathbf{x}_n^*$, $\mathbf{u}_n^*$ are, respectively, the corresponding optimal trajectory and optimal control. Similarly, let $\mathbf{p}_n$ be the corresponding adjoint variable. Passing to a subsequence, if necessary, the boundary values $\mathbf{x}_n(t_1)$ and $\mathbf{p}_n(t_1)$ converge, respectively, for some x_0 and p_0. The optimal trajectories $\mathbf{x}_n^*$ and corresponding optimal controls $\mathbf{u}_n^*$ converge uniformly, by using Ascoli-Arzela theorem, to optimal trajectories and controls of the limit problem. Let

$$\mathbf{p}(s) = \lim_{n \to \infty} \mathbf{p}_n(s).$$

Then, for almost every s,

$$\mathbf{u}^* \cdot \mathbf{p}(s) + L(\mathbf{x}^*(s), \mathbf{u}^*(s)) = \inf_v \left[v \cdot \mathbf{p}(s) + L(\mathbf{x}^*(s), v) \right],$$

which implies

$$\mathbf{p}(s) = -D_v L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)),$$

for almost all s. But, in the previous equation both terms are continuous functions thus the identity holds for all s.

LEMMA 26. *For $t < s \leq t_1$ we have*

$$(\mathbf{p}(s), H(\mathbf{p}(s), \mathbf{x}^*(s))) \in D^- V(\mathbf{x}^*(s), s).$$

PROOF. Let $\mathbf{x}^*$ be an optimal trajectory and $\mathbf{u}^*$ the corresponding optimal control. For $r \leq t_1$ and $y \in \mathbb{R}^n$, define $x_r = \mathbf{x}^*(r)$ and consider the sub-optimal control

$$\mathbf{u}^\sharp(s) = \mathbf{u}^*(s) + \frac{y - x_r}{r - t},$$

whose trajectory we denote by $\mathbf{x}^\sharp$, $\mathbf{x}^\sharp(t) = x$. Note that $\mathbf{x}^\sharp(r) = y$.

We have

$$V(x,t) = \int_t^s L(\mathbf{x}^*(\tau), \mathbf{u}^*(\tau))d\tau + V(\mathbf{x}^*(s), s)$$

and, by the sub-optimality of $x^\sharp$,

$$V(\mathbf{x}^*(t),t) \le \int_t^r L(\mathbf{x}^\sharp(\tau), \mathbf{u}^\sharp(\tau))d\tau + V(y,r).$$

This implies

$$V(\mathbf{x}^*(s),s) - V(y,r) \le \phi(y,r),$$

with

$$\phi(y,r) = \int_t^r L(\mathbf{x}^\sharp(\tau), \mathbf{u}^\sharp(\tau))d\tau - \int_t^s L(\mathbf{x}^*(\tau), \mathbf{u}^*(\tau))d\tau.$$

Since ϕ is differentiable at y and r,

$$(-D_y\phi(\mathbf{x}^*(s),s), -D_r\phi(\mathbf{x}^*(s),s)) \in D^-V(\mathbf{x}^*(s),s).$$

Observe that

$$\mathbf{x}^\sharp(\tau) = \mathbf{x}^*(\tau) + \frac{y - x_r}{r-t}(\tau - t),$$

and, therefore,

$$D_y\phi(\mathbf{x}^*(s),s) = \int_t^s \left[D_xL\frac{\tau-t}{s-t} + D_vL\frac{1}{s-t}\right] d\tau.$$

Integrating by parts and using (25), we obtain

$$D_y\phi(\mathbf{x}^*(s),s) = D_vL(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)) = -\mathbf{p}(s).$$

Similarly,

$$\begin{aligned} D_r\phi(y,r) = L(y, \mathbf{u}^\sharp(r)) + \int_t^s \Bigg[&-D_xL\frac{y-x_r}{(r-t)^2}(\tau - t) \\ &+ D_xL\frac{-\mathbf{u}^*(r)}{(r-t)}(\tau-t) - D_vL\frac{y-x_r}{(r-t)^2} + D_vL\frac{-\mathbf{u}^*(r)}{r-t}\Bigg] d\tau. \end{aligned}$$

Integrating by parts and evaluating at $y = \mathbf{x}^*(s)$, $r = s$, we obtain

$$\begin{aligned} D_r\phi(\mathbf{x}^*(s),s) &= L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)) - \mathbf{u}^*(s)D_vL(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)) \\ &= -H(\mathbf{p}(s), \mathbf{x}^*(s)), \end{aligned}$$

as we needed. □

LEMMA 27. *The value function V is Lipschitz.*

PROOF. Let $t < t_1$ be fixed and x, y arbitrary. We suppose first that $t_1 - t < 1$. Then

$$V(y,t) - V(x,t) \le J(y,t;\mathbf{u}^*) - V(x,t),$$

where $V(x,t) = J(x,t;\mathbf{u}^*)$. Therefore, there exists a constant C, depending only on the Lipschitz constant of ψ and of the supremum of $|D_x L|$, such that

$$V(y,t) - V(x,t) \leq C|x-y|.$$

Suppose that $t_1 - t > 1$. Let

$$\begin{cases} \tilde{\mathbf{u}}(s) = \mathbf{u}^* + (x-y) \text{ if } t < s < t+1 \\ \tilde{\mathbf{u}}(s) = \mathbf{u}^*(s) \text{ if } t+1 \leq s \leq t_1. \end{cases}$$

Then

$$V(y,t) - V(x,t) \leq J(y,t;\tilde{\mathbf{u}}) - V(x,t) \leq C|x-y|,$$

where the constant C depends only on $D_x L$ and on $D_v L$, and not on the Lipschitz constant of ψ. Reverting the roles of x and y we conclude

$$|V(y,t) - V(x,t)| \leq C|x-y|.$$

Without loss of generality we can suppose that $t < \hat{t}$. Note that

$$|V(x,t) - V(\mathbf{x}^*(\hat{t}),\hat{t})| \leq C|t-\hat{t}|.$$

To prove that V is Lipschitz in t it is enough to check that

$$|V(\mathbf{x}^*(\hat{t}),\hat{t}) - V(x,\hat{t})| \leq C|t-\hat{t}|. \tag{26}$$

But since $\dot{\mathbf{x}}^*$ is uniformly bounded

$$|\mathbf{x}^*(\hat{t}) - x| \leq C|t-\hat{t}|$$

thus, the previous Lipschitz estimate implies (26). □

Lemma 28. *V is differentiable almost everywhere.*

Proof. Since V is Lipschitz, the almost everywhere differentiability follows from Rademacher theorem. □

In general, the value function is Lipschitz and not C^1 or C^2. However we can prove an one-side estimate for second derivatives, i.e. that V is semiconcave.

Lemma 29. *Suppose that $|D^2_{xv}L|, |D^2_{vv}L| \leq C(R)$ whenever $|v| \leq R$. Then, for each $t < t_1$, $V(x,t)$ is semiconcave in x.*

Proof. Fix t and x. Choose $y \in \mathbb{R}^n$ arbitrary. We claim that

$$V(x+y,t) + V(x-y,t) \leq 2V(x,t) + C|y|^2,$$

for some constant C. Clearly,

$$V(x+y,t)+V(x-y,t)-2V(x,t)$$
$$\leq \int_t^{t_1} \left[L(\mathbf{x}^*+\mathbf{y},\dot{\mathbf{x}}^*+\dot{\mathbf{y}})+L(\mathbf{x}^*-\mathbf{y},\dot{\mathbf{x}}^*-\dot{\mathbf{y}})-2L(\mathbf{x}^*,\dot{\mathbf{x}}^*)\right]ds,$$

where

$$\mathbf{y}(s)=y\frac{t_1-s}{t_1-t}.$$

Since $|D^2_{xx}L|\leq C_1(R)$,

$$L(\mathbf{x}^*+\mathbf{y},\dot{\mathbf{x}}^*+\dot{\mathbf{y}})\leq L(\mathbf{x}^*,\dot{\mathbf{x}}^*+\dot{\mathbf{y}})+D_xL(\mathbf{x}^*,\dot{\mathbf{x}}^*+\dot{\mathbf{y}})\mathbf{y}+C|\mathbf{y}|^2$$

and, in a similar way for the other term. We also have

$$L(\mathbf{x}^*,\dot{\mathbf{x}}^*+\dot{\mathbf{y}})+L(\mathbf{x}^*,\dot{\mathbf{x}}^*-\dot{\mathbf{y}})\leq 2L(\mathbf{x}^*,\dot{\mathbf{x}}^*)+C|\dot{\mathbf{y}}|^2+C|\mathbf{y}||\dot{\mathbf{y}}|.$$

Thus

$$L(\mathbf{x}^*+\mathbf{y},\dot{\mathbf{x}}^*+\dot{\mathbf{y}})+L(\mathbf{x}^*-\mathbf{y},\dot{\mathbf{x}}^*-\dot{\mathbf{y}})\leq 2L(\mathbf{x}^*,\dot{\mathbf{x}}^*)+C|\mathbf{y}|^2+C|\dot{\mathbf{y}}|^2.$$

This inequality implies the lemma. □

Lemma 30. *We have*

$$(\mathbf{p}(s),H(\mathbf{p}(s),\mathbf{x}^*(s)))\in D^+V(\mathbf{x}^*(s),s)$$

for $t\leq s<t_1$. Therefore $DV(\mathbf{x}^*(s),s)$ exists for $t<s<t_1$.

Proof. Let $\mathbf{u}^*$ be an optimal control at (x,s) and let $\mathbf{p}$ be the corresponding adjoint variable. Define W by

$$W(y,r)=J\left(y,r;\mathbf{u}^*+\frac{\mathbf{x}^*(r)-y}{t_1-r}\right)-V(x,s).$$

Hence, for each $y\in\mathbb{R}^n$ and $t\leq r<t_1$,

$$V(y,r)-V(x,s)\leq W(y,r),$$

with equality at $(y,r)=(x,s)$. Since W is C^1, it is enough to check that

$$D_yW(\mathbf{x}^*(s),s)=\mathbf{p}(s),$$

and

$$D_rW(\mathbf{x}^*(s),s)=H(\mathbf{p}(s),\mathbf{x}^*(s)).$$

The first identity follows from

$$D_yW(s,\mathbf{x}^*(s))=\int_s^{t_1} D_xL\varphi+D_vL\frac{d\varphi}{d\tau}d\tau,$$

where $\varphi(\tau)=\frac{t_1-\tau}{t_1-s}$. Using the Euler-Lagrange equation

$$\frac{d}{dt}D_vL-D_xL=0$$

and integration by parts we obtain

$$D_y W(s, \mathbf{x}^*(s)) = -D_v L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)) = \mathbf{p}(s).$$

On the other hand,

$$D_r W(s, \mathbf{x}^*(s)) = -L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s)) + \int_s^{t_1} D_x L\phi + D_v L\frac{d\phi}{d\tau} d\tau,$$

where

$$\phi(\tau) = \frac{\tau - t_1}{t_1 - s}\dot{\mathbf{x}}^*(s).$$

Using again the Euler-Lagrange equation and integration by parts, we obtain

$$D_r W(s, \mathbf{x}^*(s)) = -L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s), s) + D_v L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s))\dot{\mathbf{x}}^*(s),$$

or equivalently

$$D_r W(s, \mathbf{x}^*(s)) = H(\mathbf{p}(s), \mathbf{x}^*(s)).$$

The last part of the lemma follows from proposition 18. ☐

This ends the proof of the theorem. ☐

In what follows we prove that the value function is differentiable at points of uniqueness of optimal trajectory.

A point (x, t) is *regular* if there exists a unique optimal trajectory $\mathbf{x}^*(s)$ such that $\mathbf{x}^*(t) = x$ and

$$V(x, t) = \int_t^{t_1} L(\mathbf{x}^*(s), \dot{\mathbf{x}}^*(s))ds + \psi(\mathbf{x}^*(t_1)).$$

THEOREM 31. *V is differentiable with respect to x at (x, t) if and only if (x, t) is a regular point.*

PROOF. The next lemma shows that differentiability at a point x implies that x is a regular point:

LEMMA 32. *If V is differentiable with respect to x at a point (x, t), then there exists a unique optimal trajectory*

PROOF. Since V is differentiable with respect to x at (x, t), then any optimal trajectory satisfies

$$\dot{\mathbf{x}}^*(t) = -D_p H(\mathbf{p}(t), \mathbf{x}^*(t)),$$

since $\mathbf{p}(t) = D_x V(x)$. Therefore, once $D_x V(\mathbf{x}^*(t), t)$ is given, the velocity $\dot{\mathbf{x}}^*(t)$ is uniquely determined. The solution of the Euler-Lagrange equation (23) is determined by the initial condition and velocity: $\mathbf{x}^*(t)$ and $\dot{\mathbf{x}}^*(t)$. Thus, the optimal trajectory is unique. ☐

To prove the other implication we need an auxiliary lemma:

LEMMA 33. *Let p such that*

$$\|D_x V(\cdot, t) - p\|_{L^\infty(B(x,2\epsilon))} \to 0$$

when $\epsilon \to 0$. Then V is differentiable with respect to x at (x,t) and $D_x V(x,t) = p$.

PROOF. Since V is Lipschitz, it is differentiable almost everywhere. By Fubinni theorem, for almost every point with respect to the Lebesgue measure induced in S^{n-1}, V is differentiable $y = x + \lambda k$, with respect to the Lebesgue measure in $\mathbb{R}$. For these directions

$$\frac{V(y,t) - V(x,t) - p \cdot (y-x)}{|x-y|}$$
$$= \int_0^1 \frac{(D_x V(x + s(y-x), t) - p) \cdot (y-x)}{|x-y|} ds.$$

Suppose $0 < |x-y| < \epsilon$. Then

$$\left| \frac{V(x,t) - V(y,t) - p \cdot (x-y)}{|x-y|} \right| \leq \|D_x V(\cdot, t) - p\|_{L^\infty(B(x,\epsilon))}.$$

In principle, the last identity only holds almost everywhere. However, for $y \neq x$, the left-hand side is continuous in y. consequently, the inequality holds for all $y \neq x$. Therefore, when $y \to x$,

$$\left| \frac{V(x,t) - V(y,t) - p \cdot (x-y)}{|x-y|} \right| \to 0,$$

which implies $D_x V(x,t) = p$. □

Suppose that V is not differentiable at (x,t). We claim that (x,t) is not regular. By contradiction, suppose that (x,t) is regular. Then if V fails to be differentiable, the previous lemma implies that for each p,

$$\|D_x V(\cdot, t) - p\|_{L^\infty(B(x,\epsilon))} \not\to 0.$$

Thus, we could choose two sequences x_n^1 and x_n^2 such that $x_n^i \to x$ but whose corresponding optimal trajectories $\mathbf{x}_n^i$ satisfy

$$\lim \dot{\mathbf{x}}_n^1(t) \neq \lim \dot{\mathbf{x}}_n^2(t).$$

However, this shows that (x,t) is not regular. Indeed if (x,t) were regular, and x_n were any sequence converging to x, and $\mathbf{x}_n^*(\cdot)$ the corresponding optimal trajectory then

$$\dot{\mathbf{x}}_n^*(t) \to \dot{\mathbf{x}}^*(t).$$

If this were not true, by Ascoli-Arzela theorem, we could extract a convergent subsequence $\dot{\mathbf{x}}_{n_k}(\cdot) \to \dot{\mathbf{y}}(\cdot)$, and for which

$$\dot{\mathbf{x}}^*_{n_k}(t) \to v \neq \dot{\mathbf{x}}^*(t).$$

Let $\mathbf{y}(\cdot)$ be the solution to the Euler-Lagrange equation with initial condition $\mathbf{y}(t) = \mathbf{x}(t)$ and $\dot{\mathbf{y}}(t) = v$. Note that $\mathbf{x}^*_n(\cdot) \to \mathbf{y}(\cdot)$ and $\dot{\mathbf{x}}^*_n(\cdot) \to \dot{\mathbf{y}}(\cdot)$, uniformly in compact sets, and, therefore,

$$\begin{aligned} V(x,t) = \lim_{n\to\infty} V(x_n,t) = \lim_{n\to\infty} J(x_n,t;\dot{\mathbf{x}}_n) \\ = J(x,t;\dot{\mathbf{y}}) > J(x,t;\dot{\mathbf{x}}^*) = V(x,t), \end{aligned}$$

since the trajectory $\mathbf{y}$ cannot be optimal, by regularity, which is a contradiction. □

Remark. This theorem implies that all points of the form $(\mathbf{x}^*(s), s)$, in which $\mathbf{x}^*$ is and optimal trajectory are regular for $t < s < t_1$.

