An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton–Jacobi Equations and Applications

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Abstract In this course, we first present an elementary introduction to the concept of viscosity solutions for first-order Hamilton–Jacobi Equations: definition, stability and comparison results (in the continuous and discontinuous frameworks), boundary conditions in the viscosity sense, Perron's method, Barron–Jensen solutions ... etc. We use a running example on exit time control problems to illustrate the different notions and results. In a second part, we consider the large time behavior of periodic solutions of Hamilton–Jacobi Equations: we describe recents results obtained by using partial differential equations type arguments. This part is complementary of the course of H. Ishii which presents the dynamical system approach ("weak KAM approach").

1 Introduction

This text contains two main parts: in the first one, we present an elementary introduction of the notion of viscosity solutions in which we restrict ourselves to the case of first-order Hamilton–Jacobi Equations (we do not present the uniqueness arguments for second-order equations). We recall that this notion of solutions was introduced in the 1980s by Crandall and Lions [22] (see also Crandall et al. [21]). In the second part, we describe recent results on the large time behavior of solutions of Hamilton–Jacobi Equations which are obtained by using partial differential equations type arguments: this part is complementary of the course of H. Ishii which presents the dynamical system approach ("weak KAM approach").

Despite the main focus of this article will be on first-order equations, we point out that the natural framework for presenting viscosity solutions' theory is to

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consider fully nonlinear degenerate elliptic equations (and even equations with integro-differential operators under suitable assumptions); we will use this natural framework when there will be no additional difficulty.

We refer the reader to the book of Bardi and Capuzzo Dolcetta [2] for a more complete presentation of this notion of solutions including applications to deterministic optimal control problems and differential games, to the "Users guide" of Crandall et al. [23] for extensions to second-order equations and to the book of Fleming and Soner [26] where the applications to deterministic and stochastic optimal control are also described. An introduction to the notion of viscosity solutions as well as applications in various directions can also be found in the 1995 CIME course [3].

By *"fully nonlinear degenerate elliptic equations"*, we mean equations which can be written as

$$F(y, u, Du, D^2u) = 0 \quad \text{in } \mathcal{O} , \tag{1}$$

where $\mathcal{O}$ is a domain in $\mathbb{R}^N$ and F is, say, a continuous, real-valued function defined on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$, $\mathcal{S}^N$ being the space of $N \times N$ symmetric matrices, and which satisfies the *(degenerate) ellipticity condition*

$$F(y, r, p, M_1) \leq F(y, r, p, M_2) \quad \text{if } M_1 \geq M_2 , \tag{2}$$

for any $y \in \mathcal{O}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, $M_1, M_2 \in \mathcal{S}^N$. The solution u is a scalar function and Du, D^2u denote respectively its gradient and Hessian matrix.

Of course, first-order equations obviously enter in this framework since, in that case, F does not depend on D^2u and is therefore elliptic. We also point out that parabolic/first-order evolution equations like

$$u_t + H(x, t, u, D_x u) - \varepsilon \Delta^2_{xx} u = 0 \quad \text{in } \Omega \times (0, T) ,$$

are also degenerate elliptic equations if $\varepsilon \geq 0$ (including $\varepsilon = 0$) with the domain $\mathcal{O} = \Omega \times (0, T)$ and the variable $y = (x, t)$; in other words, a classical (possibly degenerate) parabolic equation is a degenerate elliptic equation.

The ellipticity property is a key property for defining the notion of viscosity solutions: this fact will become clear in Sect. 3. From now on, we will always assume it is satisfied by the equations we consider.

In fact, the notion of viscosity solutions applies naturally to (a priori) any type of equations modelling *monotone phenomenas*. A famous result in this direction is given by Alvarez et al. [1] for image analysis (see also Biton [19]): a multiscale analysis which satisfies some locality, regularity, causality and *monotonicity* properties is given by a fully nonlinear parabolic pde, and even by the viscosity solution of this pde. Furthermore, one has a geometrical counterpart of this result in [14] for front propagation problems, where monotonicity has to be understood in the inclusion sense. We will emphasize this monotonicity feature, starting, in Sect. 2, with a running example on exit time control problems.

The article is organized as follows: in Sect. 3, we provide the definition of *continuous* viscosity sub and supersolutions and their first properties (different formulations, connections with classical properties, changes of variables, ... etc); we also provide a first stability result for continuous solutions (Sect. 4). Section 5 describes what is called (improperly) "uniqueness results": in fact, these are "comparison results" of Maximum Principle type which (roughly speaking) implies that subsolutions are below supersolutions. After describing the basic arguments (doubling of variables and basic estimates), we show how to obtain such comparison results in various situations (in particular for problems set in $\mathbb{R}^N \times (0,T)$ with or without "finite speed of propagation" type properties). In Sect. 6, we describe the notion of viscosity solutions for discontinuous solutions and equations: the main motivation comes from the discontinuous stability result ("half relaxed limit method") which allows passage to the limit with only a uniform (L^∞) bound on the solutions. This last result leads us to the existence properties for viscosity obtained by the Perron's method (Sect. 7). In Sect. 8, we show how to prove regularity results: Lipschitz continuity, semi-concavity, ... etc and we conclude by the Barron–Jensen's approach for first-order equations with convex Hamiltonians (Sect. 9).

In a second part, in Sect. 10, we provide an application of the presented tools to the study (by pde methods) of the large time behavior of solutions of Hamilton–Jacobi Equations: we present the various difficulties and key results for these problems (basic estimates, ergodic problem, ... etc.) and we describe the two main convergence results, namely the Namah–Roquejoffre framework [42] and what we name as the "strictly convex" framework, even if the Hamiltonians do not really need to be strictly convex, related to the result by Souganidis and the author [15]; while the Namah–Roquejoffre result relies on rather classical viscosity solutions' methods, the "strictly convex" one uses a more surprising asymptotic monotone property of the solutions in t.

2 Preliminaries: A Running Example

In this section, we present an example which is used in the sequel to illustrate several concepts or results related to viscosity solutions. This example concerns deterministic control problems and, more precisely, exit time control problems. We describe it now.

We consider a controlled system whose state is described by the solution y_x of the ordinary differential equation (the "dynamic")

$$\begin{cases} \dot{y}_x(s) = b(y_x(s), \alpha(s)) & \text{for } s > 0, \\ y_x(0) = x \in \Omega \ . \end{cases} \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^N$ (Ω or its closure $\overline{\Omega}$ represents the possible "states of the system"), $\alpha(\cdot)$, the control, is a measurable function which takes its value in a compact metric space $\mathscr{A}$ and $b : \mathbb{R}^N \times \mathscr{A} \to \mathbb{R}^N$ is a function satisfying, for some constant $C > 0$ and for any $x, y \in \overline{\Omega}$, $\alpha \in \mathscr{A}$

$$\begin{cases} b \text{ is a continuous function from } \mathbb{R}^N \times \mathscr{A} \text{ into } \mathbb{R}^N \ . \\ |b(x,\alpha) - b(y,\alpha)| \leq C|y-x| \ , \quad |b(x,\alpha)| \leq C \ , \end{cases} \tag{4}$$

Because of this assumption, the ordinary differential equation (3) has a unique solution which is defined for all $s > 0$.

The trajectories y_x depend both on the starting point x but also on the choice of the control $\alpha(\cdot)$. We omit this second dependence for the sake of simplicity of notations.

The "value function" is then defined, for $x \in \Omega$ (or $\overline{\Omega}$) and $t \in [0, T]$, by

$$\mathbf{U}(x,t) = \inf_{\alpha(.)} \left\{ \int_0^\tau f(y_x(s), \alpha(s))ds + \varphi(y_x(\tau))\mathbf{1}_{\{\tau \leq t\}} + u_0(y_x(t))\mathbf{1}_{\{\tau > t\}} \right\} , \tag{5}$$

where f, φ, u_0 are continuous functions defined respectively on $\overline{\Omega} \times \mathscr{A}$, $\partial\Omega$ and $\overline{\Omega}$ which takes values in $\mathbb{R}$. We denote by τ the first exit time of the trajectory y_x from Ω, i.e.

$$\tau = \inf\{t \geq 0 \ ; \ y_x(t) \notin \Omega \ \} \ .$$

Of course, τ depends on x and $\alpha(\cdot)$ but we drop this dependence, again for the sake of simplicity of notations. Finally, for any set A, $\mathbf{1}_A$ denotes the indicator function of the set A. For reasons which will be clear later on, we assume the compatibility condition

$$u_0 = \varphi \text{ on } \overline{\Omega} \ . \tag{6}$$

In the sequel, we will say that the "control assumptions", and we will write **(CA)**, are satisfied if (4) holds, if f, φ, u_0 are continuous functions and if we have (6).

The first remark that we can make on this example concerns the monotonicity: keeping the same dynamic, if we consider different costs f_1, φ_1, u_0^1 and f_2, φ_2, u_0^2 with

$$f_1 \leq f_2 \quad \text{on } \overline{\Omega} \times \mathscr{A}, \quad \varphi_1 \leq \varphi_2 \quad \text{on } \partial\Omega, \quad u_0^1 \leq u_0^2 \quad \text{on } \overline{\Omega} \ ,$$

then the associated value functions satisfy $\mathbf{U}_1 \leq \mathbf{U}_2$ on $\overline{\Omega} \times [0, T]$. In other words, the value functions depends in a monotone way of the data.

We will see that the value function $\mathbf{U}$ is a solution of

$$\mathbf{U}_t + H(x, D\mathbf{U}) = 0 \quad \text{in } \Omega \times (0, T) \ , \tag{7}$$

where $H(x, p) := \sup_{\alpha \in \mathscr{A}} \{-b(x,\alpha) \cdot p - f(x,\alpha)\}$, with the Dirichlet boundary condition

$$\mathbf{U}(x,t) = \varphi(x,t) \quad \text{on } \partial\Omega \times (0, T) \ , \tag{8}$$

and the initial condition

$$\mathbf{U}(x,0) = u_0(x) \quad \text{on } \overline{\Omega} . \tag{9}$$

We have to answer to several questions in the sequel:

- A priori, the value function **U** is not regular: in which sense can it be a solution of (7)–(9)?
- How is the boundary data achieved? In which sense?
- Is the value function the unique solution of (7)–(9)?
- Are we able to prove directly that a solution of (7)–(9) satisfies the monotonicity property?

We conclude this section by (very) few some references on exit time control problems. The work of Soner [44] on state constraints problems is the first article which studies this kind of problems in connections with viscosity solutions, uses boundary conditions in the viscosity solutions' sense and provides a general argument to prove uniqueness results. Boundary conditions in the viscosity solutions' sense have been considered previously for Neumann/reflection problems by Lions [37]. Pushing their ideas a little bit further, Perthame and the author [8–10] (see also [5]) systematically study Dirichlet/exit time control problems (including state constraints problems). For stochastic control, we refer the reader to [12] and references therein.

3 The Notion of Continuous Viscosity Solutions: Definition(s) and First Properties

3.1 Why a "Good" Notion of Weak Solution is Needed?

We give now few concrete examples of equations where there will be a unique viscosity solution but either no smooth solutions or with several *generalized* solutions (i.e. solutions which are locally Lipschitz continuous and satisfy the equation almost everywhere). We refer to Sect. 5 for the proof of the uniqueness results we are going to use.

The first example is

$$\frac{\partial u}{\partial t} + \left|\frac{\partial u}{\partial x}\right| = 0 \quad \text{in } \mathbb{R}\times(0,+\infty) . \tag{10}$$

We first remark that (10) enters into our framework with $\mathcal{O} = \mathbb{R}\times(0,+\infty)$, the variable is $y = (x,t)$, $Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)$[1] and

$$F(y,u,p,M) = p_t + |p_x| ,$$

[1] Here we use the notation Du for the full gradient of u in space and time but, in general, we will use it for the gradient in space of u.

with $p = (p_x, p_t)$.

It can be shown that the function u defined in $\mathbb{R} \times (0, +\infty)$ by

$$u(x,t) = -(|x| + t)^2 ,$$

is the unique viscosity solution of (10) in $C(\mathbb{R} \times (0, +\infty))$ (see Sect. 5.3). It is worth remarking in this example that u is only locally Lipschitz continuous for $t > 0$ despite the initial data

$$u(x,0) = -x^2 \quad \text{in } \mathbb{R} ,$$

is in $C^\infty(\mathbb{R})$. In particular, this problem has no smooth solution as it is generally the case for such nonlinear hyperbolic equations.

Moreover, if we consider (10) together with the initial data

$$u(x,0) = |x| \quad \text{in } \mathbb{R} , \tag{11}$$

then the functions $u_1(x,t) = |x| - t$ and $u_2(x,t) = (|x| - t)^+$ are two "generalized" solutions in the sense that they satisfy the equation almost everywhere (at each of their points of differentiability). This problem of nonuniqueness is solved by the notion of viscosity solutions since it can be shown that u_2 is the unique continuous viscosity solution of (10)–(11) (see again Sect. 5.3). In that case, the notion of viscosity solutions selects the "good" solution which is here the value-function of the associated deterministic control problem (cf. Bardi and Capuzzo Dolcetta [2] and Fleming and Soner [26]). An other remark (or interpretation) is that the notion of viscosity solutions selects the solution which satisfies the right monotonicity property: indeed the initial data is positive and therefore the solution has to be positive since 0 is a (natural) solution.

For second-order equations, non-smooth solutions appear generally as a consequence of the degeneracy of the equation. We refer to [23] for details in this direction.

3.2 Continuous Viscosity Solutions

As we already mention it in the introduction, we are going to present the different definitions of viscosity solutions in the framework of *fully nonlinear degenerate elliptic equations* i.e. equations like (1) which satisfies the *ellipticity condition* (2).

In order to introduce the notion of viscosity solutions and to show the importance of the ellipticity condition, we first give an equivalent definition of the notion of classical solution which only uses the Maximum Principle.

Theorem 3.1 (Classical Solutions and Maximum Principle). *$u \in C^2(\mathcal{O})$ is a classical solution of* (1) ***if and only if***
for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local maximum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \leq 0,$$

and, *for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local minimum point of $u - \varphi$, one has*

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0\ .$$

Proof. The proof of this result is very simple: the first part of the equivalence just comes from the classical properties $Du(y_0) = D\varphi(y_0)$, $D^2u(y_0) \leq D^2\varphi(y_0)$, at a maximum point y_0 of $u - \varphi$ (recall that u and φ are smooth) or $Du(y_0) = D\varphi(y_0)$, $D^2u(y_0) \geq D^2\varphi(y_0)$, at a minimum point y_0 of $u - \varphi$. One has just to use these properties together with the ellipticity property (2) of F to obtain the inequalities of the theorem.

The second part is a consequence of the fact that we can take $\varphi = u$ as test-function and therefore $F(y_0, u(y_0), Du(y_0), D^2u(y_0))$ is both positive and negative at any point y_0 of $\mathcal{O}$ since any $y_0 \in \mathcal{O}$ is both a local maximum and minimum point of $u - u$.

Now we simply remark that the equivalent definition of classical solutions which is given here in terms of test-functions φ does not require the existence of first and second derivatives of u. For example, the continuity of u is sufficient to give a sense to this equivalent definition; therefore we use this formulation to define viscosity solutions.

Definition 3.1 (Continuous Viscosity Solutions). The function $u \in C(\mathcal{O})$ is a viscosity solution of (1) **if and only if**
for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local maximum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \leq 0,$$

and, for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local minimum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0\ .$$

If u only satisfies the first property of Definition 3.1 (with maximum points), we will say that u is a viscosity subsolution of the equation, while it is called a viscosity supersolution if it only satisfies the second one. From now on, we will talk only of subsolution, supersolution and solution considering that they will be anytime taken in the viscosity sense. This notion of solution was called "viscosity solution" because for first-order equations, as we will see it below, viscosity solutions were first obtained as limits in the "vanishing viscosity method", i.e. by an approximation procedure involving a $-\varepsilon\Delta$ term.

For first-order equations (otherwise this remark makes no sense), it is worth pointing out that a solution of $F = 0$ is not necessarily a solution of $-F = 0$: the sign of the nonlinearity plays a role. This phenomena can be understood in the following way: the viscosity solution of the equation $F = 0$ when unique can be thought as being obtained through the vanishing viscosity approximation $-\varepsilon\Delta + F = 0$ and there is no reason why the other vanishing approximation

$\varepsilon\Delta + F = 0$ (which leads in fact to a solution of $-F = 0$) converges to the same solution.

Finally we remark that parabolic equations are just a particular case of (degenerate) elliptic equations: the y—variable is just the (x,t)—variable and, of course, Du, D^2u have to be understood as the gradient and Hessian matrix of u with respect to the variable (x,t).

3.3 Back to the Running Example (I): The Value Function U *is a Viscosity Solution of (7)*

The key result is the *Dynamic Programming Principle*

Theorem 3.2. *Under the hypothesis* **(CA)***, if $x \in \Omega$, $0 < t \leq T$, the value function satisfies, for $S > 0$ small enough*

$$\mathbf{U}(x,t) = \inf_{\alpha(.)} \left[\int_0^S f(y_x(s),\alpha(s))ds + \mathbf{U}(y_x(S),t-S) \right]. \tag{12}$$

We leave the proof of this result to the reader and show how it implies that $\mathbf{U}$ is a viscosity solution of (7). To do so, we assume that $\mathbf{U}$ is continuous (an assumption which will be removed later on). We only prove that it is a supersolution, the subsolution property being easier to obtain.

Let $\phi \in C^1(\Omega \times (0,T))$ and assume that $(x,t) \in \Omega \times (0,T)$ is a local minimum point of $\mathbf{U} - \phi$. There exists $r > 0$ such that, if $|x'-x| \leq r$ and $|t'-t| \leq r$, then $x' \in \Omega$, $t' > 0$ and

$$\mathbf{U}(x',t') - \phi(x',t') \geq \mathbf{U}(x,t) - \phi(x,t)\ .$$

Using the Dynamic Programming Principle with S small enough in order to have $S \leq r$ and $|y_x(S) - x| \leq r$ (recall that b is uniformly bounded), we obtain

$$\phi(x,t) \geq \inf_{\alpha(.)} \left[\int_0^S f(y_x(s),\alpha(s))ds + \phi(y_x(S),t-S) \right]\ .$$

But, by standard calculus

$$\begin{aligned}
&\phi(y_x(S),t-S) \\
&\quad = \phi(x,t) + \int_0^S \left(D\phi(y_x(s),t-s)\cdot b(y_x(s),\alpha(s)) - \phi_t(y_x(s),t-s)\right) ds\ .
\end{aligned}$$

And therefore

$$0 \geq \inf_{\alpha(.)} \left[\int_0^S (D\phi(y_x(s), t-s) \cdot b(y_x(s), \alpha(s)) - \phi_t(y_x(s), t-s) \right.$$
$$\left. + f(y_x(s), \alpha(s)))\, ds \right] ,$$

or

$$\sup_{\alpha(.)} \left[\int_0^S (-D\phi(y_x(s), t-s) \cdot b(y_x(s), \alpha(s)) \right.$$
$$\left. + \phi_t(y_x(s), t-s) - f(y_x(s), \alpha(s)))\, ds \right] \geq 0 .$$

Next, we remark that the integrand can be replaced by (the larger quantity)

$$\phi_t(y_x(s), t-s) + H(y_x(s), D\phi(y_x(s), t-s))$$

and then, because of the regularity of ϕ and the continuity property of H, by $\phi_t(x,t) + H(x, D\phi(x,t)) + o(1)$ where $o(1)$ denotes a quantity which tends to 0 as $S \to 0$, uniformly with respect to the control. Finally

$$\sup_{\alpha(.)} \left[\int_0^S (\phi_t(x,t) + H(x, D\phi(x,t)) + o(1))\, ds \right] \geq 0 ,$$

and the conclusion follows by dividing by S and letting S tends to 0, noticing that the sup can be dropped.

