NONCLASSICAL SOLUTIONS OF FULLY NONLINEAR ELLIPTIC PDE's

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These notes are intended as a **brief** introduction and survey of the theory of viscosity solutions as applied to second-order fully nonlinear partial differential equations. They are certainly far from complete, but hopefully sufficient references are given to allow the reader to pursue any specific issues further.

1 Introduction

We consider in this lecture the partial differential equation of second-order

$$F(x, u(x), Du(x), D^2u(x)) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where Ω is some bounded domain in $\mathbb{R}^n$, and $F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$. We will always assume that F is (possibly degenerate) elliptic, so

$$F(x, z, p, r + \eta) \geq F(x, z, p, r), \tag{1.2}$$

for all $x \in \Omega$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and $r, \eta \in \mathcal{S}^n$, with $\eta \geq 0$ in the partial ordering for $\mathcal{S}^n$. We will not talk about boundary conditions yet, but keep in mind that in applications one will be imposed.

A *classical solution* of the above equation is a function $u : \Omega \to \mathbb{R}$ which at every point in Ω has all the partial derivatives necessary to interpret (1.1) literally, and for which the equality holds at each point $x \in \Omega$.

There are two basic limitations on our use of such classical solutions. In brief they are that:

- Many problems do not allow the existence of a classical solution.
- Many problems are too complicated or unpleasant for us to know if classical solutions exist or not.

For an example of the first situation, there are fully nonlinear PDE's for which 'bad' (i.e. insufficiently smooth) Dirichlet boundary data implies there cannot be a solution any better than $C^{1,1-2/n}(\Omega)$ (see [19]). In the second class are problems such as PDE's which are neither concave nor convex in D^2u (or a wide variety of problems involving degenerate ellipticity), where classical solutions may or may not exist, but current results cannot be applied.

Given the above restrictions on the classical theory, we must either say nothing until more is known (and say nothing at all about the first class of problems!), or compromise by seeking some partial answer. This is done by considering functions which may not have the smoothness of a classical solution, but which display *some* of the desirable features and properties of a classical solution. Not only is this a sensible compromise, but such partial results are often very useful as a 'stepping-stone' to a classical theory, in that one begins by proving the existence of a weakened form of solution, and then proceeding to show that it is in fact a classical solution.

One such approach for *linear* equations is to relax the differentiability requirements on a solution by using integration by parts to 'transfer' one or more orders of differentiation from the solution to an arbitrary smooth test function (required to have suitable support). A function u which behaves in this context exactly as a classical solution would is called a *weak solution*, although the name is often used to refer to any type of non-classical solution. Such functions need not be even once differentiable, but possess many of the properties of a classical solution (see [4]).

Unfortunately, the above-mentioned process using integration by parts is inappropriate for fully nonlinear equations. We must choose some other desirable feature of classical solutions to focus upon. It must be a feature which doesn't require solutions to be $C^2(\Omega)$, or else we are back where we started.

One such feature of classical solutions of elliptic PDE's is the following: If u is a classical solution (or subsolution), and another $C^2(\Omega)$ function ϕ touches u from above at some point x_0, then $D\phi(x_0) = Du(x_0)$ and $D^2\phi(x_0) \geq D^2u(x_0)$. From ellipticity, this implies that

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq 0$$

where the important information is simply that

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \tag{1.3}$$

For u a classical supersolution, if a function $\phi \in C^2(\Omega)$ touches u from below at x_0, by similar arguments we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0 \tag{1.4}$$

Notice that in the above equations the emphasis is not upon the derivatives of u but instead upon the derivatives of the smooth test function ϕ. This means one can require a function u to behave in the above manner without imposing *any* regularity conditions upon the function u other than simple continuity (and even continuity is not strictly necessary). It is this behavior which we shall attempt to preserve in our notion of a weak solution.

The following ideas were introduced in 1983 by Michael Crandall and Pierre-Louis Lions [3] for first-order (Hamilton-Jacobi) equations, and extended by Lions [13] to second-order elliptic equations.