Exercise 23. *Show that in the optimal control "bounded control space" setting, the value function is Lipschitz continuous if the terminal cost is Lipschitz continuous.*

Exercise 24. *In the optimal control "bounded control space" setting, show that if ψ is Lipschitz, for any (x,t) there exists $\mathbf{p}$ such that*

$$(\mathbf{p}(s), H(\mathbf{p}(s), \mathbf{x}^*(s))) \in D^- V(\mathbf{x}^*(s), s)$$

for $t < s \leq t_1$ and

$$(\mathbf{p}(s), H(\mathbf{p}(s), \mathbf{x}^*(s))) \in D^+ V(\mathbf{x}^*(s), s)$$

for $t \leq s < t_1$.

4. Viscosity solutions - the calculus of variations setting

In this section we discuss the viscosity solutions in the calculus of variations setting.

Theorem 34. *Consider the calculus of variations setting for the optimal control problem. Suppose that the value function V is differentiable at (x,t). Then, at this point, V satisfies the Hamilton-Jacobi equation*

$$-V_t + H(D_x V, x, t) = 0. \tag{27}$$

Proof. If V is differentiable at (x,t) then the result follows by using statement 6 in theorem 21. □

Note that that (27) also holds in the "bounded control case" setting, by Theorem 14.

COROLLARY 35. *Consider the calculus of variations setting for the optimal control problem. Then the value function V satisfies the Hamilton-Jacobi equation almost everywhere.*

PROOF. Since the value function V is differentiable almost everywhere, by theorem 21, theorem 34 implies this result. ☐

EXERCISE 25. *Show that the previous corollary also holds in the "bounded control case" setting.*

However, it is not true that a Lipschitz function satisfying the Hamilton-Jacobi equation almost everywhere is the value function of the terminal value problem, as shown in the next example.

EXAMPLE 8. Consider the Hamilton-Jacobi equation

$$-V_t + |D_x V|^2 = 0$$

with terminal data $V(x, 1) = 0$. The value function is $V \equiv 0$, which is a (smooth) solution of the Hamilton-Jacobi equation However, there are other solutions, for instance,

$$V(x,t) = \begin{cases} 0 \quad \text{if} \quad |x| \geq 1 - t \\ |x| - 1 + t \quad \text{if} \quad |x| < 1 - t \end{cases}$$

which satisfy the same terminal condition $t = 1$ and is solution almost everywhere. ◀

An useful characterization of viscosity solutions is given in the next proposition:

PROPOSITION 36. *Let V be a bounded uniformly continuous function. Then V is a viscosity subsolution of (27) if and only if for each $(p, q) \in D^+V(x, t)$,*

$$-q + H(p, x, t) \leq 0.$$

Similarly, V is a viscosity supersolution if and only if for each $(p, q) \in D^-V(x, t)$,

$$-q + H(p, x, t) \geq 0.$$

PROOF. This result is an immediate corollary of proposition 19. ☐

EXAMPLE 9. In example 8 we have found two different almost everywhere solutions to

$$-V_t + |D_x V|^2 = 0$$

satisfying $V(x,T) = 0$.

It is easy to check that the value function $V = 0$ is viscosity solution (it is smooth, satisfies the equation and the terminal condition) and it agrees with the value function of the corresponding optimal control problem. The other solution, which is an almost everywhere solution is not a viscosity solution (check!).

Now we will show that the definition of viscosity solution is consistent with classical solutions.

PROPOSITION 37. *Let V be a C^1 viscosity solution of (27). Then V is a classical solution.*

PROOF. If V is differentiable then

$$D^+V = D^-V = \{(D_x V, D_t V)\}.$$

Since V is a viscosity solution, we obtain immediately

$$-D_t V + H(D_x V, x, t) \leq 0, \quad \text{and} \quad -D_t V + H(D_x V, x, t) \geq 0,$$

therefore $-D_t V + H(D_x V, x, t) = 0$. □

THEOREM 38. *Let V be the value function of the terminal value problem. Then V is a viscosity solution to*

$$-V_t + H(D_x V, x) = 0.$$

PROOF. Let $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $\varphi(x,t)$, be a C^∞ function, and let $(x_0, t_0) \in \operatorname{argmax}(V - \varphi)$. We must show that

$$-\varphi_t(x_0, t_0) + H(D_x\varphi(x_0, t_0), x_0) \leq 0,$$

or equivalently, that for all $v \in \mathbb{R}^d$ we have

$$-\varphi_t(x_0, t_0) - v \cdot D_x\varphi(x_0, t_0) - L(x_0, v) \leq 0.$$

Fix $v \in \mathbb{R}^d$. Let $\mathbf{x} = x_0 + v(t - t_0)$. Then, for any $h > 0$

$$\int_{t_0}^{t_0+h} \varphi_t + v D_x \varphi(\mathbf{x}(s), s) ds = \varphi(\mathbf{x}(t_0 + h), t_0 + h) - \varphi(x_0, t_0)$$

$$\geq V(\mathbf{x}(t_0 + h), t_0 + h) - V(x_0, t_0) \geq -\int_{t_0}^{t_0+h} L(\mathbf{x}, \dot{\mathbf{x}}) dt.$$

Dividing by h and letting $h \to 0$ we obtain the result.

Now let $(x_0, t_0) \in \text{argmin}(V - \varphi)$. We must show that

$$-\varphi_t(x_0, t_0) + H(D_x\varphi(x_0, t_0), x_0) \geq 0,$$

that is, there exists $v \in \mathbb{R}^d$ such that

$$-\varphi_t(x_0, t_0) - v \cdot D_x\varphi(x_0, t_0) - L(x_0, v) \geq 0.$$

By contradiction assume that there exists $\theta > 0$ such that

$$-\varphi_t(x_0, t_0) - v \cdot D_x\varphi(x_0, t_0) - L(x_0, v) < -\theta,$$

for all v. Because the mapping $v \mapsto L$ is superlinear and φ is C^1, there exists a $R > 0$ and $r_1 > 0$ such that for all $(x, t) \in B_{r_1}(x_0, t_0)$ and all $v \in B_R^c(0) = \mathbb{R}^d \setminus B_R(0)$ we have

$$-\varphi_t(x, t) - v \cdot D_x\varphi(x, t) - L(x, v) < -\frac{\theta}{2}.$$

By continuity, for some $0 < r < r_1$ and all $(x, t) \in B_r(x_0, t_0)$ we have

$$-\varphi_t(x, t) - v \cdot D_x\varphi(x, t) - L(x, v) < -\frac{\theta}{2},$$

for all $v \in B_R(0)$.

Therefore, for any trajectory $\mathbf{x}$ with $\mathbf{x}(0) = x_0$ and any $T \geq 0$ such that the trajectory $\mathbf{x}$ stays near x_0 on $[t_0, t_0+T]$, i.e., $(\mathbf{x}(t), t) \in B_r(x_0, t_0)$ for $t \in [t_0, t_0+T]$ we have

$$\begin{aligned}
&V(\mathbf{x}(t_0+T), t_0+T) - V(x_0, t_0) \geq \varphi(\mathbf{x}(t_0+T), t_0+T) - \varphi(x_0, t_0) \\
&= \int_{t_0}^{t_0+T} \varphi_t(\mathbf{x}(t), t) + \dot{\mathbf{x}}(t) \cdot D_x\varphi(\mathbf{x}(t))\big) dt \\
&\geq \frac{\theta}{2} \int_{t_0}^{t_0+T} dt - \int_{t_0}^{t_0+T} L(\mathbf{x}, \dot{\mathbf{x}}) dt.
\end{aligned}$$

This yields

$$V(x_0, t_0) \leq -\frac{\theta}{2}T + \int_{t_0}^{t_0+T} L(\mathbf{x}, \dot{\mathbf{x}}) dt + V(\mathbf{x}(t_0+T), t_0+T)$$

Choose $\epsilon < \frac{\theta T}{4}$. Let $\mathbf{x}^\epsilon$ be such that:

$$V(x_0, t_0) \geq \int_{t_0}^{t_0+T} L(\mathbf{x}^\epsilon, \dot{\mathbf{x}}^\epsilon) dt + V(\mathbf{x}^\epsilon(t_0+T), t_0+T) - \epsilon$$

This then yields a contradiction.

$\square$

EXERCISE 26. *Show that the function $V(x,t)$ given by the Lax-Hopf formula is Lipschitz in x for each $t < t_1$, regardless of the smoothness of the terminal data (note, however that the constant depends on t).*

EXERCISE 27. *Use the Lax-Hopf formula to determine the viscosity solution of*

$$-u_t + u_x^2 = 0,$$

para $t < 0$ *and* $u(x, 0) = \pm x^2 - 2x$.

EXERCISE 28. *Use the Lax-Hopf formula to determine the viscosity solution of*

$$-u_t + u_x^2 = 0,$$

for $t < 0$ *and*

$$u(x, 0) = \begin{cases} 0 \text{ if } x < 0 \\ x^2 \text{ if } 0 \le x \le 1 \\ 2x - 1 \text{ if } x > 1. \end{cases}$$

5. Uniqueness of viscosity solutions

To establish uniqueness of viscosity solutions we need the following technical lemma:

LEMMA 39. *Let* V *be a viscosity solution of*

$$-V_t + H(D_x V, x) = 0$$

in $[0, T] \times \mathbb{R}^n$ *and* ϕ *a* C^1 *function. If* $V - \phi$ *has a maximum (resp. minimum) at* $(x_0, t_0) \in \mathbb{R}^d \times [0, T)$ *then*

$$-\phi_t(x_0, t_0) + H(D_x\phi(x_0, t_0), x_0) \le 0 \text{ (resp. } \ge 0) \quad \text{at} \quad (x_0, t_0). \tag{28}$$

REMARK: The important point is that the inequality is valid even for some non-interior points ($t_0 = 0$).

PROOF. Only the case $t_0 = 0$ requires proof since in the other case the maximum is interior and then the viscosity property (the definition of viscosity solution) yields the inequality. So suppose $(x_0, 0)$ is a strict maximum point. Consider

$$\tilde{\phi} = \phi + \frac{\epsilon}{t}.$$

Then $V - \tilde{\phi}$ has an interior local maximum at (x_ϵ, t_ϵ) with $t_\epsilon > 0$. Furthermore, $(x_\epsilon, t_\epsilon) \to (x_0, 0)$, as $\epsilon \to 0$. At the point (x_ϵ, t_ϵ) we have

$$-\phi_t(x_\epsilon, t_\epsilon) + \frac{\epsilon}{t_\epsilon^2} + H(D_x\phi(x_\epsilon, t_\epsilon), x_\epsilon) \le 0,$$

that is, since $\frac{\epsilon}{t_\epsilon^2} \ge 0$,

$$-\phi_t(x_0, 0) + H(D_x\phi(x_0, 0), x_0) \le 0.$$

Analogously we obtain the opposite inequality for the case of local minimum, using $\tilde{\phi} = \phi - \frac{\epsilon}{t}$. □

Finally we establish uniqueness of viscosity solutions:

THEOREM 40 (Uniqueness). *Suppose H satisfies*

$$\begin{aligned}
&|H(p,x) - H(q,x)| \leq C(|p| + |q|)|p - q| \\
&|H(p,x) - H(p,y)| \leq C|x - y|(C + H(p,x))
\end{aligned}$$

Then there exits a unique viscosity solution to the Hamilton-Jacobi equation

$$-V_t + H(D_x V, x) = 0$$

that satisfies the terminal condition $V(x,T) = \psi(x)$.

PROOF. Let V and $\tilde{V}$ be two viscosity solutions with

$$\sup_{0 \leq t \leq T} V - \tilde{V} = \sigma > 0.$$

For $0 < \epsilon, \lambda < 1$ we define

$$\begin{aligned}
\psi(x,y,t,s) = &V(x,t) - \tilde{V}(y,s) - \lambda(2T - t - s) \\
&- \frac{1}{\epsilon^2}(|x-y|^2 + |t-s|^2) - \epsilon \ln(1 + |x|^2 + |y|^2).
\end{aligned}$$

When ϵ, λ are sufficiently small there exists points $x_{\epsilon,\lambda}, y_{\epsilon,\lambda}, t_{\epsilon,\lambda}$, and $s_{\epsilon,\lambda}$ such that

$$\max \psi(x,y,t,s) = \psi(x_{\epsilon,\lambda}, y_{\epsilon,\lambda}, t_{\epsilon,\lambda}, s_{\epsilon,\lambda}) > \frac{\sigma}{2}.$$

Since $\psi(x_{\epsilon,\lambda}, y_{\epsilon,\lambda}, t_{\epsilon,\lambda}, s_{\epsilon,\lambda}) \geq \psi(0,0,-T,-T)$, and both V and $\tilde{V}$ are bounded, we have

$$|x_{\epsilon,\lambda} - y_{\epsilon,\lambda}|^2 + |t_{\epsilon,\lambda} - s_{\epsilon,\lambda}|^2 \leq C\epsilon^2$$

and

$$\epsilon \ln(1 + |x_{\epsilon,\lambda}|^2 + |y_{\epsilon,\lambda}|^2) \leq C.$$

Observe that

$$\psi(x_{\epsilon,\lambda}, x_{\epsilon,\lambda}, t_{\epsilon,\lambda}, t_{\epsilon,\lambda}) \geq \psi(x_{\epsilon,\lambda}, y_{\epsilon,\lambda}, t_{\epsilon,\lambda}, s_{\epsilon,\lambda})$$

Thus from the fact that V and $\tilde{V}$ are continuous, it then follows that

$$\frac{|x_{\epsilon,\lambda} - y_{\epsilon,\lambda}|^2 + |t_{\epsilon,\lambda} - s_{\epsilon,\lambda}|^2}{\epsilon^2} = o(1),$$

as $\epsilon \to 0$.