Remark 3.1. The above argument is a key one and it is worth pointing out that it just uses the fact that

$$u(x,t) = G(S, x, t, u(\cdot)) ,$$

where G is *monotone in* $u(\cdot)$ and *consistent* with the equation, in the sense that

$$\frac{\phi(x,t) - G(S, x, t, \phi(\cdot))}{S} \to \phi_t(x,t) + H(x, D\phi(x,t) \text{ as } S \to 0 ,$$

for any smooth function ϕ.[2] Therefore it is a rather general argument which connects "monotonicity" and "viscosity solutions": it appears in various situations such as the convergence of numerical scheme (see in particular [13]), the connection of monotone semi-group with viscosity solutions (see, for instance, [1,19,36]), ... etc.

[2] Here we have also used a less important (but simplifying) property, namely the commutation with constants: for any $c \in \mathbb{R}$, S, x, t and for any function $u(\cdot)$, $G(S,x,t,u(\cdot)+c) = G(S,x,t,u(\cdot)) + c$.

3.4 An Equivalent Definition and Its Consequences

We continue by giving some equivalent definitions which may be useful.

Proposition 3.1. *An equivalent definition of subsolution, supersolution and solution is obtained by replacing in Definition 3.1:*

1. *"$\phi \in C^2(\mathcal{O})$" by "$\phi \in C^k(\mathcal{O})$" ($2 < k < +\infty$) or by "$\phi \in C^\infty(\mathcal{O})$"*
2. *"$\phi \in C^2(\mathcal{O})$" by "$\phi \in C^1(\mathcal{O})$" in the case of first-order equations*
3. *"local maximum" or "local minimum" by "strict local maximum" or "strict local minimum" or by "global maximum" or "global minimum" or by "strict global maximum" or "strict global minimum".*

This proposition is useful since, in general, the proofs are simplified by a right choice of the definition. In particular the definition with "global maximum points" or "global minimum points" in order to avoid heavy localisation arguments.

The proof of this proposition is left as an exercise (despite it is not obvious at all): it is based on classical Analysis type arguments, some of them being rather delicate.

We give now a more "pointwise" definition using generalized derivatives ("sub and super-differential" or "semi-jets") which plays a central role for second-order equations.

Definition 3.2 (Second-order sub and super-differential of a continuous function). The second-order superdifferential of $u \in C(\mathcal{O})$ at $y \in \mathcal{O}$ is the, possibly empty, convex subset of $\mathbb{R}^N \times \mathcal{S}^N$, denoted by $D^{2,+}u(y)$, of all couples $(p, M) \in \mathbb{R}^N \times \mathcal{S}^N$ satisfying

$$u(y+h) - u(y) - (p, h) - \frac{1}{2}(Mh, h) \leq o(|h|^2) ,$$

for $h \in \mathbb{R}^N$ small enough.

The second-order subdifferential of $u \in C(\mathcal{O})$ at $y \in \mathcal{O}$ is the, possibly empty, convex subset of $\mathbb{R}^N \times \mathcal{S}^N$, denoted by $D^{2,-}u(y)$, of all couples $(p, M) \in \mathbb{R}^N \times \mathcal{S}^N$ satisfying

$$u(y+h) - u(y) - (p, h) - \frac{1}{2}(Mh, h) \geq o(|h|^2) ,$$

for $h \in \mathbb{R}^N$ small enough.

As indicated in the definition, these subsets can be empty, even both as it is the case, at the point $y = 0$, for the function $y \mapsto \sqrt{|y|}\sin(\frac{1}{y^2})$ extended at 0 by 0.

If u is twice differentiable at y then

$$D^{2,+}u(y) = \{(Du(y), M);\ M \geq D^2u(y)\}\ ,$$

$$D^{2,-}u(y) = \{(Du(y), M);\ M \leq D^2u(y)\}\ ,$$

Now we turn to the connections between sub and super-differentials with viscosity solutions.

Theorem 3.3. *(i)* *$u \in C(\mathcal{O})$ is a subsolution of* (1) *iff, for any $y \in \mathcal{O}$ and for any $(p, M) \in D^{2,+}u(y)$*

$$F(y, u(y), p, M) \leq 0. \tag{13}$$

(ii) *$u \in C(\mathcal{O})$ is a supersolution of* (1) *iff, for any $y \in \mathcal{O}$ and for any $(p, M) \in D^{2,-}u(y)$*

$$F(y, u(y), p, M) \geq 0\ . \tag{14}$$

Before giving some elements of the proof of Theorem 3.3, we provide some easy (but useful) consequences.

Corollary 3.1. *(i)* *If $u \in C^2(\mathcal{O})$ satisfies $F(y, u(y), Du(y), D^2u(y)) = 0$ in $\mathcal{O}$ then u is a viscosity solution of* (1).

(ii) *If $u \in C(\mathcal{O})$ is a viscosity solution of* (1) *and if u is twice differentiable at $y_0 \in \mathcal{O}$ then*

$$F(y_0, u(y_0), Du(y_0), D^2u(y_0)) = 0\ .$$

(iii) *If $u \in C(\mathcal{O})$ is a viscosity solution of* (1) *and if $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2—function such that $\varphi' > 0$ on $\mathbb{R}$ then the function v defined by $v = \varphi(u)$ is a viscosity solution of*

$$K(y, v, Dv, D^2v) = 0 \quad in\ \mathcal{O}\ ,$$

where $K(y, z, p, M) = F(y, \psi(z), \psi'(z)p, \psi'(z)M + \psi''(z)p \otimes p)$ and $\psi = \varphi^{-1}$.

The proof of this Corollary is based on the classical technics of calculus and is left as an exercise.

This corollary is formulated in terms of "solution" but, of course analogous results hold for subsolutions and supersolutions.

A lot of different changes can be considered instead of the one in the result (iii): as long as signs are preserved in order to keep the inequalities satisfied by the sub or superdifferentials or, if the minima are not transformed in maxima and vice-versa, such result remains true. Let us mention, for example, the transformations: $v = u + \psi$, with ψ being of classe C^2 or $v = \chi u + \psi$, χ, ψ being of classe C^2 and $\chi \geq \alpha > 0 \ldots$ etc.

In the case when "signs are changed", we have the following proposition.

Proposition 3.2. *$u \in C(\mathcal{O})$ is a subsolution (resp. supersolution) of* (1) *iff $v = -u$ is a supersolution (resp. subsolution) of*

$$-F(y, -v, -Dv, -D^2v) = 0 \quad in\ \mathcal{O}\ .$$

The *proof of Theorem 3.3* (that Proposition 3.2 allows us to do only in the subsolution case) relies only on two arguments; the first is elementary: if ϕ a C^2 test-function and if y_0 a local maximum point of $u - \phi$ then, by combining the regularity of ϕ and the property of local maximum, we get

$$\begin{aligned} u(y) &\leq \phi(y) + u(y_0) - \phi(y_0) \\ &\leq u(y_0) + (D\phi(y_0), y - y_0) + \frac{1}{2}D^2\phi(y_0)(y - y_0)\cdot(y - y_0) + o(|y - y_0|^2)\,. \end{aligned}$$

Therefore $(D\phi(y_0), D^2\phi(y_0))$ is in $D^{2,+}u(y_0)$.

The second one is not as simple as the first one and is described in the following lemma.

Lemma 3.1. *If $(p, M) \in D^{2,+}u(y_0)$, there exists a C^2—function $\phi : \mathcal{O} \to \mathbb{R}$ such that $D\phi(y_0) = p$, $D^2\phi(y_0) = M$ and such that y_0 is a local maximum point of $u - \phi$.*

The proof of this lemma uses classical but rather tricky Analysis tools, in particular regularization arguments. We skip it since it is rather long and not in the central scope of this course. We refer to Crandall et al. [21] or Lions [36] for a complete proof.

4 The First Stability Result for Viscosity Solutions

There is no need to recall here that problems involving passage to the limit in nonlinear equations when we have only a weak convergence is one of the fundamental problem of nonlinear Analysis. We call "stability result" a result showing under which conditions a limit of a sequence of sub or supersolutions is still a sub or a supersolution.

We present in theses notes two types of stability results which are of different natures: the first one looks rather classical since it requires compactness (or convergence) properties on the considered sequences. It may be a priori of a rather difficult use since the needed estimates on the solutions are not so easy to obtain in concrete situations. The second one, on the contrary, will be far less classical and requires only easy estimates but rather strong uniqueness properties for the limiting equation: we present this second stability result in Sect. 6 since it requires the notion of discontinuous viscosity solutions. We state both results in the framework of second-order equations since there are no additional difficulties.

The first result is the

Theorem 4.1. *Assume that, for $\varepsilon > 0$, $u_\varepsilon \in C(\mathcal{O})$ is a subsolution (resp. a supersolution) of the equation*

$$F_\varepsilon(y, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0 \quad \text{in } \mathcal{O}\,, \tag{15}$$

where $(F_\varepsilon)_\varepsilon$ *is a sequence of continuous functions satisfying the ellipticity condition. If* $u_\varepsilon \to u$ *in* $C(\mathcal{O})$ *and if* $F_\varepsilon \to F$ *in* $C(\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N)$ *then u is a subsolution (resp. a supersolution) of the equation*

$$F(y,u,Du,D^2u) = 0 \quad \textit{in } \mathcal{O}\ .$$

We first recall that the convergence in the spaces of continuous functions $C(\mathcal{O})$ or $C(\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N)$ is the uniform convergence on compact subsets.

This result allows to pass to the limit in a nonlinear equation (and in particular with a nonlinearity on the gradient and the Hessian matrix of the solutions) with only the local uniform convergence of the sequence $(u_\varepsilon)_\varepsilon$, which, of course, does not imply any strong convergence (for example, a convergence in the almost everywhere sense) neither on the gradient nor a fortiori on the Hessian matrix of the solutions.

An unusual characteristic of this result is to consider separately the convergence of the equation—or more precisely of the nonlinearities F_ε—and of the solutions u_ε. Classical arguments would lead to a question like: is the convergence of u_ε strong enough in order to pass to the limit in the equality $F_\varepsilon(y,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon) = 0$?". In this case, the necessary convergence on u_ε would have depended strongly on the equations through the properties of the F_ε. Here this is not at all the case: the required convergences for F_ε and u_ε are fixed a priori.

The most classical example of application of this result is the vanishing viscosity method

$$-\varepsilon\Delta u_\varepsilon + H(y,u_\varepsilon,Du_\varepsilon) = 0 \quad \text{in } \mathcal{O}.$$

This explains why we present the above result in the second-order framework. In this case, the nonlinearity F_ε is given by

$$F_\varepsilon(y,u,p,M) = -\varepsilon\mathrm{Tr}(M) + H(y,u,p)\ ,$$

and its convergence in $C(\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N)$ to $H(y,u,p)$ is obvious. If u_ε converges uniformly to u, then Theorem 4.1 implies that u is a solution of

$$H(y,u,Du) = 0 \quad \text{in } \mathcal{O}\ .$$

The above example shows that the solutions of Hamilton–Jacobi Equations—and more generally of nonlinear elliptic equations—obtained by the vanishing viscosity method are viscosity solutions of these equations, and this justifies the terminology.

In practical use, most of the time, Theorem 4.1 is applied to a subsequence of $(u_\varepsilon)_\varepsilon$ instead of the sequence itself. When one wants to pass to the limit in an equation of the type (15), one proceeds, in general, as follows:

1. One proves that u_ε is locally bounded in L^∞, uniformly w.r.t $\varepsilon > 0$.
2. One shows that u_ε is locally bounded in some Hölder space $C^{0,\alpha}$ for some $0 \le \alpha < 1$ or in $W^{1,\infty}$, uniformly w.r.t $\varepsilon > 0$.
3. Because of the two first steps, by Ascoli's Theorem, the sequence $(u_\varepsilon)_\varepsilon$ is in a compact subset of $C(K)$ for any $K \subset\subset \mathcal{O}$.

4. One applies the stability result to a converging subsequence of $(u_\varepsilon)_\varepsilon$ which is obtained by a diagonal extraction procedure.

This method will be really complete only when we will have a uniqueness result: indeed, the above argument shows that all converging subsequence of the sequence $(u_\varepsilon)_\varepsilon$ converges to A viscosity solution of the limiting equation. If there exists only one solution of this equation then all the converging subsequences converge to THE viscosity solution of the limiting equation that we denote by u. A classical compactness and separation argument then implies that all the sequence $(u_\varepsilon)_\varepsilon$ converge to u (exercise!).

But, in order to have uniqueness and to justify this argument, one has to impose boundary conditions and also to be able to pass to the limit in these boundary conditions ... (to be continued!).

We now give an example of application of this method.

Example. This example is unavoidably a little bit formal since our aim is to show a mechanism of passage to the limit by viscosity solutions' methods and we do not intend to obtain the estimates we need in full details. In particular, we are to use the Maximum Principle in $\mathbb{R}^N$ without justification.

For $\varepsilon > 0$, let $u_\varepsilon \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be the unique solution of the equation

$$-\varepsilon \Delta u_\varepsilon + H(Du_\varepsilon) + u_\varepsilon = f(x) \quad \text{in } \mathbb{R}^N,$$

where H is a locally Lipschitz continuous function on $\mathbb{R}^N$, $H(0) = 0$ and $f \in W^{1,\infty}(\mathbb{R}^N)$. By the Maximum Principle, we have

$$-||f||_\infty \le u_\varepsilon \le ||f||_\infty \quad \text{in } \mathbb{R}^N,$$

because $-||f||_\infty$ and $||f||_\infty$ are respectively sub- and supersolution of the equation. Moreover, if $h \in \mathbb{R}^N$, since $u_\varepsilon(. + h)$ is a solution of an analogous equation where $f(.)$ is replaced by $f(. + h)$ in the right-hand side, the Maximum Principle also implies

$$||u_\varepsilon(. + h) - u_\varepsilon(.)||_\infty \le ||f(. + h) - f(.)||_\infty \quad \text{in } \mathbb{R}^N,$$

and, since f is Lipschitz continuous, the right-hand side is estimated by $C|h|$ where C is the Lipschitz constant of f. This yields

$$||u_\varepsilon(. + h) - u_\varepsilon(.)||_\infty \le C|h| \quad \text{in } \mathbb{R}^N \ .$$

Since this inequality is true for any h, it implies that u_ε est Lipschitz continuous with Lipschitz constant C.

Using the Ascoli's Theorem and a diagonal extraction procedure, we can extract a subsequence still denoted by $(u_\varepsilon)_\varepsilon$ which converges to a continuous function u which is, by Theorem 4.1, a solution of the equation

$$H(Du) + u = f(x) \quad \text{in } \mathbb{R}^N .$$

In this example, we perform a passage to the limit in a singular perturbation problem without facing much difficulties; again this example will be complete only when we will know that u is the unique solution of the limiting equation since it will imply that the whole sequence $(u_\varepsilon)_\varepsilon$ converges to u by a classical compactness and separation argument.

Now we turn to the *Proof of Theorem 4.1*. We prove the result only in the subsolution case, the other case being shown in an analogous way.

We consider $\phi \in C^2(\mathcal{O})$ and $y_0 \in \mathcal{O}$ a local maximum point of $u-\phi$. Subtracting if necessary a term like $\chi(y) = |y - y_0|^4$ to $u-\phi$, one can always assume that y_0 is a *strict* local maximum point. We then use the following lemma (left as an exercise).

Lemma 4.1. *Let $(v_\varepsilon)_\varepsilon$ be a sequence of continuous functions on an open subset $\mathcal{O}$ which converge in $C(\mathcal{O})$ to v. If $y_0 \in \mathcal{O}$ is a strict local maximum point of v, there exists a sequence of local maximum points of v_ε, denoted by $(y_\varepsilon)_\varepsilon$, which converges to y_0.*

One uses Lemma 4.1 with $v_\varepsilon = u_\varepsilon - (\phi + \chi)$ and $v = u - (\phi + \chi)$. Since u_ε is a subsolution of (15) and since y^ε is a local maximum of $u_\varepsilon - (\phi + \chi)$, we have, by definition

$$F_\varepsilon\Big(y^\varepsilon, u_\varepsilon(y^\varepsilon), D\phi(y^\varepsilon) + D\chi(y^\varepsilon), D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon)\Big) \le 0 .$$

Now we have just to pass to the limit in this inequality: since $y^\varepsilon \to y_0$, we use the regularity of the test-functions ϕ and χ which implies

$$D\phi(y^\varepsilon) + D\chi(y^\varepsilon) \to D\phi(y_0) + D\chi(y_0) = D\phi(y_0) ,$$

and

$$D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon) \to D^2\phi(y_0) + D^2\chi(y_0) = D^2\phi(y_0) .$$

Moreover, because of the local uniform convergence of u_ε, we have $u_\varepsilon(y^\varepsilon) \to u(y_0)$, and the convergence of F_ε finally yields

$$F_\varepsilon\Big(y^\varepsilon, u_\varepsilon(y^\varepsilon), D\phi(y^\varepsilon) + D\chi(y^\varepsilon), D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon)\Big)$$
$$\to F\Big(y_0, u(y_0), D\phi(y_0), D^2\phi(y_0)\Big) .$$

Therefore

$$F\Big(y_0, u(y_0), D\phi(y_0), D^2\phi(y_0)\Big) \le 0.$$

And the proof is complete.

5 Uniqueness: The Basic Arguments and Additional Recipes

5.1 A First Basic Result

In this section, we present the basic arguments to obtain "comparison results" for viscosity solutions. In order to simplify the presentation, we begin with a simple result and then we show (few) additional arguments which are needed in order to extend it to different situations.

We consider the equation

$$u_t + H(x,t,Du) = 0 \quad \text{in } \Omega \times (0,T) \;, \tag{16}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $T > 0$ and, here, Du denotes the gradient of u in the space variable x and H is a continuous function. We use the (standard) notations

$$Q = \Omega \times (0,T) \;\text{ and }\; \partial_p Q = \partial\Omega \times [0,T] \cup \overline{\Omega} \times \{0\} \;.$$

$\partial_p Q$ is called the parabolic boundary of Q.

By "comparison result", we mean the following

If $u, v \in C(\overline{Q})$ are respectively subsolution and supersolution of (16) *and if $u \le v$ on $\partial_p Q$ then*

$$u \le v \quad \text{on } \overline{Q}.$$

To state and prove the main result, we use the following assumption

(H1) There exists a modulus $m : [0,+\infty) \to [0,+\infty)$ such that, for any $x, y \in \overline{\Omega}$, $t \in (0,T]$ and $p \in \mathbb{R}^N$

$$|H(x,t,p) - H(y,t,p)| \le m\big(|x-y|(1+|p|)\big) \;.$$

We recall that a modulus m is an increasing, positive function, defined on $[0,+\infty)$ such that $m(r) \to 0$ when $r \downarrow 0$.

The result is the following.

Theorem 5.1. *If (H1) holds, we have a comparison result for* (16)*. Moreover, the result remains true if we replace the hypothesis* (H1) *by either "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.*

This result means that the Maximum Principle, which is classical for elliptic and parabolic equations, extends to viscosity solutions of first-order Hamilton–Jacobi Equations.

