Viscosity solutions of elliptic equations in $\mathbb{R}^n$: existence and uniqueness results

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References

Results presented in this school are extracted from:

G. Galise and A. Vitolo, *Viscosity Solutions of Uniformly Elliptic Equations without Boundary and Growth Conditions at Infinity*, International Journal of Differential Equations (2011).

Some previous results

- H. Brezis[1]

$$\Delta u - |u|^{s-1}u = f \qquad s > 1,\ f \in L^1_{loc}(\mathbb{R}^n)$$

  $\exists!$ solution (distributional sense)

- M. J. Esteban, P. Felmer, A. Quaas[2]

$$F(D^2u) - |u|^{s-1}u = f \qquad s > 1,\ f \in L^n_{loc}(\mathbb{R}^n)$$

  $\exists!$ solution (L^n-viscosity solution)

[1] H.Brezis, Semilinear equations in $\mathbb{R}^n$ without conditions at infinity, *Appl. Math. Optim.* 12 (1984), 271–282

[2] M.J.Esteban., P.L.Felmer and A.Quaas, Superlinear elliptic equations for fully nonlinear operators without growth restrictions for the data, *Proc. Edinb. Math. Soc.* **53** (2010), 125–141

Structure conditions (SC)

$$F(x,u,Du,D^2u) = f(x) \quad \text{in } \mathbb{R}^n$$

Assumptions on F:

- $\mathcal{P}^-_{\lambda,\Lambda}(Y-X) - \gamma|\eta-\xi| \leq F(x,u,\eta,Y) - F(x,u,\xi,X) \leq \mathcal{P}^+_{\lambda,\Lambda}(Y-X) + \gamma|\eta-\xi|$
- $F(x,u,\xi,X) - F(x,v,\xi,X) \leq -\delta(u-v)^s$ if $v < u$, $s > 1$
- $F(x,0,0,0) = 0$

Example

$$F(x,u,Du,D^2u) = \mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| - |u|^{s-1}u$$

Structure conditions (SC)

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \mathbb{R}^n$$

Assumption on f:

- $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $p > p_0 = p_0(n, \Lambda/\lambda) \in (n/2, n)$.

p_0 is the exponent such that the generalized maximum principle (GMP) holds true:

GMP

If $f \in L^p(\Omega)$ with $p > p_0$ and $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ is an L^p-strong solution of the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| \geq f,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + Cd^{2-\frac{n}{p}} \|f^-\|_{L^p(\Omega)} \tag{1}$$

with $d = \text{diam}(\Omega)$ and C a positive constant depending on $n, \lambda, \Lambda, p, \gamma d$.

Lemma1

Let Ω be a domain of $\mathbb{R}^n$ such that $\Omega_R := \Omega \cap B_R \neq \emptyset$. Suppose that F satisfy structure conditions (**SC**) a.e. $x \in \Omega_R$. If $u \in C(\overline{\Omega}_R)$ is an L^p-viscosity solution ($p > p_0$) of the equation

$$F(x, u, Du, D^2u) \geq f(x)$$

with $f \in L^p(\Omega_R)$, then for each $r \in (0, R)$ we have

$$\sup_{\Omega_r} u \leq u^+_{\partial\Omega} + \frac{C_0(1+R)^{\mu/2}R^\mu}{(R^2 - r^2)^\mu} + C\|f^-\|_{L^p(\Omega_R)} \tag{2}$$

with $\mu = 2/(s-1)$, $C_0 = C_0(n, \Lambda, \gamma, s, \delta)$ and $C = C(n, p, \lambda, \Lambda, \gamma R)$ are positive constants. Here

$$u^+_{\partial\Omega} = \begin{cases} \sup\limits_{B_R \cap \partial\Omega} u^+ & \text{if } B_R \cap \partial\Omega \neq \emptyset \\ 0 & \text{if } B_R \subset \Omega\,. \end{cases}$$

Sketch of the proof:

- **Osserman's barrier function**

$$\Phi(x) = \frac{C_R R^\mu}{(R^2 - |x|^2)^\mu}, \ |x| < R$$

$$\mu = 2/(s-1), \ \ C_R^{s-1} = 2\mu\delta^{-1}(\Lambda(n + 2(1+\mu)) + \gamma R)$$

- (SC)$\Rightarrow F(x, \Phi(x), D\Phi(x), D^2\Phi(x)) \leq 0$ a.e. in Ω_R
- $w = u - \Phi$ is an L^p-viscosity solution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) + \gamma|Dw| \geq f(x)$ in $\Omega_R \cap \{u > \Phi\}$
- (GMP)$\Rightarrow$ conclusion.

Lemma2

Let Ω_R, F and f as in Lemma1. If $u \in C(\overline{\Omega}_R)$ is an L^p-viscosity solution ($p > p_0$) of the equation

$$F(x, u, Du, D^2u) = f(x),$$

then for each $r \in (0, R)$ we have

$$\sup_{\Omega_r} |u| \leq |u|_{\partial\Omega} + \frac{C_0(1+R)^{\mu/2}R^{\mu}}{(R^2 - r^2)^{\mu}} + C\|f\|_{L^p(\Omega_R)} \tag{3}$$

with C_0, C and $|u|_{\partial\Omega} = \max(u^+_{\partial\Omega}, u^-_{\partial\Omega})$ as defined in Lemma1.

Assumption:
$\forall R > 0\ \exists \omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_R(t) \to 0$ as $t \to 0^+$ and

$$|F(x, v, \xi, X) - F(x, u, \xi, X)| \leq \omega_R(|v - u|) \tag{4}$$

a.e. in x for $|u| + |v| + |\xi| + \|X\| \leq R$.

$$(\mathbf{SC})' = (\mathbf{SC}) + (4)$$

Theorem

Let $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be measurable in x and satisfy the structure condition $(\mathbf{SC})'$ a.e. $x \in \mathbb{R}^n$ for all $(u, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$. If $f \in L^p_{loc}(\mathbb{R}^n)$, then equation

$$F(x, u, Du, D^2u) = f(x)$$

has an L^p-viscosity solution in $\mathbb{R}^n$ for any $p > p_0$.

Sketch of the proof:

- $f_k \in C^\infty(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|f_k - f\|_{L^p(\Omega)} = 0$
- (4)$\Rightarrow$ solvability in the ball B_{2^k} of (DP) $F = f_k$ + continuous boundary condition
- Uniform estimates $\Rightarrow$ for $h > k$

$$\sup_{B_{2^k}} |u_h| \leq C_0 + C\|f\|_{L^p(B_{2^{k+1}})}$$

- (**SC**)'+C^α - estimates $\Rightarrow$

$$\|u_h\|_{C^\alpha(B_{2^k})} \leq C_1(1 + \|f\|_{L^p(B_{2^{k+1}})})$$

- Diagonal argument $u_{h_k} \to u \in C(\mathbb{R}^n)$ uniformly on every bounded domain
- Stability results $\Rightarrow$ conclusion.

Maximum Principle

Let $\delta > 0$, $s > 1$ and Ω be a domain of $\mathbb{R}^n$.
Suppose for a.e. $x \in \Omega$ that

$$F(x, u, \xi, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X) + \gamma|\xi| - \delta|u|^{s-1}u$$

for all $(u, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $u \in C(\overline{\Omega})$ is an L^p-viscosity solution $(p > p_0)$ of the equation

$$F(x, u, Du, D^2u) \geq 0 \quad \text{in } \Omega.$$

- If $\Omega = \mathbb{R}^n$, then $u \leq 0$ in $\mathbb{R}^n$.
- If $\Omega \subsetneq \mathbb{R}^n$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.