Denote by ω and $\tilde{\omega}$ the modulus of continuity of V and $\tilde{V}$. Then

$$\begin{aligned}\frac{\sigma}{2} &\leq V(x_{\epsilon,\lambda},t_{\epsilon,\lambda})-\tilde{V}(y_{\epsilon,\lambda},s_{\epsilon,\lambda})\\ &= V(x_{\epsilon,\lambda},t_{\epsilon,\lambda})-V(x_{\epsilon,\lambda},T)+V(x_{\epsilon,\lambda},T)-\tilde{V}(x_{\epsilon,\lambda},T)+\\ &\qquad+\tilde{V}(x_{\epsilon,\lambda},T)-\tilde{V}(x_{\epsilon,\lambda},s_{\epsilon,\lambda})+\tilde{V}(x_{\epsilon,\lambda},s_{\epsilon,\lambda})-\tilde{V}(y_{\epsilon,\lambda},s_{\epsilon,\lambda})\leq\\ &\leq \omega(T-t_{\epsilon,\lambda})+\tilde{\omega}(T-s_{\epsilon,\lambda})+\tilde{\omega}(o(\epsilon)).\end{aligned}$$

Therefore, if ϵ is sufficiently small $T-t_{\epsilon,\lambda}>\mu>0$, uniformly in ϵ.

Let ϕ be given by

$$\begin{aligned}\phi(x,t) &= \tilde{V}(y_{\epsilon,\lambda},s_{\epsilon,\lambda})+\lambda(2T-t-s_{\epsilon,\lambda})+\\ &\qquad+\frac{1}{\epsilon^2}(|x-y_{\epsilon,\lambda}|^2+|t-s_{\epsilon,\lambda}|^2)+\epsilon\ln(1+|x|^2+|y_{\epsilon,\lambda}|^2).\end{aligned}$$

Then, the difference

$$V(x,t)-\phi(x,t)$$

achieves a maximum at $(x_{\epsilon,\lambda},t_{\epsilon,\lambda})$.

Similarly, for $\tilde{\phi}$ given by

$$\begin{aligned}\tilde{\phi}(y,s) &= V(x_{\epsilon,\lambda},t_{\epsilon,\lambda})-\lambda(2T-t_{\epsilon,\lambda}-s)-\\ &\qquad-\frac{1}{\epsilon^2}(|x_{\epsilon,\lambda}-y|^2+|t_{\epsilon,\lambda}-s|^2)-\epsilon\ln(1+|x_{\epsilon,\lambda}|^2+|y|^2),\end{aligned}$$

the difference

$$\tilde{V}(y,s)-\tilde{\phi}(y,s)$$

has a minimum at $(y_{\epsilon,\lambda},s_{\epsilon,\lambda})$.

Therefore

$$-\phi_t(x_{\epsilon,\lambda},t_{\epsilon,\lambda})+H(D_x\phi(x_{\epsilon,\lambda},t_{\epsilon,\lambda}),x_{\epsilon,\lambda})\leq 0,$$

and

$$-\tilde{\phi}_s(y_{\epsilon,\lambda},s_{\epsilon,\lambda})+H(D_y\tilde{\phi}(y_{\epsilon,\lambda},s_{\epsilon,\lambda}),y_{\epsilon,\lambda})\geq 0.$$

Simplifying, we have

$$\lambda-2\frac{t_{\epsilon,\lambda}-s_{\epsilon,\lambda}}{\epsilon^2}+H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}+2\epsilon\frac{x_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\leq 0, \tag{29}$$

and

$$-\lambda-2\frac{t_{\epsilon,\lambda}-s_{\epsilon,\lambda}}{\epsilon^2}+H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},y_{\epsilon,\lambda})\geq 0. \tag{30}$$

From (29) we gather that

$$H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}+2\epsilon\frac{x_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\leq -\lambda+\frac{o(1)}{\epsilon}. \tag{31}$$

By subtracting (29) to (30) we have

$$
\begin{aligned}
2\lambda &\leq H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},y_{\epsilon,\lambda})-H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}+2\epsilon\frac{x_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\\
&\leq H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},y_{\epsilon,\lambda})-H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\\
&+H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})-H(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}+2\epsilon\frac{x_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\\
&\leq \left(C+CH(2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2},x_{\epsilon,\lambda})\right)|x_{\epsilon,\lambda}-y_{\epsilon,\lambda}|\\
&+C\epsilon\left(\left|2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}+2\epsilon\frac{x_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2}\right|+\left|2\frac{x_{\epsilon,\lambda}-y_{\epsilon,\lambda}}{\epsilon^2}-2\epsilon\frac{y_{\epsilon,\lambda}}{1+|x_{\epsilon,\lambda}|^2+|y_{\epsilon,\lambda}|^2}\right|\right)|x_{\epsilon,\lambda}-y_{\epsilon,\lambda}|\\
&\leq \left(\frac{o(1)}{\epsilon}+C\right)|x_{\epsilon,\lambda}-y_{\epsilon,\lambda}|\to 0,
\end{aligned}
$$

when $\epsilon \to 0$, which is a contradiction. □

6. Bibliographical notes

The main references for this section are [**FS06**], [**Lio82**], [**BCD97**], [**Bar94**]. Introductory material can be found in [**Eva98**]. A very nice introduction to viscosity solutions written in Portuguese is the book [**LLF**].

5

Differential Games

This chapter is a brief introduction to deterministic differential games and its connection with viscosity solutions of Hamilton-Jacobi equations.

1. Dynamic programming principle

Consider a problem where two players have conflicting objectives. Each of them partially controls a dynamical system, and one of the players wants to maximize a pay-off functional, whereas the other one wishes to minimize the same pay-off functional. To set-up this problem, let U^+ and U^- be two convex closed subsets of, respectively, $\mathbb{R}^{m_+}$ and $\mathbb{R}^{m_-}$. The $+$ sign stands for the controls or variables available for the maximizing player, whereas the $-$ sign corresponds to the minimizing player.

Consider a differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}^+, \mathbf{u}^-), \tag{32}$$

where $\mathbf{u}^\pm$ are controls for the two players taking values on $U^\pm$. To simplify, we suppose that $U^\pm$ are compact sets, that f is globally bounded and satisfies the Lipschitz estimate

$$|f(x, u^+, u^-) - f(y, u^+, u^-)| \leq C|x - y|.$$

Let T be a terminal time. To each pair of controls $(\mathbf{u}^+, \mathbf{u}^-)$ on (t, T), consider the corresponding solution to (32) with initial condition $\mathbf{x}(t) = x$. We are given a running cost $L(x, \mathbf{u}^+, \mathbf{u}^-)$ and a terminal cost $\psi(x)$. Associated to the controls and these costs we define the cost

$$J[x, t; \mathbf{u}^+, \mathbf{u}^-] = \int_t^T L(\mathbf{x}, \mathbf{u}^+, \mathbf{u}^-) ds + \psi(\mathbf{x}(T)),$$

where $\mathbf{x}$ solves (32) with the initial condition $\mathbf{x}(t) = x$. The objective of the $+$ player is to maximize this cost, whereas the $-$ player wishes to minimize this cost. Of course the players are not allowed to foresee the future and we must therefore discuss the appropriate strategies.

Denote by $\mathcal{U}^{\pm}([t,T])$ the set of all mappings from $[t,T]$ into $U^{\pm}$. A non-anticipating strategy $\mu^{\pm}$ is a mapping

$$\mu^{\pm} : \mathcal{U}^{\mp}([t,T]) \to \mathcal{U}^{\pm}([t,T])$$

such that for any $\mathbf{u}^{\mp}, \tilde{\mathbf{u}}^{\mp} \in \mathcal{U}^{\mp}([t,T])$ and any $t < s < T$ such that, for all $t \leq \tau \leq s$,

$$\mathbf{u}^{\mp}(\tau) = \tilde{\mathbf{u}}^{\mp}(\tau)$$

we have

$$\mu^{\pm}(\mathbf{u}^{\mp})(\tau) = \mu^{\pm}(\tilde{\mathbf{u}}^{\mp})(\tau),$$

for all $t \leq \tau \leq s$. Denote by $\Lambda^{\pm}$ the set of all non-anticipating strategies.

The upper V^{+} value functions are defined to be

$$V^{+}(x,t) = \sup_{\mu^{+} \in \Lambda^{+}([t,T])} \inf_{u^{-} \in \mathcal{U}^{-}([t,T])} J(x,t;\mu^{+}(\mathbf{u}^{-}),\mathbf{u}^{-}),$$

whereas the lower value function is

$$V^{-}(x,t) = \inf_{\mu^{-} \in \Lambda^{-}([t,T])} \sup_{u^{+} \in \mathcal{U}^{+}([t,T])} J(x,t;\mathbf{u}^{+},\mu^{-}(\mathbf{u}^{+})).$$

THEOREM 41 (Dynamic programming principle). *For any $t' < T$ we have*

$$V^{+}(x,t) = \sup_{\mu^{+} \in \Lambda^{\pm}([t,t'])} \inf_{u^{-} \in \mathcal{U}^{-}([t,t'])} \int_{t}^{t'} L(\mathbf{x},\mu^{+}(\mathbf{u}^{-}),\mathbf{u}^{-})ds + V^{+}(\mathbf{x}(t'),t').$$

Note that a similar result holds for the lower value, with a identical proof.

PROOF. Define

$$\tilde{V}(x,t) = \sup_{\mu^{+} \in \Lambda^{+}([t,t'])} \inf_{u^{-} \in \mathcal{U}^{-}([t,t'])} \int_{t}^{t'} L(\mathbf{x},\mu^{+}(\mathbf{u}^{-}),\mathbf{u}^{-})ds + V^{+}(\mathbf{x}(t'),t').$$

Fix $\epsilon > 0$ and choose $\mu_{\epsilon}^{+} \in \Lambda^{\pm}([t,t'])$ so that

$$\tilde{V}(x,t) \leq \inf_{u^{-} \in \mathcal{U}^{-}([t,t'])} \int_{t}^{t'} L(\mathbf{x},\mu_{\epsilon}^{+}(\mathbf{u}^{-}),\mathbf{u}^{-})ds + V^{+}(\mathbf{x}(t'),t') + \epsilon.$$

Choose now $\tilde{\mu}_{\epsilon}^{+} \in \Lambda^{+}([t',T])$ so that

$$V(\mathbf{x}(t'),t') \leq \inf_{u^{-} \in \mathcal{U}^{-}([t',T])} \int_{t}^{t'} L(\mathbf{x},\tilde{\mu}_{\epsilon}^{+}(\mathbf{u}^{-}),\mathbf{u}^{-})ds + \psi(\mathbf{x}(T)) + \epsilon.$$

By considering the concatenation of the non-anticipating strategies μ_ϵ^+ and $\tilde\mu_\epsilon^+$ we obtain a non-anticipating strategy $\bar\mu_\epsilon^+$ such that

$$\begin{aligned}\tilde V(x,t) &\le \inf_{u^-\in\mathcal{U}^-([t,T])}\int_t^T L(\mathbf{x},\bar\mu_\epsilon^+(\mathbf{u}^-),\mathbf{u}^-)ds+\psi(\mathbf{x}(T))+2\epsilon\\ &\le V^+(x,t)+2\epsilon.\end{aligned}$$

Sending $\epsilon\to 0$ we obtain $\tilde V\le V^+$.

To obtain the opposite inequality, fix again $\epsilon>0$ and choose a non-anticipating strategy $\bar\mu_\epsilon^+$ so that

$$V^+(x,t)\le \inf_{u^-\in\mathcal{U}^-([t,T])}\int_t^T L(\mathbf{x},\bar\mu_\epsilon^+(\mathbf{u}^-),\mathbf{u}^-)ds+\psi(\mathbf{x}(T))+\epsilon.$$

Note that

$$\inf_{u^-\in\mathcal{U}^-([t',T])}\int_{t'}^T L(\mathbf{x},\bar\mu_\epsilon^+(\mathbf{u}^-),\mathbf{u}^-)ds+\psi(\mathbf{x}(T))\le V^+(\mathbf{x}(t'),t').$$

Therefore

$$\begin{aligned}V^+(x,t) &\le \inf_{u^-\in\mathcal{U}^-([t,T])}\int_t^{t'} L(\mathbf{x},\bar\mu_\epsilon^+(\mathbf{u}^-),\mathbf{u}^-)ds+V^+(\mathbf{x}(t'),t')+\epsilon\\ &\le \tilde V(x,t)+\epsilon.\end{aligned}$$

$\square$

2. Viscosity solutions

We define the upper and lower Hamiltonians to be, respectively,

$$H^-(p,x)=\inf_{u^+\in U^+}\sup_{u^-\in U^-}-p\cdot f(u^+,u^-,x)-L(x,u^+,u^-),$$

and

$$H^+(p,x)=\sup_{u^-\in U^-}\inf_{u^+\in U^+}-p\cdot f(u^+,u^-,x)-L(x,u^+,u^-).$$

Note that in general $H^+\le H^-$ and the inequality may be strict. When equality holds we say that the Isaac's condition holds.

Before stating and proving the main result of this section, we will prove two auxiliary results.

Lemma 42. *Suppose φ satisfies*

$$-\varphi_t+H^+(D_x\varphi,x)\le-\theta,$$

at a point (x,t) and for some $\theta>0$. Then, for all h sufficiently small there exists $\mu^+\in\Lambda^+([t,t+h])$ such that for all $u^-\in\mathcal{U}^-([t,t+h])$ we have

$$\int_t^{t+h}\left[L(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)+f(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)D_x\varphi(\mathbf{x}(s),s)\right.$$
$$\left.+\varphi_t(\mathbf{x}(s),s)\right]ds\geq h\frac{\theta}{2}.$$

Proof. It is clear by the compactness assumption on $U^\pm$ that there exists a compact K_h such that $\mathbf{x}(s)\in K_h$ for all $t\leq s\leq t+h$. Furthermore, for h small enough we can assume that

$$-\varphi_t(y,s)+H^+(D_x\varphi(y,s),y)\leq-\frac{3}{4}\theta,$$

for all $y\in K_h$ and $t\leq s\leq t+h$. Define

$$\Theta(y,s;u^+,u^-)=-\varphi_t(y,s)-D_x\varphi(y,s)\cdot f(u^+,u^-,y)-L(y,u^+,u^-).$$

Then, for any $u^+\in U^+$ and $(y,s)\in K_h\times(t,t+h)$ there exists $\beta^-(u^+,y,s)$ such that

$$\Theta(y,s;u^+,\beta^-(u^+,y,s))\leq-\frac{3}{4}\theta.$$

By the compactness of $U^\pm$ and uniform continuity we conclude that for all there exists $r>0$ such that

$$\Theta(\tilde{y},\tilde{s};v,\beta^-(v,\tilde{y},\tilde{s}))\leq-\frac{1}{2}\theta$$

if $|\tilde{y}-y|+|\tilde{s}-s|+|v-u^+|\leq r$. By compactness we can find a finite cover of $K\times[t,t+h]\times U^+$ by "balls", B_i of the form $|\tilde{y}-y_i|+|\tilde{s}-s_i|+|v-u_i^+|\leq r_i$, $1\leq i\leq n$. We now define a non-anticipating strategy in the following way: if $(x,s,u^-(s))\in B_i\cap\cup_{j=1}^{i-1}B_j^C$ we set $\mu^+(u^-)(s)=\beta(u^-(s),\tilde{y},\tilde{s})$. Then it is clear that for any strategy $\mathbf{u}^-$ and for $\mathbf{x}$ satisfying

$$\dot{\mathbf{x}}=f(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)$$

we have

$$L(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)+f(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)D_x\varphi(\mathbf{x}(s),s)+\varphi_t(\mathbf{x}(s),s)\leq\frac{\theta}{2}.$$

h ☐

We should observe that an analogous results for H^- can be established in exactly the same way.