At first glance, assumption (H1) does not seem to be a very natural assumption. We first remark that, if H is a locally Lipschitz continuous function in x for any

$t \in (0,T]$ and for any $p \in \mathbb{R}^N$, (H1) is satisfied if there exists a constant $C > 0$, such that, for any $t \in (0,T]$ and $p \in \mathbb{R}^N$

$$|\frac{\partial H}{\partial x}(x,t,p)| \leq C(1+|p|) \quad \text{a.e. in } \mathbb{R}^N .$$

This version of (H1) is perhaps easier to understand.

In order to justify (H1), let us consider the case of the transport equation

$$u_t - b(x)\cdot Du = f(x) \quad \text{in } Q \ . \tag{17}$$

It is clear that the hypothesis (H1) is satisfied if b is a Lipschitz continuous vector field on $\overline{\Omega}$ and the function f has to be continuous on $\overline{\Omega}$.

In this example, the Lipschitz assumption on b is the most restrictive and important in order to have (H1): we will see in the proof of Theorem 5.1 the central role of the term $|x-y|.|p|$ in (H1) which comes from this hypothesis. But it is well-known that the properties of (17) are connected to those of the dynamical system

$$\dot{x}(t) = b(x(t)) \ . \tag{18}$$

Indeed, one can compute the solutions of (17) by solving this ode through the *Method of Characteristics*. Therefore the Lipschitz assumption on b appears as being rather natural since it is also the standard assumption to have existence and uniqueness for (18) by the Cauchy–Lipschitz Theorem.[3]

Remark 5.1. It is worth pointing out that, in Theorem 5.1, no assumption is made on the behavior of H en p (except indirectly with the restrictions coming from (H1)). For example, one has a uniqueness result for the equation

$$u_t + H(Du) = f(x,t) \quad \text{in } Q \ ,$$

if f is continuous on Q, for *any* continuous function H, without any growth condition.

There are a lot of variations for Theorem 5.1: for example, one can play with (H1) and the regularity of the solutions (as it is already the case in the statement of Theorem 5.1).

A classical and useful corollary of Theorem 5.1 is the one when we do not assume anything on the sub and supersolution on the parabolic boundary of Q

[3] In Biton [19], a non-trivial counterexample to the uniqueness for (17) is given in a situation where the Cauchy–Lipschitz Theorem cannot be applied to (18).

Corollary 5.1. *Under the assumptions of Theorem 5.1, if $u, v \in C(\overline{Q})$ are respectively sub and supersolutions of (16) then*

$$\max_{\overline{Q}} (u-v)^+ \leq \max_{\partial_p Q} (u-v)^+ \ .$$

Moreover, the result remains true if we replace (H1) by "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.

The proof of Corollary 5.1 is immediate by remarking that, if we set $C = \max_{\partial_p Q} (u-v)^+$, $v + C$ is still a supersolution of (16) and $u \leq v + C$ on $\partial_p Q$. Theorem 5.1 implies then $u \leq v + C$ on $\overline{Q}$, which is the desired result.

Remark 5.2. As the above proof shows it, this type of corollary is an immediate consequence of all comparison results with a suitable change on the sub or supersolution which may be more complicated depending on the dependence of H in u. We can have also more precise results by applying the comparison property on sub-intervals.

Now we turn to the *Proof of Theorem 5.1*. The aim of is to show that $M = \max_{\overline{Q}} \big(u - v\big)$ is less or equal to 0. We argue by contradiction assuming that $M > 0$.

In order to simplify the proof, we are going to make some reductions and to give preliminary results.

First, changing u in $u_\eta(x,t) := u(x,t) - \eta t$ for some $\eta > 0$ (small), we may assume without loss of generality that u a strict subsolution of (16) since u_η is a subsolution of

$$(u_\eta)_t + H(x,t,Du_\eta) \leq -\eta < 0 \quad \text{in } \Omega \times (0,T) \tag{19}$$

To complete the proof, it suffices to show that $u_\eta \leq v$ on $\overline{Q}$ for any η and then to let η tends to 0. Notice also that we still have $u_\eta \leq v$ on $\partial_p Q$. To simplify the notations and since the proof is clearly reduced to compare u_η and v, we drop the η and use the notation u instead of u_η.

Next, we consider the difficulty with $\Omega \times \{T\}$: a priori, we do not know if $u \leq v$ on this part of the boundary and a maximum point of $u - v$ (or related functions) can be located there. It is solved by the

Lemma 5.1. *If $u, v \in C(\overline{Q})$ are respectively sub and supersolutions of (16) in Q, they are also sub and supersolutions in $\Omega \times (0,T]$. More precisely the viscosity inequalities hold if the maximum or minimum points are on $\Omega \times \{T\}$.*

We leave the simple checking of this result to the reader: if (x_0, T) is a strict maximum point of $u - \varphi$, where φ is a smooth function, we consider the function $u(x,t) - \varphi(x,t) - \frac{\eta}{T-t}$ for $\eta > 0$ small enough. By Lemma 4.1, this function has a maximum point at a nearby point (x_η, t_η) ($t_\eta < T$) and $(x_\eta, t_\eta) \to (x_0, T)$; in order to conclude, it suffices to pass to the limit in the viscosity inequality at the point (x_η, t_η), remarking that the term $\frac{\eta}{T-t}$ has a positive derivative which can be dropped.

Next, since u and v are not smooth, we need an argument in order to be able to use the definition of viscosity solutions. This argument is the "doubling of variables". For $0 < \varepsilon, \alpha \ll 1$, we introduce the "test-function"

$$\psi_{\varepsilon,\alpha}(x,t,y,s) = u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} .$$

The function $\psi_{\varepsilon,\alpha}$ being continuous on $\overline{Q} \times \overline{Q}$, it achieves its maximum at a point which we denote by $(\overline{x},\overline{t},\overline{y},\overline{s})$ and we set $\overline{M} := \psi_{\varepsilon,\alpha}(\overline{x},\overline{t},\overline{y},\overline{s})$; we have dropped the dependences of $\overline{x},\overline{t},\overline{y},\overline{s}$ and $\overline{M}$ in all the parameters in order to avoid heavy notations.

Because of the "penalisation" terms $\left(\frac{|x-y|^2}{\varepsilon^2} \text{ and } \frac{|t-s|^2}{\alpha^2}\right)$ which imposes to the maximum points $(\overline{x},\overline{t},\overline{y},\overline{s})$ of $\psi_{\varepsilon,\alpha}$ to verify $(\overline{x},\overline{t}) \sim (\overline{y},\overline{s})$ if ε, α are small enough, one can think that the maximum of $\psi_{\varepsilon,\alpha}$ looks like the maximum of $u - v$. This idea is justified by the following lemma which plays a key role in the proof.

Lemma 5.2. *The following properties hold*

1. *When* $\varepsilon, \alpha \to 0$, $\overline{M} \to M$.
2. $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) \to M$ *when* $\varepsilon, \alpha \to 0$.
3. *We have*

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}, \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to 0 \quad \textit{when } \varepsilon, \alpha \to 0 .$$

Moreover, if u or v is Lipschitz continuous in x, then $\overline{p} := \dfrac{2(\overline{x}-\overline{y})}{\varepsilon^2}$ *is bounded by twice the (uniform in t) Lipschitz constant of u or v.*

4. $(\overline{x},\overline{t})$, $(\overline{y},\overline{s}) \in \Omega \times (0,T]$ *if* ε, α *are sufficiently small.*

We conclude the proof of the theorem by using the lemma. We assume that ε, α are sufficiently small in order that the last point of the lemma holds true. Since $(\overline{x},\overline{t},\overline{y},\overline{s})$ is a maximum point of $\psi_{\varepsilon,\alpha}$, $(\overline{x},\overline{t})$ is a maximum point of the function

$$(x,t) \mapsto u(x,t) - \varphi^1(x,t) ,$$

where

$$\varphi^1(x,t) = v(\overline{y},\overline{s}) + \frac{|x-\overline{y}|^2}{\varepsilon^2} + \frac{|t-\overline{s}|^2}{\alpha^2} ;$$

but u is viscosity subsolution of (19) and $(\overline{x},\overline{t}) \in \Omega \times (0,T]$, therefore

$$\frac{\partial \varphi^1}{\partial t}(\overline{x},\overline{t}) + H\left(\overline{x},\overline{t}, D\varphi^1_\varepsilon(\overline{x},\overline{t})\right) = \frac{2(\overline{t}-\overline{s})}{\alpha^2} + H\left(\overline{x},\overline{t}, \frac{2(\overline{x}-\overline{y})}{\varepsilon^2}\right) \le -\eta .$$

In the same way, $(\overline{y},\overline{s})$ is a maximum point of the function

$$(y,s) \mapsto -v(y,s) + \varphi^2(y,s) ,$$

where

$$\varphi^2(y,s) = u(\overline{x},\overline{t}) - \frac{|\overline{x}-y|^2}{\varepsilon^2} - \frac{|\overline{t}-s|^2}{\alpha^2} \; ;$$

hence $(\overline{y},\overline{s})$ is a minimum point of the function $v - \varphi^2$; but v is viscosity supersolution of (16) and $(\overline{y},\overline{s}) \in \Omega \times (0,T]$, therefore

$$\frac{\partial \varphi^2}{\partial s}(\overline{y},\overline{s}) + H\left(\overline{y},\overline{s}, D\varphi^2(\overline{y},\overline{s})\right) = \frac{2(\overline{t}-\overline{s})}{\alpha^2} + H\left(\overline{y},\overline{s}, \frac{2(\overline{x}-\overline{y})}{\varepsilon^2}\right) \geq 0 \, .$$

Then we subtract the two viscosity inequalities: recalling that $\overline{p} := \dfrac{2(\overline{x}-\overline{y})}{\varepsilon^2}$, we obtain

$$H\left(\overline{x},\overline{t},\overline{p}\right) - H\left(\overline{y},\overline{s},\overline{p}\right) \leq -\eta \, .$$

We can remark that a formal proof where we would assume that u et v are C^1 and where we could directly consider a maximum point of $u - v$, would have lead us to an analogous situation, the term $\overline{p}$ playing the role of "$Du = Dv$" at the maximum point; the fact that we keep such equality here is a key point in the proof. The only -rather important- difference is the one corresponding to the current points: $(\overline{x},\overline{t})$ for u, $(\overline{y},\overline{s})$ for v. This is where (H1) is going to play a central role.

We add and subtract the term $H(\overline{x},\overline{s},\overline{p})$ which allows us to rewrite the inequality as

$$\left(H(\overline{x},\overline{t},\overline{p}) - H(\overline{x},\overline{s},\overline{p})\right) - \left(H(\overline{y},\overline{s},\overline{p}) - H(\overline{x},\overline{s},\overline{p})\right) \leq -\eta \, .$$

In the left-hand side, the first term is related to the regularity of H in t and the second one to the regularity of H in x, namely (H1). For fixed ε, $\overline{p}$ remains bounded (say, by at most a K/ε for some constant $K > 0$) and denoting by m_H^ε the modulus of continuity of H on $\overline{Q} \times \overline{B(0,K/\varepsilon)}$, we are lead, using (H1) to

$$m_H^\varepsilon(|\overline{t}-\overline{s}|) + m\left(|\overline{x}-\overline{y}|(1+|\overline{p}|)\right) \leq -\eta \, .$$

But, on one hand, $|\overline{t}-\overline{s}| \to 0$ as $\alpha \to 0$ since the maximum point property implies that the penalisation term $\frac{|\overline{t}-\overline{s}|^2}{\alpha^2}$ is less than $R := \max(||u||_\infty, ||v||_\infty)$ (see the proof of Lemma 5.2 below) and therefore $|\overline{t}-\overline{s}| \leq (2R)^{1/2}\alpha$ while, on the other hand,

$$|\overline{x}-\overline{y}|(1+|\overline{p}|) = |\overline{x}-\overline{y}| + \frac{2|\overline{x}-\overline{y}|^2}{\varepsilon^2} \to 0 \quad \text{when } \varepsilon,\alpha \to 0 \, .$$

In order to conclude, we first fix ε and let α tend to 0 and then we let ε tend to 0. The above inequality and the properties we just recall lead us to a contradiction.

In the case when u or v is Lipschitz continuous in x, uniformly w.r.t. t, Lemma 5.2 implies that $|\overline{p}|$ is uniformly bounded and the contradiction just follows from the uniform continuity of H on $\overline{Q} \times \overline{B(0,2\tilde{K})}$, where $\tilde{K}$ denotes the Lipschitz constant of u or v, and the proof is complete.

Now we prove Lemma 5.2. Since $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ is a maximum point of $\psi_{\varepsilon,\alpha}$, we have, for any $(x,t), (y,s) \in \overline{Q}$

$$\psi_{\varepsilon,\alpha}(x,t,y,s) \leq \psi_{\varepsilon,\alpha}(\overline{x}, \overline{t}, \overline{y}, \overline{s}) = u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} = \overline{M} \ . \tag{20}$$

Choosing $x = y$ and $t = s$ in the left-hand side yields

$$u(x,t) - v(x,t) \leq \overline{M} \ , \quad \text{for all } (x,t) \in \overline{Q} \ ,$$

and, by considering the supremum in x, we obtain the inequality $M \leq \overline{M}$.

Since u, v are bounded, we can set as above $R := \max(||u||_\infty, ||v||_\infty)$ and we also have by arguing in an analogous way

$$M \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq 2R - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \ .$$

Recalling that we assume $M > 0$, we deduce

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq 2R \ .$$

In particular, $|\overline{x}-\overline{y}|, |\overline{t}-\overline{s}| \to 0$ as $\varepsilon, \alpha \to 0$.
Now we use again the inequality

$$M \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) \ . \tag{21}$$

Since $\overline{Q}$ is compact, we may assume without loss of generality that $(\overline{x}, \overline{t}), (\overline{y}, \overline{s})$ converge and this is to the same point because $|\overline{x}-\overline{y}|, |\overline{t}-\overline{s}| \to 0$ as $\varepsilon, \alpha \to 0$. We deduce from this property and (21) that

$$M \leq \liminf(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) \leq \limsup(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) \leq M \ . \tag{22}$$

As a consequence $\lim(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) = M$ and using again (21)

$$\overline{M} = u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to M \ .$$

But, since $u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) \to M$, we immediately deduce that

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to 0 \ ,$$

and we have proved the two first points of the lemma.

For the last one, it is enough to remark that, if (x,t) is a limit of a subsequence of $(\overline{x},\overline{t}), (\overline{y},\overline{s})$, then $u(x,t) - v(x,t) = M > 0$ and therefore (x,t) cannot be on $\partial_p Q$.

It just remains to prove the estimate on $\overline{p}$ if u or v is Lipschitz continuous in x, uniformly w.r.t. t. We assume, for instance, that u has this property with Lipschitz constant $\tilde{K}$, the proof with v being analogous.

We come back to (20) and we choose $x = y = \overline{y}$, $t = \overline{t}$ and $s = \overline{s}$; after straightforward computations, this yields

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} \leq u(\overline{y},\overline{t}) - u(\overline{x},\overline{t}) \leq \tilde{K}|\overline{x}-\overline{y}| \; .$$

Therefore $|\overline{p}| \leq 2\tilde{K}$. This concludes the proof of lemma.

5.2 Several Variations

The first one concerns equations with a dependence in u

$$u_t + H(x,t,u,Du) = 0 \quad \text{in } \Omega \times (0,T) \; . \tag{23}$$

Of course, an assumption is needed in order to avoid Burgers type equations which do not fall into this kind of framework. The classical one is

(H2) For any $0 < R < +\infty$, there exists $\gamma_R \in \mathbb{R}$ such that, for any $(x,t) \in Q$, $-R \leq v \leq u \leq R$ and $p \in \mathbb{R}^N$

$$H(x,t,u,p) - H(x,t,v,p) \geq \gamma_R(u-v) \; .$$

If $\gamma_R \geq 0$ for any R, then the proof follows exactly from the same arguments. Otherwise, the simplest way to reduce to this case is to make a change of variable $u \to u\exp(\gamma t)$ for some well-chosen $\gamma \in \mathbb{R}$, typically some γ_R for large enough R (larger than $||u||_\infty$). Finally we point out that, in general, (H1) is modified by allowing the modulus m to depend on R as γ_R in (H2).

Next we consider problems set in the whole space $\mathbb{R}^N$ where the lack of compactness of the domain creates additional problems. The following assumption is needed

(H3) H is uniformly continuous on $\mathbb{R}^N \times [0,T] \times \overline{B}_R$ for any $R > 0$.

We also introduce the space $BUC(\mathbb{R}^N \times [0,T])$ of the functions which are bounded, uniformly continuous on $\mathbb{R}^N \times [0,T]$. The result for (16) is the

Theorem 5.2. *Assume (H1) and (H3). If $u, v \in BUC(\mathbb{R}^N \times [0,T])$ are respectively sub and supersolution of (16) with $\Omega = \mathbb{R}^N$, then*

$$\sup_{\mathbb{R}^N\times[0,T]} (u-v) \leq \sup_{\mathbb{R}^N} (u(x,0)-v(x,0)) \ .$$

Moreover, the result remains true if we replace the hypothesis (H1) *by either "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.*

We just sketch the proof since it follows the same ideas as the proof of Theorem 5.1: for $0 < \varepsilon, \alpha, \beta \ll 1$, we introduce the test-function

$$\psi(x,t,y,s) = u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \beta(|x|^2+|y|^2) \ .$$

The main change is with the β-term: because of the non-compactness of the domain, such term is needed for the maximum of ψ to be achieved. Two technical remarks are enough to complete the proof:

1. From the proof of Lemma 5.2, it is clear that $\beta(|x|^2+|y|^2) \leq R = \max(||u||_\infty, ||v||_\infty)$ and these terms produces derivatives which are small since $|2\beta x| = 2\beta^{1/2}(\beta(|x|^2)^{1/2} \leq 2\beta^{1/2}R^{1/2}$ and the same is (of course) true for $2\beta y$. (H3) takes care of these small perturbations.
2. The proof of Lemma 5.2 is not as simple as in the compact case because the result is not true in general for any continuous functions u and v. In fact, the behavior of the maximum of ψ depends on the way we play with the different parameters. The two extreme cases are:
   - If we fix β and let first ε and α tend to 0, the maximum of ψ actually converges to $\max_{\overline{Q}}(u(x,t) - v(x,t) - 2\beta|x|^2))$ and then, if we send β tend to 0, this maximum converges to the supremum of $u - v$.
   - But, if, on the contrary, we first let β tend to 0 by fixing ε and α and then we let ε and α tend to 0, the maximum of ψ does not converges to the supremum of $u - v$ but to $\limsup_{h\downarrow 0} \sup_{|(x,t)-(y,s)|\leq h} (u(x,t) - v(y,s))$.

In general these limits are different and therefore playing with the parameters may be delicate. This explains the assumption "u or v is in $BUC(\mathbb{R}^N \times [0,T])$" in Theorem 5.2: indeed all these limits are the same in this case. In the $BUC(\mathbb{R}^N \times [0,T])$ framework, the proof follows the one of Theorem 5.1 since (21) leads to

$$\begin{aligned}
\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} &\leq u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) - M \\
&\leq u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) + u(\overline{y},\overline{s}) - v(\overline{y},\overline{s}) - M \\
&\leq u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) \ ,
\end{aligned}$$

because $u(\overline{y},\overline{s}) - v(\overline{y},\overline{s}) \leq M$. If m_u denotes a modulus of continuity of u, we have $u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) \leq m_u(|(\overline{x},\overline{t}) - (\overline{y},\overline{s})|)$ and therefore

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}+\frac{|\overline{t}-\overline{s}|^2}{\alpha^2}\leq m_u(|(\overline{x},\overline{t})-(\overline{y},\overline{s})|)\ .$$

Finally using that $|\overline{x}-\overline{y}|\leq(2R)^{1/2}\varepsilon$ and $|\overline{t}-\overline{s}|\leq(2R)^{1/2}\alpha$, we have a complete estimate of the penalisation terms.