Definition 1.1 A function $u \in C^0(\Omega)$ is said to be a *viscosity subsolution* of (1.1) if, for all $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that ϕ touches u from above at x_0, one has

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \tag{1.5}$$

Definition 1.2 A fnction $u \in C^0(\Omega)$ is said to be a *viscosity supersolution* of (1.1) if, for all $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that ϕ touches u from below at x_0, one has

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0 \tag{1.6}$$

Definition 1.3 A function $u \in C^0(\Omega)$ is a *viscosity solution* if it is both a viscosity subsolution and supersolution.

The effect of the above definition is to ensure that the differential operator F is evaluated only using the smooth comparison function ϕ. It is easy to see that

- a classical solution of an elliptic PDE is always a viscosity solution,
- a $C^2(\Omega)$ viscosity solution is a classical solution (just take $\phi = u$).

Note: The name 'viscosity solution' harks back to the approach for first-order equations in which $\epsilon \triangle u$ was added to the differential operator, a viscosity term in the fluid dynamics context. As $\epsilon \searrow 0$, the original equation was obtained in the limit, while the limit of the classical solutions was called a 'vanishing viscosity solution' of the original equation.

Having defined such solutions, the natural questions are: What use are they? Under what conditions do they exist? Are they unique? How smooth are they? What do they look like?

One of the most important features of viscosity theory is the following result. It gives one way of answering many of the above questions, and can be generalized in a number of ways.

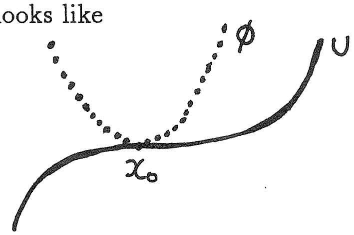
Convergence Theorem [13]. *If the functions* $u_i \in C^0(\Omega)$ *are viscosity solutions of* $F_i[u_i] = 0$ *in* Ω, *and* $F_i \to F$ *in* $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$ *and* $u_i \to u$ *in* $C^0(\Omega)$, **then** *u is a viscosity solution of* $F[u] = 0$ *in* Ω.

Note: The above theorem is strongly nonclassical, since the convergence of the u_i's to u is only in $C^0(\Omega)$, not $C^2(\Omega)$.

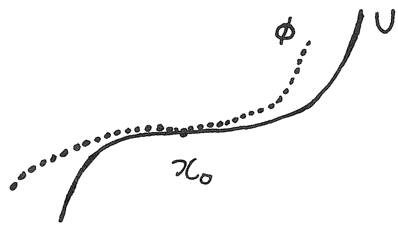
Outline of Proof We need only prove the result holds if one replaces 'solution' with 'subsolution', because the same will then hold for supersolutions, giving us the full result.

We therefore consider ϕ a $C^2(\Omega)$ function which touches u from above at x_0, and wish to show that $F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$.

Let us begin by assuming that ϕ 'pulls away' from u at x_0 fairly rapidly, so it looks like



rather than



Since the u_i converge locally uniformly to u, for i large, ϕ must (after a vertical shift if necessary) touch u_i from above at some point x_i very close to x_0. Because u_i is a viscosity solution of $F_i[v] = 0$, we have

$$F_i(x_i, u_i(x_i), D\phi(x_i), D^2\phi(x_i)) \geq 0.$$

As $i \to \infty$, we have $x_i \to x_0$ by our imposed assumption, so by locally uniform convergence and the smoothness of ϕ we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

as desired.

The above argument has difficulties if $\phi - u$ is zero somewhere other than at x_0. This can be avoided by replacing ϕ with $\phi + \delta|x - x_0|^2$ and following the above arguments to obtain $F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0) + \delta) \geq 0$, then sending $\delta \to 0$. □

The above result is far more useful than first appearances might suggest, because *we get to actively chose the operators* F_i. The best example of an application of the above result is a technique called *elliptic regularization.*

Elliptic Regularization. Many problems in PDE theory are difficult because the differential operator is lacking some highly desirable feature. The best and most obvious example of this is PDE's which are degenerate and/or nonuniformly elliptic, usually making them far more difficult to deal with than uniformly elliptic equations.

It is a standard part of PDE theory to try and prove the existence of classical solutions by deriving *a priori* bounds on the $C^{2,\alpha}(\overline{\Omega})$ norm of solutions. For very complicated or unpleasant problems it is common to find that one can get 'halfway', and derive bounds on solutions in a space like $C^1(\overline{\Omega})$, but that the more complicated higher-order bounds are too difficult for known techniques to handle. If only the equation were nicer!