Theorem 43. *The upper and lower values are viscosity solutions to the Isaacs-Bellman-Hamilton-Jacobi equation*

$$-V_t^\pm+H^\pm(D_xV^\pm,x)=0,$$

with the terminal value $V(x,T)=\psi(x)$.

PROOF. We will do the proof for the upper value V^+ as the case of the lower value is similar. Suppose $V^+ - \varphi$ has a strict local maximum at (x_0, t_0) but, by contradiction, there exists $\theta > 0$ such that

$$-\varphi_t^+ + H^+(D_x\varphi^+, x) \geq \theta.$$

Using the dynamic programming principle we have

$$V(x_0, t_0) = \sup_{\mu^+ \in \Lambda^\pm([t_0, t_0+h])} \inf_{u^- \in \mathcal{U}^-([t_0, t_0+h])} \int_{t_0}^{t_0+h} L(\mathbf{x}, \mu^+(\mathbf{u}^-), \mathbf{u}^-) ds + V(\mathbf{x}(t_0+h), t_0+h).$$

Choose μ_h^+ such that

$$V(x_0, t_0) \leq \inf_{u^- \in \mathcal{U}^-([t_0, t_0+h])} \int_{t_0}^{t_0+h} L(\mathbf{x}, \mu_h^+(\mathbf{u}^-), \mathbf{u}^-) ds + V(\mathbf{x}(t_0+h), t_0+h) + h^2.$$

Then the local maximum property, we have

$$\inf_{u^- \in \mathcal{U}^-([t_0, t_0+h])} \int_{t_0}^{t_0+h} L(\mathbf{x}, \mu_h^+(\mathbf{u}^-), \mathbf{u}^-) ds + \varphi(\mathbf{x}(t_0+h), t_0+h) - \varphi(x_0, t_0) + h^2 \geq 0.$$

Thus

$$\inf_{u^- \in \mathcal{U}^-([t_0, t_0+h])} \int_{t_0}^{t_0+h} L(\mathbf{x}, \mu_h^+(\mathbf{u}^-), \mathbf{u}^-) + f(\mathbf{x}, \mu_h^+(\mathbf{u}^-), \mathbf{u}^-) D_x\varphi + \varphi_t ds + h^2 \geq 0.$$

As before, is clear by the compactness assumption on $U^\pm$ that there exists a compact K_h such that $\mathbf{x}(s) \in K_h$ for all $t \leq s \leq t+h$. Furthermore, for h small enough we can assume that

$$-\varphi_t(y, s) + H^+(D_x\varphi(y, s), y) \geq \frac{3}{4}\theta,$$

for all $y \in K_h$ and $t \leq s \leq t+h$. Define

$$\Theta(y, s; u^+, u^-) = -\varphi_t(y, s) - D_x\varphi(y, s) \cdot f(u^+, u^-, y) - L(y, u^+, u^-).$$

Then in K_h we have

$$\sup_{u^- \in U^-} \inf_{u^+ \in U^+} \Theta(y, s; u^+, u^-) \geq \frac{3}{4}\theta.$$

Hence we can choose $\nu^-(y, s)$ such that in K_h

$$\inf_{u^+ \in U^+} \Theta(y, s; u^+, \nu^-(y, s)) \geq \frac{1}{2}\theta.$$

And so

$$\Theta(y,s;u^+,\nu^-(y,s)) \geq \frac{1}{2}\theta,$$

for any $u^+ \in U^+$. Therefore for $\mathbf{u}^-(s) = \nu^-(\mathbf{x}(s),s)$ we have

$$\int_{t_0}^{t_0+h} L(\mathbf{x},\mu_h^+(\mathbf{u}^-),\mathbf{u}^-) + f(\mathbf{x},\mu_h^+(\mathbf{u}^-),\mathbf{u}^-)D_x\varphi + \varphi_t \leq -\frac{1}{2}\theta h.$$

But then this gives a contradiction.

Now suppose $V^+ - \varphi$ has a strict local minimum at (x_0,t_0) but, by contradiction, there exists $\theta > 0$ such that

$$-\varphi_t^+ + H^+(D_x\varphi^+, x) \leq -\theta.$$

Using the dynamic programming principle and the local maximum property, we have

$$\sup_{\mu^+\in\Lambda^\pm([t_0,t_0+h])} \inf_{u^-\in\mathcal{U}^-([t_0,t_0+h])} \int_{t_0}^{t_0+h} L(\mathbf{x},\mu^+(\mathbf{u}^-),\mathbf{u}^-)ds + \varphi(\mathbf{x}(t_0+h),t_0+h) - \varphi(x_0,t_0) \leq 0.$$

This then contradicts lemma 42. $\square$

3. Bibliographical notes

The main reference for this chapter is the book [**BCD97**]. The reader may also want to consult [**FS06**] (the second edition of the book) for additional material.

6

An introduction to mean field games

This introduction to mean field games based upon the author's joint work with Joana Mohr and Rafael Souza from the Universidade Federal do Rio Grande do Sul.

Mean field games is a recent area of research started by Pierre Louis Lions and Jean Michel Lasry [**LL06a, LL06b, LL07a, LL07b**] which attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play dynamic games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, Lions and Lasry introduced a class of models in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. A key question is how to derive such effective or mean field equations that drive the system as well as to show convergence as the number of agents increases to infinity. The literature on mean field games and its applications is growing fast, for a recent survey see [**LLG10b**] and reference therein. Applications of mean field games arise in the study of growth theory in economics [**LLG10a**] or environmental policy [**ALT**], for instance, and it is likely that in the future they will play an important rôle in economics and population models. There is also a growing interest in numerical methods for these problems [**ALT**], [**AD10**]. In [**GMS10**] was studied the discrete time, finite state problem.

In this paper we consider the mean field limit of games between a large number of players that are allowed to switch between two states. We are particularly interested in understanding the limit as the number of players increases to infinity. We should stress the the fact that we are considering only two states plays no special rôle and we could easily generalize our results to any finite number of states.

In his PhD thesis, [**Gue09**], O. Guéant considered a problem with two states, modeling the labor market. In this work he considered a continuum of individuals and a labor market consisting of 2 sectors. Each individual has to decide on which sector he or she is going to work. This model consists in a coupled systems of ordinary differential equations of the type that will be derived in section 2. Another possible application of our models concerns the adoption or change of a technology or services. For instance, a single agent faced with different social networks will have a incentive to move to the network with more potential contacts, however other effects play a role in this player decision, such as the level of services, trouble of changing network, loss of contacts and so on. Another similar example concerns switching between cell phone companies.

We start in Section 1 to model the $N+1$ player problem as a Markov decision process. We assume that N of the players have a fixed Markov switching strategy β and then look at a reference player which looks to minimize a certain performance criterion by choosing a suitable switching strategy $\alpha(\beta)$. This is a well know Markov decision problem. The key novelty in this section consists in showing the existence of a Nash equilibrium such that $\alpha(\beta) = \beta$ and its characterization through a non-linear ordinary differential equation. In fact, this is a continuous time, partial information, symmetric version of the Markov perfect equilibrium notion that has been studied (mostly in discrete time or stationary setting) in [**PS09, Liv02, MT01, Str93**], and references therein. In [**PM01, Sle01**] symmetric Markov perfect equilibrium are also considered, and in the last paper the case with an infinite number of players is studied. In [**Kap95**] the passage from discrete time to continuous time is considered for N players in a war of attrition problem.

In Section 2 we derive a mean field model for the optimal switching policy of a reference player given the fraction $\theta(t)$ of players in one of the states. This model turns out to be a coupled system of ordinary differential equations, where one equation governs the evolution of θ, and is subjected to initial conditions, whereas the other equation models the evolution of the value function and has terminal data. We call this problem the initial-terminal value problem. Initial terminal value problems are in fact a general feature in many mean field game problems, see for instance [**LL06a, LL06b, LL07a**]. Of course, existence and uniqueness of solutions is not immediate from the general ODE theory but, adapting the methods of Lions and Lasry we were successful in establishing both.

Our main result, theorem 57, is discussed in Section 3 where we prove the convergence as the number of players $N \to \infty$ to a mean field model.

1. The $N+1$ player game

In this section we consider symmetric games between $N+1$ players under a symmetric partial information pattern. We start by discussing the framework of this problem, namely controlled Markov Dynamics, §1.1, admissible controls §1.3, and the individual player problem §1.4. Then in §1.5 we discuss the main assumptions on running and terminal cost that allow us to use Hamilton-Jacobi ODE methods, in §1.6 to solve the $N+1$ player problem. Maximum principle type estimates are considered in §1.7 which are then applied to establishing the existence of Nash equilibrium solutions, §1.8. This section ends with an example §1.9.

1.1. Controlled Markov Dynamics. We consider a dynamic game between $N+1$ players that are allowed to switch between two states denoted by 0 and 1. We suppose that all players are identical and so the game is symmetric with respect to permutation of the players. To describe the game we will use a reference player, which could be chosen as any one of the players.

If we fix any player as the reference player, we will suppose that he knows his own state at time t, given by $i(t)$, and also knows the number $n(t)$ of remaining players that are in state 0. $i(t)$ and $n(t)$ are stochastic processes that we will describe in the following. No further information is available to the reference player. Because the game is symmetric, the identity of the reference player is not important, and all other players have access to the same kind of information, i.e., its own state and the fraction of other players in state 0.

We suppose the process $(n(t), i(t))$ is a continuous time Markov process: the reference player follows a controlled Markov process $i(t)$ with transition rates from state i to the other state $1-i$ given by $\beta = \beta(i, n, t)$. More precisely we have

$$\mathbb{P}\Big(i(t+h) = 1-i \| n(t) = n, i(t) = i\Big) = \beta(i, n, t).h + o(h)\,,$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$. Because of the symmetry of the game, all other players follow their own Markov process controlled by the same transition rate function $\beta : \{0,1\} \times \{0, ..., N\} \times [0, +\infty) \to [0, +\infty)$. Note that the rate function β is a deterministic time-dependent function, which makes $(n(t), i(t))$ a non-time homogeneous Markov process. We will suppose that β is bounded and continuous as a function of time. We will refer to any Markov control with rate function which is bounded and continuous on time, as an *admissible control.*

The transition rates of the process $n(t)$ are given by

$$\begin{aligned} \gamma_\beta^+(i, n, t) &= (N-n)\beta(1, n+1-i, t)\,, \\ \gamma_\beta^-(i, n, t) &= n\beta(0, n-i, t)\,, \end{aligned} \tag{33}$$

where γ_β^+ stands for the transition rate from n to $n+1$, and γ_β^- is the transition rate from n to $n-1$. Note that $n+1-i$ is the total number of players in state 0, as seen by a player (distinct from the reference player) in state 1 whereas $n-i$

is the number of players in state 0 as seen by a player (distinct from the reference player) in state 0.

More precisely, we have

$$\mathbb{P}\Big(n(t+h) = n+1 \| n(t) = n, i(t) = i\Big) = \gamma_\beta^+(i,n,t).h + o(h)\,,$$
$$\mathbb{P}\Big(n(t+h) = n-1 \| n(t) = n, i(t) = i\Big) = \gamma_\beta^-(i,n,t).h + o(h)\,,$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$.

We assume further that the state transitions of the different players are independent, conditioned on i and n. Note that no information is available to any player concerning the state of any other individual player. All each player knows is its position and the number of other players in state 0, which mean, the fraction of other players in each one of the states 0 and 1.

1.2. A control problem. Let now $T > 0$, and let $c : \{0,1\} \times [0,1] \times \mathbb{R}_0^+ \to \mathbb{R}$ and $\psi : \{0,1\} \times [0,1] \to \mathbb{R}$ be two (non-negative) functions. We will discuss the precise hypothesis on c and ψ in section 1.5. We suppose $c\left(i, \frac{n}{N}, \beta\right)$ represents a running cost incurred by the reference player when he is in state i, n of the remaining N players are in state 0 and this player has a transition rate β from i to $1-i$. We also suppose $\psi\left(i, \frac{n}{N}\right)$ represents a terminal cost incurred by the reference player at the terminal time T, if he ends up at time T in state i and at that time n of the other players are in state 0.

If $A_t(i,n)$ denotes the event $i(t) = i$ and $n(t) = n$, the expected total cost of the reference player, giving the control β and conditioned on the event $A_t(i,n)$, will be

$$V^\beta(i,n,t) = \mathbb{E}^\beta_{A_t(i,n)}\left[\int_t^T c\left(i(s), \frac{n(s)}{N}, \beta(s)\right) ds + \psi\left(i(T), \frac{n(T)}{N}\right)\right]\,.$$

We could be interested in finding an admissible control β that minimizes, for each (i,n,t), the function V defined above. This however would require a cooperative behavior between players and it would be an usual stochastic optimal control problem. Instead, we are interested in finding an admissible control β that is a symmetric Nash equilibria for the game which we will soon describe.

1.3. The Dynkin formula. Given two admissible controls β and α, we can define a non-time homogeneous Markov process $(n(t), i(t))$ where the transition rates for n are given by (33) and the transition rate for i is given by α as

$$\mathbb{P}\Big(i(t+h) = 1-i \| n(t) = n, i(t) = i\Big) = \alpha(i,n,t).h + o(h)\,,$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$. The idea here is that, while other players use the control β, the reference player can choose another control α.

Furthermore, we have that, for any function $\varphi : \{0,1\} \times \{0,1,2,...,N\} \times [0,+\infty) \to \mathbb{R}$, smooth in the last variable, and any $s > t$,

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\varphi(i(s),n(s),s) - \varphi(i,n,t)\right] \tag{34}$$

$$= \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^s \frac{d\varphi}{dt}(i,n,r) + A^{\beta,\alpha}\varphi(i,n,r)dr\right]\,, \tag{35}$$

where $A_t(i,n)$ still denotes the event $i(t) = i$ and $n(t) = n$, and

(36)

$$A^{\beta,\alpha}\varphi(i,n,r) = \alpha(i,n,r)(\bar{\varphi} - \varphi)(i,n,r) +$$
$$+ \gamma^+_\beta(i,n,r)(\varphi(i,n+1,r) - \varphi(i,n,r)) + \gamma^-_\beta(i,n,r)(\varphi(i,n-1,r) - \varphi(i,n,r))\,,$$

where γ^+_β and γ^-_β are defined by (33), and $\bar{\varphi}(i,n,t) = \varphi(1-i,n,t)$.

We call $A^{\alpha,\beta}$ the generator of the process and (34) the Dynkin's formula in analogy to the Dynkin's formula in stochastic calculus.

