Sufficient structure conditions for uniqueness of viscosity solutions of semilinear and quasilinear equations

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Abstract

In this article, we introduce a new approach for proving Maximum Principle type results for viscosity solutions of second-order, fully nonlinear possibly degenerate elliptic equations. This approach leads, in particular, to a better understanding of the conditions on the equation which are necessary to obtain such results. It allows us to provide new comparison results for semilinear and quasilinear equations.

1 Introduction

In this work, we are interested in Maximum Principle type results for viscosity solutions of fully nonlinear elliptic equations

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \tag{1.1}$$

where Ω is a bounded domain of $I\!\!R^n$, the solution $u : \overline{\Omega} \to I\!\!R$ is a real-valued function, Du, D^2u denote respectively its gradient and Hessian matrix. The nonlinearity $F(x, u, p, M)$ is a locally Lipschitz continuous function defined on $\Omega \times I\!\!R \times I\!\!R^n \times S^n$ taking values in $I\!\!R$, where S^n is the space of $n \times n$ symmetric matrices; Throughout this paper, the nonlinearities are always assumed to satisfy the ellipticity condition

$$F_M(x, u, p, M) \leq 0 \quad \text{a.e. in } \Omega \times I\!\!R \times I\!\!R^n \times S^n .$$

More specifically, we are looking for structure conditions on F which insure that we have a comparison result between upper semicontinuous (USC) subsolutions and lower semicontinuous (LSC) supersolutions of (1.1). Such results allow to derive, in particular, uniqueness

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properties for (1.1) but also existence ones through the Perron's method introduced in the framework of viscosity solutions by H. Ishii[I]. We put an emphasis in this work on the semilinear and quasilinear cases.

General Maximum Principle type results for viscosity solutions of equations like (1.1) are described, for example, in the "Users'guide to viscosity solutions" of M. Crandall, H. Ishii and P.L. Lions [CIL] (See also H. Ishii and P.L. Lions [IL] and the book of W. Fleming and H.M. Soner [FS]). The main assumptions are the following : first, there exists $\nu > 0$ such that

$$\nu(u - v) \leq F(x, u, p, X) - F(x, v, p, X) \quad \text{for any } u \geq v, \ (x, p, X) \in \Omega \times I\!\!R^n \times S^n \quad (1.2)$$

and, for any $R > 0$, there is a modulus $m_R : [0, \infty] \to [0, \infty]$ that satisfies $m_R(0+) = 0$ such that

$$F(y, u, \alpha(x - y), Y) - F(x, , u, \alpha(x - y), X) \leq m_R(\alpha|x - y|^2 + |x - y|) \quad (1.3)$$

whenever $x, y \in \Omega$, $|u| \leq R, X, Y \in S^n$ satisfying the matricial inequality for $\alpha > 0$.

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (1.4)$$

(See Theorem 3.3 in [CIL]).

Our motivation to do this work comes from two remarks : first, the condition (1.3) is not easily checkable and it would be better to have an hypothesis which is easier to verify. Of course, the difficulty is that X, Y in (1.3) have to satisfy the inequalities (1.4) and it is not completely obvious to see what are all the consequences on X and Y of these inequalities. The second remark is that (1.2) which is an assumption on the behavior of F in r is independent of (1.3) which is (essentially) an assumption on the behavior of F in x. The remark is that one would get better results by connecting them and this is what we are going to do.

The key step for solving the first above mentioned difficulty is to get a precise estimate on $X - Y$ for X, Y satisfying (1.4) : it is known that these inequalities imply at least $X \leq Y$ but we prove below that we have the sharper estimate

$$X - Y \leq -\frac{1}{6\alpha}(tX + (1 - t)Y)^2 \quad \text{for all } t \in [0, 1].$$

This property is then used to derive a rather simple structure condition on F, which ensures that the Maximum Principe holds for (1.1). This structure condition consists in saying that a suitable combination of the derivatives of F has to be positive and this is easily verifiable.

Then, in the same spirit as in [BA] where gradient bounds are studied, we systematically study the effect of changes of variables $u = \phi(v)$ for obtaining more general results in the semilinear and quasilinear cases where F is given by

$$F(x, u, p, X) = -\operatorname{Tr}(A(x, p)X) + H(x, u, p).$$

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In the semilinear case (i.e. when A does not depend on p), we are able to obtain general comparison results for discontinuous solutions by using this strategy. Unfortunately, in the quasilinear case (i.e. when A depends on p), this is not the case in general because there is no way that the equation obtained after a change of variable can satisfy a property like (1.2) and, for this reason, we have to restrict ourselves to Hölder continuous solutions.

This paper is organized as follows : in Section 2, we present a general structure condition on the nonlinearity F which allows to obtain a comparison result in the case when F satisfies (1.2); in particular a proof of the above estimate on $X - Y$ is given. Section 3 is devoted to a systematic study of the changes of variable which can be used to transform an equation which does not satisfy the structure condition of Section 2 into a new one which does it. Finally the quasilinear case is studied in Section 4 which can be seen as following the same strategy described in Section 2 and 3 but in a framework where (1.2) does not hold.

2 A General Structure Condition for the Maximum Principle

We first give a comparison result for equation (1.1) using a structure condition on F.

Theorem 2.1 : Assume that F is a locally Lipschitz continuous function which satisfies (1.2) and assume that, for any $R > 0$, there exists $C_R > 0$ such that

$$F_x(x, u, p, M) \cdot p - \frac{1}{6} F_M(x, u, p, M) M^2 \geq -C_R F_u(x, u, p, M)(1 + |p|^2) \qquad (2.1)$$

for almost all $(x, u, p, M) \in \Omega \times [-R, R] \times {I\!\!R}^n \times S^n$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are subsolution and supersolution of (1.1) respectively with $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\overline{\Omega}$.

As we mentioned it in the Introduction, the key step in the proof of this theorem, is the

Lemma 2.2 : If (1.4) holds for $X, Y \in S^n$, then we have

$$X - Y \leq -\frac{1}{6\alpha}(tX + (1-t)Y)^2 \quad \text{for all } t \in [0, 1]. \qquad (2.2)$$

Proof. In this proof, for the sake of clarity, we denote by $\langle \cdot, \cdot \rangle$ the usual salar product in ${I\!\!R}^n$. From (1.4), we have for any $\xi, \eta \in {I\!\!R}^n$,

$$\left\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \leq 3\alpha \left\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \leq 3\alpha \left\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi & -\eta \\ -\xi & +\eta \end{pmatrix} \right\rangle,$$

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which reduces to

$$\langle X\xi, \xi \rangle - \langle Y\eta, \eta \rangle \le 3\alpha |\xi - \eta|^2. \tag{2.3}$$

We make a special choice for ξ and η :

$$\xi = p, \quad \eta = p - q.$$

With this choice of ξ, η, (2.3) becomes

$$\langle (X - Y)p, p \rangle \le 3\alpha |q|^2 + \langle Yq, q \rangle - 2\langle Yq, p \rangle \tag{2.4}$$

similarly with a choice of $\xi = p + q$ and $\eta = p$, (2.3) becomes

$$\langle (X - Y)p, p \rangle \le 3\alpha |q|^2 - 2\langle Xq, p \rangle - \langle Xq, q \rangle. \tag{2.5}$$

For $\theta \in [0, 1]$, we multiply (2.4) by $(1 - \theta)$ and (2.5) by θ and add both the inequalities to get

$$\langle (X - Y)p, p \rangle \le 3\alpha |q|^2 - 2\langle (\theta X + (1 - \theta)Y)q, p \rangle + \langle (-\theta X + (1 - \theta)Y)q, q \rangle. \tag{2.6}$$