What one can do is *make* the equation nicer, or more accurately, turn one's attention to a nicer equation. For example, if the difficult is that the PDE is degenerate elliptic, one might simply look at the new equation

$$F_i(x,u,Du,D^2u) \equiv F(x,u,Du,D^2u) + i^{-1}\triangle u = 0 \quad \text{in } \Omega \tag{1.7}$$

which is now strictly elliptic, while keeping whatever boundary condition was originally imposed. (There is often a better way of improving the equation, but the above is a nice simple example.)

If one chooses the right way of improving the original equation, not only does one gain the benefits of uniform ellipticity, but the results for the original equation (such as the $C^1(\overline{\Omega})$ estimates) still hold for all i. It is now a much simpler task to derive *classical* solutions u_i for (1.7) with whatever boundary condition one originally had.

We now have viscosity (even better, classical) solutions u_i of $F_i[v] = 0$ with $F_i \to F$ by construction. Since we have the original $C^1(\overline{\Omega})$ bounds holding regardless of i, by the Arzela-Ascoli theorem the u_i converge uniformly to some function u with $u \in C^{0,1}(\overline{\Omega})$. By the above theorem, this u is a viscosity solution of our original problem, and we know it is Lipschitz.

The above ideas also explain why viscosity theory is also very useful when dealing with difficulties such as low-regularity Dirichlet boundary data. For example, think back to the problem described earlier where poor regularity of the boundary data implied there could not be a solution in $C^{1,1-2/n}(\Omega)$. In that case, what did we mean by 'solution'? The obvious answer is that we can still obtain a viscosity solution.

To give an explicit example of the kind of differential operator we are thinking of, take $n \geq 3$, Ω a suitable domain, and consider the Monge-Ampére equation with Dirichlet boundary condition

$$\begin{aligned} F(x,u,Du,D^2u) = det\, D^2u - f(x) &= 0 \qquad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.8}$$

for some f smooth and positive on $\overline{\Omega}$.

In this particular example it is well known that it is *only* the low regularity of g which stops the problem having a classical solution.

Therefore, if we look instead at the problem

$$\begin{aligned} F(x,u,Du,D^2u) = det\, D^2u - f(x) &= 0 && \text{in } \Omega \\ u(x) &= g_\epsilon(x) && \text{on } \partial\Omega \end{aligned} \tag{1.9}$$

where g_ϵ is a smooth approximation to g, we obtain classical solutions u_ϵ to these approximating problems. If we can obtain some convergence result for the u_ϵ from a compactness argument, we again obtain a viscosity solution of the original problem.

Comparison principles for viscosity solutions.

Comparison principles play a vital role in the classical theory of PDE's, since they lead to uniqueness results and the all-important *a priori* estimates used to prove existence results. In viscosity theory comparison principles have even greater significance, since they lead more directly to existence results. The natural question is therefore: What comparison principles hold in the viscosity theory of second-order elliptic PDE?

The natural first step is to try and transplant the well-known classical results over to the viscosity context. Let's look at the basic background of most comparison principles, where u and v are two viscosity solutions of some given problem, with $w = u - v$ attaining a positive interior supremum. The automatic first step is to say '$w(\cdot)$ has an interior supremum at some point x_0, so $Dw(x_0) = 0$, and $D^2w(x_0) \leq 0$...' The trouble is that w is just some continuous function, so we **cannot** talk about derivatives of any order! Even for linear PDE's the whole language of comparison principles is suddenly illegal.

So what do we do? One fairly reasonable option is to approximate u and v with $C^2(\Omega)$ functions $\overline{u}$, $\overline{v}$ (with mollification being an obvious way of doing so), allowing us to talk about the derivatives of $\overline{w}$ at the interior supremum. This is fine, but the next step in a comparison principle is to take this nice information about the derivatives and substitute it into the differential equations. This gives another major obstacle. For nonlinear PDE, the mollification of a solution does not necessarily satisfy *any* sort of 'approximate' PDE. We've gained smoothness, but at the cost of throwing away effectively *all* our PDE information!