1.4. Individual player point of view - introducing the game. Now we suppose the reference player decides unilaterally to use a different control, trying to improve its value function.

We will suppose the other players continue to follow the Markov Chain with transition rate $\beta(i,n,t)$, bounded and continuous on time. Therefore $n(t)$, the number of such players that are in state 0, is a process to which correspond transition rates γ^+_β and γ^-_β as in (33).

The reference player looks for an admissible control α, possibly different from β, that minimizes

$$u(i,n,t,\beta,\alpha) = \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T c\left(i(s), \frac{n(s)}{N}, \alpha(s)\right)ds + \psi\left(i(T), \frac{n(T)}{N}\right)\right]\,.$$

That is, reference player looks for the control α which is a solution to the minimization problem

$$u(i,n,t;\beta) = \inf_\alpha u(i,n,t,\beta,\alpha),$$

where the minimization is performed over the set of all admissible controls α. We will call the function $u(i,n,t;\beta)$ above the value function for the reference player associated to the strategy β of the remaining N players. The control α that attains the minimum above can be called the best response of any player to a control β.

1.5. Assumptions on running and terminal cost. We discuss now the hypothesis used in this paper concerning the running and terminal costs. We suppose that both the running cost $c = c(i, \theta, \alpha) : \{0,1\} \times [0,1] \times \mathbb{R}_0^+ \to \mathbb{R}$ and the terminal cost $\psi = \psi(i, \theta) : \{0,1\} \times [0,1] \to \mathbb{R}$ are non-negative functions, as mentioned in the previous section, and also that they are Lipschitz continuous in θ. Of course, our results would still be valid without any change if c and ψ are simply bounded below, instead of being non-negative.

We assume that $c(i, \theta, \alpha)$ is uniformly convex on $\alpha \geq 0$ and superlinear. We assume further that c is differentiable, and $c'(\theta, \alpha)$ is Lipschitz in the variable θ.

For $p \in \mathbb{R}$ we define

$$h(p, \theta, i) = \min_{\alpha \geq 0} \left[c(i, \theta, \alpha) + \alpha p \right].$$

Note that h is an increasing concave function of p, Lipschitz in θ, and, hence, bounded below by

$$\min_{\theta \in [0,1], i \in \{0,1\}} h(0, \theta, i).$$

Because of the uniform convexity the minimum is achieved at a single point, and the function

$$\alpha^*(p, \theta, i) = \operatorname{argmin}_{\alpha \geq 0} \left[c(i, \theta, \alpha) + \alpha p \right].$$

is well defined. Furthermore we have

PROPOSITION 44. *The function α^* is locally Lipschitz in p, uniformly in $\theta \in [0,1]$. Furthermore it is uniformly Lipschitz in θ.*

PROOF. We will use the following inequalities, which are consequence of the uniform convexity of c: for all $\theta, \alpha', \alpha, p$ and p', we have

$$c(\theta, \alpha') + \alpha' p' \geq c(\theta, \alpha) + \alpha p' + (c'(\theta, \alpha) + p')(\alpha' - \alpha) + \gamma |\alpha' - \alpha|^2, \tag{37}$$

and because $\alpha^*(p, \theta)$ is a minimizer,

$$(c'(\theta, \alpha^*(p, \theta)) + p)(\alpha' - \alpha^*(p)) \geq 0. \tag{38}$$

We will first prove that α^* is uniformly Lipschitz in p : for that, we suppose that θ is fixed. By the definition of α^* and equation (37) we have

$$c(\alpha^*(p)) + \alpha^*(p) p' \geq c(\alpha^*(p')) + \alpha^*(p') p' \geq$$

$$\geq c(\alpha^*(p)) + \alpha^*(p) p' + (c'(\alpha^*(p)) + p')(\alpha^*(p') - \alpha^*(p)) + \gamma |\alpha^*(p') - \alpha^*(p)|^2,$$

hence

$$0 \geq (c'(\alpha^*(p))+p)(\alpha^*(p')-\alpha^*(p))+(p'-p)(\alpha^*(p')-\alpha^*(p))+\gamma|\alpha^*(p')-\alpha^*(p)|^2.$$

Now using equation (38) we obtain

$$0 \geq (p'-p)(\alpha^*(p')-\alpha^*(p))+\gamma|\alpha^*(p')-\alpha^*(p)|^2.$$

Therefore

$$|p'-p|\,|\alpha^*(p')-\alpha^*(p)| \geq \gamma|\alpha^*(p')-\alpha^*(p)|^2\,,$$

which implies

$$|\alpha^*(p')-\alpha^*(p)| \leq \frac{1}{\gamma}|p'-p|.$$

This shows that α^* is uniformly Lipschitz in p.

Now we prove that α^* is Lipschitz in θ: for that, we suppose that p is fixed. Again by the definition of α^* and by equation (37) we have

$$c(\theta',\alpha^*(\theta))+\alpha^*(\theta)p \geq c(\theta',\alpha^*(\theta'))+\alpha^*(\theta')p$$

$$\geq c(\theta',\alpha^*(\theta))+\alpha^*(\theta)p+c'(\theta',\alpha^*(\theta))(\alpha^*(\theta')-\alpha^*(\theta))+\gamma|\alpha^*(\theta')-\alpha^*(\theta)|^2,$$

and then

$$0 \geq c'(\theta',\alpha^*(\theta))(\alpha^*(\theta')-\alpha^*(\theta))+\gamma|\alpha^*(\theta')-\alpha^*(\theta)|^2.$$

Using equation (38) we get

$$0 \geq [c'(\theta',\alpha^*(\theta))-c'(\theta,\alpha^*(\theta))](\alpha^*(\theta')-\alpha^*(\theta))+\gamma|\alpha^*(\theta')-\alpha^*(\theta)|^2.$$

As $c'(\theta,\alpha)$ is Lipschitz in the variable θ we have

$$0 \geq -K|\theta'-\theta|\,|\alpha^*(\theta)-\alpha^*(\theta')|+\gamma|\alpha^*(\theta')-\alpha^*(\theta)|^2.$$

Therefore

$$|\alpha^*(\theta)-\alpha^*(\theta')| \leq \frac{K}{\gamma}|\theta-\theta'|\,,$$

which implies that α^* is Lipschitz in θ. $\square$

In section 2.3 we will present and discuss monotonicity assumptions on ψ and h, namely conditions (50) and (52), which will be necessary to prove uniqueness of solutions of the mean field model that will be presented in section 2.

1.6. The Hamilton-Jacobi ODE. Fix a admissible control β. Consider the system of ODE´s indexed by i and n given by

$$\begin{aligned}-\frac{d\varphi}{dt}(i,n,t) =&\gamma_\beta^+(i,n,t)(\varphi(i,n+1,t)-\varphi(i,n,t))+\gamma_\beta^-(i,n,t)(\varphi(i,n-1,t)-\varphi(i,n,t))\\ &+h\left(\bar\varphi(i,n,t)-\varphi(i,n,t),\frac{n}{N},i\right),\end{aligned}$$

where $\bar\varphi_\beta(i,n,t)=\varphi_\beta(1-i,n,t)$, and γ_β^+ and γ_β^- are given by (33). Since $\gamma_\beta^-(i,0,t)=0$ and $\gamma_\beta^+(i,N,t)=0$, the evaluation of φ at $n+1$ and $n-1$ does not cause problems outside the range, resp. when $n=N$ or $n=0$). By setting $\varphi_n(i,t)=\varphi(i,n,t)$ we write the previous ODE in compact notation:

$$-\frac{d\varphi_n}{dt}=\gamma_\beta^+(\varphi_{n+1}-\varphi_n)+\gamma_\beta^-(\varphi_{n-1}-\varphi_n)+h\left(\bar\varphi_n-\varphi_n,\frac{n}{N},i\right). \tag{39}$$

This system of ODE is called the Hamilton-Jacobi (HJ) ODE for player $N+1$ associated to the strategy β of the remaining N players. We start by proving a verification theorem, which is completely analogous to the optimal control verification theorem, see [**FS06**] for instance.

THEOREM 45. *Let φ_β be a solution to* (39) *satisfying the terminal condition $\varphi_\beta(i,n,T)=\psi\left(i,\frac{n}{N}\right)$. Then*

$$u(i,n,t;\beta)=\varphi_\beta(i,n,t).$$

Also, the control

$$\bar\alpha(\beta)(i,n,t)\equiv\alpha^*\left(\bar\varphi_\beta(i,n,t)-\varphi_\beta(i,n,t),\frac{n}{N},i\right), \tag{40}$$

is admissible and satisfies

$$u(i,n,t;\beta)=u(i,n,t,\beta,\bar\alpha(\beta)).$$

Thus a classical solution to the HJ equation associated to β is the value function corresponding to β and determines an optimal admissible control $\bar\alpha(\beta)$, for the reference player.

PROOF. Let α be an admissible control. By (34) we have

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\varphi_\beta(i(T),n(T),T)\right]-\varphi_\beta(i,n,t)=\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T\frac{d\varphi_\beta}{dt}(i,n,r)+A^{\beta,\alpha}\varphi_\beta(i,n,r)dr\right],$$

where $A^{\beta,\alpha}$ is given by (36). Adding

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T c\left(i(r),\frac{n(r)}{N},\alpha(r)\right)dr\right]+\varphi_\beta(i,n,t),$$

to both sides of the previous identity, where $\alpha(r) = \alpha(i(r), n(r), r)$, and using the definition of $A^{\beta,\alpha}\varphi_\beta(i,n,r)$, we have

$$
\begin{aligned}
&u(i,n,t;\beta,\alpha) = \\
&= \varphi_\beta(i,n,t) + \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r) + \gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r) - \varphi_\beta(i,n,r))\right. \\
&\left. + \gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r) - \varphi_\beta(i,n,r)) + c\left(i, \frac{n}{N}, \alpha\right) + \alpha(r)(\bar\varphi_\beta - \varphi_\beta)(i,n,r)dr\right].
\end{aligned}
$$

The equation above is valid for all admissible controls α. Now we can define

$$
\bar\alpha(\beta)(i,n,r) = \alpha^*\left(\bar\varphi_\beta(i,n,r) - \varphi_\beta(i,n,r), \frac{n}{N}, i\right),
$$

which is a bounded continuous Markov control and therefore admissible. We have

$$
\begin{aligned}
u(i,n,t;\beta) \leq &u(i,n,t,\beta,\alpha^*) = \varphi_\beta(i,n,t) \\
&+ \mathbb{E}^{\beta,\alpha^*}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r) + \gamma^+_\beta(n,r)(\varphi_\beta(i,n+1,r) - \varphi_\beta(i,n,r))\right. \\
&\left. + \gamma^-_\beta(n,r)(\varphi_\beta(i,n-1,r) - \varphi_\beta(i,n,r)) + h\left(\bar\varphi_\beta(i,n,r) - \varphi_\beta(i,n,r), \frac{n}{N}, i\right) dr\right].
\end{aligned}
$$

Now, we see that the integrand vanishes since φ_β is a solution to HJ, and therefore we have $u(i,n,t;\beta) \leq \varphi_\beta(i,n,t)$.

Now we prove the other inequality:

$$
\begin{aligned}
&u(i,n,t;\beta) = \inf_\alpha u(i,n,t,\beta,\alpha) = \varphi_\beta(i,n,t) \\
&+ \inf_\alpha \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r) + \gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r) - \varphi_\beta(i,n,r))\right. \\
&\left. + \gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r) - \varphi_\beta(i,n,r)) + c\left(i, \frac{n}{N}, \alpha\right) + \alpha(r)(\bar\varphi_\beta - \varphi_\beta)(i,n,r)dr\right] \\
&\geq \varphi_\beta(i,n,t) + \mathbb{E}^{\beta}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r) + \gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r) - \varphi_\beta(i,n,r))\right. \\
&\left. + \gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r) - \varphi_\beta(i,n,r)) + h\left(\bar\varphi_\beta(i,n,r) - \varphi_\beta(i,n,r), \frac{n}{N}, i\right) dr\right] \\
&= \varphi_\beta(i,n,t),
\end{aligned}
$$

where the last equation holds because the integrand vanishes since φ is a solution to HJ.

Thus we have proved that $u(i,n,t;\beta) = \varphi_\beta(i,n,t)$. □

1.7. Maximum principle. Here we prove that the solutions to the Hamilton-Jacobi equations are uniformly bounded independently on the control β. We denote by

$$\|u(t)\|_\infty = \max_{n,i} |u_n(i,t)|,$$

and

$$M = \max_{(i,\theta)\in\{0,1\}\times[0,1]} |h(0,\theta,i)|.$$

PROPOSITION 46. *Let u be a solution to* (39). *For all $0 \le t \le T$ we have*

$$\|u(t)\|_\infty \le \|u(T)\|_\infty + 2M(T-t).$$

PROOF. Let u be a solution to (39). Let $\tilde{u} = u + \rho(T-t)$. Then

$$-\frac{d\tilde{u}_n}{dt} = \rho + \gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) + \gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) + h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right).$$

Let (i,n,t) be a minimum point of $\tilde{u}$ on $\{0,1\} \times \{0,1,\cdots,N\} \times [0,T]$. We have $\tilde{u}_n(i,t) \le \tilde{u}_{n-1}(i,t)$ and $u_n(i,t) \le u_{n+1}(i,t)$. This implies $\gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) \ge 0$ and $\gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) \ge 0$. We also have $\tilde{u}_n(i,t) \le \tilde{u}_n(1-i,t) = \bar{\tilde{u}}_n(i,t)$, which implies $(\bar{\tilde{u}}_n - \tilde{u}_n)(i,t) \ge 0$. Hence

$$-\frac{d\tilde{u}_n}{dt}(i,t) \ge h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right) + \rho \ge h\left(0, \frac{n}{N}, i\right) + \rho,$$

because $h(p,\theta,i)$ is monotone increasing in p. Furthermore, if we take $M < \rho < 2M$ we get

$$-\frac{d\tilde{u}_n}{dt}(i,t) > 0.$$

This shows that the minimum of $\tilde{u}$ is achieved at T hence

$$u_n(t,i) \ge -\|u(T)\|_\infty - 2M(T-t).$$

Similarly, let (i,n,t) be a maximum point of $\tilde{u}$ on $\{0,1\} \times \{0,1,\cdots,N\} \times [0,T]$. We have $\tilde{u}_n(i,t) \ge \tilde{u}_{n-1}(i,t)$ and $u_n(i,t) \ge u_{n+1}(i,t)$, and this implies $\gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) \le 0$ and $\gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) \le 0$. We also have $\tilde{u}_n(i,t) \ge \tilde{u}_n(1-i,t) = \bar{\tilde{u}}_n(i,t)$, which implies $(\bar{\tilde{u}}_n - \tilde{u}_n)(i,t) \le 0$. Hence

$$-\frac{d\tilde{u}_n}{dt}(i,t) \le h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right) + \rho \le h\left(0, \frac{n}{N}, i\right) + \rho,$$

because $h(p,\theta,i)$ is monotone increasing in p. Furthermore, if we take $-2M < \rho < -M$ we get

$$-\frac{d\tilde{u}_n}{dt}(i,t) < 0.$$

This shows that the maximum of $\tilde{u}$ is achieved at T hence

$$u_n(t,i) \le \|u(T)\|_\infty + 2M(T-t).$$

∎

1.8. Equilibrium solutions. We now consider the equilibrium situation in which the best response of any player to a control β is β itself.