Notice that from the first part of the inequality (1.4) , we have

$$X \ge -3\alpha I \ \text{ and } - Y \ge -3\alpha I$$

Hence we have

$$\langle (-\theta X + (1 - \theta)Y)q, q \rangle \le \langle (3\alpha\theta + (1 - \theta)3\alpha)q, q \rangle \le 3\alpha\langle q, q \rangle.$$

Calling $(\theta X + (1 - \theta)Y) = M_\theta$, we now choose $q = tM_\theta p$, to reduce (2.6) as

$$
\begin{aligned}
\langle (X - Y)p, p \rangle &\le 3\alpha t^2 \langle M_\theta^2 p, p \rangle - 2t\langle M_\theta^2 p, p \rangle + t^2 3\alpha\langle M_\theta^2 p, p \rangle \\
&\le 6\alpha t^2 \langle M_\theta^2 p, p \rangle - 2t\langle M_\theta^2 p, p \rangle
\end{aligned}
$$

Now we choose the optimal $t = \frac{1}{6\alpha}$ and get

$$\langle (X - Y)p, p \rangle \le \frac{1}{6\alpha}\langle M_\theta^2 p, p \rangle - \frac{1}{3\alpha}\langle M_\theta^2 p, p \rangle \le \frac{-1}{3\alpha}\langle M_\theta^2 p, p \rangle,$$

for all $\theta \in [0, 1]$. ∎

Proof of Theorem 2.1. The beginning of the proof follows along the lines of [CIL]. We argue by contradiction assuming that $M := \max_{\overline{\Omega}}(u - v) > 0$ and, for $\alpha > 0$ large, we introduce the function $(x, y) \mapsto u(x) - v(y) - \frac{\alpha}{2}|x - y|^2$. This function is upper semicontinuous on $\overline{\Omega}$ and therefore it attains its maximum at some $(\hat{x}_\alpha, \hat{y}_\alpha) \in \overline{\Omega} \times \overline{\Omega}$. Moreover, since $M > 0$ and $u \le v$ on $\partial\Omega$, then we know that $\hat{x}_\alpha, \hat{y}_\alpha \in \Omega$ for α large enough.

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Since u is a subsolution and v is a supersolution of (1.1), classical arguments lead to the existence of matrices X and Y, satisfying (1.4), with $(\alpha(\hat{x}_\alpha - \hat{y}_\alpha), X) \in \overline{D}^{2,+} u(\hat{x}_\alpha)$ and $(\alpha(\hat{x}_\alpha - \hat{y}_\alpha), Y) \in \overline{D}^{2,-} v(\hat{y}_\alpha)$, such that the following viscosity inequalities hold

$$F(\hat{x}_\alpha, u(\hat{x}_\alpha), \alpha(\hat{x}_\alpha - \hat{y}_\alpha), X) \le 0 \,,$$

$$F(\hat{y}_\alpha, v(\hat{y}_\alpha), \alpha(\hat{x}_\alpha - \hat{y}_\alpha), Y) \ge 0 \,.$$

We refer the reader to the proof of Theorem 3.2 in [CIL] for a justification of this claim.

From now, we assume for simplicity that F is a C^1 function, the case of locally Lipschitz functions F can be treated by tedious but straightforward arguments.

Dropping the α's for the sake of notational simplicity, we can write

$$
\begin{aligned}
0 \;\ge\; & F(\hat{x}, u(\hat{x}), \alpha(\hat{x} - \hat{y}), X) - F(\hat{y}, v(\hat{y}), \alpha(\hat{x} - \hat{y}), Y) \\
= \;& \int_0^1 \frac{d}{dt} \left\{ F(t\hat{x} + (1-t)\hat{y}, tu(\hat{x}) + (1-t)v(\hat{y}), \alpha(\hat{x} - \hat{y}), tX + (1-t)Y \right\} dt \\
= \;& \int_0^1 \left\{ F_x(z, w, p, M)(\hat{x} - \hat{y}) + F_u(z, w, p, M)(u(\hat{x}) - v(\hat{y})) + F_M(z, w, p, M)(X - Y) \right\} dt
\end{aligned}
$$

where $z = t\hat{x} + (1-t)\hat{y}$, $w = tu(\hat{x}) + (1-t)v(\hat{y})$, $p = \alpha(\hat{x} - \hat{y})$ and $M = tX + (1-t)Y$. Using the lemma and the fact that $F_M \le 0$, we get

$$F_M(z, w, p, M)(X - Y) \ge \frac{-1}{6\,\alpha} F_M M^2.$$

¿From this and (1.2), it follows that

$$0 \ge \int_0^1 (F_u M_\alpha + F_x(\hat{x} - \hat{y}) - \frac{1}{6\alpha} F_M M^2) dt = \int_0^1 (F_u M_\alpha + \frac{1}{\alpha}(F_x \cdot p - \frac{1}{6} F_M M^2)) dt. \quad (2.7)$$

¿From [CIL], (see Lemma 3.1 there), we know that $\alpha|\hat{x}_\alpha - \hat{y}_\alpha|^2 \to 0$ as $\alpha \to +\infty$. Hence

$$\frac{|p|^2}{\alpha} \to 0 \quad \text{and} \quad \frac{1 + |p|^2}{\alpha} \to 0 \text{ as } \alpha \to +\infty.$$

Thus (2.7) reduces in view of our assumption (2.1), to

$$0 \ge \int_0^1 F_u \left(M_\alpha - C_R \frac{(1 + |p|^2)}{\alpha} \right) dt.$$

But by (1.2), $F_u \ge \nu > 0$ and since $M_\alpha \to M > 0$ as $\alpha \to +\infty$, this leads to a contradiction.

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Example 2.3 We consider the model equation,

$$-\mathrm{Tr}(A(x, Du)D^2u) + H(x, u, Du) = 0$$

and we check when (2.1) holds. Here $A(x, p)$ is a nonnegative matrix with

$$A(x, p) = \sigma(x, p)\sigma(x, p)^T$$

Then we have

$$
\begin{aligned}
F_x \cdot p - \frac{1}{6}F_M M^2 &= -\mathrm{Tr}(A_x \cdot pM) + \frac{1}{6}\mathrm{Tr}(AM^2) + H_x \cdot p \\
&= -\mathrm{Tr}(\sigma_x \cdot p\sigma^T M) - \mathrm{Tr}(\sigma\sigma_x^T \cdot pM) + \frac{1}{6}\mathrm{Tr}(AM^2) + H_x \cdot p
\end{aligned}
$$

Thanks to Cauchy-Schwarz inequality, we have for any matrices A and B,

$$\mathrm{Tr}(AB) \le \frac{\theta}{2}\mathrm{Tr}(AA^T) + \frac{1}{2\theta}\mathrm{Tr}(BB^T)$$

Using this, we obtain

$$
\begin{aligned}
\mathrm{Tr}(\sigma_x \cdot p\sigma^T M) &\le \frac{\theta}{2} \cdot \mathrm{Tr}((\sigma_x \cdot p)(\sigma_x \cdot p)^T) + \frac{1}{2\theta}\mathrm{Tr}(\sigma^T M(\sigma^T M)^T), \\
\mathrm{Tr}(\sigma\sigma_x^T \cdot pM) &\le \frac{\theta}{2}\mathrm{Tr}((\sigma_x^T \cdot p)(\sigma_x^T \cdot p)^T) + \frac{1}{2\theta}\mathrm{Tr}(M\sigma(M\sigma)^T).
\end{aligned}
$$