The question of uniqueness and comparison principles was probably the central issue in viscosity theory in the mid 1980's, and as explained above, can be approached by asking: Can we approximate viscosity solutions (or rather: subsolutions and supersolutions) in such a way that we gain some sort of regularity, while retaining the information contained in the differential inequalities? In other words: Can we regularize subsolutions and have them remain as subsolutions of the same PDE?

The answer came in a very clever but complicated paper by Jensen in 1988 [9]. He showed that (under some restrictions) one could approximate sub- and supersolutions in such a way that the approximations were sub- and supersolutions *of very slightly perturbed equations.* We give the definitions here, but will more on quickly to later, more refined, versions of Jensen's results.

Definition 1.4 Given $u \in C^0(\overline{\Omega})$, $\Omega \in \mathbb{R}^n$, define $Q \in C(\mathbb{R}^{n+1})$ by

$$Q(x,y) \stackrel{\text{def}}{=} \mathit{distance}\Big((x,y),\, \mathit{graph}\,(u)\Big) \tag{1.10}$$

Theorem 1.5 If $Du \in L^\infty(\Omega;\mathbb{R}^n)$ then there is an $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ there exist functions $u_\epsilon^+,\, u_\epsilon^- \in C(\overline{\Omega}_\epsilon)$ with the following properties

$$\begin{aligned} u_\epsilon^- < u < u_\epsilon^+ \quad &\text{in } \Omega_\epsilon \\ Q(x, u_\epsilon^\pm(x)) = \epsilon \quad &\text{for } x \in \Omega_\epsilon \\ \|Du_\epsilon^\pm\|_{L^\infty} \le \|Du\|_{L^\infty} & \\ D_\lambda^2 u_\epsilon^+ \ge -c_0/\epsilon \quad &\text{in the sense of distributions} \\ D_\lambda^2 u_\epsilon^- \le c_0/\epsilon \quad &\text{in the sense of distributions} \end{aligned} \tag{1.11}$$

where λ is any direction and the constant c_0 depends only on $\|Du\|_{L^\infty}$.

Theorem 1.6 Assume $u \in C^0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ is a viscosity subsolution (supersolution) of (1.1), where F is independent of x, degenerate elliptic and nonincreasing in u. **Then** $u_\epsilon^+ - \epsilon$ (resp. $u_\epsilon^- + \epsilon$) is also a viscosity subsolution (resp. supersolution) of (1.1) for all $0 \le \epsilon < \epsilon_0$.

The assumption that u is Lipschitz can be removed in later versions of the above approximation scheme.

The techniques used by Jensen were improved and significantly simplified by a result due to Jensen, Lions and Souganidis in 1988 (see [11]). The main feature of this paper was the replacing of Jensen's original approximation process by another type, called the *sup and inf convolutions*, or the *Moreau-Yoshida approximations.*

Definition 1.7 Let $u \in C^0(\overline{\Omega})$, and define the functions u_ϵ^+, u_ϵ^- on Ω_ϵ by

$$\begin{aligned} u_\epsilon^+(x) &\stackrel{\text{def}}{=} \sup_{y\in\overline{\Omega}} \left\{ u(y) - \frac{|x-y|^2}{\epsilon^2} \right\} \\ u_\epsilon^-(x) &\stackrel{\text{def}}{=} \inf_{y\in\overline{\Omega}} \left\{ u(y) + \frac{|x-y|^2}{\epsilon^2} \right\} \end{aligned} \tag{1.12}$$

Note that we use here an ϵ^2 term in the definition, rather than the 2ϵ used by many. This has no real effect other than to change some of powers of ϵ in various background calculations. More information about these approximations can be found in Lasry and Lions [12] or Jensen [10].

In parallel to the above results for Jensen's original approximation, we now have the following results.

Lemma 1.8 The above approximations have the following properties:

1. The supremum (infimum) in the above definition is attained at points $x^{\pm}$ such that $|x - x^{\pm}| \leq \epsilon\,(osc\,u)^{1/2}$.

2. $|Du_\epsilon^{\pm}| \leq 2(osc\,u)^{1/2}\epsilon^{-1}$ in the sense of distributions.

3. $\pm\, D^2u_\epsilon^{\pm} \geq -2\epsilon^{-2}$ in the sense of distributions.