Mimum Principle

Let $\delta > 0$, $s > 1$ and Ω be a domain of $\mathbb{R}^n$.
Suppose for a.e. $x \in \Omega$

$$F(x, v, \xi, X) \geq \mathcal{P}^-_{\lambda,\Lambda}(X) - \gamma|\xi| - \delta|v|^{s-1}v$$

for all $(v, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $v \in C(\overline{\Omega})$ an L^p-viscosity solution ($p > p_0$) of the equation

$$F(x, v, Dv, D^2v) \leq 0 \quad \text{in } \Omega.$$

- If $\Omega = \mathbb{R}^n$, then $v \geq 0$ in $\mathbb{R}^n$.
- If $\Omega \subsetneq \mathbb{R}^n$ and $v \geq 0$ on $\partial\Omega$, then $v \geq 0$ in Ω.

$$F \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n), \; f \in C(\mathbb{R}^n)$$

Theorem

If F is indipendent of x and satisfies (SC) then the equation

$$F(u, Du, D^2u) = f \quad \text{in } \mathbb{R}^n$$

has a unique C-viscosity solution.

Sketch of the proof:

- u, v solution, $\Omega = \{u > v\}$. Jensen's approximations $\Rightarrow$

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2(u-v)) + \gamma|D(u-v)| - \delta(u-v)^s \geq 0 \quad \text{in } \Omega$$

- Maximum Principle $\Rightarrow u \leq v$...

$$F \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n),\ f \in C(\mathbb{R}^n)$$

Theorem

Suppose that F satisies (SC) and that for all $R > 0$ there exist a constant $K_R > 0$ and a function $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t \to 0^+} \omega_R(t) = 0$ and

$$|F(y,u,\xi,X) - F(x,u,\xi,X)| \leq K_R \|X\| \, |y-x| + \omega_R((1+|\xi|)|y-x|) \tag{A2.1}$$

as $x, y \in \mathbb{R}^n$, $u \in (-R,R)$ and $(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n$. If $p > p_0$ and

$$\|f\|_{M^p} := \sup_{x \in \mathbb{R}^n} \|f\|_{L^p(B_1(x))} < +\infty\,, \tag{A2.2}$$

then equation $F(x,u,Du,D^2u) = f$ has a unique C-viscosity solution.

Sketch of the proof:

u, v solutions, (A2.1)+(A2.2)$\Rightarrow$ $u - v$ satisfies a maximal equation...

$$x \mapsto F(x,\cdot,\cdot,\cdot) \text{ measurable}, \quad f \in L^p_{\mathrm{loc}}(\mathbb{R}^n),\ p > p_0$$

We suppose that for every $R > 0$ there exists $c_R > 0$ such that

$$\begin{aligned} &\mathcal{P}^-_{\lambda,\Lambda}(Y - X) - \gamma|\eta - \xi| - c_R|v - u| \\ &\le F(x,v,\eta,Y) - F(x,u,\xi,X) \le \mathcal{P}^+_{\lambda,\Lambda}(Y - X) + \gamma|\eta - \xi| + c_R|v - u| \end{aligned} \tag{5}$$

for $x \in \mathbb{R}^n$ and any $R > 0$, $u, v \in (-R, R)$, $\xi, \eta \in \mathbb{R}^n$, $X, Y \in \mathcal{S}^n$

(SC)"=(SC)+(5)

$$\beta_F(x,x_0) := \sup_{\substack{X\in\mathcal{S}^n \\ X\neq 0}} \frac{|F(x,0,0,X) - F(x_0,0,0,X)|}{\|X\|}.$$

Theorem

Suppose:

- (SC)" holds true
- $F(\cdot,\cdot,\cdot,X)$ convex
- $$\sup_{r\in(0,r_0)}\left(\fint_{B_r(x)}|\beta_F(x,y)|^n\,dy\right)^{1/n}\le\theta$$
  for every $x\in\mathbb{R}^n$, with $\theta=\theta(n,p,\lambda,\Lambda,r_0)$.

Then the equation $F(x,u,Du,D^2u)=f(x)$ has a unique L^p-strong solution $u\in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$.

Sketch of the proof:
u,v solutions are L^p-strong solution and by using (SC)" we get a maximal equation for $u-v$. We conclude from maximum principle.