Since M is symmetric,

$$\mathrm{Tr}(\sigma^T M(\sigma^T M)^T) = \mathrm{Tr}(M\sigma(M\sigma)^T) = \mathrm{Tr}(\sigma\sigma^T \cdot M^2) = \mathrm{Tr}(AM^2).$$

Further

$$\mathrm{Tr}(\sigma_x \cdot p)(\sigma_x \cdot p)^T) = \mathrm{Tr}((\sigma_x)^T \cdot p)((\sigma_x)^T \cdot p)^T) \le (\|\sigma_x\|)^2|p|^2$$

Here

$$\|\sigma_x\|^2 = \sum_i \sum_j |\nabla(\sigma_{i,j})|^2$$

If we choose $\theta = 6$, then

$$H_x \cdot p - \mathrm{Tr}(A_x \cdot pM) + \frac{1}{6}\mathrm{Tr}(AM^2) \ge -C\left\{\mathrm{Tr}(\sigma_x \cdot p(\sigma_x \cdot p)^T) + \mathrm{Tr}((\sigma_x^T \cdot p)(\sigma_x^T \cdot p)^T)\right\}$$

Hence condition (2.1) holds for this model equation if

$$
\begin{aligned}
\|\sigma_x(x, p)\|^2 &\le C_R H_u(x, u, p), \\
|H_x(x, u, p)| &\le C_R H_u(x, u, p)(1 + |p|)
\end{aligned}
\tag{2.8}
$$

for all $(x, p) \in \Omega \times \mathbb{R}^n$ and $|u| \le R$.

In particular, for the semilinear case i.e. when A does not depend on p, and when H_u is expected to be bounded, we recover the "natural" conditions under which the Maximum Principle is known to hold, namely $A = \sigma\sigma^T$ with σ Lipschitz continuous and H satisfying $H_u(x, u, p) \geq \nu$ and $|H_x(x, u, p)| \leq \tilde{C}_R(1 + |p|)$ for some constant $\tilde{C}_R$.

Our approach provides new results in the case when, typically, H_u depends on p. For example, the above conditions show that the Maximum Principle holds when H is given by

$$H(x, u, p) = g(x, u)|p|^k + u - f(x),$$

where $k \geq 1$ and g, f are locally Lipschitz continuous function with $g_u(x, u) \geq \eta > 0$. If $k > 1$, this example does not enter into the classical framework.

3 Semilinear Elliptic equations

In this section, we address the following question in the case of semilinear elliptic equation : if we have an equation which does not satisfy the structure condition (2.1), does there exist a change of variable $u = \phi(v)$ such that we can apply Theorem 2.1 to the transformed equation?

We consider the equation

$$F(x, u, Du, D^2u) = -\mathrm{Tr}(A(x)D^2u) + B(x, u, Du) = 0. \tag{3.1}$$

where

$$A(x) = \sigma(x)\sigma(x)^T \geq 0 \ \text{ and } B_u > \gamma > 0. \tag{3.2}$$

In the example at the end of Section 2, we give conditions on σ and B such that we can compare an upper semicontinuous subsolutions and lower semicountinuous supersolutions of (3.1). We are going to show that by an appropriate change of variable, we can have the comparison under much weaker conditions on B.

To do so, we consider the change of function $u = \phi(v)$. The transformed equation is given by $G = 0$ where

$$
\begin{aligned}
G(x, v, Dv, D^2v) &= \frac{1}{\phi'(v)}\left\{F(x, \phi(v), \phi'(v)Dv, D^2(v)\phi'(v) + \phi''(v)Dv \otimes Dv)\right\}, \\
&= \frac{1}{\phi'(v)}\left\{-\mathrm{Tr}(A(x)(\phi'(v)D^2v + \phi''(v)Dv \otimes Dv) + B(x, \phi(v), \phi'(v)Dv).\right\}.
\end{aligned}
$$

In the sequel, we use the following notations : q is the variable corresponding to Du, p for Dv, M for D^2u and N for D^2v. With these notations, we have

$$G(x, v, p, N) = -\mathrm{Tr}(A(x)N) - \frac{\phi''(v)}{\phi'(v)}\mathrm{Tr}(A(x)(p \otimes p)) + \frac{1}{\phi'(v)}B(x, \phi(v), \phi'(v)p) \tag{3.3}$$

We are going to compute the derivatives of G w.r.t v, p and N and, in order to choose the change of variable, we are going to transform them back in the old variables (u, q, M) as in [BA].

We first consider G_v.

$$
\begin{aligned}
G_v(x, v, p, N) &= \frac{1}{\phi'(v)} \left[B_u \phi' + B_q p \phi''(v) - \mathrm{Tr}(A(x)(\phi'''(v)p \otimes p)) \right] \\
&\quad - \frac{\phi''(v)}{(\phi'(v))^2} \left[-\mathrm{Tr}(A(x)(\phi''(v)p \otimes p + B] \right. \\
&= B_u + \frac{\phi''(v)}{(\phi'(v))^2}(B_q q - B) - \mathrm{Tr}(A(x)q \otimes q)\left(\frac{\phi'''(v)\phi'(v) - (\phi''(v))^2}{\phi'(v)^4} \right) \\
&= B_u + \frac{\phi''(v)}{(\phi'(v))^2}(B_q q - B) - \frac{|\sigma(x)q|^2}{(\phi'(v))^2}\left\{ \frac{\phi'''(v)\phi'(v) - \phi''(v)^2}{(\phi'(v))^2} \right\}. \quad (3.4)
\end{aligned}
$$

Now we look for a function $w(u)$ which will transform the coefficients, which are nonlinear function of derivatives of $\phi(v)$, into ones which are linear in the derivatives of $w(u)$ so that $w(u)$ can be chosen easily according to our convenience. Let

$$
w(u) = \phi'(v).
$$

Then one can check that

$$
\begin{aligned}
w'(u) &= \frac{\phi''(v)}{\phi'(v)}, \\
w''(u) &= \frac{\phi'(v)\phi'''(v) - (\phi''(v))^2}{(\phi'(v))^3}
\end{aligned}
$$

Hence now we have

$$
G_v(x, v, p, N) = B_u + \frac{w'(u)}{w(u)}(B_q q - B) - \frac{w''(u)}{w(u)}|\sigma(x)q|^2. \quad (3.5)
$$

The idea is first to look for a function $w(u) > 0$ so that $G_v > 0$ and then to look at the other terms in condition (2.1). For this, we need to consider 2 cases separately and choose w differently in each case:
(i) The matrix $A(x)$ is uniformly elliptic refered below as the "subquadratic case",
(ii) The matrix $A(x)$ is only non-negative refered below as the "superquadratic case".

The Subquadratic Case. In this case we choose $w''(u) < 0$ so that the positivity of $|\sigma(x)q|^2$ can be used to get a positive sign for G_v.

Assume that, for any $R > 0$, there exist $C_1, C_{2,R} > 0$, satisfying

$$
\begin{aligned}
(i) &\quad \sigma(x)\sigma^T(x) \geq C_1 I \quad \text{for all } x \in \mathbb{R}^n, \\
(ii) &\quad (B_q \cdot q - B) \geq -C_{2,R}(1 + |q|^2), \quad\quad\quad\quad (3.6)
\end{aligned}
$$

for almost all $|u| \leq R, (X,p) \in S^n \times \mathbb{R}^n$. The condition (ii) in (3.6) explains the terminology "subquadratic case".