4. If u satisfies $F[u] = 0$ in Ω in the viscosity sense, then at any point in Ω_ϵ where u_ϵ^+ (u_ϵ^-) is twice differentiable, we have

$$\begin{gathered} F(x^+, u(x^+), Du_\epsilon^+(x), D^2u_\epsilon^+(x)) \geq 0 \\ (\ F(x^-, u(x^-), Du_\epsilon^-(x), D^2u_\epsilon^-(x)) \leq 0\ ) \end{gathered} \tag{1.13}$$

The second last of these properties follows from the definition by adding $\pm 2\epsilon^{-2}|x|^2$ to the right-hand side of (1.12) before taking the sup or inf. We prove here the last of the above properties since the proof is quite simple and constitutes the main advance upon the work of Jensen.

Proof of last property. To prove the result for u_ϵ^+, we consider a function $\phi \in C^2(\Omega)$ which touches u_ϵ^+ from above at $x_0 \in \Omega_\epsilon$. Since $dist(x_0, \partial\Omega)$ is greater than an upper bound for $|x - x^+|$, it follows that $x^+ \in \Omega$. We also have the inequality

$$u(x^+) - \frac{1}{\epsilon^2}|x_0 - x^+|^2 - \phi(x_0) \geq u(y) - \frac{1}{\epsilon^2}|x - y|^2 - \phi(x) \tag{1.14}$$

for x near x_0 and y near x^+ respectively. From this it follows that the mapping $y \to u(y) - \phi(x_0 - x^+ + y)$ admits a maximum at x^+. Since u is a viscosity subsolution of (1.1), it follows that

$$F(x^+, u(x^+), D\phi(x_0), D^2\phi(x_0)) \geq 0 \tag{1.15}$$

At points x_0 where u_ϵ^+ is twice differentiable, we may take ϕ such that it matches u_ϵ^+ up to second derivatives, giving the desired result. The case for v and v_ϵ^- follows similarly. □

Note: The difference in the spatial variable is a very common situation in viscosity theory. Because of this, structure conditions designed to control the effect of this difference are imposed in nearly all viscosity problems, as illustrated below.

Jensen, Lions and Souganidis showed that using this method of approximation, the results of Jensen could be obtained more simply, and without the assumption that u is Lipschitz. In the same paper it was shown that some dependence upon x was allowable, giving a result of the form:

Theorem 1.9 Let $u, v \in BUC(\overline{\Omega})$ be respectively viscosity subsolution and supersolution of (1.1). We assume that in addition to the ellipticity condition, F also satisfies

1. F is uniformly continuous with respect to p, uniformly for $r \in S^n$, $x \in \Omega$, and p, z bounded.

2. For every $R > 0$ there exists a γ_R such that $F(x,z,p,r) - F(x,t,p,r) \geq \gamma_R(z-t)$ for all $-R \leq t \leq z \leq R, r, p$ and x.

3. For every $R > 0$ there exists a $\omega_R : \mathbb{R} \to \mathbb{R}$ such that $\omega_R(s) \searrow 0$ as $s \searrow 0$, and
$$F(x,z,\lambda(x-y),r) - F(y,z,\lambda(x-y),r) \geq -\omega_R\left(\lambda|x-y|^2 + |x-y|\right) \quad (1.16)$$
for all $r, \lambda \geq 1, x, y$ and $|z| \leq R$.

then
$$\sup_{\overline{\Omega}}(u-v)^+ = \sup_{\partial\Omega}(u-v)^+ \quad (1.17)$$

Note: As is often the case, this paper uses 'ellipticity' in the opposite sense to the standard. (i.e. $-\triangle$ is called an elliptic operator). This has the effect of reversing the sign associated with sub- and supersolutions.

The almost-everywhere twice differentiability of the approximations often allows one to work directly with the sup- and inf-convolutions, rather than using smooth test functions. One problem with such an approach is that since the second derivatives need only exist almost everywhere, there is no guarantee that if $u_\epsilon^+ - v_\epsilon^-$ attains an interior maximum at x_0 then u_ϵ^+ and v_ϵ^- will be twice differentiable there. Jensen produced the following result to help avoid such problems.

Lemma 1.10 Let U be a bounded open set in $\mathbb{R}^n$, and let $w(\cdot)$ be Lipschitz continuous, semi-convex function on $\overline{U}$. Assume that there exists a $y \in U$ such that $w(y) = \sup_U w > \sup_{\partial U} w$. **Then** for any $\eta > 0$, there exist points $p \in \mathbb{R}^n$ and $z \in U$ such that $|p| < \eta$ and the function $x \to w(x) + < p, x >$ on U attains a maximum at z and has the second differential at z.