Remark 5.3. In fact, there is a technical way which allows to avoid (partially) the above mentioned difficulty, assuming only that there exists $u_0\in BUC(\mathbb{R}^N)$ such that

$$u(x,0)\leq u_0(x)\leq v(x,0)\quad \text{in } \mathbb{R}^N\ .$$

By a standard result (exercise!), the modulus m given by (H1) satisfies: for any $\eta>0$, there exists C_η such that $m(\tau)\leq C_\eta\tau+\eta/2$. We then change the test-function into

$$\psi(x,t,y,s)=u(x,t)-v(y,s)-\exp(C_\eta t)\frac{|x-y|^2}{\varepsilon^2}-\frac{|t-s|^2}{\alpha^2}-\beta(|x|^2+|y|^2)\ .$$

The effect of the new "$\exp(C_\eta t)$"-term is to produce a positive $C_\eta\exp(C_\eta t)\frac{|x-y|^2}{\varepsilon^2}$ term in the inequality which allows to control the "bad" dependence in $\frac{|x-y|^2}{\varepsilon^2}$ and therefore allows to treat cases where we do not know that this quantity tends to 0. Clearly the α-penalisation term does not create any difficulty.

5.3 *Finite Speed of Propagation*

An important feature of time-dependent equations is the possibility of having "finite speed of propagation" type results which can be stated in the following way for $u,v\in C(\mathbb{R}^N\times[0,T])$ which are respectively sub and supersolution of (16) in $\mathbb{R}^N\times[0,T]$

There exists a constant $c>0$ such that, if $u(x,0)\leq v(x,0)$ in $B(0,R)$ for some R then $u(x,t)\leq v(x,t)$ for any x in $B(0,R-ct)$, $ct\leq R$.

The constant c is the "speed of propagation" and, of course, $B(0,R)$ can be replaced by any other ball $B(z,R)$. The key assumption for having such result is the

(H4) For any $x\in\mathbb{R}^N$, $t\in[0,T]$ and $p,q\in\mathbb{R}^N$

$$|H(x,t,p)-H(x,t,q)|\leq C|p-q|\ .$$

Theorem 5.3. *Assume (H1) and (H4). Then we have a "finite speed of propagation" type results for* (16) *in* $\mathbb{R}^N\times[0,T]$ *with a speed of propagation equal to* C.

Before giving the proof of this result, we want to point out that such result may also be obtained for sub and supersolutions which are Lipschitz continuous in space, uniformly w.r.t. t by assuming only H to be locally Lipschitz continuous in p: indeed, in that case, only bounded p and q play a role and the inequality in (H4) is satisfied if H is locally Lipschitz continuous.

Proof of Theorem 5.3. We just sketch it since it is a long but easy proof which borrows a lot of arguments from the proof of Theorem 5.1.

Lemma 5.3. *If $u, v \in C(\mathbb{R}^N \times [0,T])$ are respectively sub and supersolution of (16) in $\mathbb{R}^N \times [0,T]$, the function $w := u - v$ is a subsolution of*

$$w_t - C|Dw| = 0 \quad \text{in } \mathbb{R}^N \times (0,T)\ . \tag{24}$$

Formally the result is obvious since it suffices to subtract the inequalities for u and v and use (H4). But to show it in the viscosity sense is a little bit more technical. Again we just sketch the proof: if (x_0, t_0) is a *strict* maximum point of $w - \varphi$ where φ is a smooth test-function, we introduce the function

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \varphi(x,t)\ .$$

If (x_0, t_0) is a *strict* maximum point of $w - \varphi$ in $\overline{B((x_0,t_0),r)}$, we look at maximum points of this function in $\overline{B((x_0,t_0),r)} \times \overline{B((x_0,t_0),r)}$. Because of the compactness of the domain, the maximum is achieved at a point $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ and one easily shows that $(\overline{x}, \overline{t}), (\overline{y}, \overline{s}) \to (x_0, t_0)$ as $\varepsilon, \alpha \to 0$; in particular $(\overline{x}, \overline{t}), (\overline{y}, \overline{s})$ are in $B((x_0,t_0),r)$ for ε, α small enough. Writing the viscosity inequalities, following the arguments of the proof of Theorem 5.1 and using (H4), one concludes easily.

The next step consists in showing that, if $w(x,0) \leq 0$ in $B(0,R)$ for some R, then $w(x,t) \leq 0$ for any x in $B(0, R - Ct)$, $Ct \leq R$, which is equivalent to the "finite speed of propagation" type results. To do so, it is enough to build a suitable sequence of (smooth) supersolutions.

We introduce smooth functions $\chi_\delta : \mathbb{R} \to \mathbb{R}$ such that $\chi_\delta(r) \equiv 0$ for $r \leq R - \delta$, $\chi_\delta(r) \equiv M$ for $r \geq R$, where $M = \max_{\overline{B(0,R)} \times [0,T]} w(x,t)$ and χ_δ is increasing in $\mathbb{R}$. Next we consider the functions $\chi_\delta(|x| + Ct)$; it is immediate to check that this function is a smooth solution of (24) for $Ct \leq R - \delta$, i.e. for $t \leq t_\delta := (R-\delta)/C$ and that, on $\partial B(0,R) \times [0, t_\delta]$ and $B(0,R) \times \{0\}$, $w(x,t) \leq \chi_\delta(|x| + Ct)$. Applying Theorem 5.1 in $B(0,R) \times [0,t_\delta]$, we obtain that $w(x,t) \leq \chi_\delta(|x| + Ct)$ in $B(0,R) \times [0,t_\delta]$ and therefore, by the properties of χ_δ, $w(x,t) \leq 0$ for $|x| + Ct \leq R + \delta$. Letting δ tend to 0 gives the complete answer.

Remark 5.4. In fact, we do not really need a comparison result, namely Theorem 5.1, to conclude: the last part of the proof follows from the definition of viscosity (sub)solution. Indeed the function $\chi_\delta(|x| + Ct) + \delta t$ is a smooth *strict* supersolution in $B(0,R) \times (0, t_\delta)$; this shows that $w(x,t) - (\chi_\delta(|x| + Ct) + \delta t)$ cannot achieve a maximum point in $B(0,R) \times (0,T]$, which immediately leads to the conclusion.

6 Discontinuous Viscosity Solutions, Discontinuous Nonlinearities and the "Half-Relaxed Limits" Method

The main objective of this section is to present a general method, based on the notion of discontinuous viscosity solutions, which allows passage to the limit in (fully) nonlinear pdes with just an L^∞-bounds on the solutions. To do so, we have to extend the notion of viscosity solution to the discontinuous setting. We refer to Ishii [30,31], Perthame and the author [8,9] for the notion of discontinuous viscosity solutions, the half-relaxed limits method being introduced in [8].

We use the following notations: if z is a locally bounded function (possibly discontinuous), we denote by z^* its upper semicontinuous (usc) envelope

$$z^*(x) = \limsup_{y \to x}\ z(y)\,,$$

and by z_* its lower semicontinuous (lsc) envelope

$$z_*(x) = \liminf_{y \to x}\ z(y)\,.$$

6.1 Discontinuous Viscosity Solutions

The definition is the following.

Definition 6.1 (Discontinuous Viscosity Solutions). A locally bounded upper semicontinuous (usc in short) function u is a viscosity subsolution of the equation

$$G(y, u, Du, D^2u) = 0 \quad \text{on } \overline{\mathcal{O}} \tag{25}$$

if and only if, for any $\varphi \in C^2(\overline{\mathcal{O}})$, if $y_0 \in \overline{\mathcal{O}}$ is a maximum point of $u - \varphi$, one has

$$G_*(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \le 0.$$

A locally bounded lower semicontinuous (lsc in short) function v is a viscosity supersolution of the (25) **if and only if**, for any $\varphi \in C^2(\overline{\mathcal{O}})$, if $y_0 \in \overline{\mathcal{O}}$ is a minimum point of $u - \varphi$, one has

$$G^*(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \ge 0\,.$$

A (discontinuous) solution is a function whose usc and lsc envelopes are respectively viscosity sub and supersolution of the equation.

The first reason to introduce such a complicated formulation is to unify the convergence result we present in the next section: in fact, when $\mathcal{O}$ is an open subset different from $\mathbb{R}^N$, the function G may contain both the equation and the boundary condition. With such general formulation, we avoid to have a different result for each type of boundary conditions. The possibility of handling discontinuous sub and supersolutions is also a key point in the convergence proof.

To be more specific, we consider the problem

$$\begin{cases} F(y,u,Du,D^2u) = 0 & \text{in } \mathscr{O}, \\ B(y,u,Du) = 0 & \text{on } \partial\mathscr{O}, \end{cases}$$

where F, B are a given continuous functions.

In order to solve it, a classical idea consists in considering the vanishing viscosity method

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + F(y,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon) = 0 & \text{in } \mathscr{O}, \\ B(y,u_\varepsilon,Du_\varepsilon) = 0 & \text{on } \partial\mathscr{O}. \end{cases}$$

Indeed, by adding a $-\varepsilon\Delta$ term, we regularize the equation in the sense that one can expect to have more regular solutions for this approximate problem—typically in $C^2(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$.

If we assume that this is indeed the case, i.e. that this regularized problem has a smooth solution u_ε and that, moreover, $u_\varepsilon \to u$ in $C(\overline{\mathscr{O}})$. It is easy to see, by the arguments of Theorem 4.1, that the continuous function u satisfies in the viscosity sense

$$\begin{cases} F(y,u,Du,D^2u) = 0 & \text{in } \mathscr{O}, \\ \min(F(y,u,Du,D^2u), B(y,u,Du)) \le 0 & \text{on } \partial\mathscr{O}, \\ \max(F(y,u,Du,D^2u), B(y,u,Du)) \ge 0 & \text{on } \partial\mathscr{O}, \end{cases}$$

where, for example, the "min" inequality on $\partial\mathscr{O}$ means: *for any $\varphi \in C^2(\overline{\mathscr{O}})$, if $y_0 \in \partial\mathscr{O}$ is a maximum point of $u-\varphi$ on $\overline{\mathscr{O}}$, one has*

$$\min(F(y_0,u(y_0),D\varphi(y_0),D^2\varphi(y_0)), B(y,u(y_0),Du(y_0))) \le 0.$$

The interpretation of this new problem can be done by setting the equation in $\overline{\mathscr{O}}$ instead of $\mathscr{O}$. To do so, we introduce the function G defined by

$$G(y,u,p,M) = \begin{cases} F(y,u,p,M) & \text{if } y \in \mathscr{O}, \\ B(y,u,p) & \text{if } y \in \partial\mathscr{O}. \end{cases}$$

The above argument shows that the function u is a viscosity solution of

$$G(y,u,Du,D^2u) = 0 \quad \text{on } \overline{\mathscr{O}}\ ,$$

and in particular on $\overline{\mathscr{O}}$, if

$$G_*(y,u,Du,D^2u) \le 0 \quad \text{on } \overline{\mathscr{O}}$$

$$G^*(y,u,Du,D^2u) \ge 0 \quad \text{on } \overline{\mathscr{O}}$$

where G_* and G^* stand respectively for the lower semicontinuous and upper semicontinuous envelopes of G. Indeed, the "min" and the "max" above are nothing but G_* and G^* on $\partial\mathcal{O}$.

6.2 Back to the Running Example (II): The Dirichlet Boundary Condition for the Value-Function

In this subsection, we show that the value function of the exit time control problem actually satisfy the Dirichlet boundary condition in the viscosity sense.

To do so, we use a more sophisticated version of the Dynamic Programming Principle.

Theorem 6.1. *Under the assumptions* **(CA)***, the value-function satisfies, for any* $x \in \overline{\Omega}$*,* $t > 0$ *and* $0 < S < t$

$$\mathbf{U}(x,t) = \inf_{v(.)}\left[\int_0^{S\wedge\tau} f(y_x(s),\alpha(s))ds + \mathbf{1}_{\{S<\tau\}}\mathbf{U}(y_x(S),t-S) + \mathbf{1}_{\{S\geq\tau\}}\varphi(y_x(\tau))\right]. \tag{26}$$

In order to understand why this formulation leads naturally to boundary conditions in the viscosity solutions sense, we consider $x \in \partial\Omega$, $0 < t < T$ and a sequence $(x_\varepsilon, t_\varepsilon)$ converging to (x,t) such that $\mathbf{U}(x_\varepsilon,t_\varepsilon) \to \mathbf{U}_*(x,t)$. We apply the Dynamic Programming Principle at the point $(x_\varepsilon,t_\varepsilon)$. We argue formally assuming that there exists an optimal control $\alpha_\varepsilon(\cdot)$ in such a way that we have

$$\begin{aligned}\mathbf{U}(x_\varepsilon,t_\varepsilon) = &\int_0^{S\wedge\tau_\varepsilon} f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{1}_{\{S<\tau_\varepsilon\}}\mathbf{U}(y_{x_\varepsilon}(S),t_\varepsilon - S)\\ &+ \mathbf{1}_{\{S\geq\tau_\varepsilon\}}\varphi(y_{x_\varepsilon}(\tau_\varepsilon))\ .\end{aligned}$$

Here there are two cases:

(i) Either $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$ and letting ε tends to 0, we obtain (formally) $\mathbf{U}_*(x,t) = \varphi(x)$.

(ii) Or τ_ε remains bounded away from 0 and by choosing S small enough, we have

$$\mathbf{U}(x_\varepsilon,t_\varepsilon) = \int_0^S f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{U}(y_{x_\varepsilon}(S),t_\varepsilon - S)\ ,$$

which, since $\mathbf{U} \geq \mathbf{U}_*$ on $\overline{\Omega}$ can be rewritten as

$$\mathbf{U}_*(x,t) + o_\varepsilon(1) \geq \int_0^S f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{U}_*(y_{x_\varepsilon}(S),t_\varepsilon - S)\ ,$$

a similar situation to the case when $x \in \Omega$. Playing with ε and S (or fixing S and using relaxed controls to pass to the limit $\varepsilon \to 0$), it is easy to show that the supersolution inequality holds.

In conclusion, boundary conditions in the viscosity solutions sense are natural from the optimal control point of view since they take into account the strategy of the controller and/or the controlability properties of the system. Indeed, we obtain $\mathbf{U}_*(x,t) \geq \varphi(x)$ [i.e. we are in the case (i)] if either it is interesting in term of cost to pay φ (and if we can exit the domain to do it) or, on the contrary, if we are obliged to exit the domain, even if this cost is high. Case (ii) may arise either if we want to avoid paying the cost φ (and if some control allows to do it) or if we have no choice but to go away from the boundary.

These interpretations for the "min" and "max" inequalities are important since they connect the control problem and its properties with the equation and the boundary conditions.

6.3 The Half-Relaxed Limit Method

The first key point is a stability result for discontinuous viscosity solutions. To state it we use the following notations: if $(z_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions, the half-relaxed limits of $(z_\varepsilon)_\varepsilon$ are defined by

$$\limsup{}^* z_\varepsilon(y) = \limsup_{\substack{\tilde{y}\to y\\ \varepsilon\to 0}} z_\varepsilon(\tilde{y}) \quad \text{and} \quad \liminf{}_* z_\varepsilon(y) = \liminf_{\substack{\tilde{y}\to y\\ \varepsilon\to 0}} z_\varepsilon(\tilde{y}) \;.$$

Theorem 6.2. *Assume that, for $\varepsilon > 0$, u_ε is an usc viscosity subsolution (resp. a lsc supersolution) of the equation*

$$G_\varepsilon(y, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0 \quad \text{on } \overline{\mathscr{O}} \;,$$

where $(G_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions in $\overline{\mathscr{O}} \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N$ which satisfy the ellipticity condition. If the functions u_ε are uniformly locally bounded on $\overline{\mathscr{O}}$, then $\overline{u} = \limsup^*\ u_\varepsilon$ (resp. $\underline{u} = \liminf_*\ u_\varepsilon$) is a subsolution (resp. a supersolution) of the equation

$$\underline{G}(y, u, Du, D^2u) = 0 \quad \text{on } \overline{\mathscr{O}} \;,$$

where $\underline{G} = \liminf_*\ G_\varepsilon$.
(resp. of the equation

$$\overline{G}(y, u, Du, D^2u) = 0 \quad \text{on } \overline{\mathscr{O}} \;,$$

where $\overline{G} = \limsup^*\ G_\varepsilon$).

Of course, the main interest of this result is to allow the passage to the limit in fully nonlinear, degenerate elliptic pdes with only a uniform local L^∞—bound on the solutions. This is a striking difference with Theorem 4.1 which requires far more informations on the u_ε's. The counterpart is that we do not have anymore a limit but two half-limits $\overline{u}$ and $\underline{u}$ which have to be connected in order to obtain a real convergence result.

This is the aim of the *half-relaxed limit method*:

1. One proves that the u_ε are uniformly bounded in L^∞ (locally or globally).
2. One applies the above discontinuous stability result.
3. By definition, we have $\underline{u} \leq \overline{u}$ on $\overline{\mathcal{O}}$.
4. To obtain the converse inequality, one uses a *Strong Comparison Result* (SCR in short) i.e. a comparison result which is valid for *discontinuous* sub and supersolutions. It yields
$$\overline{u} \leq \underline{u} \quad \text{in } \mathcal{O} \text{ (or on } \overline{\mathcal{O}}).$$
5. From the SCR, we deduce $\overline{u} = \underline{u}$ in $\mathcal{O}$ (or on $\overline{\mathcal{O}}$). If we set $u := \overline{u} = \underline{u}$, then u is continuous (because $\overline{u}$ is usc and $\underline{u}$ is lsc) and it is easy to show that, on one hand, u is *the unique solution* of the limiting equation (using again the SCR) and, on the other hand, we have the convergence of u_ε to u in $C(\mathcal{O})$ (or in $C(\overline{\mathcal{O}})$).

It is clear that, in this method, SCR play a central role: we give in the next subsection few indications on how to prove such results and references on the existing SCR.

We first describe a typical example of the use of Theorem 6.2.

Example 6.1. We consider the problem

$$\begin{cases} -\varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = 1 \text{ in } (0,1) \\ \quad u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases}$$

Of course, it is expected that the solution of this problem converges to the solution of

$$\begin{cases} \quad u'(x) = 1 \quad & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

But the solution of this problem does not seem to exist.

The solution u_ε can be computed explicitly

$$u_\varepsilon(x) = x - \frac{\exp(\varepsilon^{-1}(x-1)) - \exp(-\varepsilon^{-1})}{1 - \exp(-\varepsilon^{-1})} ,$$

and therefore we can also compute the half-relaxed limits of the sequence $(u_\varepsilon)_\varepsilon$

$$\overline{u}(x) = x \quad \text{and} \quad \underline{u}(x) = \begin{cases} x & \text{if } x \in [0,1) \\ 0 & \text{for } x = 1 \ . \end{cases}$$

By Theorem 6.2, these half-relaxed limits are respectively sub and supersolution of

$$u'(x) - 1 = 0 \quad \text{in } (0,1)\ ,$$
$$\min(u'(x) - 1, u) \le 0 \quad \text{at } x = 0 \text{ and } 1\ ,$$
$$\max(u'(x) - 1, u) \ge 0 \quad \text{at } x = 0 \text{ and } 1\ .$$

The problem is, of course, at the point $x = 1$ where $\overline{u}$ is 1 while $\underline{u}$ is 0. Several remarks: this fact is a consequence of the *boundary layer* near 1 since u_ε looks like x but it has also to satisfies the Dirichlet boundary condition $u_\varepsilon(1) = 0$. A clear advantage of Theorem 6.2 is that we can pass to the limit despite of this boundary layer. Of course, there is no hope here to apply Theorem 4.1. But the price to pay is that $\overline{u}(1)$ is different from $\underline{u}(1)$.