DEFINITION 1. *Let β be an admissible control. This control β is a Nash equilibrium if $\bar{\alpha}(\beta) = \beta$.*

THEOREM 47. *There exists a Nash equilibrium, i.e, an admissible Markov control β_*, which satisfies $\bar{\alpha}(\beta_*) = \beta_*$. Moreover, the Nash equilibrium is unique.*

PROOF. It suffices to observe that, by (40)

$$\beta_*(i, n, t) = \alpha^* \left(\bar{\varphi}_{\beta^*} - \varphi_{\beta^*}, \frac{n}{N}, i \right),$$

and hence the Markov control can be obtained by solving the system of nonlinear differential equations

$$-\frac{du_n}{dt} = \gamma_n^+ (u_{n+1} - u_n) + \gamma_n^- (u_{n-1} - u_n) + h\left(\bar{u}_n - u_n, \frac{n}{N}, i \right), \tag{41}$$

with terminal condition $u(i, n, T) = \psi\left(i, \frac{n}{N}\right)$, where $\gamma_n^\pm$ are given by

$$\begin{aligned} \gamma_n^+(i,t) &= (N-n)\alpha^* \left(\bar{u}_{n+1-i} - u_{n+1-i}, \frac{n+1-i}{N}, 1 \right) \\ \gamma_n^-(i,t) &= n\alpha^* \left(\bar{u}_{n-i} - u_{n-i}, \frac{n-i}{N}, 0 \right). \end{aligned} \tag{42}$$

Note that (41) is well posed because u_n is bounded and the righthand side is Lipschitz. Hence it follows the existence and uniqueness of a Nash equilibrium. □

For the record we give here some properties of $\gamma_n^\pm$:

$$|\gamma_n^\pm| \le CN,$$

and

$$|\gamma_{n+1}^\pm - \gamma_n^\pm| \le C + CN\|u_{n+1} - u_n\|_\infty.$$

1.9. An example. Let $f : \{0,1\} \times [0,1] \to \mathbb{R}$ and $g : \{0,1\} \times [0,1] \to \mathbb{R}$ be two continuous function. We take

$$c(i, \theta, \alpha) = f(i, \theta) + \frac{\alpha^2}{2} - \alpha g(i, \theta).$$

This example could model, for instance, the marketshare of cellular companies where there are only two competitors and N individual costumers. If the state of the player represents the company he uses, we can think of $g(i, \theta)$ as a bonus the company i offers customers of company $1-i$ in case they decide to switch. If there are no such bonus, we set $g = 0$.

Then

$$h(p,\theta,i) = \min_{\alpha\geq 0}\left[c(i,\theta,\alpha)+\alpha p\right] = f(i,\theta) - \frac{((g(i,\theta)-p)^+)^2}{2}\,,$$

and

$$\alpha^*(p,\theta,i) = \operatorname{argmin}_{\alpha\geq 0}\left[c(i,\theta,\alpha)+\alpha p)\right] = (g(i,\theta)-p)^+\,.$$

Therefore (41) becomes

(43)

$$-\frac{du}{dt} = f - \frac{((u-\bar{u}+g)^+)^2}{2} + (N-n)(u-\bar{u}+g)^+_{1,n+1-i}(u_{n+1}-u_n) + n(u-\bar{u}+g)^+_{0,n-i}(u_{n-1}-u_n).$$

By the results of section 1.7 we know that any solution to (43) is bounded a-priori. Hence, if f and g are Lipschitz, (43) has a unique solution u. Therefore, there exists a unique Nash equilibrium.

2. A mean field model

This section is dedicated to a mean field model which, as we will see in the next section, corresponds to the limit as the number of players $N+1\to\infty$. We start in §2.1 by discussing the model and its derivation under the mean field hypothesis. Then, in §2.2 we address existence of solutions. Uniqueness of solutions (under a monotonicity hypothesis similar to the ones in [**LL06a, LL06b**]) is established in §2.3. Finally, in §2.4, we continue the study of the model problem from §1.9.

2.1. The control problem in the mean field model and Nash equilibria. If the number of players is very large, we expect their distribution between the two states to be a deterministic function of the time t, as it would happen if we could somehow apply the law of large numbers. So, we suppose the fraction of players in state 0 is given by a deterministic function $\theta(t)$. If all players use the same Markovian control $\beta=\beta(i,t)$, which now only depends on i and t, then θ is a solution to

(44)
$$\frac{d\theta}{dt} = (1-\theta)\beta_1 - \theta\beta_0 \qquad\qquad \theta(0)=\bar{\theta}\,,$$

where β_i denotes the function $t\to\beta(i,t)$, and $0\leq\bar{\theta}\leq 1$ is given and represents the initial distribution. We suppose here that β_i are continuous and bounded, for $i=0$ and $i=1$, and call such controls admissible controls.

We can now consider the optimization problem from a single player point of view. As before, we fix an individual player as the reference player and assume he can choose any admissible control α, while other players have a probability distribution among states determined by (44). Let

$$u(i,t,\alpha) = \mathbb{E}^{\alpha}_{i(t)=i}\left[\int_t^T c(i(s),\theta(s),\alpha(i(s),s))ds + \psi(i(T),\theta(T))\right] ,$$

where $i(t)$ is a controlled Markov chain switching between state 0 and 1 with rate α. We assume this player looks for an admissible control α which solves

$$u(i,t) = \inf_{\alpha} u(i,t,\alpha).$$

Note that the situation is now simpler than in the $N+1$-player game, because θ is deterministic and the only stochastic process is $i(t)$ whose switching rate is controlled by α. We call $u(i,t)$ the value function associated to the mean field distribution θ.

Consider the following HJ equation:

$$-\frac{du}{dt} = h(\bar{u}-u,\theta,i)\,. \tag{45}$$

As in the verification theorem of §1.6, any solution u to the equation above, with the terminal condition $u(i,T) = \psi(i,\theta(T))$, is the value function associated to θ. Furthermore, the optimal control is $\alpha^*(\bar{u}-u,\theta,i)$.

Under the symmetry hypothesis, all players must use the same control when the Nash equilibria is attained. In other words, Nash equilibria is the fixed point to the operator described above, i.e., the operator that uses the control β to calculate θ as a solution to (44), and after that determines the control $\alpha^*(\bar{u}-u,\theta,i)$ where u is the solution to the HJ equation (45) determined by θ, making the control $\alpha^*(\bar{u}-u,\theta,i)$ the image of β under this operator.

This leads then to the following system of ordinary differential equations

$$\begin{cases} -\frac{du}{dt} = h(\bar{u}-u,\theta,i) \\ \frac{d\theta}{dt} = (1-\theta)\alpha^*(u(0,t)-u(1,t),\theta,1) - \theta\alpha^*(u(1,t)-u(0,t),\theta,0), \end{cases} \tag{46}$$

with the boundary data

$$\begin{cases} u(i,T) = \psi(i,\theta(T)) \\ \theta(0) = \bar{\theta}\,. \end{cases} \tag{47}$$

Note that from the ODE point of view this problem is somewhat non-standard as some of the variables have initial conditions whereas other variables have prescribed terminal data. We call this the initial-terminal value problem.

2.2. Existence of Nash Equilibria in the MFG. We now address the existence of solutions to (46) satisfying the initial-terminal conditions (47). The proof of existence will be based upon a fixed point argument, using the operator ξ described in the following, which is the analogous of the operator acting on the controls described in the last section, but now acting on distributions.

PROPOSITION 48. *There exists a solution to* (46) *satisfying the initial-terminal conditions* (47).

PROOF. We need to solve (46) and (47) which can be rewritten as

$$\frac{d\theta}{dt} = (1-\theta)\alpha_1 - \theta\alpha_0 \qquad\qquad \theta(0) = \bar{\theta} \tag{48}$$

$$-\frac{du}{dt} = h(\bar{u} - u, \theta, i) \qquad\qquad u(i,T) = \psi(i, \theta(T)) \tag{49}$$

where

$$\alpha = \alpha^*(\bar{u} - u, \theta, i).$$

Let $\mathcal{F}$ be the set of continuous functions defined on $[0,T]$ and taking values in $[0,1]$, with the C^0 norm. Consider the function $\xi : \mathcal{F} \to \mathcal{F}$ that is obtained in the following way: given $\theta \in \mathcal{F}$, let u^θ be the solution of equation (49). Let $\beta^\theta = \alpha^*(\bar{u}^\theta - u^\theta, \theta, i)$, and then let $\xi(\theta)$ be the solution to $\frac{d\theta}{dt} = (1-\theta)\beta_1^\theta - \theta\beta_0^\theta$ and $\theta(0) = \bar{\theta}$.

From standard ODE theory we know ξ is a continuous function from $\mathcal{F}$ to $\mathcal{F}$. Moreover, as β is bounded, $\xi(\theta)$ is Lipschitz, with Lipschitz constant Λ independent of θ.

Now consider the set $\mathcal{C}$ of all Lipschitz continuous function in $\mathcal{F}$ with Lipschitz constant bounded by Λ. This is a set of uniformly bounded and equicontinuous functions. Thus, by Arzela-Ascoli, it is a relatively compact set. It is also clear that it is a convex set. Hence, by Brouwer fixed point theorem, ξ has a fixed point in $\mathcal{C}$. □

2.3. Uniqueness of Equilibria. To establish uniqueness we need to use the monotonicity method of [**LL06a, LL06b**].

We will suppose the following monotonicity hypothesis on ψ:

$$(x-y)[\psi(0,x) - \psi(0,y)] + (y-x)[\psi(1,x) - \psi(1,y)] \geq 0\,, \tag{50}$$

for any x and y in $[0,1]$. This hypothesis holds, for instance, if we suppose that ψ is differentiable on its second variable, and

$$\frac{d\psi}{d\theta}(0,\theta) - \frac{d\psi}{d\theta}(1,\theta) \geq 0\,,$$

or if we suppose that $\psi(0,\theta)$ is non-decreasing as function of θ and $\psi(1,\theta)$ is non-increasing as function of θ, which could be interpreted as a penalization on crowded states.

Now, from the concavity of h in p we have, for all p, q, θ and i

$$h(q,\theta,i) - h(p,\theta,i) - \alpha^*(p,\theta,i)(q-p) \leq 0, \tag{51}$$

because $\alpha^*(p,\theta,i) \in \partial_p^+ h(p,\theta,i)$. We suppose the additional monotonicity property

$$\begin{aligned} &\theta\Big(h(q,\tilde\theta,0) - h(q,\theta,0)\Big) + \tilde\theta\Big(h(p,\theta,0) - h(p,\tilde\theta,0)\Big) \\ &+(1-\theta)\Big(h(-q,\tilde\theta,1) - h(-q,\theta,1)\Big) + (1-\tilde\theta)\Big(h(-p,\theta,1) - h(-p,\tilde\theta,1)\Big) \leq -\gamma|\theta-\tilde\theta|^2, \end{aligned} \tag{52}$$

for all $p, q \in \mathbb{R}$, for some $\gamma > 0$. This property will hold, for instance, if

$$h(p,\theta,i) = h_0(p) + f(i,\theta), \tag{53}$$

with f satisfying

$$(\theta-\tilde\theta)(f(0,\tilde\theta) - f(0,\theta)) + (\tilde\theta-\theta)(f(1,\tilde\theta) - f(1,\theta)) \leq -\gamma|\theta-\tilde\theta|^2. \tag{54}$$

Note that the example of section 1.9 easily fits the previous conditions (53) and (54) provided we suppose g is a constant function and the functions $\theta \mapsto f(0,\theta)$ and $\theta \mapsto f(1,\theta)$ satisfy

$$(\tilde\theta-\theta)(f(0,\tilde\theta) - f(0,\theta)) \geq \frac{\gamma}{2}|\theta-\tilde\theta|^2$$

and

$$(\tilde\theta-\theta)(f(1,\tilde\theta) - f(1,\theta)) \leq -\frac{\gamma}{2}|\theta-\tilde\theta|^2,$$

which could be seen as a consequence of the fact that the running cost is greater when the reference player is in the more crowded state (i.e. when $\theta = 1$ if $i = 0$ and when $\theta = 0$ if $i = 1$).

Then , using (51) and (52) we obtain

$$\begin{aligned} &\theta\Big(h(q,\tilde\theta,0) - h(p,\theta,0) - \alpha^*(p,\theta,0)(q-p)\Big) + \tilde\theta\Big(h(p,\theta,0) - h(q,\tilde\theta,0) - \alpha^*(q,\tilde\theta,0)(p-q)\Big) \\ &+ (1-\theta)\Big(h(-q,\tilde\theta,1) - h(-p,\theta,1) - \alpha^*(-p,\theta,1)(p-q)\Big) \\ &+ (1-\tilde\theta)\Big(h(-p,\theta,1) - h(-q,\tilde\theta,1) - \alpha^*(-q,\tilde\theta,1)(q-p)\Big) \leq -\gamma|\theta-\tilde\theta|^2. \end{aligned} \tag{55}$$

THEOREM 49. *Under the monotonicity hypothesis* (50) *and* (52), *the system (48) and (49) has a unique solution* (θ, u).

PROOF. To establish uniqueness we will use monotonicity argument from [**LL06a, LL06b**].