Let us choose

$$\hat{u} = (u + ||u||_\infty + 1), \quad w(u) := 1 - e^{-k\hat{u}},$$

for a suitable k to be chosen more precisely later on. Then

$$\begin{aligned}
G_v\phi'(v) &= B_u(1 - e^{-k\hat{u}}) + ke^{-k\hat{u}}(B_q q - B) + |\sigma(x)q|^2 k^2 e^{-k\hat{u}} \\
&\geq \gamma(1 - e^{-k\hat{u}}) + ke^{-k\hat{u}}(k|\sigma(x)q|^2 + (B_q q - B)) \\
&= \gamma(1 - e^{-k\hat{u}}) - C_{2,R}ke^{-k\hat{u}} + ke^{-k\hat{u}}(kC_1 - C_{2,R})|q|^2.
\end{aligned}$$

If

$$k > \frac{C_{2,R}}{C_1} \quad \text{and} \quad \frac{ke^{-k\hat{u}}}{1 - e^{-k\hat{u}}} < \frac{\gamma}{C_{2,R}},$$

then $G_v \geq 0$. This will hold, if we choose k large, satisfying

$$k > \frac{C_{2,R}}{C_1} \quad \text{and} \quad \frac{ke^{-k}}{1 - e^{-k}} < \frac{\gamma}{C_{2,R}},$$

since $\hat{u} \geq 1$. Then it follows that, for this transformation $\phi(v) = u$,

$$\begin{aligned}
G_v(x,v,p,N)\phi'(v) \geq \tilde{C}_R(1 + |q|^2) &= \tilde{C}_R(1 + w^2|p|^2) \\
&\geq \hat{C}_R(1 + |p|^2)
\end{aligned}$$

Remark : We point out that, actually, the change of variable may depend on R since we compare bounded solutions.

The Superquadratic Case. Now we assume that, for any $R > 0$, there exists constants $C_{1,R}, C_{2,R} > 0$ such that

$$\begin{aligned}
&(i) \ \sigma(x)\sigma^T(x) \geq 0 \ \text{for all } x \in \mathbb{R}^n \\
&(ii) \ B_q \cdot q - B \geq C_{1,R}|q|^2 - C_{2,R}
\end{aligned} \tag{3.7}$$

for almost all $(q,M) \in \mathbb{R}^n \times S^n$ and $|u| \leq R$. The condition (3.7)(ii) explains the terminology "superquadratic case".

Let us choose again $w(u) := 1 - e^{-k\hat{u}}$. where $\hat{u} = (u + ||u||_\infty + 1)$. Then we have

$$\begin{aligned}
G_v \cdot \phi'(v) &= B_u(1 - e^{-k\hat{u}}) + ke^{-k\hat{u}}(B_q q - B) + |\sigma(x)q|^2 k^2 e^{-k\hat{u}} \\
&\geq \gamma(1 - e^{-k\hat{u}}) + ke^{-k\hat{u}}(k|\sigma(x)q|^2 + (B_q q - B)) \\
&= \gamma(1 - e^{-k\hat{u}}) + ke^{-k\hat{u}}(C_{1,R}|q|^2 - C_{2,R})
\end{aligned}$$

Now choose k such that

$$\frac{ke^{-k}}{(1 - e^{-k})} < \frac{\gamma}{C_{2,R}}.$$

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Then

$$G_v \cdot \phi'(v) \geq \tilde{C}_R(1+|q|^2) \geq \hat{C}_R(1+|p|^2) \quad \text{for all } (p,N) \in I\!\!R^n \times S^n \text{ and } \text{ for all } |u| \leq R.$$

Thus, in either of the cases, equation (3.1) under the assumptions (3.6) or (3.7) can be transformed to the equation (3.3) satisfying

$$G_v(x,v,p,N) \geq \hat{C}_R(1+|p|^2) \tag{3.8}$$

for all $(x,p,N) \in \Omega \times I\!\!R^n \times S^n$ and $|v| \leq R$.

Because of this property and (2.1), we observe that a subsolution v_1 and a supersolution v_2 of equation (3.3) can be compared under the further assumption

$$G_x \cdot p - \frac{1}{6}G_N \cdot N^2 \geq -C_R(1+|p|^4) \tag{3.9}$$

for all $(x,p,N) \in \Omega \times I\!\!R^n \times S^n$ and $|v| \leq R$. This is because the assumption (2.1) follows from (3.8) and (3.9) and the comparison holds by Theorem 2.1.

It remains to calculate the conditions on F which will lead to (3.9) for the transformed equation (3.3). We have

$$G_x \cdot p - \frac{1}{6}G_N \cdot N^2 = \frac{1}{\phi'(v)}[-\text{Tr}((A_x \cdot p)M) + B_x \cdot p] + \frac{1}{6}\text{Tr}(A(x)N^2) \tag{3.10}$$

Using the expression for N we deduce that

$$
\begin{aligned}
N^2 &= \left(\frac{M - p \otimes p\phi''(v)}{\phi'(v)}\right)^2 = \frac{1}{(\phi'(v))^2}(M - \frac{q \otimes q\phi''(v)}{(\phi'(v))^2})^2 \\
&= \frac{1}{(\phi'(v))^2}\left(M^2 + (q \otimes q)^2\frac{(\phi''(v))^2}{(\phi'(v))^4} - M\frac{(q \otimes q)\phi''(v)}{(\phi'(v))^2} - \frac{q \otimes qM\phi''(v)}{(\phi'(v))^2}\right) \\
&= \frac{1}{(\phi'(v))^2}[M^2 + \left(\frac{\phi''(v)}{\phi'(v)}\right)^2\frac{(q \otimes q)|q|^2}{(\phi'(v))^2} - \frac{\phi''(v)}{(\phi'(v))^2}[M(q \otimes q) + (q \otimes q)M],
\end{aligned}
$$

since $(q \otimes q)^2 = (q \otimes q)|q|^2$. Substituting this in (3.10),

$$
\begin{aligned}
G_x \cdot p - \frac{1}{6}G_N \cdot N^2 &= \frac{1}{\phi'(v)^2}\{-\text{Tr}((A_x \cdot q)M) + B_x \cdot q\} \\
&\quad + \frac{1}{6\phi'(v)^2}\text{Tr}\left\{\left(A(x)\left(M^2 + \left(\frac{\phi''(v)}{\phi'(v)}\right)^2\frac{q \otimes q|q|^2}{(\phi'(v))^2} - \frac{\phi''(v)}{\phi'(v)^2}(Mq \otimes q + q \otimes qM)\right)\right\} \\
&= \frac{1}{w^2}\{-\text{Tr}(A_x \cdot qM) + B_x \cdot q\} \\
&\quad + \frac{1}{6w^2}\text{Tr}\{A(x)[M^2 + \frac{(w')^2}{w^2}(q \times q)|q|^2 - \frac{w'}{w}(Mq \otimes q + q \otimes qM)]\}
\end{aligned}
$$

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Now using the expression for w,

$$
\begin{aligned}
G_x \cdot p - \frac{1}{6} G_N \cdot N^2 \; = \;\; & \frac{1}{w^2} \Big(-\text{Tr}(A_x \cdot qM) + \frac{1}{6}\text{Tr}(AM^2) + \frac{k^2 e^{-2k\hat{u}}}{6w^2}|q|^2 \text{Tr}(A(x)q \otimes q) \\
& - \frac{ke^{-k\hat{u}}}{6w}\text{Tr}(A(Mq \otimes q + q \otimes qM)) + B_x \cdot q \Big)
\end{aligned}
\tag{3.11}
$$