In order to recover the right result, namely the convergence in $[0,1)$ of u_ε to x, the SCR has to take care of this difference and this is done by "erasing" the "wrong" value of $\underline{u}$ at 1. This explains why we wrote above that we can compare $\overline{u}$ and $\underline{u}$ either in $\mathcal{O}$ or on $\overline{\mathcal{O}}$: here we can do it only in $\mathcal{O} := (0,1)$ (and even in $[0,1)$).

Now we give the *Proof of Theorem 6.2*. We do it only for the subsolution case, the supersolution one being analogous.

It is based on the

Lemma 6.1. *Let $(v_\varepsilon)_\varepsilon$ be a sequence of uniformly bounded usc functions on $\overline{\mathcal{O}}$ and $\overline{v} = \limsup^* v_\varepsilon$. If $y \in \overline{\mathcal{O}}$ is a strict local maximum point of $\overline{v}$ on $\overline{\mathcal{O}}$, there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ of $(v_\varepsilon)_\varepsilon$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ of points in $\overline{\mathcal{O}}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $v_{\varepsilon'}$ in $\overline{\mathcal{O}}$, the sequence $(y_{\varepsilon'})_{\varepsilon'}$ converges to y and $v_{\varepsilon'}(y_{\varepsilon'}) \to \overline{v}(y)$.*

We first prove Theorem 6.2 by using the lemma. Let $\varphi \in C^2(\overline{\mathcal{O}})$ and let $y \in \overline{\mathcal{O}}$ be a strict local maximum point de $\overline{u} - \varphi$. We apply Lemma 6.1 to $v_\varepsilon = u_\varepsilon - \varphi$ and $\overline{v} = \overline{u} - \varphi = \limsup^* (u_\varepsilon - \varphi)$. There exists a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $u_{\varepsilon'} - \varphi$ on $\overline{\mathcal{O}}$. But $u_{\varepsilon'}$ is a subsolution of the $G_{\varepsilon'}$-equation, therefore

$$G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'}), D^2\varphi(y_{\varepsilon'})) \le 0.$$

Since $y_{\varepsilon'} \to x$ and since φ is smooth $D\varphi(y_{\varepsilon'}) \to D\varphi(y)$ and $D^2\varphi(y_{\varepsilon'}) \to D^2\varphi(y)$; but we have also $u_{\varepsilon'}(y_{\varepsilon'}) \to \overline{u}(y)$, therefore by definition of $\underline{G}$

$$\underline{G}(x, \overline{u}(y), D\varphi(y), D^2\varphi(y)) \le \liminf G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'}), D^2\varphi(y_{\varepsilon'}))\ .$$

This immediately yields

$$\underline{G}(x, \overline{u}(y), D\varphi(y), D^2\varphi(y)) \le 0,$$

and the proof is complete.

Now we turn to the *Proof of Lemma 6.1*: since y is a strict local maximum point of $\overline{v}$ on $\overline{\mathscr{O}}$, there exists $r > 0$ such that

$$\forall z \in \overline{\mathscr{O}} \cap \overline{B}(y,r)\ , \quad \overline{v}(z) \leq \overline{v}(y)\ ,$$

the inequality being strict for $z \neq y$. But $\overline{\mathscr{O}} \cap \overline{B}(y,r)$ is compact and v_ε is usc, therefore, for all $\varepsilon > 0$, there exists a maximum point y^ε of v_ε on $\overline{\mathscr{O}} \cap \overline{B}(y,r)$. In other words

$$\forall z \in \overline{\mathscr{O}} \cap \overline{B}(y,r)\ , \quad v_\varepsilon(z) \leq v_\varepsilon(y^\varepsilon)\ . \tag{27}$$

Now we take the lim sup for $z \to y$ and $\varepsilon \to 0$: by the definition of the lim sup* , we obtain

$$\overline{v}(y) \leq \limsup_{\varepsilon}\ v_\varepsilon(y^\varepsilon)\ .$$

Next we consider the right-hand side of this inequality: extracting a subsequence denoted by ε', we have $\limsup_\varepsilon\ v_\varepsilon(y^\varepsilon) = \lim_{\varepsilon'} v_{\varepsilon'}(y_{\varepsilon'})$ and since $\overline{\mathscr{O}} \cap \overline{B}(y,r)$ is compact, we may also assume that $y_{\varepsilon'} \to \overline{y} \in \overline{\mathscr{O}} \cap \overline{B}(y,r)$. But using again the definition of the lim sup* at $\overline{y}$, we get

$$\overline{v}(y) \leq \limsup_{\varepsilon}\ v_\varepsilon(y^\varepsilon) = \lim_{\varepsilon'} v_{\varepsilon'}(y_{\varepsilon'}) \leq \overline{v}(\overline{y})\ .$$

Since y is a strict maximum point of $\overline{v}$ in $\overline{\mathscr{O}} \cap \overline{B}(y,r)$ and that $\overline{y} \in \overline{\mathscr{O}} \cap \overline{B}(y,r)$, this inequality implies that $\overline{y} = y$ and that $v_{\varepsilon'}(y_{\varepsilon'}) \to \overline{v}(y)$ and the proof is complete.

We conclude this subsection by the

Lemma 6.2. *If $\mathscr{K}$ is a compact subset of $\overline{\mathscr{O}}$ and if $\overline{u} = \underline{u}$ on $\mathscr{K}$ then u_ε converges uniformly to the function $u := \overline{u} = \underline{u}$ on $\mathscr{K}$.*

Proof of Lemma 6.2. Since $\overline{u} = \underline{u}$ on $\mathscr{K}$ and since $\overline{u}$ is usc and $\underline{u}$ is lsc on $\overline{\mathscr{O}}$, u is continuous on $\mathscr{K}$.

We first consider $M_\varepsilon = \sup\limits_{\mathscr{K}} \left(u_\varepsilon^* - u\right)$. The function u_ε^* being usc and u being continuous, this supremum is in fact a maximum and is achieved at a point y^ε. The sequence $(u_\varepsilon)_\varepsilon$ being locally uniformly bounded, the sequence $(M_\varepsilon)_\varepsilon$ is also bounded and, $\mathscr{K}$ being compact, we can extract subsequences such that $M_{\varepsilon'} \to \limsup_\varepsilon M_\varepsilon$ and $y_{\varepsilon'} \to \overline{y} \in \mathscr{K}$. But by the definition of the lim sup* , $\limsup u_{\varepsilon'}^*(y_{\varepsilon'}) \leq \overline{u}(\overline{y})$ while we have also $u(y_{\varepsilon'}) \to u(\overline{y})$ by the continuity of u. We conclude that

$$\limsup_{\varepsilon}\ M_\varepsilon = \lim_{\varepsilon'}\ M_{\varepsilon'} = \lim_{\varepsilon'}\ u_{\varepsilon'}^*(y_{\varepsilon'}) - u(y_{\varepsilon'}) \leq \overline{u}(\overline{y}) - u(\overline{y}) = 0\ .$$

This part of the proof gives half of the uniform convergence, the other part being obtained analogously by considering $\tilde{M}_\varepsilon = \sup\limits_{\mathscr{K}}\ (u - (u_\varepsilon)_*)$.

6.4 Strong Comparison Results

In general, this is clearly THE difficulty when applying the half-relaxed limit method.

The basic comparison result we have already proved, namely Theorem 5.1, is in fact a SCR: we use the continuity of u and v only once to obtain that $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) \to M$ and then an estimate on the penalization terms through the inequality

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \le u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) - M \to 0 .$$

But, if $(\overline{x},\overline{t}), (\overline{y},\overline{s}) \to (x_0,t_0)$, we have $\limsup u(\overline{x},\overline{t}) \le u(x_0,t_0)$ because u is usc and $\liminf v(\overline{y},\overline{s}) \ge v(x_0,t_0)$ because v is lsc, and therefore $\limsup(u(\overline{x},\overline{t}) - v(\overline{y},\overline{s})) \le M$, which is enough to obtain both the convergence of $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s})$ to M and the right property for the penalization terms.

For problem with boundary conditions:

(a) One has general SCR for Neumann BC (even for second-order equations): see [6, 34].
(b) Dirichlet boundary conditions present more difficulties, at least when they are not assumed in a classical sense: we refer to [5, 9, 10] for first-order problems and [12] for second-order problems.

We come back again to our running example and provide a Strong Comparison Result for the Dirichlet problem of the exit time control problem.

Theorem 6.3. *Under the above assumptions, if Ω is a $W^{2,\infty}$-domain and if there exists $\nu > 0$ such that, for any $x \in \partial\Omega$, there exists $\alpha_x^1, \alpha_x^2 \in V$ such that*

$$b(x,\alpha_x^1)\cdot n(x) \ge \nu \quad \text{and} \quad b(x,\alpha_x^2)\cdot n(x) \le -\nu , \tag{28}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x, then we have a Strong Comparison Result for (7)–(9), namely if u and v are respectively sub and supersolution of (7)–(9), then

$$u \le v \quad \text{on } \Omega .$$

We first comment Assumption (28): it is a (partial) controlability assumption on the boundary; roughly speaking, it means that, in a neighborhood of each point $x \in \partial\Omega$, the controller has both the possibility to leave Ω by using α_x^1 or to stay inside Ω by using α_x^2.

It is also worth pointing out that we can compare u and v only in Ω: unfortunately, as Example 6.1 shows it, the boundary conditions in the viscosity sense (at least in the Dirichlet case) do not impose strong enough constraints on the boundary and one may have "artificial" values for u and/or v. This is why we have to redefine u and/or v on the boundary in the proof of the SCR and also why the result holds only in Ω.

The program to study such control problems and obtain that the value-function is continuous and the unique solution of the associated Bellman problem is the following:

(a) Show that one has a dynamic programming principle for the control problem: in general, this is easy for deterministic problems, more technical for stochastic ones because of measurability issues. An alternative solution consists in arguing by approximation.
(b) Deduce that, if $\mathbf{U}$ is the value function, then $\mathbf{U}^*$ and $\mathbf{U}_*$ are respectively viscosity sub and supersolution of the Bellman problem.
(c) Use the Strong Comparison Result to prove that $\mathbf{U}^* \leq \mathbf{U}_*$ which shows that $\mathbf{U} := \mathbf{U}^* = \mathbf{U}_*$ is continuous since it is both upper and lower semicontinuous.
(d) Use again the Strong Comparison Result to obtain the uniqueness result.

7 Existence of Viscosity Solutions: Perron's Method

Perron's method was introduced in the context of viscosity solutions by Ishii [31]. We present the main arguments in the case of (16) together with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{29}$$

where $u_0 \in BUC(\mathbb{R}^N)$.

The result is the

Theorem 7.1. *Assume (H1), (H3) and that $u_0 \in BUC(\mathbb{R}^N)$. For any $T > 0$, there exists a unique viscosity solution u of (16)–(29) in $BUC(\mathbb{R}^N \times [0,T])$.*

Proof of Theorem 7.1. We denote by $M = ||u_0||_\infty$ and $C = \sup_{\mathbb{R}^N \times [0,T]} H(x,t,0)$.

The functions $\underline{u}(x,t) := -M - Ct$ and $\overline{u}(x,t) := M + Ct$ are respectively sub and supersolution of (16); moreover

$$\underline{u}(x,0) \leq u_0(x) \leq \overline{u}(x,0) \quad \text{in } \mathbb{R}^N \ .$$

We denote by $\mathscr{S}$ the set of all usc subsolutions w of (16) such that $\underline{u} \leq w \leq \overline{u}$ in $\mathbb{R}^N \times [0,T]$ and which satisfies $w(x,0) \leq u_0(x)$ in $\mathbb{R}^N$. Then we set

$$u(x,t) = \sup\{w(x,t) : \ w \in \mathscr{S}\} \, .$$

The first step consists in showing that u^* is a (possibly discontinuous) viscosity subsolution of (16). The proof of this claim comes from three types of arguments:

1. If u_1 and u_2 are usc functions then $D^{2,+}[\sup(u_1,u_2)] \subset D^{2,+}u_1 \cap D^{2,+}u_2$, a property which immediately yields that the supremum of two subsolutions (and then of a finite number of subsolutions) is a subsolution.

2. Next the discontinuous stability result allows to extend this result to a countable number of subsolutions. In this case, the supremum of a countable number of usc functions is not necessarily usc and one has to use an usc envelope: this is done automatically by the $\limsup^*$ operation.
3. In order to prove that u^* is a subsolution of (16), we have to extend Point 2 to any set of subsolutions. We remark that, for a given point (x,t), there exists a sequence $(w_n)_n$ of elements of $\mathcal{S}$ such that, if

$$v_n(y,s) := \sup_{0\le k\le n} w_k(y,s) ,$$

then

$$u^*(x,t) = \limsup{}^* \ v_n(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} v_n(y,s) = \left(\sup_{k\in\mathbb{N}} w_k(y,s)\right)^* .$$

This leads us to introduce the function $\tilde{u} := \limsup^* \ v_n$ which is a subsolution of (16) by Point 2. To conclude, we use an analogous argument to the one of Point 1. If u_1 and u_2 are usc functions such that $u_1 \le u_2$ and $u_1(x,t) = u_2(x,t)$ for some point (x,t) then $D^{2,+}u_2(x,t) \subset D^{2,+}u_1(x,t)$. Applying this result with $u_1 = \tilde{u}$ and $u_2 = u^*$ shows that u^* satisfies the subsolution inequalities at (x,t) since $\tilde{u}$ does. Since this is true for any point (x,t), we have proved that u^* is a subsolution of (16) and also that u is usc since, by definition, $u \ge u^*$ because $u^* \in \mathcal{S}$.

The next step consists in showing that u_* is a viscosity supersolution of (16). To do so, we argue by contradiction assuming that there exists a smooth function ϕ such that $u_* - \phi$ has a global minimum point at some $(\overline{x},\overline{t})$ for $\overline{t} > 0$ and

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t}, D\phi(\overline{x},\overline{t})) < 0 . \tag{30}$$

We may assume without loss of generality that $u_*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t})$. For $\varepsilon > 0$, we consider the functions

$$w_\varepsilon(x,t) = \max\{u(x,t), \phi_\varepsilon(x,t)\},$$

where $\phi_\varepsilon(x,t) := \phi(x,t) + \varepsilon - |x-\overline{x}|^4 - |t-\overline{t}|^4$.

Since $\phi \le u_* \le u$ and $u_*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t})$, w_ε can differ from u only in a small neighborhood of $(\overline{x},\overline{t})$ and more precisely where $|x-\overline{x}|^4 + |t-\overline{t}|^4 \le \varepsilon$. And we point out that this neighborhood becomes smaller and smaller with ε. Using (30), we see that ϕ and therefore ϕ_ε are subsolution of (16) in a small neighborhood of $(\overline{x},\overline{t})$. This implies that w_ε is still a subsolution of (16) as the supremun of two subsolutions, if we choose ε small enough.

Next we want to prove that $w_\varepsilon \in \mathscr{S}$ and to do so, it remains to show that $w_\varepsilon \leq \overline{u}$, at least if ε is small enough. Since this is true for u, we have just to check it for ϕ_ε and for $|x-\overline{x}|^4 + |t-\overline{t}|^4 \leq \varepsilon$, i.e. close enough to $(\overline{x}, \overline{t})$.

By the same argument as in Point 3 above, we cannot have $u_*(\overline{x}, \overline{t}) = \overline{u}(\overline{x}, \overline{t})$: otherwise, since $u_* \leq \overline{u}$, $D^{2,-}u_*(\overline{x}, \overline{t}) \subset D^{2,+}\overline{u}(\overline{x}, \overline{t})$ and u_* would satisfies the supersolutions inequalities at $(\overline{x}, \overline{t})$. Therefore $u_*(\overline{x}, \overline{t}) = \phi(\overline{x}, \overline{t}) = \phi_\varepsilon(\overline{x}, \overline{t}) < \overline{u}(\overline{x}, \overline{t})$ and, for ε small enough, the last inequality remains true in a neighborhood by the continuity of ϕ_ε and $\overline{u}$. Hence $w_\varepsilon \in \mathscr{S}$.

This fact is a contradiction with the definition of u: indeed,

$$u_*(\overline{x}, \overline{t}) := \liminf_{(y,s)\to(\overline{x},\overline{t})} u(y,s) = \lim_k u(y_k, t_k) \ .$$

But, $w_\varepsilon(\overline{x}, \overline{t}) = u_*(\overline{x}, \overline{t}) + \varepsilon$ and by the continuity of w_ε, it is clear that, for k large enough, $u(y_k, t_k) < w_\varepsilon(y_k, t_k)$.

In fact, the above argument is not completely correct since we do not take into account the initial data. There are two ways to do it, the first one being simpler, the second one being more general.

The first solution consists in showing that u is, in fact, continuous at time $t = 0$ and that $u(x, 0) = u_0(x)$ for any $x \in \mathbb{R}^N$. To do so, we remark that, thanks to the property on the modulus of continuity recalled in Remark 5.3, since u_0 is uniformly continuous in $\mathbb{R}^N$, we have, for any $x, y \in \mathbb{R}^N$ and $\eta > 0$

$$u_0(x) - \eta/2 - C_\eta|x-y| \leq u_0(y) \leq u_0(x) + \eta/2 + C_\eta|x-y| \ ,$$

for some large constant $C_\eta > 0$. Then choosing a constant $\tilde{C}_\eta > 0$ large enough, the functions

$$u_\pm(y,t) = u_0(x) \pm \eta/2 \pm C_\eta|x-y| \pm \tilde{C}_\eta t \ ,$$

are respectively viscosity subsolution and supersolution of (16). We use these functions in the following way: on one hand, if $w \in \mathscr{S}$, $w \leq u_+$ in $\mathbb{R}^N \times [0, T]$; this inequality can be easily obtained by smoothing the term $|x-y|$ and remarking that u^+ being a strict supersolution of (16) for $\tilde{C}_\eta$ large enough, $w - u_+$ cannot achieved a maximum in $\mathbb{R}^N \times (0, T]$ (remark also that such maximum is achieved because $u_+(y,t) \to +\infty$ as $|y| \to +\infty$) and therefore it is achieved for $t = 0$ where $w \leq u_+$. On the other hand, $\max(u_-, \underline{u}) \in \mathscr{S}$. Therefore, combining these properties with the definition of u, we have

$$u_- \leq \max(u_-, \underline{u}) \leq u \leq u_+ \quad \text{ in } \mathbb{R}^N \times [0, T] \ ,$$

and, since $u_\pm$ are continuous, this yields $u_-(x,0) \leq u_*(x,0) \leq u^*(x,0) \leq u_+(x,0)$, i.e

$$u_0(x) - \eta/2 \leq u_*(x,0) \leq u^*(x,0) \leq u_0(x) + \eta/2 \ .$$

This property being true for any $\eta > 0$ and $x \in \mathbb{R}^N$, we have $u^*(x,0) \le u_0(x)$ and $u_*(x,0) \ge u_0(x)$ in $\mathbb{R}^N$, which are the desired properties since they imply that u is continuous at $(x,0)$ and $u(x,0) = u_0(x)$.