Suppose (θ, u) and $(\tilde{\theta}, \tilde{u})$ are solutions of (48) and (49). At the initial point $t = 0$ we have that $(\theta - \tilde{\theta})(u - \tilde{u}) = 0$ and $((1-\theta) - (1-\tilde{\theta}))(\bar{u} - \bar{\tilde{u}}) = 0$, where $u(t) = u(0, t)$ and $\bar{u} = u(1, t)$, and similarly for $\tilde{u}$. Then

$$(\theta - \tilde{\theta})(u - \tilde{u})_t = (\theta - \tilde{\theta})[-h(\bar{u} - u, \theta, 0) + h(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0)],$$

and

$$((1-\theta) - (1-\tilde{\theta}))(\bar{u} - \bar{\tilde{u}})_t = ((1-\theta) - (1-\tilde{\theta}))[-h(u - \bar{u}, \theta, 1) + h(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1)].$$

Furthermore,

$$\begin{aligned}(u - \tilde{u})(\theta - \tilde{\theta})_t =& (u - \tilde{u})[(1-\theta)\alpha^*(u - \bar{u}, \theta, 1) - \theta\alpha^*(\bar{u} - u, \theta, 0) \\ & - (1-\tilde{\theta})\alpha^*(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1) + \tilde{\theta}\alpha^*(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0)],\end{aligned}$$

and

$$\begin{aligned}(\bar{u} - \bar{\tilde{u}})((1-\theta) - 1 + \tilde{\theta})_t =& (\bar{u} - \bar{\tilde{u}})[\theta\alpha^*(\bar{u} - u), \theta, 0) - (1-\theta)\alpha^*(u - \bar{u}, \theta, 1) \\ & - \tilde{\theta}\alpha^*(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0) + (1-\tilde{\theta})\alpha^*(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1)].\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\Big((\theta - \tilde{\theta})(u - \tilde{u}) + ((1-\theta) - (1-\tilde{\theta}))(\bar{u} - \bar{\tilde{u}})\Big) = \\ =& \theta\Big(- h(\bar{u} - u, \theta, 0) + h(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0) + [(\bar{u} - \bar{\tilde{u}}) - (u - \tilde{u})]\alpha^*(\bar{u} - u, \theta, 0)\Big) \\ & + \tilde{\theta}\Big(h(\bar{u} - u, \theta, 0) - h(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0) + [-(\bar{u} - \bar{\tilde{u}}) + (u - \tilde{u})]\alpha^*(\bar{\tilde{u}} - \tilde{u}, \tilde{\theta}, 0)\Big) \\ & + (1-\theta)\Big(- h(u - \bar{u}, \theta, 1) + h(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1) + [(u - \tilde{u}) - (\bar{u} - \bar{\tilde{u}})]\alpha^*(u - \bar{u}, \theta, 1)\Big) \\ & + (1-\tilde{\theta})\Big(h(u - \bar{u}, \theta, 1) - h(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1) + [-(u - \tilde{u}) + (\bar{u} - \bar{\tilde{u}})]\alpha^*(\tilde{u} - \bar{\tilde{u}}, \tilde{\theta}, 1)\Big).\end{aligned}$$

Then, by using (55), with $p = \bar{u} - u$ and $q = \bar{\tilde{u}} - \tilde{u}$, we obtain

$$\frac{d}{dt}\Big((\theta - \tilde{\theta})(u - \tilde{u}) + ((1-\theta) - (1-\tilde{\theta}))(\bar{u} - \bar{\tilde{u}})\Big) \leq -\gamma|\theta - \tilde{\theta}|^2. \tag{56}$$

Integrating the previous equation between 0 and T, and using the terminal conditions, we have that

$$(\theta(T) - \tilde{\theta}(T))[\psi(0, \theta(T)) - \psi(0, \tilde{\theta}(T))] + (\tilde{\theta}(T) - \theta(T))[\psi(1, \theta(T)) - \psi(1, \tilde{\theta}(T))] \leq -\gamma\int_0^T |\theta(s) - \tilde{\theta}(s)|^2 ds.$$

Hence by the monotonicity condition (50) we get

$$0 \leq -\gamma \int_0^T |\theta(s) - \tilde{\theta}(s)|^2 ds,$$

which implies that $\theta(s) = \tilde{\theta}(s)$ for all $s \in [0, T]$. Therefore, we have the uniqueness for θ. Then, once θ is known to be unique, we obtain by a standard ODE argument that $u = \tilde{u}$. □

2.4. Back to the example. Just to illustrate, equations (46), in the special case of the example of section 1.9, and supposing g is a constant function, becomes

$$\frac{d\theta}{dt} = (1-\theta)(u-\bar{u})_1^+ - \theta(u-\bar{u})_0^+ ,$$

and

$$-\frac{du}{dt} = f(i, \theta) - \frac{((u - \bar{u} + g)^+)^2}{2} .$$

As we have already seen, provided the condition (54) holds and given the initial-terminal condition

$$\theta(0) = \bar{\theta}, \quad u(i, T) = \psi(i, \theta(T))$$

the system above has a unique solution.

3. Convergence

This last section addresses the convergence as the number of players tends to infinity to the mean field model derived in the previous section.

We start this section by discussing some preliminary estimates in §3.1. Then, in §3.2 we establish uniform estimates for $|u_{n+1} - u_n|$, which are essential to prove our main result, theorem 57, which is discussed in §3.3. This theorem shows that the model derived in the previous section can be obtained as an appropriate limit of the model with $N + 1$ players discussed in section 1.

3.1. Preliminary results. Consider the system of ordinary differential equations

$$-\dot{z}_n = a_n(t)(z_{n+1} - z_n) + b_n(t)(z_{n-1} - z_n) + \mu_n(t)(\bar{z}_n - z_n), \tag{57}$$

with $a_n(t), b_n(t), \mu_n(t) \geq 0$. Here $z_n = (z_n^0, z_n^1)$, $a_n = (a_n^0, a_n^1)$, etc. We assume further that $a_N = 0$ and $b_0 = 0$.

We write (57) in compact form as

$$-\dot{z}(t) = M(t)z(t). \tag{58}$$

The solution to this equation with terminal data $z(T)$ can be written as

$$z(t) = K(t,T)z(T), \tag{59}$$

where $K(t,T)$ is the fundamental solution to (58) with $K(T,T) = I$. Note that equations (58) and (59) imply

$$\frac{d}{dt}K(t,T) = -M(t)K(t,T). \tag{60}$$

LEMMA 50. *For $t < T$ we have*

$$\|z(t)\|_\infty \leq \|z(T)\|_\infty.$$

Furthermore, if $z(T) \leq 0$ then $z(t) \leq 0$.

PROOF. Let z be a solution of (58), and fix $\epsilon > 0$. We define $\tilde{z} = z + \epsilon(t - T)$. Hence $\tilde{z}$ satisfies

$$-\dot{\tilde{z}}_n = -\epsilon + a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) + b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) + \mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n).$$

Let (i,n,t) be a maximum point of $\tilde{z}$ on $\{0,1\} \times \{0,1,\cdots,N\} \times [0,T]$. We have $\tilde{z}_n(i,t) \geq \tilde{z}_{n-1}(i,t)$ and $z_n(i,t) \geq z_{n+1}(i,t)$, also $\tilde{z}_n(i,t) \geq \tilde{z}_n(1-i,t) = \bar{\tilde{z}}_n(i,t)$, this implies $b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) \leq 0$ and $a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) \leq 0$ and $\mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n)(i,t) \leq 0$. Hence

$$-\frac{d\tilde{z}_n}{dt}(i,t) \leq -\epsilon.$$

This shows that the maximum of $\tilde{z}$ is achieved at T. Therefore, for all (j,m,t),

$$z_m(j,t) + \epsilon(t-T) = \tilde{z}_m(j,t) \leq \tilde{z}_n(i,T) = z_n(i,T)$$

Letting $\epsilon \to 0$, we get

$$z_m(j,t) \leq \max_{n,i} z_n(i,T).$$

From this equation we have the following conclusions:

1. if $z(T) \leq 0$, we then have $z_m(j,t) \leq 0$, for all (j,m,t), and so $z(t) \leq 0$;
2. for all (j,m,t),
$$z_m(j,t) \leq \|z(T)\|_\infty.$$

Now we define $\tilde{z} = z + \epsilon(T - t)$. Hence $\tilde{z}$ satisfies

$$-\dot{\tilde{z}}_n = \epsilon + a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) + b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) + \mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n).$$

Let (i,n,t) be a minimum point of $\tilde{z}$ on $\{0,1\} \times \{0,1,\cdots,N\} \times [0,T]$. We have $\tilde{z}_n(i,t) \leq \tilde{z}_{n-1}(i,t)$ and $z_n(i,t) \leq z_{n+1}(i,t)$, also $\tilde{z}_n(i,t) \leq \tilde{z}_n(1-i,t) = \bar{\tilde{z}}_n(i,t)$.

This implies $b_n(t)(\tilde{z}_{n-1}-\tilde{z}_n) \geq 0$, and $a_n(t)(\tilde{z}_{n+1}-\tilde{z}_n) \geq 0$ and $\mu_n(t)(\bar{\tilde{z}}_n-\tilde{z}_n)(i,t) \geq 0$. Therefore we have

$$-\frac{d\tilde{z}_n}{dt}(i,t) \geq \epsilon.$$

This shows that the minimum of $\tilde{z}$ is also achieved at T, hence for all (j,m,t)

$$z_m(j,t) + \epsilon(T-t) = \tilde{z}_m(j,t) \geq \tilde{z}_n(i,T) = z_n(i,T).$$

Letting $\epsilon \to 0$, we get

$$z_m(j,t) \geq \min_{n,i} z_n(i,T).$$

Hence

$$z_m(j,t) \geq -\|z(T)\|_\infty.$$

Therefore we have $\|z(t)\|_\infty \leq \|z(T)\|_\infty$. □

Note: let $z(t) = K(t,s)z(s)$ be a solution of (58) with terminal data $z(s) = b$, then lemma 50 implies that $\|z(t)\|_\infty \leq \|z(s)\|_\infty$, and therefore

$$\|K(t,s)b\|_\infty \leq \|b\|_\infty, \forall\, b. \tag{61}$$

From the previous lemma we also conclude

Lemma 51. *If $p_1 \leq p_2$, and $t \leq s$, then we have*

$$K(t,s)p_1 \leq K(t,s)p_2.$$

Proof. Observe that if $p_1 - p_2 \leq 0$ then $K(t,s)(p_1-p_2) \leq 0$, by lemma 50. □

We note now that if $t \leq s \leq T$ we have $K(t,s)K(s,T) = K(t,T)$, which implies

$$\frac{d}{ds}\Big(K(t,s)K(s,T)\Big) = 0.$$

Hence, using equation (60) we get

$$-K(t,s)M(s)K(s,T) + \left(\frac{d}{ds}K(t,s)\right)K(s,T) = 0,$$

and therefore, by taking $T = s$ we conclude that

$$\frac{d}{ds}K(t,s) = K(t,s)M(s). \tag{62}$$

We now prove the main technical lemma:

LEMMA 52. *Suppose z is a solution to*

$$-\dot{z}(s) \leq M(s)z(s) + f(z(s)). \tag{63}$$

Then

$$z(t) \leq \|z(T)\|_\infty + \int_t^T \|f(z(s))\|_\infty ds.$$

PROOF. Multiplying (63) by the order preserving operator $K(t,s)$, we have

$$-K(t,s)\dot{z}(s) \leq K(t,s)M(s)z(s) + K(t,s)f(z(s))$$

using the identity

$$\frac{d}{ds}K(t,s)z(s) = K(t,s)\dot{z}(s) + K(t,s)M(s)z(s),$$

which follows from (62), we get

$$-\frac{d}{ds}\Big(K(t,s)z(s)\Big) + K(t,s)M(s)z(s) \leq K(t,s)M(s)z(s) + K(t,s)f(z(s)).$$

Thus, integrating between t and T, we have

$$z(t) - K(t,T)z(T) \leq \int_t^T K(t,s)f(z(s))ds.$$

So, using equation (61),

$$z(t) \leq \|z(T)\|_\infty + \int_t^T \|f(z(s))\|_\infty ds.$$

$\square$

3.2. Uniform estimates. In this section we prove "gradient estimates" for the $N+1$ player game, that is, we assume that the difference $u_{n+1} - u_n$ is of the order $\frac{1}{N}$ at time T and show that it remains so for $0 \leq t \leq T$, as long as T is sufficiently small.

We start by establishing an auxiliary result:

LEMMA 53. *Suppose $v = v(s)$ is a solution to the ODE with terminal condition*

$$\begin{cases} -\frac{dv}{ds} = Cv + CNv^2 + \frac{C}{N} \\ v(T) \leq \frac{C}{N}, \end{cases} \tag{64}$$

where N is a natural number, and $C > 0$. Then, there exists $T^\star > 0$, which does not depend on N, such that $T \leq T^\star$ implies $v(s) \leq \frac{2C}{N}$ for all $0 \leq s \leq T$.

Proof. Note that (64) implies that v is a monotone decreasing function of s and is equivalent to

$$\begin{cases} \frac{ds}{dv} = \frac{-1}{Cv+CNv^2+\frac{C}{N}} \\ s(\frac{C}{N}) \leq T. \end{cases}$$

This implies by direct integration that

$$s\left(\frac{2C}{N}\right) \leq T - \int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{dv}{Cv+CNv^2+\frac{C}{N}}.$$

Now

$$\int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{dv}{Cv+CNv^2+\frac{C}{N}} \geq \int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{N}{2C^2+4C^3+C}dv = \frac{1}{2C+4C^2+1}.$$

Therefore if we define $T^\star = \frac{1}{2C+4C^2+1}$, we have that $s\left(\frac{2C}{N}\right) \leq 0$ if $T \leq T^\star$. Hence this implies $v(0) \leq \frac{2C}{N}$, which yields the desired result when we take into account that v is a decreasing function of s. □

Proposition 54. *Suppose that*

$$\|u_{n+1}(T) - u_n(T)\|_\infty \leq \frac{C}{N}. \tag{65}$$

for $C > 0$. *Let* u *be a solution of* (41). *Then there exists* $T^\star > 0$ *such that, for* $0 < T < T^\star$ *,we have*

$$\|u_{n+1}(t) - u_n(t)\|_\infty \leq \frac{2C}{N},$$

for all $0 \leq t \leq T$.

Proof. Let

$$z_n = u_{n+1} - u_n.$$

Note that, as usual, $z_n = (z_n^0, z_n^1)$. We have

$$-\dot{z}_n = \gamma_{n+1}^+ z_{n+1} - \gamma_n^+ z_n - \gamma_{n+1}^- z_n + \gamma_n^- z_{n-1} + h\left(\frac{n+1}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) - h\left(\frac{n}{N}, i, \bar{u}_n - u_n\right).$$

We can write

$$\begin{aligned} \gamma_{n+1}^+ z_{n+1} - &\gamma_n^+ z_n - \gamma_{n+1}^- z_n + \gamma_n^- z_{n-1} \\ &= \frac{\gamma_{n+1}^+ + \gamma_n^+}{2}(z_{n+1} - z_n) + \frac{\gamma_{n+1}^+ - \gamma_n^+}{2}(z_{n+1} + z_n) \\ &\quad + \frac{\gamma_{n+1}^- + \gamma_n^-}{2}(z_{n-1} - z_n) + \frac{\gamma_n^- - \gamma_{n+1}^-}{2}(z_{n-1} + z_n). \end{aligned}$$

We must now observe that

$$\left|\frac{\gamma_n^- - \gamma_{n+1}^-}{2}\right| \leq C + CN\|z\|_\infty,$$

as well as

$$\left|\frac{\gamma_{n+1}^{+}-\gamma_{n}^{+}}{2}\right| \leq C+CN\|z\|_{\infty}.$$

Furthermore, we have

$$\begin{aligned}
& h\left(\frac{n+1}{N}, i, \bar{u}_{n+1}-u_{n+1}\right)-h\left(\frac{n}{N}, i, \bar{u}_{n}-u_{n}\right) \\
& = h\left(\frac{n+1}{N}, i, \bar{u}_{n+1}-u_{n+1}\right)-h\left(\frac{n}{N}, i, \bar{u}_{n+1}-u_{n+1}\right) \\
& + h\left(\frac{n}{N}, i, \bar{u}_{n+1}-u_{n+1}\right)-h\left(\frac{n}{N}, i, \bar{u}_{n}-u_{n}\right) \\
& \leq \frac{C}{N}+h_{p}\left(\frac{n}{N}, i, \bar{u}_{n}-u_{n}\right)\left(\left(\bar{u}_{n+1}-u_{n+1}\right)-\left(\bar{u}_{n}-u_{n}\right)\right) \\
& \leq \frac{C}{N}+\mu_{n}\left(\bar{z}_{n}-z_{n}\right),
\end{aligned}$$

where $\mu_{n}=h_{p}\left(\frac{n}{N}, i, \bar{u}_{n}-u_{n}\right) \geq 0$.