The idea behind this result is remarkable simple. If w is Lipschitz and semi-convex, then it is twice differentiable almost everywhere. If w attains its supremum over $\overline{U}$ in the interior, and by misfortune the maximum occurs at a point where the second differential does not exist, then one may 'tilt' the function by a small amount to move the supremum to a different point where the second differential does exist. It is intuitively reasonable that one may use as small a tilt as one wishes, and also that the point z may be taken as close as desired to y.

Note: The greater complexity of Jensen's original graph approximation when compared to the sup and inf convolutions is in part due simply to the fact that it was the first such result. For some geometric problems, it may be more appropriate to use the original idea than to use the sup and inf convolutions.

2 Uniformly Elliptic Equations

In the case where the differential operator F is uniformly elliptic, in other words, there exist constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda\|\eta\| \leq F(x,z,p,r) - F(x,z,p,r+\eta) \leq \Lambda\|\eta\|, \tag{2.1}$$

then many results originally associated with classical solutions can be generalized to apply to viscosity solutions under quite reasonable assumptions. This statement is also true to a lesser extent when one is assured only of strict ellipticity.

One of the main technique used to exploit the uniform ellipticity of F is to re-write the differential inequalities (1.13) in the form

$$a_0^{ij} D_{ij} u_\epsilon^+ \geq -F(x^+, u(x^+), Du_\epsilon^+, 0) \tag{2.2}$$

where

$$a_0(x) \stackrel{\text{def}}{=} \int_0^1 F_r(x^+, u(x^+), Du_\epsilon^+, tD^2u_\epsilon^+)\, dt$$

This formulation then allows the application of many results constructed for linear equations. For example, if one imposes additional structure conditions such as

$$\begin{aligned} F2: \quad & |F(x,z,p,0)| \leq \mu_0 + \mu_1|p| \\ F3: \quad & |F(x,z,p,r) - F(y,t,q,r)| \leq \mu_0 + \mu_1(|p|+|q|) + \omega(|x-y|+|z-t|)|r| \end{aligned} \tag{2.3}$$

for all $x, y \in \Omega$, $|z|, |t| < K_0$, $p, q \in \mathbb{R}^n$, $r \in S^n$, the μ_i positive constants and ω a nondecreasing real function such that $\omega(a)\searrow 0$ as $a \searrow 0$, **then** Trudinger [16], proved the following result.

Theorem 2.1 Let u be a continuous viscosity solution of (1.1) where F is uniformly elliptic and satisfies (F2) and (F3) using the modulus $\omega(a) \equiv \mu_2 a^\tau$, for some μ_2, τ positive constants. **Then** u is continuously differentiable in ω, with first derivatives locally Hölder continuous with exponent α depending only on $n, \Lambda/\lambda$ and τ.

Brief Description of Proof. The proof proceeds by considering the function

$$v_\epsilon(x,\xi) \stackrel{\text{def}}{=} u_\epsilon^+(x+h\xi) - u_\epsilon^-(x) \tag{2.4}$$

for some small h. The function v is therefore a function of the $2n$ variables, (x,ξ). The differential inequalities (1.13) can be manipulated to give the differential inequality

$$a^{ij}D_{ij}v_\epsilon + \lambda\sigma^{ij}D_{i\xi_j}v_\epsilon \geq -\mu_1\left(|Dv_\epsilon| + \frac{2\lambda}{\omega}|D_\xi v_\epsilon|\right) - \mu_0 \tag{2.5}$$

after a minor coordinate transformation and using

$$\sigma^{ij}(x) \stackrel{\text{def}}{=} \frac{D_{ij}u_\epsilon^+(x+h\xi)}{|D^2u_\epsilon^+(x+h\xi)|}$$