Notice that by using Cauchy-Schwartz inequality

$$
\begin{aligned}
\text{Tr}(A_x \cdot qM) \; = \;\; & \text{Tr}(\sigma_x \cdot q\sigma^T M) + \text{Tr}(\sigma\sigma_x^T qM) \\
\leq \;\; & 2\theta\text{Tr}(\sigma_x \cdot q)(\sigma_x \cdot q)^T) + \frac{1}{2\theta}\text{Tr}(\sigma^T M)\sigma^T M)^T) \\
& + 2\theta\text{Tr}((\sigma_x)^T \cdot q)((\sigma_x)^T \cdot q)^T) + \frac{1}{2\theta}\text{Tr}(M\sigma)(M\sigma)^T) \\
\leq \;\; & 4\theta(\|\sigma_x\|)^2 |q|^2 + \frac{1}{\theta}\text{Tr}(AM^2)
\end{aligned}
$$

for some suitable $\theta > 0$. Here

$$
\|\sigma_x\|^2 = \sum_i \sum_j |\nabla(\sigma_{i,j})|^2
$$

Similarly we get

$$
\begin{aligned}
-C\text{Tr}(AMq \otimes q) + \frac{1}{\theta}\text{Tr}(AM^2) \; \geq \; & -\theta C^2 \text{Tr}\{(q \otimes q\sigma)(q \otimes q\sigma)^T\} \\
-C\text{Tr}(Aq \otimes qM) + \frac{1}{\theta}\text{Tr}(AM^2) \; \geq \; & -\theta C^2 \text{Tr}\{(q \otimes q\sigma^T)(q \otimes q\sigma^T)^T\}
\end{aligned}
$$

Notice that

$$
\text{Tr}\{(q \otimes q\sigma)(q \otimes q\sigma)^T\} = \text{Tr}\{(q \otimes q\sigma^T)(q \otimes q\sigma^T)^T\} = (|\sigma(x)q|^2)|q|^2.
$$

If we choose θ such that

$$
\frac{3}{\theta} \leq \frac{1}{6} \quad \text{i.e.} \quad \theta \geq 18
$$

then

$$
\begin{aligned}
G_x \cdot p - \frac{1}{6}G_N \cdot N^2 \; \geq \; & \frac{1}{w^2}\Big\{ -4\theta(\|\sigma_x\|)^2|q|^2 + B_x \cdot q \\
& + \; |q|^2(|\sigma(x)q|^2)(-2\theta C_1 + C_2)\Big\} \\
\geq \; & -C_R(1 + |q|^4)
\end{aligned}
$$

provided A is Lipschitz continuous in x and B satisfies

$$
B_x(x, u, q) \cdot q \geq -\tilde{C}_R(1 + |q|^4) ,
\tag{3.12}
$$

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for all $x \in \Omega$, $|u| \leq R$ and $q \in I\!\!R^n$ in both the cases.

Thus we have proved the

Theorem 3.1 : Assume there exists a Lipschitz continuous function σ such that $A(x) = \sigma(x)\sigma(x)^T$ and that B is a locally Lipschitz continuous function satisfying (3.12) and

$$B_u(x, u, p) > \gamma_R > 0 \quad \text{a.e. for } |u| \leq R, \ (x, p) \in \Omega \times I\!\!R^n. \tag{3.13}$$

Further assume that either one set of the following conditions, (3.14) or (3.15) hold:

there exists constants $C_1, C_{2,R} > 0$ such that,

$$
\begin{aligned}
(i) \quad & \sigma(x)\sigma(x)^T \ \geq \ C_1 I \quad \text{for all } x \in \Omega, \\
(ii) & (B_q \cdot q - B) \ \geq \ -C_{2,R}(1 + |q|^2) \quad \text{a.e. for } |u| \leq R, \ (x, p) \in \Omega \times I\!\!R^n
\end{aligned}
\tag{3.14}
$$

or there exists constants $C_{1,R}, C_{2,R} > 0$ such that,

$$B_q \cdot q - B \geq C_{1,R}|q|^2 - C_{2,R} \quad \text{a.e. for } |u| \leq R, \ (x, p) \in \Omega \times I\!\!R^n. \tag{3.15}$$

If $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ are respectively sub and supersolutions of the equation

$$-\text{Tr}(A(x)D^2u) + B(x, u, Du) = 0 \quad \text{in } \Omega \ ,$$

with $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\overline{\Omega}$.

Example 3.2 As concrete examples of equations satisfying the conditions of Theorem 3.1, we have

$$-\text{Tr}(A(x)D^2u) + b(x)|Du|^\alpha + u = f(x) \quad \text{in } \Omega \ ,$$

where $A = \sigma\sigma^T$, σ, b, f being Lipschitz continuous functions. Condition (3.14) is satisfied if $\sigma \geq \nu I$ in Ω and if $1 \leq \alpha \leq 2$, while (3.15) holds if $\alpha \geq 2$.

4 Quasilinear Equations

We consider in this section quasilinear equation of the following form

$$-\text{Tr}(A(x, Du)D^2u) + B(x, u, Du) = 0 \tag{4.1}$$

The key difference with the previous section is that, after a change of variable $u = \phi(v)$, we have no hope to obtain a nonlinearity which is increasing in v. We are anyway going to follow the same type of ideas as in Section 2 and 3 but in order to obtain our results, we will be obliged to restrict our study to Hölder continuous solutions.

We assume again that

$$A(x, p) = \sigma(x, p)\sigma(x, p)^T \geq 0 \text{ in } \Omega \times \mathbb{R}^n \text{ and } B_u > \nu > 0 \text{ a.e. in } \Omega \times \mathbb{R} \times \mathbb{R}^n. \qquad (4.2)$$

To follow the same program, we start by an analogue of Theorem 2.1 but we drop the condition "$F_u > 0$".

Theorem 4.1: Let G be a continuous function satisfying the ellipticity condition and the following structure conditions:

(i) $G_v(x, v, p, N) = G_v^1(x, v, p, N) + G_v^2(x, v, p, N)$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n$ with

$$G_v^1(x, v, p, N) > 0 \quad \text{for all } |v| \leq R \text{ and } (x, p, N) \in \Omega \times \mathbb{R}^n \times S^n \, ;$$

(ii) $|G_v^2(x, v, p, N)| \leq \chi(p)\sqrt{-G_N N^2 + G_v^1 |p|^2}$ for some continuous function $\chi : \mathbb{R}^n \to \mathbb{R}$,

(iii) for every $0 < \nu < \frac{1}{6}$, there exists $C_\nu < 1/2$, such that

$$G_x \cdot p - \nu G_N . N^2 \geq -C_\nu G_v^1 (1 + |p|^2).$$

Let $v_1, v_2 \in C^{0,\delta}(\Omega)$ with $\delta > 1/2$ satisfy in the viscosity sense, for some $\varepsilon > 0$,

$$\begin{aligned} G(x, v_1, Dv_1, D^2v_1) &\leq -\varepsilon & \text{in } \Omega, \\ G(x, v_2, Dv_2, D^2v_2) &\geq 0 & \text{in } \Omega, \end{aligned}$$

and we assume in addition that, either $\delta = 1$ or the function χ satisfies : for any constant K, we have for $k_2 := (2\delta - 1)^{-1}$

$$G_v^1(x, v, p, N)|p|^2 - K|p|^{2\delta k_2} \chi^{2k_2}(p) \geq o(|p|^{\frac{2-\delta}{1-\delta}}) \, ,$$

uniformly for $x \in \overline{\Omega}$, v bounded and $N \in S^n$. Then, we cannot have at the same time $\max_{\overline{\Omega}}(v_1 - v_2) = 0$ and $\max_{\partial \Omega}(v_1 - v_2) < 0$.