At this point we are in position to apply lemma 52 from the previous section. We obtain

$$z_{n}(t)=\left(u_{n+1}-u_{n}\right)(t) \leq\|z(T)\|_{\infty}+\int_{t}^{T} C\|z(s)\|_{\infty}+C\|z(s)\|_{\infty}^{2}+\frac{C}{N}\, ds\ .$$

We can also use the same argument applied to

$$\tilde{z}_{n}=u_{n}-u_{n+1}\ .$$

Finally, if we set $w=\|u_{n+1}-u_{n}\|_{\infty}$ we conclude that

$$w(t) \leq w(T)+\int_{t}^{T} C w(s)+C N w(s)^{2}+\frac{C}{N} ds.$$

Now we define

$$\eta(t)=w(T)+\int_{t}^{T} C w(s)+C N w(s)^{2}+\frac{C}{N} ds.$$

We have that

$$w(t) \leq \eta(t), \tag{66}$$

and also that

$$\frac{d \eta}{d t}(t)=-g(w(t)),$$

where g is the nondecreasing function $g(w)=C w+C N w^{2}+\frac{C}{N}$. Thus

$$\begin{cases}\frac{d \eta}{d t}(t) \geq-g(\eta(t)) \\ \eta(T)=w(T).\end{cases}$$

A standard argument from the basic theory of differential inequalities can now be used to prove that $\eta(t) \le v(t)$ for $0 \le t \le T$ if $v(t)$ is the solution of

$$\begin{cases} \frac{dv}{dt}(t) = -g(v(t)) \\ v(T) = w(T). \end{cases}$$

This last result can be combined with lemma 53, the hypothesis $w(T) \le \frac{C}{N}$ and the inequality (66), to prove that $w(t) \le \frac{2C}{N}$ for all $0 \le t \le T$, which ends the proof of the proposition. $\square$

3.3. Convergence. In this section we prove theorem 57, which implies the convergence of both distribution and value function of the $N+1$-player game to the mean field game, for small times.

We start by assuming that at the initial time the N players distinct from the reference player distribute themselves between states 0 and 1 according to a Bernoulli distribution with probability $\bar{\theta}$ of being in state 0.

Let

$$\begin{cases} V_N(t) \equiv \mathbb{E}\left[\left(\frac{n(t)}{N} - \theta(t)\right)^2\right], \\ \\ W_N(t) \equiv \mathbb{E}\left[\left(u(0,t) - u_{n(t)}(0,t)\right)^2\right], \\ \\ \bar{W}_N(t) \equiv \mathbb{E}\left[\left(u(1,t) - u_{n(t)}(1,t)\right)^2\right], \\ \\ Q_N(t) \equiv W_N(t) + \bar{W}_N(t), \end{cases} \tag{67}$$

where $\theta(t)$ is the solution of (44), $0 \le n(t) \le N$ is the number of players (distinct from the reference player) which are in state 0 at time t, and $u = u(i,t)$ and $u_n = u_n(i,t)$ are respectively the solution of the HJ equation and terminal conditions for the MFG (46) and $N+1$ player game (41).

We have

$$V_N(0) = \mathrm{Var}\left[\frac{n(0)}{N}\right] = \frac{\bar{\theta}(1-\bar{\theta})}{N},$$

because $n(0)$ is the sum of N iid rv with Bernoulli distribution.

In this section $\alpha = \alpha(i,t)$ is the optimal control for the MFG, while $\alpha^N = \alpha^N(i,n,t)$ is the optimal control for the $N+1$ player game. We know from sections 1.5, 1.6 and 2.1 that $\alpha^N = \alpha^*\left(\bar{u}_n - u_n, \frac{n}{N}, i\right)$ and $\alpha = \alpha^*(\bar{u} - u, \theta, i)$.

Lemma 55. *There exists $C_1 > 0$ such that*

$$V_N(t) \le \int_0^t C_1(V_N(s) + Q_N(s))ds + \frac{C_1}{N}.$$

Proof. Using Dynkin's Formula (34) with $\varphi(i,n,s) = \left(\frac{n(s)}{N} - \theta(s)\right)^2$, we have

$$V_N(t) - \frac{\theta_0(1-\theta_0)}{N} = \mathbb{E}\int_0^t \omega_N(s) + \varsigma_N(s)ds$$

where

$$\omega_N(s) = (N-n)\alpha_1^N \left[\left(\frac{n+1}{N} - \theta\right)^2 - \left(\frac{n}{N} - \theta\right)^2\right] + n\alpha_0^N \left[\left(\frac{n-1}{N} - \theta\right)^2 - \left(\frac{n}{N} - \theta\right)^2\right],$$

$$\alpha_0^N = \alpha^*\left(\bar{u}_{n-i} - u_{n-i}, \frac{n-i}{N}, 0\right),$$

$$\alpha_1^N = \alpha^*\left(\bar{u}_{n+1-i} - u_{n+1-i}, \frac{n+1-i}{N}, 1\right),$$

$$u_n = u_N(i,n,t),$$

and

$$\varsigma_N(s) = \frac{d\varphi}{dt}(i,n,r) = -2\left(\frac{n}{N} - \theta\right)((1-\theta)\alpha_1 - \theta\alpha_0).$$

We have

$$\begin{aligned}\omega_N(s) =& \left(1 - \frac{n}{N}\right)\alpha_1^N\left(\frac{2n+1}{N} - 2\theta\right) - \frac{n}{N}\alpha_0^N\left(\frac{2n-1}{N} - 2\theta\right)\\ =& 2\alpha_1^N\left(1 - \frac{n}{N}\right)\left(\frac{n}{N} - \theta\right) - 2\alpha_0^N\frac{n}{N}\left(\frac{n}{N} - \theta\right) + \tau_N(s),\end{aligned}$$

where $\tau_N(s) = \frac{\alpha_1^N}{N} + \frac{n}{N^2}(\alpha_0^N - \alpha_1^N)$. Now

$$\begin{aligned}\omega_N(s) + \varsigma_N(s) =& 2\left(\frac{n}{N} - \theta\right)\left[\alpha_1^N\left(1 - \frac{n}{N}\right) - \alpha_0^N\frac{n}{N} - ((1-\theta)\alpha_1 - \theta\alpha_0)\right] + \tau_N(s)\\ =& 2\left(\frac{n}{N} - \theta\right)\left[(\alpha_1^N + \alpha_0^N)\left(-\frac{n}{N}\right) + (\alpha_1 + \alpha_0)\theta + (\alpha_1^N - \alpha_1)\right] + \tau_N(s)\\ =& 2\left(\frac{n}{N} - \theta\right)\left[(\alpha_1^N + \alpha_0^N)\left(\theta - \frac{n}{N}\right) + (\alpha_1 - \alpha_1^N + \alpha_0 - \alpha_0^N)\theta + (\alpha_1^N - \alpha_1)\right] + \tau_N(s)\\ =& -2(\alpha_0^N + \alpha_1^N)\left(\frac{n}{N} - \theta\right)^2 + 2\left(\frac{n}{N} - \theta\right)\left((\alpha_1 - \alpha_1^N + \alpha_0 - \alpha_0^N)\theta + (\alpha_1^N - \alpha_1)\right) + \tau_N(s).\end{aligned}$$

Then

$$\begin{aligned}V_N(t) - \frac{\theta_0(1-\theta_0)}{N} =& -2\mathbb{E}\left[\int_0^t (\alpha_0^N + \alpha_1^N)\left(\frac{n}{N} - \theta\right)^2 ds\right]\\ &+ \mathbb{E}\left[\int_0^t 2\left(\frac{n}{N} - \theta\right)\left((\alpha_1 - \alpha_1^N + \alpha_0 - \alpha_0^N)\theta + (\alpha_1^N - \alpha_1)\right)ds\right]\\ &+ \mathbb{E}\left[\int_0^t \tau_N(s)ds\right].\end{aligned}$$

Now we see that

$$
\begin{aligned}
|\alpha_0 - \alpha_0^N| &= \left|\alpha^*(\bar u - u, \theta, 0) - \alpha^*\left(\bar u_{n-i} - u_{n-i}, \frac{n-i}{N}, 0\right)\right| \\
&< K\left(\left|\theta - \frac{n-i}{N}\right| + |\bar u - \bar u_{n-i}| + |u - u_{n-i}|\right) \\
&< K\left(\left|\theta - \frac{n}{N}\right| + |\bar u - \bar u_n| + |\bar u_{n-i} - \bar u_n| + |u - u_n| + |u_{n-i} - u_n| + \frac{1}{N}\right) \\
&< K\left(\left|\theta - \frac{n}{N}\right| + |\bar u - \bar u_n| + |u - u_n| + \frac{3}{N}\right),
\end{aligned}
$$

where we used that α^* is Lipschitz in both variables, and u and u^N are bounded, and the uniform bounds on $|u_{n+1} - u_n|$ obtained in proposition 54 of §3.2. Similarly

$$
|\alpha_1 - \alpha_1^N| < K\left(\left|\theta - \frac{n}{N}\right| + |\bar u - \bar u_n| + |u - u_n| + \frac{3}{N}\right).
$$

Thus

$$
\begin{aligned}
V_N(t) \leq & K_1 \int_0^t V_N(s) ds + 2\mathbb{E} \int_0^t \left(\frac{n}{N} - \theta\right) K\left(\left|\theta - \frac{n}{N}\right| + |\bar u - \bar u_n| + |u - u_n| + \frac{3}{N}\right) ds + \frac{K_2}{N} \\
\leq & (K_1 + 2K) \int_0^t V_N(s) ds + 2\mathbb{E} \int_0^t \left(\frac{n}{N} - \theta\right) K\left(|\bar u - \bar u_n| + |u - u_n| + \frac{3}{N}\right) ds + \frac{K_2}{N} \\
\leq & (K_1 + 2K) \int_0^t V_N(s) ds + 2K \int_0^t 2V_N(s) + (W_N(s) + \bar W_N(s)) ds + \frac{K_2 + 6T}{N} \\
= & \int_0^t K_3 V_N(s) + 2K Q_N(s) ds + \frac{K_2 + 6T}{N} \\
\leq & \int_0^t C_1 (V_N(s) + Q_N(s)) ds + \frac{C_1}{N}.
\end{aligned}
$$

□

LEMMA 56. *There exists $C_2 > 0$ such that*

$$
Q_N(t) \leq \int_t^T C_2 (V_N(s) + Q_N(s)) ds + \frac{C_2}{N}.
$$

PROOF. In this proof, $u_n(s)$ or simply u_n will denote the expected minimum cost of player $N+1$ conditioned on its state being equal to 0 at time s, i.e., $u_{n(s)}(0, s)$. We will also use, here, $u(s)$ or simply u to denote $u(0, s)$.

Using Dynkin formula (34) with $\varphi(i,n,s) = \left(u_{n(s)}(0,s) - u(0,s)\right)^2$, and equations (41) and (45), we have

$$
\begin{aligned}
&W_N(t) - W_N(T) = -\mathbb{E}[(u_n(t)-u(t))^2] + \mathbb{E}[(u_n(T)-u(T))^2]\\
=&\mathbb{E}\int_t^T 2(u_n-u)\frac{d}{ds}(u_n-u)ds\\
&\quad+\mathbb{E}\int_t^T \gamma_n^+\left[(u_{n+1}-u)^2-(u_n-u)^2\right]+\gamma_n^-\left[(u_{n-1}-u)^2-(u_n-u)^2\right]ds\\
=&\mathbb{E}\int_t^T 2(u_n-u)\left(-\gamma_n^+(u_{n+1}-u_n)-\gamma_n^-(u_{n-1}-u_n)-h\Big(\bar{u}_n-u_n,\frac{n}{N},0\Big)+h(\bar{u}-u,\theta,0)\right)ds\\
&\quad+\mathbb{E}\int_t^T \gamma_n^+\left[(u_{n+1}-u)^2-(u_n-u)^2\right]+\gamma_n^-\left[(u_{n-1}-u)^2-(u_n-u)^2\right]ds\\
=&\mathbb{E}\int_t^T \gamma_n^+(u_{n+1}-u_n)^2+\gamma_n^-(u_{n-1}-u_n)^2-2\left(h\left(\bar{u}_n-u_n,\frac{n}{N},0\right)-h(\bar{u}-u,\theta,0)\right)(u_n-u)ds,
\end{aligned}
$$

where $\gamma_n^\pm = \gamma_n^\pm(0,n(s),s)$. In the last equation we used the fact that

$$-2(u_n-u)\gamma_n^+(u_{n+1}-u_n)+\gamma_n^+\left[(u_{n+1}-u)^2-(u_n-u)^2\right]=\gamma_n^+\left(u_{n+1}-u_n\right)^2\ ,$$

and a similar calculation for γ_n^-.

Now, using results from §3.2, proposition 54, we have that $\gamma_n^+(u_{n+1}-u_n)^2$, $\gamma_n^-(u_{n-1}-u_n)^2$ and $W_N(T)$ are bounded by $\frac{K_5}{N}$, which implies

$$W_N(t) \le \frac{K_6}{N} + 2\mathbb{E}\int_t^T \left(h\left(\bar{u}_n-u_n,\frac{n}{N},0\right)-h\left(\bar{u}-u,\theta,0\right)\right)(u_n-u)ds\,.$$

Using the fact that h is Lipschitz in both variables, we have

$$\left|h\left(\bar{u}_n-u_n,\frac{n}{N},0\right)-h(\bar{u}-u,\theta,0)\right| < K\left(\left|\theta-\frac{n}{N}\right|+|\bar{u}-\bar{u}_n|+|u-u_n|\right).$$

Thus

$$W_N(t) \le \frac{K_6}{N} + K_7\int_t^T V_N(s)+W_N(s)+\bar{W}_N(s)ds\,.$$

With a similar calculation we have a analogous inequality for $\bar{W}_N(t)$, which ends the proof. □

Now we can state and prove our main result that establishes the convergence of the $N+1$ player game to the mean field model as $N\to\infty$.

Theorem 57. *If $\rho = TC < 1$, where $C = \max\{C_1, C_2\}$, and $Q_N(t)+V_N(t)$ is given in (67) then*

$$Q_N(t)+V_N(t) \le \frac{C}{1-\rho}\frac{1}{N} \quad \forall t\in[0,T]\,.$$

Proof. Adding both inequalities given in the two last lemmas, we have

$$Q_N(t) + V_N(t) \leq C \int_0^T (V_N(s) + Q_N(s))ds + \frac{C}{N}.$$

Now suppose $\rho = TC < 1$. Defining

$$Q_N + V_N = \max_{0 \leq t \leq T} Q_N(t) + V_N(t),$$

we have

$$Q_N + V_N \leq \rho(Q_N + V_N) + \frac{C}{N},$$

which proves the theorem. □

4. Bibliographical notes

Mean field games is a recent area of research started by Pierre Louis Lions and Jean Michel Lasry in a series of seminal papers [**LL06a, LL06b, LL07a, LL07b**]. The literature on mean field games and its applications is growing fast, for a recent survey see [**LLG10b**] and reference therein. Applications of mean field games arise in the study of growth theory in economics [**LLG10a**] or environmental policy [**ALT**]. Two references on numerical methods for these problems are [**ALT**], [**AD10**]. The discrete time, finite state problem was studied in [**GMS10**].

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