The second method to treat the initial data consists in understanding this initial data in the viscosity solution sense, i.e.

$$\min(w_t + H(x,0,Dw), w - u_0) \le 0 \quad \text{in } \mathbb{R}^N, \tag{31}$$

and

$$\max(w_t + H(x,0,Dw), w - u_0) \ge 0 \quad \text{in } \mathbb{R}^N\ . \tag{32}$$

With few modifications, the above arguments can take into account, at the same time, the equation in the domain and the initial data in this viscosity sense.

Hence u satisfies (31)–(32) but then we use the

Lemma 7.1. *If w is an usc subsolution of* (16) *satisfying* (31) *(resp. a lsc supersolution of* (16) *satisfying* (32)*), we have $w(x,0) \le u_0(x)$ (resp. $u_0(x) \le w(x,0)$) in $\mathbb{R}^N$.*

Therefore, in non-singular situations, initial data in the viscosity sense always reduce to initial data in the classical sense.

Using this lemma, Remark 5.3 shows that we can compare the subsolution u^* and the supersolution u_*; therefore

$$u^*(x,t) \le u_*(x,t) \quad \text{in } \mathbb{R}^N \times [0,T]\ .$$

But, by definition, the opposite inequality holds and we can conclude that u is continuous, the BUC-property for u coming from a careful examination of the uniqueness proof. And the existence result is complete.

Proof of Lemma 7.1. We prove the result only in the subsolution case, the supersolution one being analogous. For $x \in \mathbb{R}^N$, we introduce the function

$$\chi(y,t) = w(y,t) - \frac{|y-x|^2}{\varepsilon} - C_\varepsilon t\ ,$$

where $\varepsilon > 0$ is a parameter devoted to tend to 0 and $C_\varepsilon > 0$ is a large constant to be chosen later on.

Standard argument shows that χ has a maximum point $(\overline{y},\overline{t})$ near $(x,0)$ for small enough ε and large enough C_ε. Since w is a subsolution of (16) satisfying (31), if $\overline{t} > 0$, we have

$$C_\varepsilon + H\left(\overline{y},\overline{t},\frac{2(\overline{y}-x)}{\varepsilon}\right) \le 0.$$

But this inequality cannot hold if C_ε is chosen large enough (the size depending on ε and H but neither on $\overline{y}$ nor on $\overline{t}$ since the term $\frac{|\overline{y}-x|^2}{\varepsilon}$ is bounded). Therefore $\overline{t} = 0$ and (31) holds. But since the above inequality cannot hold, (31) implies

$w(\overline{y},0) \leq u_0(\overline{y})$. We conclude by remarking that, as $\varepsilon \to 0$, $w(\overline{y},0) \to w(x,0)$ by using the maximum point property and the upper-semicontinuity of w, while $u_0(\overline{y}) \to u_0(x)$ by the continuity of u_0.

8 Regularity Results

The aim of this section is to investigate further regularity properties for the solutions obtained through Theorem 7.1. To do so, we first strengthen assumption (H1) into

(H1-s) There exists $L_1, L_2 > 0$ such that, for any $x, y \in \Omega$, $t \in (0,T]$ and $p \in \mathbb{R}^N$

$$|H(x,t,p) - H(y,t,p)| \leq L_1|x-y||p| + L_2|x-y| \ .$$

Theorem 8.1. *Assume (H1-s), (H3) and that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then the solution of u of* (16)–(29) *given by Theorem 7.1 is Lipschitz continuous in x for any $t \in [0,T]$ and*

$$||Du(\cdot,t)||_\infty \leq \exp(L_1 t)||Du_0||_\infty + \frac{L_2}{L_1}(\exp(L_1 t) - 1) \ .$$

Proof of Theorem 8.1. The proof is similar to the proof of the comparison result and we just sketch it to avoid repeating the same arguments. We introduce the function $(x,y,t) \mapsto u(x,t) - u(y,t) - C(t)|x-y|$: the aim is to show that this function is negative for some well-chosen (smooth) function $C(\cdot)$; at least for $t = 0$, we can choose $C(0) = ||Du_0||_\infty$ to have this property.

To do so, we argue by contradiction, assuming that its supremum is strictly positive and in order to use viscosity solutions' arguments, we double the variables in time, namely

$$\psi(x,t,y,s) = u(x,t) - u(y,s) - C(t)|x-y| - \frac{|t-s|^2}{\alpha^2} - \beta(|x|^2 + |y|^2) \ .$$

For $\alpha, \beta > 0$ small enough, the maximum of ψ is still strictly positive and we denote by $(\overline{x},\overline{t},\overline{y},\overline{s})$ a maximum point of ψ. We notice that we cannot have $\overline{x} = \overline{y}$, otherwise $\psi(\overline{x},\overline{t},\overline{y},\overline{s})$ would be negative. Dropping the β-terms which are not going to play any role and performing the same arguments as in the proof of Theorem 5.1, we are lead to the inequality

$$\frac{dC}{dt}(\overline{t})|\overline{x}-\overline{y}| + H\left(\overline{x},\overline{t},\overline{p}\right) - H\left(\overline{y},\overline{s},\overline{p}\right) \leq 0 \ ,$$

with $\overline{p} = C(\overline{t})\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}$. Writing this inequality as

$$\frac{dC}{dt}(\overline{t})|\overline{x}-\overline{y}| + H\left(\overline{x},\overline{t},\overline{p}\right) - H\left(\overline{y},\overline{t},\overline{p}\right) + H\left(\overline{y},\overline{t},\overline{p}\right) - H\left(\overline{y},\overline{s},\overline{p}\right) \leq 0 \ ,$$

and using (H1-s), we obtain

$$\frac{dC}{dt}(\bar{t})|\bar{x}-\bar{y}| - L_1 C(\bar{t})|\bar{x}-\bar{y}| - L_2|\bar{x}-\bar{y}| + o_\alpha(1) \leq 0 \,,$$

where $o_\alpha(1) \to 0$ as $\alpha \to 0$. If $\frac{dC}{dt}(\bar{t}) - L_1 C(\bar{t}) - L_2 > 0$, we get the contradiction by letting α tend to 0.

Therefore it is enough to solve $\frac{dC}{dt}(\bar{t}) - L_1 C(\bar{t}) - L_2 = \delta$ for some $\delta > 0$ and with $C(0) = ||Du_0||_\infty$. This yields

$$C_\delta(\bar{t}) = \exp(L_1 t)||Du_0||_\infty + \frac{L_2+\delta}{L_1}(\exp(L_1 t) - 1) \,.$$

The above proof shows that $u(x,t) - u(y,t) - C_\delta(t)|x-y| \leq 0$ for all x, y, t and $\delta > 0$. Letting δ tends to 0, we obtain the right bound on $||Du(\cdot,t)||_\infty$.

An other way to get Lipschitz regularity is, for coercive Hamiltonians, through an estimate of u_t when H is independent of t. We recall that $H(x,p)$ is said to be coercive if it satisfies

(H5) $H(x,p) \to +\infty$ as $|p| \to +\infty$, uniformly in x.

Theorem 8.2. *Assume that H is independent of t and satisfies (H1), (H3) and (H5). If $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then the solution of u of* (16)–(29) *given by Theorem 7.1 is Lipschitz continuous in x for any $t \in [0,T]$ and*

$$||Du(\cdot,t)||_\infty \leq K(H,u_0) \,.$$

Proof of Theorem 8.2. By the comparison result, since $u(x,t)$ and $u(x,t+h)$ for $h > 0$ are solutions of the same equation, we have

$$||u(x,t+h) - u(x,t)||_\infty \leq ||u(x,h) - u(x,0)||_\infty \,.$$

But u_0 being Lipschitz continuous, if we set

$$R := ||Du_0||_\infty \quad \text{and} \quad C := \max_{\mathbb{R}^N \times B(0,R)} |H(x,p)| \,,$$

then $u_0(x) - Ct$ and $u_0(x) + Ct$ are respectively viscosity sub and supersolution of the equation and therefore

$$u_0(x) - Ct \leq u(x,t) \leq u_0(x) + Ct \quad \text{in } \mathbb{R}^N \times [0,T] \,.$$

In particular, $||u(x,h) - u(x,0)||_\infty \leq Ch$ and therefore $||u(x,t+h) - u(x,t)||_\infty \leq Ch$, which implies that $||u_t||_\infty \leq C$.

In order to deduce the gradient bound in space, we consider any point (x,t), $t>0$ and we want to show that $u(y,t)\le u(x,t)+K|y-x|$ for some large enough constant K. To do so, we consider the function

$$(y,s)\mapsto u(y,s)-u(x,t)-K|y-x|-\frac{(t-s)^2}{\alpha^2}\ .$$

The maximum of this function is achieved at some point $(\overline{y},\overline{s})$ since u is bounded and $K|y-x|+\frac{(t-s)^2}{\alpha^2}\to+\infty$ if $|y-x|+|t-s|\to+\infty$. Moreover $\overline{s}\to t$ when $\alpha\to 0$.

If $\overline{y}\neq x$, then the function $(y,s)\mapsto u(x,t)+K|y-x|+\frac{(t-s)^2}{\alpha^2}$ is smooth at $(\overline{y},\overline{s})$ and since u is a viscosity subsolution of (16) we have

$$2\frac{(\overline{s}-t)}{\alpha^2}+H(\overline{y},\overline{p})\le 0,$$

with $\overline{p}=K\frac{\overline{y}-x}{|\overline{y}-x|}$.

Now we claim that $|2\frac{(\overline{s}-t)}{\alpha^2}|\le 2C$: this can be proved in an analogous way as in the proof of Lemma 5.2 (point (3) for the estimate on $|2\frac{(\overline{x}-\overline{y})}{\varepsilon^2}|$). Using (H5) and the fact that $|\overline{p}|=K$, the above inequality can not hold if K is large enough, namely if $H(\overline{y},\overline{p})>2C$. Therefore $\overline{y}=x$ for α small enough and also necessarily $\overline{s}=t$ (otherwise the value at the maximum would be less than the value at (x,t)). The maximum point property for $s=t$ yields

$$u(y,t)-u(x,t)-K|y-x|\le 0,$$

which is the desired property.

We provide a last result on the semi-concavity of solutions when the Hamiltonian is convex in p and satisfies some smoothness assumption in (x,p). We recall that a function $u:\mathbb{R}^N\times[0,T]\to\mathbb{R}$ is semi-concave (with a uniform constant of semi-concavity wrt t) if there exists a constant $\bar{k}$ such that, for any $x,h\in\mathbb{R}^N$ and $t\in[0,T]$

$$u(x+h,t)+u(x-h,t)-2u(x,t)\le\bar{k}|h|^2\ .$$

For the Hamiltonian H, we use the following assumption which is satisfied for example if H is $W^{2,\infty}$ in (x,p) uniformly in t and convex in p

(H6) There exists constants $k_1,k_2>0$ such that, for any $x,h,p,k\in\mathbb{R}^N$ and $t\in[0,T]$

$$H(x+h,t,p+k)+H(x-h,t,p-k)-2H(x,t,p)\ge -k_1|h|^2-k_2|h||k|\ .$$

The result is the

Theorem 8.3. *Assume that H satisfies (H1), (H3) and (H6). If $u_0\in W^{1,\infty}(\mathbb{R}^N)$ is semi-concave, then the solution of u of* (16)–(29) *given by Theorem 7.1 is semi-concave in x for any $t\in[0,T]$, with a uniform constant of semi-concavity.*

We just give a short *sketch of the proof of Theorem 8.3* which is tedious since it requires to triple the variables (each of them corresponding to either $x+h$, $x-h$ or x). Namely we introduce the function

$$u(x,t)+u(y,s)-2u(z,\tau)-\frac{|x+y-2z|^2}{\varepsilon^2}-\frac{|x-z|^2}{\varepsilon^2}-\frac{|y-z|^2}{\varepsilon^2}-\frac{(t-\tau)^2}{\alpha^2}$$
$$-\frac{(s-\tau)^2}{\alpha^2}-\cdots\ ,$$

where we have dropped the usual "β"-terms to penalize infinity. With this function, the proof follows from straightforward but tedious computations.

9 Convex Hamiltonians, Barron–Jensen Solutions

In this section, we describe additional properties of viscosity solutions of (16) in the case when H is convex in p. The main motivation is to extend the theory -and in particular the uniqueness results- to the case when the initial data is only lower semi-continuous, a natural framework for optimal control problems. The key ideas described in this section were introduced by Barron and Jensen [17, 18] who also consider the applications to optimal control. The simplified presentation we provide follows the one of [4].

Our first result is the following.

Theorem 9.1. *Assume that H is convex in p and (H3) holds. If $u\in W^{1,\infty}(\mathbb{R}^N\times(0,T))$ satisfies*

$$u_t+H(x,t,Du)\le 0 \quad a.e\ in\ \Omega\times(0,T)\ ,$$

then u is viscosity subsolution of (16).

Proof of Theorem 9.1. We are going to use a standard regularization argument. Let $(\rho_\varepsilon)_\varepsilon$ be a sequence of C^∞, positive, smoothing kernels in $\mathbb{R}^{N+1}$, with compact support in the ball of radius ε. For $\eta>0$ small enough, we are going to show that

$$u_\varepsilon(x,t):=\int_{\mathbb{R}^{N+1}}u(y,s)\rho_\varepsilon(x-y,t-s)dyds\ ,$$

is an approximate C^1 subsolution of the equation in $\mathbb{R}^N\times(\eta,T-\eta)$ if $\varepsilon<\eta$.

To do so, for $x\in\mathbb{R}^N$, $t\in(\eta,T-\eta)$, we multiply the equation at the point (y,s) by $\rho_\varepsilon(x-y,t-s)$ and we integrate over $\mathbb{R}^{N+1}$ (or, in fact, over the ball of radius ε). By the properties of the convolution, we obtain

$$(u_\varepsilon)_t(x,t)+\int_{\mathbb{R}^{N+1}}H\left(y,s,Du(y,s)\right)\rho_\varepsilon(x-y,t-s)dyds\le 0.$$

Using (H3), we can replace, in the integral, $H(y,s,Du(y,s))$ by $H(x,t,Du(y,s))$ with a small error in ε. This gives

$$(u_\varepsilon)_t(x,t) + \int_{\mathbb{R}^{N+1}} H(x,t,Du(y,s))\,\rho_\varepsilon(x-y,t-s)dyds \le o_\varepsilon(1)\;.$$

In order to conclude, we have just to apply Jensen's inequality which leads to

$$(u_\varepsilon)_t(x,t) + H(x,t,Du_\varepsilon(x,t))\,dyds \le o_\varepsilon(1)\;.$$

Therefore u_ε is a smooth subsolution of (16) in $\mathbb{R}^N \times (\eta, T-\eta)$, hence a viscosity subsolution of (16) in $\mathbb{R}^N \times (\eta, T-\eta)$ and so is u which is the uniform limit of u_ε, by Theorem 4.1. Since this is true for any η, the proof is complete.

This result has several consequences which are listed in the following

Theorem 9.2. *Assume that H is convex in p and that (H1), (H3) hold.*

(i) The function $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ is a viscosity subsolution (resp. solution) of (16) *if and only if, for any smooth function φ, if (x,t) is a local minimum point of $u-\varphi$, one has*

$$\varphi_t(x,t) + H(x,t,D\varphi(x,t)) \le 0 \quad (\textit{resp.} = 0). \tag{33}$$

(ii) If $u_1, u_2 \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ are viscosity subsolutions (resp. solutions) of (16)*, then* $\min(u_1,u_2)$ *is also a subsolution (resp. solution) of* (16).

(iii) If $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ is a viscosity subsolution of (16) *and if (H1-s) holds then*

$$u_\varepsilon(x,t) = \inf_{y\in\mathbb{R}^N}\left\{u(y,t) + e^{-L_1 t}\frac{|x-y|^2}{\varepsilon^2}\right\},$$

is a viscosity subsolution of (16) *within a $O(\varepsilon)$ error term which depends only on the L^∞-norm of u.*

In (iii), the function u_ε is obtained through an *inf-convolution* procedure on u. The connections of such inf and sup-convolution with viscosity solutions were remarked by Lasry and Lions [35]. In general, an inf-convolution is a supersolution, while sup-convolutions are subsolutions. Therefore (iii) is a priori a rather surprising result.

Proof of Theorem 9.2. The proof of (i), (ii) and (iii) are easy: for (i), we may assume that (x,t) is a *strict* local minimum point of $u-\varphi$ and we can approximate this minimum point by minimum points $(x_\varepsilon, t_\varepsilon)$ of $u_\varepsilon - \varphi$ where u_ε is the sequence of smooth approximations of u built in the proof of Theorem 9.1. By the regularity of u_ε and φ, we have $(u_\varepsilon)_t(x_\varepsilon,t_\varepsilon) = \varphi_t(x_\varepsilon,t_\varepsilon)$ and $Du_\varepsilon(x_\varepsilon,t_\varepsilon) = D\varphi(x_\varepsilon,t_\varepsilon)$ and therefore, since u_ε is a C^1 subsolution of (16)

$$\varphi_t(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon,t_\varepsilon,D\varphi(x_\varepsilon,t_\varepsilon)) \le 0.$$

The conclusion follows by letting $\varepsilon \to 0$.

For (ii), we have just to use Stampacchia's Theorem together with Theorem 9.1: indeed

$$D[\min(u_1,u_2)] = Du_1 \quad \text{if } u_1 < u_2 \text{ and } D[\min(u_1,u_2)] = Du_2 \text{ otherwise,}$$

and $Du_1 = Du_2$ a.e. on the set $\{u_1 = u_2\}$; and the same is, of course, true for the time derivative. To get the subsolution property, we have just to argue in the a.e. sense while the supersolution property always holds since the minimum of supersolutions is a supersolution (exactly in the same way as the maximum of two subsolutions is a subsolution, cf. Perron's method).

For (iii), we just sketch the proof since it requires long but straightforward computations. Using (i), we have look at what happens at a minimum point (x,t) of $u_\varepsilon - \varphi$ where φ is a smooth function. Thanks to the definition of u_ε, this leads to consider minimum point of the function

$$(y,t,z,s) \mapsto u(y,t) + e^{-L_1 t}\frac{|z-y|^2}{\varepsilon^2} - \varphi(z,t)\ .$$

We see that we are in a framework which is close to the proof of the comparison result, and in the spirit of Remark 5.3. The computations are then easy using (i).

Theorem 9.2 provides all the necessary (technical) ingredients to extend the theory and to do so, we are first going to say that a lsc function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a Barron–Jensen (BJ for short) subsolution (or solution) of (16) if and only if it satisfies (33). Theorem 9.2 (i) shows that this is equivalent to the usual notion of viscosity solution when u is Lipschitz continuous (and it is also the case when u is continuous).

The extension to lsc subsolutions and solutions, and the uniqueness result are given by the

Theorem 9.3. *Assume that H is convex in p and that (H1), (H3) hold.*

(i) If $(u_\varepsilon)_\varepsilon$ is a sequence of BJ subsolution (resp. solution) of (16) *then* $\liminf_* u_\varepsilon$ *is a subsolution (resp. solution) of* (16).