Proof: We argue by contradiction following exactly the proof of Theorem 2.1. In particular, we consider the test-function $(x, y) \mapsto v_1(x) - v_2(y) - \frac{\alpha}{2}|x - y|^2$. For $\alpha > 0$ large enough, the maximum points of this function are in $\Omega \times \Omega$ and we get inequalities similar to (2.7), namely

$$-\varepsilon \geq \int_0^1 \{G_v(v_1 - v_2) + \frac{1}{\alpha}(G_x \cdot p - \frac{1}{6}G_N . N^2)\}dt.$$

Using the assumptions, we get for some $\nu < \frac{1}{6}$,

$$-\varepsilon \geq \int_0^1 \left\{ G_v^1(v_1 - v_2) + G_v^2(v_1 - v_2) - \left(\frac{1}{6} - \nu\right)\frac{G_N N^2}{\alpha} - \frac{C_\nu}{\alpha}G_v^1(1 + |p|^2) \right\} dt$$

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Since v_1 or v_2 is in $C^{0,\delta}(\Omega)$ and $\max\limits_{\overline{\Omega}}(v_1 - v_2) = 0$, we have

$$\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 \le v_1(\hat{x}) - v_2(\hat{y}) \le C|\hat{x} - \hat{y}|^\delta \tag{4.3}$$

Using this in the above inequality,

$$
\begin{aligned}
-\varepsilon \quad \ge \quad & \int_0^1 \Big\{ G_v^1 \tfrac{\alpha}{2}|\hat{x} - \hat{y}|^2 - C|\hat{x} - \hat{y}|^\delta \chi(p)\sqrt{-G_N.N^2 + G_v^1|p|^2} \\
& - \left(\tfrac{1}{6} - \nu\right)\tfrac{G_N.N^2}{\alpha} - \tfrac{C_\nu}{\alpha}G_v^1(1 + |p|^2) \Big\}\, dt \\
\ge \quad & \int_0^1 \Big\{ G_v^1 \tfrac{|p|^2}{2\alpha} - \tfrac{C\alpha\theta}{2}|\hat{x} - \hat{y}|^{2\delta}\chi^2(p) - \tfrac{1}{2\alpha\theta}(-G_N.N^2 + G_v^1|p|^2) \\
& - \tfrac{1}{\alpha}\left(\tfrac{1}{6} - \nu\right)G_N.N^2 - \tfrac{C_\nu}{\alpha}G_v^1(1 + |p|^2) \Big\}\, dt
\end{aligned}
\tag{4.4}
$$

by the use of Cauchy-Schwarz inequality. Let us choose $\nu < \tfrac{1}{6}$ and θ large, such that

$$\left(\frac{1}{6} - \nu\right) > \frac{1}{2\theta} \quad \text{and} \quad \left(\frac{1}{2} - C_\gamma\right) > \frac{1}{2\theta}. \tag{4.5}$$

We deduce from (4.4)

$$-\varepsilon \ge \int_0^1 \Big\{ G_v^1\big(\tfrac{|p|^2}{\alpha}(\tfrac{1}{2} - C_\nu - \tfrac{1}{2\theta}) - \tfrac{C_\nu}{\alpha}\big) - \tfrac{C\alpha\theta}{2}|\hat{x} - \hat{y}|^{2\delta}\chi^2(p) \Big\}\, dt, \tag{4.6}$$

which can be written as

$$-\varepsilon \ge \int_0^1 \frac{1}{\alpha}\Big\{ G_v^1\big(|p|^2(\tfrac{1}{2} - C_\nu - \tfrac{1}{2\theta}) - C_\nu\big) - \tfrac{C\theta}{2}\alpha^{2(1-\delta)}|p|^{2\delta}\chi^2(p) \Big\}\, dt. \tag{4.7}$$

This inequality leads to a contradiction if $|p|$ remains bounded since each term in the bracket is either bounded or a $o(\alpha)$. This is the case in particular if $\delta = 1$. Therefore we can assume without loss of generality that $|p|$ tends to $+\infty$.

Then we introduce $k_1 := [2(1 - \delta)]^{-1}$; since $\delta > 1/2$, $k_1, k_2 > 1$ and $k_1^{-1} + k_2^{-1} = 1$. Using Young's inequality in the above integral, we deduce that, for some constant K, we have

$$-\varepsilon \ge \int_0^1 \alpha^{-1} \Big\{ G_v^1\big(|p|^2(\tfrac{1}{2} - C_\nu - \tfrac{1}{2\theta}) - C_\nu\big) - K|p|^{2\delta k_2}\chi^{2k_2}(p) - \tfrac{1}{2}\alpha\varepsilon \Big\}\, dt;,$$

and therefore

$$-\frac{1}{2}\varepsilon \ge \int_0^1 \alpha^{-1} \Big\{ G_v^1\big(|p|^2(\tfrac{1}{2} - C_\nu - \tfrac{1}{2\theta}) - C_\nu\big) - K|p|^{2\delta k_2}\chi^{2k_2}(p) \Big\}\, dt;.$$

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Using the assumption on χ, this leads to the final inequality

$$-\frac{1}{2}\varepsilon \geq \int_0^1 \alpha^{-1} o(|p|^{\frac{2-\delta}{1-\delta}}) dt . \tag{4.8}$$

On the other hand, it follows from (4.3) that

$$|p|^{2-\delta} \leq C\alpha^{1-\delta}$$

so that we have $\dfrac{o(|p|^{\frac{2-\delta}{1-\delta}})}{\alpha} \to 0$ as $\alpha \to \infty$ since we know that $|p| \to +\infty$.

Using this in (4.8), we are lead to a contradiction for α small. ∎

Now we proceed by looking for a transformation $\phi(v) = u$ as before, so that equation (4.1) is transformed to

$$
\begin{aligned}
G(x, v, Dv, D^2 v) &= G(x, v, p, N) \\
&= \frac{1}{\phi'(v)}\{-\text{Tr}\frac{(A(x, \phi'(v)p)M)}{(\phi'(v))^{2m}}\} + \frac{1}{(\phi'(v))^{2m+1}} B(x, v, \phi'(v)p), \quad (4.9)
\end{aligned}
$$

where the old variables (u, q, M) and the new variables (v, p, N) are defined as in Section 2. Here $\frac{1}{(\phi')^{2m}}$ is introduced to take care of possible homogeneity in A. Then we have

$$
\begin{aligned}
G(x, v, p, N) &= \frac{1}{\phi'(v)}\{-\text{Tr}\frac{(A(x, \phi'(v)p)\ N)}{(\phi'(v))^{2m}}\} - \frac{\phi''(v)}{\phi'(v)}\frac{\text{Tr}(A(x, \phi'(v)p))(p \otimes p))}{(\phi'(v))^{2m}} \\
&\quad + \frac{1}{\phi'(v)^{2m+1}} B(x, \phi(v), \phi'(v)p) \quad (4.10)
\end{aligned}
$$

If we calculate G_v as in (3.4) and transform it back to the old variables (u, q, M),

$$
\begin{aligned}
G_v(x, v, p, N) &= \frac{1}{(\phi')^{2m}}[B_u + \frac{\phi''(v)}{(\phi'(v))^2}(B_q q - (2m+1)B) \\
&\quad - \text{Tr}(A(x)q \otimes q)\left(\frac{\phi'''(v)\phi'(v) - (\phi''(v))^2}{\phi'(v)^4}\right) \\
&\quad - \frac{\phi''(v)}{(\phi'(v))^2}\text{Tr}(M(A_q q - 2mA))] \quad (4.11)
\end{aligned}
$$