This equation is **not** uniformly elliptic in the $2n$ independent variables, but using (2.2) and (F2) we obtain the uniformly elliptic inequality

$$\begin{aligned} \overline{L}v_\epsilon &\equiv a^{ij}D_{ij}v_\epsilon + \lambda\sigma^{ij}D_{i\xi_j}v_\epsilon + na_0^{ij}D_{\xi_i\xi_j}v_\epsilon \geq \\ &\geq -\mu_1\left[\left(1+\tfrac{n\omega^2}{\lambda^2}\right)|Dv_\epsilon| + \tfrac{2\lambda}{\omega}|D_\xi v_\epsilon|\right] - \mu_0\left(1+\tfrac{n\omega^2}{\lambda^2}\right) \end{aligned} \tag{2.6}$$

At this point one may apply the results of Krylov and Safonov, to obtain a weak Harnack inequality which is independent of ϵ. Sending $\epsilon \searrow 0$, one obtains a Hölder estimate. An iteration argument then shows that the Hölder exponent can be improved to give $u \in C^{1,\beta}(\Omega)$ for some $\beta > 0$. In the same paper global estimates are also given for the case where the Dirichlet boundary data is sufficiently smooth.

Another interesting result, this time taken from Trudinger [15], is the following.

Theorem 2.2 Let F satisfy

$$\begin{aligned} F(x,z,p,r+\eta) - F(x,z,p,r) &\geq \delta_0\|\eta\| \\ |F(x,z,p,r)| &\leq \mu_0(1+|p|+\|r\|) \end{aligned} \tag{2.7}$$

for all $x \in \Omega$, $|z| < M_0$, $p \in \mathbb{R}^n$, $\eta \geq 0 \in S^n$ and $M_0 > 0$, where δ_0, μ_0 are positive constants (depending on M_0). Then if $u \in C^0(\Omega)$ satisfies $Fu = 0$ in the viscosity sense, then u is twice differentiable almost everywhere in Ω.

The proof uses the sup and inf convolutions, and revolves around the backwards use of the Alexandrov maximum principle. The details can be found in [15].

3 Formulation of Boundary Conditions

It is obviously unsatisfactory to define a weak form of solution to (1.1) without establishing a similar mechanism to treat boundary conditions. This is particularly obvious for higher order boundary conditions such as

$$B(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

since it is useless to impose such a condition in the classical sense if one's current objective is to relax the implicit need for differentiability properties of the solution. The above argument applies equally to the more common first-order boundary condition, since it is a poor compromise to require weak solutions to be continuously differentiable on the boundary.

A direct translation of the formulation used for the interior would give the result:
Attempted Definition *We say $B(x, u(x), Du(x), D^2u(x)) \leq 0$ in the viscosity sense, or u is a viscosity supersolution of $Bu = 0$ if, for every $\phi \in C^2(\partial\Omega)$ and $x \in \partial\Omega$ such that $u-\phi$ has a minimum over $\partial\Omega$ at x, we have $B(x, u(x), D\phi(x), D^2\phi(x)) \leq 0$.*

This suggestion is obviously inadequate in that the boundary condition is controlled by the behavior of u *only on* $\partial\Omega$, a situation which is unsuitable for the majority of problems (for example if there is any sort of obliqueness to the boundary operator). A more reasonable definition would be the following:

Definition 3.1 We say $B(x, u(x), Du(x), D^2u(x)) \leq 0$ in the strong viscosity sense, or u is a viscosity supersolution of $Bu = 0$ in the strong sense, if, for every $\phi \in C^2(\overline{\Omega})$ and $x \in \partial\Omega$ such that $u - \phi$ has a minimum over $\overline{\Omega}$ at x, we have

$$B(x, u(x), D\phi(x), D^2\phi(x)) \leq 0$$

The obvious parallel for viscosity subsolutions in the strong sense is also used, while a function u is said to satisfy $Bu = 0$ in the strong viscosity sense if it is both a strong viscosity subsolution and a strong viscosity supersolution.

In what follows we will restrict our attention to at most first-order boundary conditions.