(ii) Assume (H1-s), (H3) and that u_0 is a bounded lsc initial data. There exists a unique lsc BJ solution u of (16)–(29) *which satisfies*

$$\liminf_{\substack{(y,s)\to(x,0)\\ s>0}} u(y,s) = u_0(x)\ . \tag{34}$$

We just give a very brief sketch of this result. The proof of (i) follows immediately from the arguments of the proof of the (discontinuous) stability results. For (ii), if u is a lsc BJ solution (or even only a subsolution) of (16) then the result of Theorem 9.2 (iii) holds (even if u is just lsc) and (34) implies that $u_\varepsilon(x,0) \le u_0(x)$ in $\mathbb{R}^N$. But now u_ε is an approximate solution of (16), which is Lipschitz continuous in x (by its definition through the "inf-convolution" formula) and also in t (by the equation). If v is an other solution, we can compare u_ε and v: clearly $u_\varepsilon(x,0) \le v(x)$

in $\mathbb{R}^N$ and the Lipschitz continuity of u_ε allows to use the arguments of the proof of Theorem 5.2 in a rather easy way.

10 Large Time Behavior of Solutions of Hamilton–Jacobi Equations

10.1 Introduction

In this second part, we are interested in the behavior, as $t \to +\infty$, of the viscosity solutions of first-order Hamilton–Jacobi Equations of the form

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) , \tag{35}$$

with the initial data

$$u = u_0 \quad \text{in } \mathbb{R}^N, \tag{36}$$

in the case when the Hamiltonian $H(x, p)$ and the initial datum u_0 are $\mathbb{Z}^N$*-periodic* in x, i.e., for all $x, p \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$,

$$H(x + z, p) = H(x, p) \quad \text{and} \quad u_0(x + z) = u_0(x) . \tag{37}$$

and when H is *coercive*, namely

$$H(x, p) \to +\infty \quad \text{when } |p| \to +\infty, \text{ uniformly wrt } x \in \mathbb{R}^N. \tag{38}$$

In the last decade, the large time behavior of solutions of Hamilton–Jacobi Equation in compact manifold $\mathscr{M}$ (or in $\mathbb{R}^N$, mainly in the periodic case) has received much attention and general convergence results for solutions have been established by using two different types of methods: in his course, H. Ishii [this volume] describes the "weak Kam approach" which is an optimal control/dynamical system approach and both uses and provides formulas of representation, the ones for the asymptotic solutions being based on the notion of Aubry–Mather sets.

Our aim is to describe a second approach which relies only on partial differential equations methods: it provides results even when the Hamiltonians are not convex but it gives a slightly less precise description of the phenomenas compared to the "weak Kam approach".

In 1999, Namah and Roquejoffre [42] are the first to obtain convergence results in a general framework, by pde arguments which we describe below. They use the following additional assumptions

$$H(x, p) \geq H(x, 0) \text{ for all } (x, p) \in \mathscr{M} \times \mathbb{R}^N \quad \text{and} \quad \max_{\mathscr{M}} H(x, 0) = 0, \tag{39}$$

where $\mathscr{M}$ is a smooth compact N-dimensional manifold without boundary.

Then Fathi in [25] proved a different type of convergence result, by dynamical systems type arguments, introducing the "weak KAM theory". Contrarily to [42], the results of [25] use strict convexity (and smoothness) assumptions on $H(x,\cdot)$, i.e., $D_{pp}H(x,p) \geq \alpha I$ for all $(x,p) \in \mathscr{M} \times \mathbb{R}^N$ and $\alpha > 0$ (and also far more regularity) but do not require (39). Afterwards Roquejoffre [43] and Davini and Siconolfi in [24] refined the approach of Fathi and they studied the asymptotic problem for Hamilton–Jacobi Equations on $\mathscr{M}$ or N-dimensional torus.

The first author and Souganidis obtained in [15] more general results, for possibly non-convex Hamiltonians, by using an approach based on partial differential equations methods and viscosity solutions, which was not using in a crucial way the explicit formulas of representation of the solutions: this is the second main type of results we (partially) describe here.

All these results (except perhaps the Namah–Roquejoffre ones) use in a crucial way the compactness of the domain: indeed either they are stated on a compact manifold or they use periodicity which means that we are looking at equations set on the torus. We also refer to the articles [11, 27–29, 33] for the asymptotic problems in the whole domain $\mathbb{R}^N$ without the periodic assumptions in various situations.

Finally there also exists results on the asymptotic behavior of solutions of convex Hamilton–Jacobi Equation with boundary conditions. Mitake [38] studied the case of the state constraint boundary condition and then the Dirichlet boundary conditions [39, 40]. Roquejoffre in [43] was also dealing with solutions of the Cauchy–Dirichlet problem which satisfy the Dirichlet boundary condition pointwise (in the classical sense): this is a key difference with the results of [39, 40] where the solutions were satisfying the Dirichlet boundary condition in a generalized (viscosity solutions) sense. These results were slightly extended in [7] by using an extension of PDE approach of [15].

10.2 Existence and Regularity of the Solution

The first result concerns the (global) existence, uniqueness and regularity of the solution.

Theorem 10.1. *Assume that H satisfies (37)–(38) and that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ is a $\mathbb{Z}^N$-periodic function. Then there exists a unique solution of (35)–(36) which is (i) periodic in x and (ii) Lipschitz continuous in x and t on $\mathbb{R}^N \times [0,+\infty)$.*

We just sketch the *proof of Theorem 10.1* since it is an easy adaptation of the results given in the previous sections, which we can simplify here.

For the existence, we use Perron's method: assuming first that $u_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, the functions $-Ct + u_0(x)$ and $Ct + u_0(x)$ are respectively sub and supersolution of (35)–(36) if C is given by

$$R := ||Du_0||_\infty \quad \text{and} \quad C := \max_{\mathbb{R}^N \times \overline{B(0,R)}} |H(x,p)| \ .$$

Truncating $H(x, p)$ by replacing it by $H_K(x, p) := \min(H(x, p), K)$ for some large constant $K > 0$, we can apply readily Perron's method. We obtain the existence of a continuous solution u_K of the H_K-equation which satisfies

$$-Ct + u_0(x) \le u_K(x,t) \le Ct + u_0(x) \quad \text{for any } x \in \mathbb{R}^N,\ t > 0\ . \tag{40}$$

Periodicity comes directly from the construction since if w is a subsolution of (35)–(36), then it is also the case for $\sup_{z\in\mathbb{Z}^N}[w(\cdot + z)] \ge w(\cdot)$. Therefore the supremum of subsolutions is clearly achieved for a periodic subsolution.

The uniqueness is proved readily by the argument of the proof of Theorem 5.1 (at least if we assume periodicity) or by the slight adaptation for having the comparison in $BUC(\overline{Q})$.

The time derivative $(u_K)_t$ is bounded since, for any $h > 0$

$$||u_K(x,t+h) - u_K(x,t)||_\infty \le ||u_K(x,h) - u_0(x)||_\infty$$

and $-Ch+u_0(x) \le u_K(x,h) \le Ch+u_0(x)$ by construction. Therefore $|(u_K)_t| \le C$ and, if $K > C$, then u_K is a solution of the H-equation. We denote it by u.

Finally, since H is coercive and $H(x, Du) = -u_t$, we deduce immediately that Du is bounded as well. Using that u is Lipschitz continuous, a (slight) variant of Theorem 5.1 implies that it is the unique solution of (35)–(36).

10.3 Ergodic Behavior

The first step in the study of the large time behavior of u is the

Theorem 10.2. *Under the assumptions of Theorem 10.1, there exists a constant $c \in \mathbb{R}$ such that*

$$\frac{u(x,t)}{t} \to c \quad \text{as } t \to +\infty \text{ uniformly w.r.t. } x \in \mathbb{R}^N\ . \tag{41}$$

Proof. We set

$$m(t) := \max_{\mathbb{R}^N} (u(x,t) - u_0(x))\ .$$

We first have

$$m(t+s) \le \max_{\mathbb{R}^N} (u(x,t+s) - u(x,t)) + \max_{\mathbb{R}^N} (u(x,t) - u_0(x))\ ,$$

and then by comparison

$$\max_{\mathbb{R}^N} (u(x,t+s) - u(x,t)) \le \max_{\mathbb{R}^N} (u(x,s) - u(x,0)) = m(s)\ .$$

Therefore $m(t+s) \leq m(t) + m(s)$ for any $t, s > 0$ but the Lipschitz continuity of u in t gives also $m(t) \geq -Ct$ for some constant C. A classical result on sub-additive functions implies

$$\frac{m(t)}{t} \to c := \inf_{t>0} \left(\frac{m(t)}{t} \right) .$$

Finally, it is easy to show that $u(x,t) - m(t)$ is bounded independently of x and t by using the periodicity and Lipschitz continuity in x of u, and the result follows.

For the convenience of the reader we sketch the proof of the result for m. Pick any $\tau > 0$. If $t > 0$, there exists $n \in \mathbb{N}$ such that $n\tau \leq t < (n+1)\tau$. Using the sub-additivity of m yields

$$m(t) \leq nm(\tau) + m(\varepsilon) ,$$

where $\varepsilon := t - n\tau \in [0, 1)$. Dividing by $t = n\tau + \varepsilon$ gives

$$\frac{m(t)}{t} \leq \frac{nm(\tau)}{n\tau + \varepsilon} + \frac{m(\varepsilon)}{n\tau + \varepsilon} ,$$

and letting $t \to +\infty$, we obtain

$$\limsup_{t \to +\infty} \frac{m(t)}{t} \leq \frac{m(\tau)}{\tau} .$$

But this is true for any τ, hence

$$\limsup_{t \to +\infty} \left(\frac{m(t)}{t} \right) \leq \inf_{\tau} \left(\frac{m(\tau)}{\tau} \right) = c .$$

But obviously $\liminf\limits_{t \to +\infty} \dfrac{m(t)}{t} \geq c$, therefore $\dfrac{m(t)}{t} \to c$.

It is worth pointing out that the assumption "$m(t) \geq -Ct$" is just used to have a well-defined constant c.

Then we are led to several natural questions:

(a) Can we have a characterization of the constant c?
(b) Can we go further in the asymptotic behavior ? Namely: is $u(x,t) - ct$ bounded? does it converge to some function?

A first remark is the following: if, for large t, $u(x,t)$ looks like $\lambda t + v(x)$, then λ and v should satisfy the equation

$$H(x, Dv) + \lambda = 0 \quad \text{in } \mathbb{R}^N . \tag{42}$$

A key question is then: does this equation, where both the constant λ and the function v are unknown, have (periodic) solutions?

The answer is given by the following result of Lions et al., Homogenization of Hamilton–Jacobi equations, unpublished work.

Theorem 10.3. *Assume that H satisfies (37)–(38). There exists a unique constant λ such that (42) has a periodic, Lipschitz continuous solution.*

An immediate consequence of Theorem 10.3 is the

Corollary 10.1. *Assume that H satisfies (37)–(38). Then $c = \lambda$ and $u(x,t) - ct$ is bounded.*

The proof of this corollary is obvious since, if (λ, v) solves (42), then $v(x) + \lambda t$ is a solution of (35) and by comparison

$$||u(x,t) - (v(x) + \lambda t)||_\infty \le ||u(x,0) - v(x)||_\infty \ .$$

Therefore $u(x,t) - \lambda t$ is bounded and dividing by t and letting $t \to +\infty$ shows that $c = \lambda$.

As a consequence, Theorem 10.3 gives a characterization of the ergodic constant c as the unique constant such that the "ergodic problem" (42) has a periodic (bounded) solution.

Proof of Theorem 10.3. For $0 < \alpha \ll 1$, we consider the equation

$$H(x, Dv_\alpha) + \alpha v_\alpha = 0 \quad \text{in } \mathbb{R}^N \ , \tag{43}$$

and we set $M := ||H(x,0)||_\infty$. In order to prove that this equation has a unique periodic solution v_α, we use Perron's method.

We first remark that $-\frac{1}{\alpha}M$ and $\frac{1}{\alpha}M$ are respectively sub and supersolution of this equation and we are looking for a solution which satisfies

$$-\frac{1}{\alpha}M \le v_\alpha \le \frac{1}{\alpha}M \quad \text{in } \mathbb{R}^N \ .$$

Since H does not a priori satisfy Assumption (H1), we have to argue either as in proof of Theorem 10.1, introducing some truncated Hamiltonians H_K or we remark that, because of (38), the subsolutions w which are bounded from below by $-\frac{1}{\alpha}M$ are equi-Lipschitz continuous: in this last case, we directly build a Lipschitz continuous solution of (43).

In any case, we build a solution v_α of (43) such that

$$||v_\alpha||_\infty \le \frac{1}{\alpha}M \ ,$$

which is Lipschitz continuous and an easy modification of the proof of Theorem 5.1 shows that v_α is the unique periodic solution of (43).

Moreover, as a consequence of (38), since αv_α is bounded, $H(x, Dv_\alpha)$ is also bounded and therefore the v_α's are equi-Lipschitz continuous.

Using this property together with the periodicity of the v_α, the functions $w_\alpha(x) := v_\alpha(x) - v_\alpha(0)$ are equi-bounded and equi-Lipschitz continuous. By Ascoli's Theorem, they converge (up to a subsequence) to some function $v \in W^{1,\infty}(\mathbb{R}^N)$. And we may assume as well that the bounded constants $\alpha v_\alpha(0)$ converges to some constant λ.

We have $H(x, Dw_\alpha) + \alpha w_\alpha + \alpha v_\alpha(0) = 0$ in $\mathbb{R}^N$ and we can pass to the limit by using Theorem 4.1: λ and v solves (42).

For the *uniqueness* of λ, if (v, λ) and (v', λ') are solutions of the ergodic problem, we compare the solutions $v(x) + \lambda t$ and $v'(x) + \lambda' t$ of (35)

$$||(v(x) + \lambda t) - (v'(x) + \lambda' t)||_\infty \leq ||v(x) - v'(x)||_\infty$$

or equivalently

$$||(v(x) - v'(x)) + (\lambda - \lambda')t)||_\infty \leq ||v(x) - v'(x)||_\infty .$$

Dividing by t and letting $t \to +\infty$ gives $\lambda = \lambda'$.

10.4 Asymptotic Behavior of $u(x, t) - ct$

By considering $H_c = H + c$ and $u_c(x, t) = u(x, t) - ct$, we may assume that $c = 0$ and the solutions u of (35) are uniformly bounded and Lipschitz continuous. We are going to do it from now on.

The main question of this section is: do the $u(x, t)$ always converge as $t \to +\infty$? or do we need additional assumptions? The following examples shows that the answer is not completely obvious.

Example 1. The function $u(x, t) := \sin(x - t)$ is a solution of the transport equation

$$u_t + u_x = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) ,$$

it satisfies very good regularity properties and uniform estimates but it does not converge as $t \to +\infty$. This shows that convergence is not only a question of estimates. But, of course, in this example the coercivity assumption is not satisfied.

Example 2. The same function is also a solution of

$$u_t + |u_x + 1| - 1 = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) .$$

In this example, the Hamiltonian is coercive and even convex but not strictly convex.

These two examples shows that the convergence as $t \to +\infty$ requires additional assumptions and/or a particular framework: we are going to show that the convergence holds in two cases:

(a) The Namah–Roquejoffre framework for which a typical example is

$$u_t + |Du| = f(x) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ ,$$

where $f(x) \geq 0$ and the set $\{x : \ f(x) = 0\}$ is non-empty.

(b) The "strictly convex" framework for which a typical example is

$$u_t + |Du + q(x)|^2 - |q(x)|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ ,$$

where q is (say) a periodic, Lipschitz continuous function.

Roughly speaking, the first framework is more restrictive on the structure of the Hamiltonians but it allows to take into account Hamiltonians $H(x, p)$ which are not strictly convex in p, contrarily to the second framework where the structure of the Hamiltonians is very general but where we have to impose strict convexity.

10.5 The Namah–Roquejoffre Framework

The main assumptions are the following. In the sequel, we refer to this assumptions as (NR).

- $H(x, p) \geq H(x, 0)$ for any $x, p \in \mathbb{R}^N$.
- $H(x, 0) \leq 0$ for any $x \in \mathbb{R}^N$ and the set $\mathcal{Z} = \{x \in \mathbb{R}^N ; \ H(x, 0) = 0\}$ is non-empty.
- For any $\alpha > 0$ (small) and for any $0 < \mu < 1$, there exists $\eta(\alpha, \mu) > 0$ such that

$$H(x, \mu p) \leq -\eta(\alpha, \mu) \quad \text{if } H(x, p) \leq 0 \text{ and if } d(x, \mathcal{Z}) \geq \alpha \ .$$

Remark 10.1. If $H(x, p) = |p| - f(x)$ where $f(x) \geq 0$ and the set $\mathcal{Z} := \{x : f(x) = 0\}$ is non-empty, these assumptions are satisfied since

$$\begin{aligned} H(x, \mu p) &= \mu|p| - f(x) = \mu(|p| - f(x)) - (1 - \mu) f(x) \ . \\ &\leq -\eta(\alpha, \mu) := -(1 - \mu) \min_{d(x, \mathcal{Z}) \geq \alpha} f(x) < 0 \quad \text{if } |p| - f(x) \leq 0. \end{aligned}$$

Theorem 10.4. *Assume that H satisfies (37)–(38) and (NR), then $c = 0$ and, for any $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, the solution u of (35)–(36) converges to a solution of the stationary equation.*

Proof of Theorem 10.4. To show that $c = 0$, we have first to solve the equation

$$H(x, Dv) = 0 \quad \text{in } \mathbb{R}^N \ .$$

We first remark that, because of (NR), 0 is a subsolution.

On the other hand, if $z \in \mathcal{Z}$, by the coercivity of H, $C|x - z|$ is a supersolution for C large enough: indeed this is obviously true for $x \neq z$ since the gradient of this function has norm C. And this is also clear for $x = z$ since, by (NR), $H(z, p) \geq H(z, 0) = 0$ for any p. As a consequence, $Cd(x, \mathcal{Z}) := \inf_{z \in \mathcal{Z}} C|x-z|$ is a (periodic) supersolution of the equation as the infimum of supersolutions.

We apply Perron's method which provides us with a discontinuous solution. To prove that this solution is continuous, we need a SCR.

Noticing that both the (continuous) sub and supersolution vanish on $\mathcal{Z}$, the value of the solution is imposed on $\mathcal{Z}$ (see the construction above) and we need a SCR for the Dirichlet problem set in the complementary of $\mathcal{Z}$, namely

$$\begin{cases} H(x, Du) = 0 & \text{dans } \mathcal{O} := \mathbb{R}^N \setminus \mathcal{Z} \\ u(x) = 0 & \text{sur } \partial\mathcal{O} \ . \end{cases}$$

To obtain it, we use ideas which are introduced in Ishii [32] (see also [5]). If v_1 is a subsolution of this problem and v_2 a supersolution with $v_1 \leq 0 \leq v_2$ on $\partial\mathcal{O}$, we pick some $\mu \in (0, 1)$, close to 1. Because of the last requirement in (NR), we have in the viscosity sense

$$H(x, D\mu v_1(\cdot)) \leq -\eta(\alpha, \mu) \quad \text{if } d(x, \mathcal{Z}) \geq \alpha \ ,$$

and following the arguments of the comparison proof, it is clear that the maximum of $\mu v_1 - v_2$ can be achieved only on $\mathcal{Z}$. Therefore $\mu v_1 - v_2 \leq 0$ and we conclude by letting μ tends to 1. Therefore we have a continuous solution of the stationary equation and $c = 0$.