Choosing as before $w(u) = \phi'(v)$, we get

$$
\begin{aligned}
G_v(x, v, p, N) &= \frac{1}{w^{2m}}\left(B_u + \frac{w'(u)}{w(u)}(B_q q - (2m+1)B) - \frac{w''(u)}{w(u)}|\sigma(x)q|^2 \right. \\
&\quad \left. - \frac{w'(u)}{w(u)}\text{Tr}(M(A_q \cdot q - 2mA))\right) \quad (4.12)
\end{aligned}
$$

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and $(G_x \cdot p - \frac{1}{6} G_N . N^2)$, the same as in (3.10) except for the factor $\frac{1}{(\phi')^{2m}}$. Let us define

$$G_v^1(x, v, p, N) = \frac{1}{w^{2m}} \left(B_u + \frac{w'}{w}(B_q . q - (2m+1)B) - \frac{w''}{w} \mathrm{Tr}(Aq \otimes q) \right)$$

$$G_v^2(x, v, p, N) = G_v(x, v, p, N) - G_v^1(x, v, p, N)$$

$$= -\frac{w'}{w^{2m+1}} \mathrm{Tr}(M(A_q \cdot q - 2mA))$$

Now the function $w(u) = 1 - e^{-k(u + \|u\|_\infty + 1)}$ can be chosen exactly in the same manner as before, so that G_v^1 satisfies condition (3.8) in both subquadratic and superquadratic cases. Since G_v^2 involves M, it has to be bounded suitably using the positive quantities G_v^1 and $(-G_N . N^2)$ in order to be able to use Theorem 4.1 to conclude uniqueness.

More precisely, the result is the following.

Theorem 4.2 : Let A and B in equation (4.1) be locally Lipschitz continuous and

$$A(x, q) = \sigma(x, q)\sigma(x, q)^T \geq 0 \quad \forall (x, q) \in \Omega \times I\!\!R^n.$$

Assume that

(i) $B_u(x, u, p) > \gamma > 0$ a.e. for $|u| \leq R, (x, p) \in \Omega \times I\!\!R^n$.

(ii) $B_x \cdot q \geq -\beta(1 + |q|^4)$ and $\|\sigma_x\| \leq \beta(1 + |q|^2)$ a.e. for $|u| \leq R, (x, p) \in \Omega \times I\!\!R^n$, with $\beta < e^{-R}\gamma$,

(iii) $\|\sigma_q(x, q) . q - m\sigma(x, q)\| \leq C(1 + |q|)^\eta$ a.e. in $\Omega \times I\!\!R^n$, for some m and $\eta \in I\!\!R$.

Further assume that one of the following set of conditions hold :
For any $R > 0$, there exists $C_1, C_{2,R} > 0$ such that,

(a1) $\sigma(x, q)\sigma(x, q)^T \geq C_1 I$ for all $(x, q) \in \Omega \times I\!\!R^n$,
(a2) $B_q \cdot q - (2m+1)B \geq -C_{2,R}(1 + |q|^2)$ a.e. for $|u| \leq R, (x, q) \in \Omega \times I\!\!R^n$.

or, for any $R > 0$, there exists $C_{1,R}, C_{2,R} > 0$ such that,

(b1) $B_q \cdot q - (2m+1)B \geq C_{1,R}|q|^2 - C_{2,R}$ a.e. for $|u| \leq R, (x, q) \in \Omega \times I\!\!R^n$.

Let $u_1 \in USC(\Omega)$ and $u_2 \in LSC(\Omega)$ be sub and supersolutions of (4.1), at least one of them being in $\in C^{0,\delta}(\Omega)$ with $\delta > 1/2$ and assume in addition that, in (iii), $\eta < \max(3\delta - 2, \frac{3\delta - 2}{2(1 - \delta)})$. Then, if $u_1 \leq u_2$ on $\partial\Omega$, we have

$$u_1 \leq u_2 \quad \text{on } \overline{\Omega}.$$

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Equivalently this result holds for $\delta > 1 - (2\eta + 3)^{-1}$ if $\eta \geq 0$ or for $\delta > \max((\eta + 2)/3, 1/2)$ if $\eta < 0$.

Proof. We argue by contradiction assuming that $\max_{\overline{\Omega}}(u_1 - u_2) = M > 0$. Let us replace u_1 by $u_1 - M$, still denoting it by u_1, so that u_1 and u_2 satisfy $\max_{\overline{\Omega}}(u_1 - u_2) = 0$, $\max_{\partial\Omega}(u_1 - u_2) < 0$ and

$$
\begin{aligned}
F(x, u_1, Du_1, D^2u_1) &\leq -\gamma_R M \quad \text{in } \Omega, \\
F(x, u_2, Du_2, D^2u_2) &\geq 0 \quad \text{in } \Omega.
\end{aligned}
$$

As explained above, we can choose a transformation $u = \phi(v), w(u) = \phi'(v)$ so that equation (4.1) is transformed into

$$
G(x, v, Dv, D^2v) = 0 \quad \text{in } \Omega,
$$

with $G_v = G_v^1 + G_v^2$ and G_v^1 satisfying the condition (i) of Theorem 4.1. In fact we can choose k large to define $w(u) = 1 - e^{-k(u + \|u\|_\infty + 1)}$ in such a way that G_v^1 satisfies condition (3.8), in both sub and super quadratic cases.

Now we check condition (ii). Notice that

$$
-G_N.N^2 = \frac{1}{\phi'(v)^{2m}}\{\text{Tr}(A(x, \phi'(v)p)\ N^2)\},
$$

and hence

$$
-G_N.N^2 = \frac{1}{\phi'(v)^{2m+2}}(\text{Tr}\{AM^2 + (\frac{w'}{w})^2 A(q \otimes q)|q|^2 - \frac{w'}{w}A(M(q \otimes q) + (q \otimes q)M)\}).
$$

Since for $\theta > 2$,

$$
-G_N.N^2 \geq \frac{1}{w^{2m+2}}(\text{Tr}(AM^2) + \frac{w'^2}{w^2}|\sigma(x,q)q|^2|q|^2 - \frac{w'^2}{w^2}2\theta|\sigma(x,q)|^2|q|^2 - \frac{2}{\theta}\text{Tr}(AM^2)),
$$

it follows that

$$
\begin{aligned}
G_v^1|p|^2 - G_N.N^2 &\geq \frac{1}{w^{2m+2}}\left[\left(1 - \frac{2}{\theta}\right)\text{Tr}(AM^2) + \frac{w'^2}{w^2}(1 - 2\theta)|\sigma q|^2|q|^2 \right. \\
&\quad \left. + |q|^2(B_u + \frac{w'}{w}(B_q.q - (2m+1)B)) + \frac{w''}{w}|\sigma q|^2)\right] \\
&\geq \frac{1}{w^{2m+2}}\left(1 - \frac{2}{\theta}\right)\text{Tr}(AM^2),
\end{aligned}
$$

if $\left(\frac{(w')^2}{w}\right)(2\theta - 1) < (-w'')$, or $e^{-k} < \frac{w}{2\theta - 1}$. This can be achieved by increasing k if necessary. Using Cauchy-Schwarz inequality as before, we get

$$
\begin{aligned}
|G_v^2| &= |\frac{w'}{w}||\text{Tr}(A_q.q - 2mA)M| \\
&= |\frac{w'}{w}||\text{Tr}(\sigma_q.q - m\sigma)(\sigma)^T M) + \text{Tr}(\sigma(\sigma_q^T q - m\sigma^T)M)| \\
&\leq C\sqrt{\text{Tr}(AM^2)}\{\sqrt{(\text{Tr}(\sigma_q.q - m\sigma)(\sigma_q.q - m\sigma)^T)} \\
&\quad + \sqrt{\text{Tr}(\sigma_q^T q - m\sigma^T)(\sigma_q^T q - m\sigma^T)^T)}\} \\
&\leq (\sqrt{\text{Tr}(AM^2)})\chi(q),
\end{aligned}
$$