Note: One important thing to note in the above definition is that for points $x \in \partial\Omega$, the restriction that "$u - \phi$ has a minimum at x" conveys less information about the nature of u than the corresponding result for an interior point. If the domain Ω has a reasonably smooth boundary, the restriction on $u - \phi$ is basically 'one-sided'. To illustrate this, consider $\partial\Omega \in C^2$, so the boundary $\partial\Omega$ may be taken locally to be the hyperplane $x_n = 0$ where the outward normal to $\partial\Omega$ is denoted by $\vec{n} = (0, 0, ..., 1)$. (In other words we may 'flatten' the boundary locally without any qualitative change to the problem using a C^2 diffeomorphism). Given a function ϕ such that $u - \phi$ has a minimum over $\overline{\Omega}$ at $(x) \in \partial\Omega$, the same result continues to hold if we replace ϕ with the mapping $x \to \phi(x) + \lambda\vec{n} \cdot x$. For the boundary condition to hold in the above sense it is essential that $B(x, u(x), D\phi(x) + \lambda\vec{n}) \leq 0$ for all $\lambda > 0$ given that $B(x, u(x), D\phi(x)) \leq 0$.

It is therefore essential to assume that the boundary condition satisfies an assumption such as

$$\lambda \to B(x, z, p + \lambda\vec{n}) \text{ is nonincreasing in } \lambda$$

otherwise a classical solution need not be a viscosity solution.

Again some caution is needed, since in the recent survey paper [2] the opposite sign is used, as is consistent with the notation there of $-\triangle$ being elliptic.

As might be suspected, the word 'strong' is used above because a weakened version is often required. For a number of problems, it is not possible to impose the boundary condition even in the strong viscosity sense. For such problems, it is frequently the case that that the impossibility of enforcing a strong viscosity boundary condition is linked to some degeneracy of the differential operator F. An example which illustrates the inapplicability of the strong viscosity boundary condition and which also demonstrates very strong degeneracy is the following:

Example 3.2 Consider the differential equation

$$\begin{aligned} F(x, u, Du, D^2u) &\stackrel{\text{def}}{=} u - f(x) = 0 \quad \text{in } \Omega \\ B(x, u, Du) &\stackrel{\text{def}}{=} u \qquad\quad\;\; = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.2}$$

where f is a given continuous function, $f(\cdot) \neq 0$ on $\partial\Omega$. Obviously any solution $u \in C^0(\overline{\Omega})$ of $Fu = 0$ does not satisfy the boundary condition $Bu = 0$ for **any** $x \in \partial\Omega$.

Note that the inapplicability of the boundary condition is obviously not due to it's complicated nature! Note also that while the boundary condition does not apply, one has $Fu(x) = 0$ for all $x \in \partial\Omega$, not just for all $x \in \Omega$.

This simple example illustrates the general situation. It is not always possible to enforce the boundary condition, but the viscosity formulation lends itself to the situation in which *either u satisfies the boundary condition <u>or</u> the differential equation for each point $x \in \partial\Omega$.*

The appropriate formulation is the following.

Definition 3.3 We say $B(x, u(x), Du(x), D^2u(x)) \leq 0$ in the weak viscosity sense, or u is a viscosity supersolution of $Bu = 0$ in the weak sense, if, for every $\phi \in C^2(\overline{\Omega})$ and $x \in \partial\Omega$ such that $u - \phi$ has a minimum over $\overline{\Omega}$ at x, we have

$$\min\left\{ B(x, u(x), D\phi(x), D^2\phi(x)),\ F(x, u(x), D\phi(x), D^2\phi(x)) \right\} \leq 0$$

Again, the obvious parallels for viscosity subsolutions and solutions in the weak sense are used.

Note: The above definition is natural if certain important results (such as the convergence theorem) are to hold in the new context. A brief study of the arguments used in the proof of the convergence theorem make this point much clearer.

Recent results for fully nonlinear oblique boundary value problems can be found in [2] or [7].

4 Concluding Remarks

The study of viscosity solutions is currently a very active area. The survey paper [2] gives over 150 references, the vast majority of which were published after the mid-1980's. The reference [2] can be used to locate papers concerning any of the topics discussed here, and also gives some speculation as to future directions for the theory.

It should be noted however that Crandall, Ishii, and Lions in [2] and in their own individual papers stress heavily one specific approach, in which the analysis of the approximating functions is replaced with matrix results involving analysis of the test functions ϕ. Other authors, most notably Jensen and Trudinger, use the completely distinct approach of making little (or more frequently, *no*) use of test functions, but instead using the properties of various approximation schemes.

The use of viscosity solutions via an elliptic regularization process is one of increasing importance in a number of PDE problems. An example of this can be found in the study of prescribed curvature equations (see [18], [19] for examples).

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