Next we examine the behavior of the solution u of the evolution equation on $\mathcal{Z}$: since $H(x, p) \geq 0$ on $\mathcal{Z}$, we have $u_t \leq 0$ on $\mathcal{Z}$ and therefore $t \mapsto u(x, t)$ is decreasing. Recalling that u is Lipschitz continuous, this implies that $u(x, t) \to \varphi(x)$ uniformly on $\mathcal{Z}$ where φ is a Lipschitz continuous function.

It remains to show the global behavior: to do so, we use the half-relaxed limit method outside $\mathcal{Z}$. For $\varepsilon > 0$, we set

$$u_\varepsilon(x, t) := u\left(x, \frac{t}{\varepsilon}\right) \quad \text{in } \mathbb{R}^N \times (0, \infty) \ .$$

The function u_ε solves

$$\varepsilon \frac{\partial u_\varepsilon}{\partial t} + H(x, Du_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) \ .$$

We introduce (as usual) the half-relaxed limits

$$\overline{u}(x,t) = \limsup{}^* \, u_\varepsilon(x,t) \text{ and } \underline{u}(x,t) = \liminf{}_* \, u_\varepsilon(x,t) \; .$$

For any $t > 0$, $\overline{u}(\cdot,t)$ and $\underline{u}(\cdot,t)$ are respectively sub and supersolution of $H(x,Dw) = 0$ in $\mathbb{R}^N$. It is worth pointing out that, here, $\overline{u}$ and $\underline{u}$ are Lipschitz continuous in x for any t, because of the uniform Lipschitz properties of u.

A priori we do not have a strong comparison result for this equation in $\mathbb{R}^N$ but we can use the additional information that we have on $\mathcal{Z}$, namely $\overline{u}(\cdot,t) = \underline{u}(\cdot,t) = \varphi(\cdot)$ on $\mathcal{Z}$. Therefore we are lead to the same Dirichlet problem as above, except that the boundary condition is now φ instead of 0. Applying readily the same arguments with a slight modification due to the Dirichlet data φ, we conclude that, for any $s,t > 0$, $\overline{u}(\cdot,t) \leq \underline{u}(\cdot,s)$ in $\mathbb{R}^N$. This implies that $\overline{u}(\cdot,t) = \underline{u}(\cdot,s)$ for any $s,t > 0$ and, setting $w(\cdot) = \overline{u}(\cdot,t) = \underline{u}(\cdot,s)$, we have the uniform convergence of $u(\cdot,t)$ as $t \to +\infty$ to the continuous function w which is the unique solution of the Dirichlet problem with φ and also solves

$$H(x,Dw) = 0 \quad \text{in } \mathbb{R}^N \; .$$

Remark 10.2. This approach does not work for the equation

$$u_t + |Du + q(x)|^2 - |q(x)|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty)$$

which does not satisfy the (NR) assumptions.

10.6 The "Strictly Convex" Framework

In fact, like in the Namah–Roquejoffre framework, the assumptions on H we are going to use in this section does not really imply that H is strictly convex; the title of this section is just to fix ideas.

Our key assumption is the following.

(SCA) There exists $\eta_0 > 0$ such that, for any $\eta \in (0,\eta_0]$, there exists a constant $\psi_\eta > 0$ such that if $H(x,p+q) \geq \eta$ and $H(x,q) \leq 0$ for some $x,p,q \in \mathbb{R}^N$, then for any $\mu \in (0,1]$,

$$\mu H\left(x, \frac{p}{\mu} + q\right) \geq H(x,p+q) + \psi_\eta(1-\mu).$$

This assumption does not implies that H is convex but it implies that, for all x, the set $\{p : \; H(x,p) \leq 0\}$ is convex (Ishii, personal communication) and imposes the behavior of H in the set $\{p : \; H(x,p) \geq 0\}$.

Remark 10.3. If H is indeed a C^2, strictly convex function of p, i.e. if $D^2_{pp}H(x,p) \geq \nu Id$ for some $\nu > 0$, have, for any $\mu \in (0,1]$, $a, b \in \mathbb{R}^N$

$$H(x, \mu a + (1-\mu)b) \leq \mu H(x,a) + (1-\mu)H(x,b) - C(\nu)\mu(1-\mu)|a-b|^2.$$

Choose $a = \frac{p}{\mu} + q$, $b = q$, $\mu a + (1-\mu)b = p + q$ and therefore

$$H(x, p+q) \leq \mu H(x, \frac{p}{\mu} + q) + (1-\mu)H(x,q) - C(\nu)\mu(1-\mu)|\frac{p}{\mu}|^2 ,$$

i.e.

$$H(x, p+q) \leq \mu H(x, \frac{p}{\mu} + q) - C(\nu)\mu(1-\mu)|\frac{p}{\mu}|^2 ,$$

since $H(x,q) \leq 0$. But p is bounded away from 0 since $H(x, p+q) \geq \eta$ and $H(x,q) \leq 0$, therefore **(SCA)** holds.

Our result is the following.

Theorem 10.5. *Assume that H satisfies (37)–(38), $c = 0$ and **(SCA)**, then, for any $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, the solution u of (35)–(36) converges to a solution of the stationary equation.*

It is worth recalling that, in this case, we actually assume that $c = 0$, it is not a consequence of the assumptions on H.

The key result is this approach is the

Theorem 10.6 (Asymptotically Monotone Property). *Under the assumption of Theorem 10.5, for any $\eta \in (0, \eta_0]$, there exists $\delta_\eta : [0,\infty) \to [0,1]$ such that*

$$\delta_\eta(s) \to 0 \quad \text{as } s \to \infty \quad \text{and}$$
$$u(x,s) - u(x,t) + \eta(s-t) \leq \delta_\eta(s)$$

for all $x \in \mathbb{R}^N$, $s, t \in [0,\infty)$ with $t \geq s$.

The meaning of Theorem 10.6 is that the solution u is becoming more and more increasing as $t \to \infty$. Why should this be true?

We can first consider the Oleinik–Lax Formula. The solution of

$$u_t + |Du|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) ,$$

is given by

$$u(x,t) := \inf_{y \in \mathbb{R}^N} \left(u_0(y) + \frac{|x-y|^2}{4t} \right) .$$

Formally, if y is a minimum point in this formula

$$Du(x,t) = \frac{2(x-y)}{4t} \quad \text{and} \quad u_t(x,t) := -\frac{|x-y|^2}{4t^2} .$$

But we know that $\frac{|x-y|^2}{4t}$ remains bounded since u_0 is bounded, hence $u_t = O(t^{-1})$.

A more general remark can be made by assuming that H is strictly convex and

$$H_p(x,p) \cdot p - H(x,p) \geq cH(x,p) \quad \text{if } H(x,p) \geq 0 ,$$

for any $x, p \in \mathbb{R}^N$ and for some $c > 0$. For example, one can think about quadratic Hamiltonians like $|p+q|^2 - |q|^2$ or $|p|^2 - f(x)^2$.

In this case, we perform the Kruzkov's change $w = -\exp(-u)$. The function w solves

$$w_t - wH(x, -\frac{Dw}{w}) = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty) .$$

Then we set $z = w_t$ and $m(t) = \|z^-\|_\infty$. Differentiating the equation with respect to t, we find that z satisfies at the same time (dropping the arguments of H and its derivatives)

$$z_t + (H_p \cdot p - H)z + H_p \cdot Dz = 0 ,$$

$$z - wH = 0 .$$

Next looking at a (negative) minimum point of z (where $Dz = 0$), it follows

$$m'(t) + (H_p \cdot p - H)m(t) = 0.$$

But $H = z/w > 0$ and therefore $(H_p \cdot p - H) \geq cH = cz/w$. Hence

$$m'(t) + c[m(t)]^2/w = 0 \quad \text{which implies} \quad m'(t) \geq \tilde{c}[m(t)]^2 .$$

Recalling that $m(t) \leq 0$, this inequality yields a behavior like $m(t) = O(t^{-1})$.

We first *prove Theorem 10.5* by using the Asymptotically Monotone Property.

(a) Since the family $(u(\cdot,t))_{t\geq 0}$ is bounded in $W^{1,\infty}(\mathbb{R}^N)$, by Ascoli's Theorem, there exists a sequence $(u(\cdot,T_n))_{n\in\mathbb{N}}$ which converges uniformly on $\mathbb{R}^N$ as $n \to \infty$.

By comparison, we have

$$\|u(\cdot,T_n+\cdot) - u(\cdot,T_m+\cdot)\|_\infty \leq \|u(\cdot,T_n) - u(\cdot,T_m)\|_\infty$$

for any $n, m \in \mathbb{N}$. Therefore, $(u(\cdot,T_n+\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}^N \times (0,+\infty))$ and therefore it converges uniformly to a function denoted by $u^\infty \in C(\mathbb{R}^N \times (0,+\infty))$. Moreover u^∞ is a solution of (35), by stability.

(b) Fix any $x \in \mathbb{R}^N$ and $s, t \in [0, \infty)$ with $t \geq s$. By the Asymptotically Monotone Property, we have

$$u(x, s + T_n) - u(x, t + T_n) + \eta(s - t) \leq \delta_\eta(s + T_n)$$

for any $n \in \mathbb{N}$ and $\eta > 0$. Sending $n \to \infty$ and then $\eta \to 0$, we get, for any $t \geq s$

$$u^\infty(x, s) \leq u^\infty(x, t).$$

The functions $x \mapsto u^\infty(x, t)$ are uniformly bounded and equi-continuous, and they are also monotone in t. This implies that $u^\infty(x, t) \to w(x)$ uniformly on $\mathbb{R}^N$ as $t \to \infty$ for some $w \in W^{1,\infty}(\mathbb{R}^N)$ which is a solution of the stationary equation.

(c) Since $u(\cdot, T_n + \cdot) \to u^\infty$ *uniformly*[4] in $\mathbb{R}^N \times (0, +\infty)$ as $n \to \infty$, we have

$$-o_n(1) + u^\infty(x, t) \leq u(x, T_n + t) \leq u^\infty(x, t) + o_n(1),$$

where $o_n(1) \to \infty$ as $n \to \infty$, uniformly in x and t.

Taking the half-relaxed semi-limits as $t \to +\infty$, we get

$$-o_n(1) + w \leq \liminf_{t \to \infty}{}_* \, u \leq \limsup_{t \to \infty}{}^* \, u \leq w + o_n(1).$$

Sending $n \to \infty$ yields

$$w(x) = \liminf_{t \to \infty}{}_* \, u(x, t) = \limsup_{t \to \infty}{}^* \, u(x, t)$$

for all $x \in \mathbb{R}^N$. Therefore $u(x, t) \to w(x)$ uniformly as $t \to \infty$ and the proof is complete.

Now we turn to the *Proof of the Asymptotically Monotone Property*. Let v be a periodic, Lipschitz continuous solution of $H(x, Dv) = 0$.

Since u is bounded and since we can change v in $v - M$ for some large constant $M > 0$, we may assume that

$$u(x, t) - v(x) \geq 1 \quad \text{for any } x \in \mathbb{R}^N \text{ and } t > 0\,.$$

We introduce the function

$$\mu_\eta(s) := \min_{x \in \mathbb{R}^N, t \geq s} \left(\frac{u(x, t) - v(x) + \eta(t - s)}{u(x, s) - v(x)} \right).$$

[4] This is a key point: the compactness of the domain (periodicity) plays a crucial role here since local uniform convergence is the same as global uniform convergence.

By the uniform continuity of u and v, $\mu_\eta \in C([0,\infty))$ and we have $0 \le \mu_\eta(s) \le 1$ for all $s \in [0,\infty)$ and $\eta \in (0,\eta_0]$.

Proposition 10.1. *Under the assumption of Theorem 10.5, $\mu_\eta(s) \to 1$ as $s \to \infty$ for any $\eta \in (0,\eta_0]$.*

As a consequence, for any $x \in \mathbb{R}^N$ and $t \ge s$,

$$\frac{u(x,t) - v(x) + \eta(t-s)}{u(x,s) - v(x)} \ge 1 + o_s(1)\ ,$$

where $o_s(1)$ depends on η and tends to 0 as $s \to \infty$.

A simple computation yields

$$u(x,t) - u(x,s) + \eta(t-s) \ge o_s(1)\ .$$

The proposition is a consequence of the following lemma.

Lemma 10.1. *Under the assumption of Theorem 10.6, for any $\eta \in (0,\eta_0]$, there exists a constant $C > 0$ such that the function μ_η is a supersolution of*

$$\max\left\{w(s) - 1, w'(s) + \frac{\psi_\eta}{C}(w(s)-1)\right\} = 0 \text{ in } (0,\infty)\ .$$

Using the lemma, it is easy to prove the proposition since the solution of the variational inequality with initial data $\mu_\eta(0)$ is given by

$$w(s) := 1 - (\mu_\eta(0) + 1)\exp\left(-\frac{\psi_\eta}{C}s\right).$$

and therefore, by comparison

$$\mu_\eta(s) \ge 1 - (\mu_\eta(0) + 1)\exp\left(-\frac{\psi_\eta}{C}s\right),$$

for any s. Recalling that $\mu_\eta(s) \le 1$, we have $\mu_\eta(s) \to 1$ as $s \to \infty$.

Proof of Lemma 10.1. We fix $\eta \in (0,\eta_0]$ and, to simplify the notations, we write μ for μ_η.

Let $\phi \in C^1((0,\infty))$ and $\overline{s} > 0$ be a strict local minimum of $\mu - \phi$.

Since there is nothing to check if $\mu(\overline{s}) = 1$, we assume that $\mu(\overline{s}) < 1$. We choose $\overline{x} \in \mathbb{R}^N$ and $\overline{t} \ge \overline{s}$ such that

$$\mu(\overline{s}) = \frac{u(\overline{x},\overline{t}) - v(\overline{x}) + \eta(\overline{t} - \overline{s})}{u(\overline{x},\overline{s}) - v(\overline{x})}\ .$$

For $0 < \varepsilon \ll 1$, we introduce the function

$$\Psi(x,y,z,t,s) := \frac{u(x,t) - v(z) + \eta(t-s)}{u(y,s) - v(z)} - \phi(s) + \frac{1}{\varepsilon^2}(|x-y|^2 + |x-z|^2) + |x-\overline{x}|^2 + |t-\overline{t}|^2$$

The function Ψ achieve its minimum at a point (x,y,z,t,s) (depending on ε) and, by classical arguments, as $\varepsilon \to 0$, we have

$$x, y, z \to \overline{x} \quad \text{and} \quad t \to \overline{t},\ s \to \overline{s}\ .$$

Moreover, by the Lipschitz continuity in x of u and v

$$\frac{|x-y|}{\varepsilon^2} + \frac{|x-z|}{\varepsilon^2} \leq C\ ,$$

for some constant C.

With the notations

$$\tilde{\mu}_1 := u(y,s) - v(z), \quad \tilde{\mu}_2 := u(x,t) - v(z) + \eta(t-s), \quad \tilde{\mu} := \frac{\tilde{\mu}_2}{\tilde{\mu}_1}$$

and if we set

$$P := \frac{\tilde{\mu}_1}{\tilde{\mu}}\left(\frac{2(y-x)}{\varepsilon^2}\right) \quad \text{and} \quad Q := \frac{\tilde{\mu}_1}{1-\tilde{\mu}}\left(\frac{2(z-x)}{\varepsilon^2}\right),$$

we have formally,

$$\begin{aligned} D_x u(x,t) &= \tilde{\mu}P + (1-\tilde{\mu})Q + o_\varepsilon(1)\ , \\ u_t(x,t) &= -\eta - 2\tilde{\mu}_1(t-\overline{t})\ , \\ D_y u(y,s) &= P\ , \\ u_s(y,s) &= -\tfrac{1}{\tilde{\mu}}(\eta + \tilde{\mu}_1\phi'(s))\ , \\ D_z v(z) &= Q. \end{aligned}$$

By the definition of viscosity solutions

$$\begin{aligned} -\eta + o_\varepsilon(1) + H(x, \tilde{\mu}P + (1-\tilde{\mu})Q + o_\varepsilon(1)) &\geq 0, \\ -\frac{1}{\tilde{\mu}}(\eta + \tilde{\mu}_1\phi'(s)) + H(y,P) &\leq 0, \\ H(z,Q) &\leq 0. \end{aligned}$$

Since P and Q are bounded, we may even let ε tend to 0 and drop the $o_\varepsilon(1)$-terms.

With

$$\mu_1 := u(\overline{x}, \overline{s}) - v(\overline{x}), \quad \mu_2 := u(\overline{x}, \overline{t}) - v(\overline{x}) + \eta(\overline{t} - \overline{s}), \quad \mu = \frac{\mu_2}{\mu_1}$$

we end up with

$$\begin{aligned} -\eta + H(\overline{x}, \mu P + (1-\mu)Q) &\geq 0, \\ -\frac{1}{\mu}(\eta + \mu_1 \phi'(\overline{s})) + H(\overline{x}, P) &\leq 0, \\ H(\overline{x}, Q) &\leq 0. \end{aligned}$$

If $p := \mu(P - Q)$ and $q = Q$, we have $H(\overline{x}, p+q) \geq \eta$ and $H(\overline{x}, q) \leq 0$, and therefore, by **(SCA)**

$$\begin{aligned} \frac{1}{\mu}(\eta + \mu_1 \phi'(\overline{s})) \geq H(\overline{x}, P) &= H(\overline{x}, \tfrac{p}{\mu} + q) \\ &\geq \frac{1}{\mu}\left(H(\overline{x}, p+q) + \psi_\eta(1-\mu)\right) \\ &\geq \frac{1}{\mu}\left(\eta + \psi_\eta(1-\mu)\right) . \end{aligned}$$

This shows

$$\phi'(\overline{s}) \geq \frac{1}{\mu_1}\psi_\eta(1-\mu) ,$$

which is the desired conclusion.

10.7 Concluding Remarks

- The Asymptotically Monotone Property is true in a more general framework (problems set in the whole space or with boundary conditions ... etc) but, in general, it does not imply the convergence as $t \to \infty$. This shows the importance of the periodic framework (compactness) where local uniform convergence is equivalent to global uniform convergence.
- In the Namah–Roquejoffre case, periodicity is less important, even if one has to avoid the infinity to play a role (by assuming that $\limsup_{|x| \to +\infty} H(x, 0) < 0$). See, for example, [11].
- For problems set in the whole space, the behavior at infinity of u_0 may determine the asymptotic behavior as $t \to \infty$ of u, even at the level of the ergodic constant c (cf. [11]).

- If H is convex and if S_H, S_{H^+} denote respectively the semi-groups associated to H and H^+, we know that these semi-groups commutes, namely

$$S_H(t)S_{H^+}(s) = S_{H^+}(s)S_H(t)$$

for any $s, t > 0$.

For any u_0, $S_{H^+}(s)u_0$ converges to the maximal subsolution of $H = 0$ which is below u_0.

If we are in a framework where we have convergence for $S_H(t)$ as $t \to \infty$, i.e. $S_H(t)u_0 \to u_\infty$ as $t \to +\infty$, then

$$S_H(\infty)S_{H^+}(s)u_0 = S_{H^+}(s)S_H(\infty)u_0 = u_\infty$$

This shows that u_∞ is the same for u_0 and for maximal subsolution of $H = 0$ which is below u_0: in other words, given u_0, $u(x,t)$ converges to the minimal solution which is above the maximal subsolution which is below u_0.

For such properties of commutations of semi-groups, we refer the reader to Cardin and Viterbo [20], Motta and Rampazzo [41] and Tourin and the author [16].

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