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where $\chi(q) = C\{\left(\mathrm{Tr}(\sigma_q.q - m\sigma)(\sigma_q.q - m\sigma)^T)\right)^{1/2} + \left(\mathrm{Tr}(\sigma_q^T q - m\sigma^T)(\sigma_q^T q - m\sigma^T)^T)\right)^{1/2}\}$. From this, it follows that condition (ii) of Theorem 4.1 holds thanks to our assumption (iii) here and the condition on η can be checked in two ways : either by requiring that

$$|q|^{2\delta k_2}\chi^{2k_2}(q) = o(G_v^1|q|^2) \quad \text{as } |q| \to +\infty \, ,$$

i.e. $|q|^{2\delta k_2}\chi^{2k_2}(q) = o(|q|^4)$ which leads to the condition $\eta < 3\delta - 2$ or that

$$|p|^{2\delta k_2}\chi^{2k_2}(p) \geq o(|p|^{\frac{2-\delta}{1-\delta}}) \quad \text{as } |q| \to +\infty \, ,$$

which leads to the condition $\eta < \frac{3\delta - 2}{2(1-\delta)}$.

To verify condition (iii) of Theorem 4.1, we proceed in the same way as in the semilinear case but we need to check now that $C_\nu < \frac{1}{2}$. Observe that using (a1) and (a2), we will have, denoting $e^{-k\hat{u}}$ by c_k, where $\hat{u} = (u + ||u||_\infty + 1)$,

$$G_v^1 \geq (\gamma(1 - c_k) - C_{2,R}kc_k) + kc_k(kC_1 - C_{2,R})|q|^2,$$

and using (b1) leads to

$$G_v^1 \geq (\gamma(1 - c_k) - kc_kC_{2,R}) + kc_kC_{1,R}|q|^2.$$

The constant k has been chosen so large that the constant term and the coefficient of $|q|^2$ are both positive in the above two estimates. Recall that

$$
\begin{aligned}
G_x \cdot p - \nu G_N.N^2 &= \frac{1}{w^{2m+2}}\{-\mathrm{Tr}(A_x \cdot qM) + B_x \cdot q\} \\
&+ \frac{\nu}{w^{2m+2}}\left\{\mathrm{Tr}\{A(x)[M^2 + \frac{(w')^2}{w^2}(q \otimes q)|q|^2 - \frac{w'}{w}(Mq \otimes q + q \otimes qM)]\}\right\} \\
&= \frac{1}{w^{2m+2}}(-\mathrm{Tr}(A_x \cdot qM) + B_x \cdot q + \nu\mathrm{Tr}\{A(x)M^2\} \\
&+ \frac{\nu k^2 e^{-2k\hat{u}}}{w^2}|q|^2\mathrm{Tr}A(q \otimes q) - \frac{\nu k e^{-k\hat{u}}}{w}\mathrm{Tr}A(Mq \otimes q + q \otimes qM)).
\end{aligned}
$$

As before, using the Cauchy-Schwarz inequality,

$$-\mathrm{Tr}(A_x \cdot qM) \geq -4\theta(||\sigma_x||)^2|q|^2 - \frac{1}{\theta}\mathrm{Tr}(AM^2),$$

for a suitable $\theta > 0$. Further,

$$
\begin{aligned}
-(\frac{\nu c_k}{w})\mathrm{Tr}(AMq \otimes q + Aq \otimes qM) \;+\; &\frac{2}{\theta}\mathrm{Tr}(AM^2) \\
&\geq\; -2\theta(\frac{\nu c_k}{w})^2\mathrm{Tr}\{(q \otimes q\sigma)(q \otimes q\sigma)^T\} \\
&\geq\; -2\theta(\frac{\nu c_k}{w})^2|\sigma q|^2|q|^2.
\end{aligned}
$$

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Combining these inequalities,

$$
\begin{aligned}
G_x \cdot p - \nu G_N \cdot N^2 \;\geq\; & \frac{1}{w^{2m+2}} \Big\{ -4\theta(\|\sigma_x\|)^2 |q|^2 + B_x \cdot q \\
& + \; |q|^2 (|\sigma(x)q|^2)\nu(c_k)^2(-2\theta\nu + 1) \Big\} \\
\geq\; & \frac{1}{w^{2m+2}} \Big\{ -4\theta\beta|q|^2(1 + |q|^2) - \beta(1 + |q|^2) - \nu(c_k)^2(-2\theta\nu + 1)C|q|^2(1 + |q|^2) \Big\}.
\end{aligned}
$$

¿From these estimates, one checks easily using our assumption (ii) that the condition (iii) of Theorem 4.1 also holds with C_ν small, after further increasing k, if necessary.

The change of variable , which is now fixed by our choice of k, $u = \phi(v)$ transforms the sub and supersolutions v_1 and v_2 into the given u_1 and u_2 with

$$
\max_{\overline{\Omega}}(v_1 - v_2) = 0
$$

because the same is true for u_1 and u_2. Then applying Theorem 4.1 to v_1 and v_2, leads to a contradiction because we have at the same time $\max_{\overline{\Omega}}(v_1 - v_2) = 0$ and $\max_{\partial\Omega}(v_1 - v_2) < 0$.

Thus we have the desired contradiction and $u_1 \leq u_2$ on $\overline{\Omega}$. ∎

We now examine few examples and concentrate on the condition (iii) and the restriction concerning η and δ, the other assumptions being similar to the ones appearing in Theorem 3.1.

We consider terms of the form $-Tr(A(q)N)$ where we take for the operator A,

(a) Mean Curvature Equation for graphs : $A(q) = \left(I - \dfrac{q \otimes q}{1 + |q|^2} \right),$

(b) m-Laplace Equation: $A(q) = |q|^{m-2}\left(I + \dfrac{(m-2)q \otimes q}{|q|^2} \right).$

Notice that the first one is uniformly elliptic satisfying condition (a1) but the second one is only degenerate elliptic. One can check that for these examples,

(a) $\quad \sigma(q) = \left(I - \dfrac{q \otimes q}{\sqrt{1 + |q|^2}(1 + \sqrt{1 + |q|^2})} \right),$

(b) $\quad \sigma(q) = |q|^{\frac{m-2}{2}}\left(I + ((\sqrt{m-1}) - 1)\dfrac{q \otimes q}{|q|^2} \right)$

and hence in each case,

(a) $\quad \sigma_q \cdot q = -\dfrac{q \otimes q}{(1 + |q|^2)^{3/2}}$

(b) $\quad \sigma_q \cdot q = \dfrac{m-2}{2}|q|^{\frac{m-2}{2}}\left(I + (\sqrt{m-1} - 1)\dfrac{q \otimes q}{|q|^2}\right).$

These computations can be done in a (relatively) simple way, in particular for the case (a), by computing $\dfrac{d}{d\lambda}(\sigma(\lambda q))$ for $\lambda = 1$.

For the case (a), we choose $k = 0$ and we see that (iii) is satisfied for $\eta = -1$ and the restriction on δ in Theorem 4.2 only reduces $\delta > 1/2$.

For the case (b), we choose $k = \frac{m-2}{2}$, we see that, σ being homogeneous, we have

$$(\sigma_q.q - k\sigma) \equiv 0$$

Hence (iii) is satisfied for every η and again just the restriction $\delta > 1/2$ has to be satisfied in Theorem 4.2.

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