FA

Lecture

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Introductory remarks

Functional Analysis (FA)

- combines linear algebra & analysis
- is a theory of infinite-dimensional vector spaces and their linear (and nonlinear) maps

FA has lots of applications

- Partial Differential Equations (PDE's)
- Quantum Mechanics (FA is its language!)
- Calculus of Variations
- Harmonic Analysis
- Approximation theory
- Numerical maths
- Probability theory (& Math. Finance)

Chapter 0: Recapitulation of basic notions

The most fundamental things on Topological & Metric spaces are treated in the handout, available online.

Please study carefully! If any questions: can (possibly) be discussed in Tutorials.

Chapter 1: Topological and metric spaces

1.1. Limits and continuity

Definition 1.1 Let $\mathbb{X}$ be a topological space.

(a) $\mathbb{X}$ is called separable $:\Leftrightarrow \exists A \subseteq \mathbb{X}$ countable with $\overline{A} = \mathbb{X}$

(b) $\mathbb{X}$ is called 1st (first) countable $:\Leftrightarrow$ Every $x \in \mathbb{X}$ has a countable neighbourhood base.

(c) $\mathbb{X}$ is called 2nd (second) countable $:\Leftrightarrow$ there exists a countable (sub)base for the topology.

[Note: countable base $\Leftrightarrow$ countable subbase (see exercise)]

Theorem 1.2 Let $\mathbb{X}$ be a topological space.

Then: $\mathbb{X}$ is 2nd countable $\Rightarrow$ $\mathbb{X}$ is 1st countable and separable.

Pf (proof): Let $\mathcal{B}$ be a countable base for the topology $\mathcal{T}$ on $\mathbb{X}$.

1st countable: Let $x \in \mathbb{X}$ and $\mathcal{N}_x := \{B \in \mathcal{B} \mid x \in B\}$. Then $\mathcal{N}_x$ is countable (clear (why?)) and a neighbourhood base at x:
Indeed, (i) every element of $\mathcal{N}_x$ is a neighbourhood of x, and
(ii) let N be any neighbourhood of x. Then $\exists G \in \mathcal{T}$ with $x \in G \subseteq N$. By the definition of a base, $G = \bigcup_{\alpha \in I} B_\alpha$, where I is an index set and $B_\alpha \in \mathcal{B}$ $\forall \alpha \in I$. Hence, $\exists \alpha_x \in I : x \in B_{\alpha_x}$, i.e., $B_{\alpha_x} \in \mathcal{N}_x$ and $B_{\alpha_x} \subseteq N$.

Separable: $\forall \emptyset \neq B \in \mathcal{B}$ choose $x_B \in B$, and let $A := \{x_B \mid \emptyset \neq B \in \mathcal{B}\}$.
We claim A is countable (trivial (why?)) and $\overline{A} = \mathbb{X}$.
For all $x \in \mathbb{X}$ and neighbourhoods U of x there exists $G \in \mathcal{T}$ such that $x \in G \subseteq U$. But $G \neq \emptyset$ is a union of sets in $\mathcal{B}$, so $\exists x_B \in G$. Thus, $A \cap U \neq \emptyset$.
Since U was arbitrary, x is an adherent point of A ∎

Definition 1.3 Let X be a topological space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. We say that $(x_n)_n$ converges to $x \in X$ $:\Leftrightarrow$ for every neighbourhood U of x there exists $n_0 \in \mathbb{N}$ so for all $n \geq n_0$: $x_n \in U$. We write: $\lim_{n\to\infty} x_n = x$, or $x_n \xrightarrow{n\to\infty} x$ or $x_n \to x$, $n \to \infty$.

Remark 1.4

(a) Convergence is harder for finer topologies.

(b) X Hausdorff $\Rightarrow$ limits are unique (see exercise)

Definition 1.5 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top. spaces and let $f: X \to Y$.

(a) f is sequentially (seq.) continuous $:\Leftrightarrow$
$x_n \xrightarrow{n\to\infty} x$ (in X) implies ($\Rightarrow$) $f(x_n) \xrightarrow{n\to\infty} f(x)$ (in Y)

(b) f is continuous $:\Leftrightarrow$ for every $A \in \mathcal{T}_Y$: $f^{-1}(A) \in \mathcal{T}_X$.

($f^{-1}(A) := \{x \in X \mid \exists y \in A : f(x) = y\}$ is the inverse image or pre-image of A)

(c) f is open (open map) $:\Leftrightarrow \forall A \in \mathcal{T}_X : f(A) \in \mathcal{T}_Y$

($f(A) := \{y \in Y \mid \exists x \in A : y = f(x)\}$ is the image of A (under f))

(d) f is a homeomorphism $:\Leftrightarrow$ f is bijective, open, and continuous

(i.e., a bijection compatible with topological structure).

Theorem 1.6 Let X, Y be topological spaces, $f: X \to Y$ a map. Then:

(a) f is continuous $\Rightarrow$ f is seq. continuous

(b) f is seq. continuous & X is 1st countable

$\Rightarrow$ f is continuous

Pf: (a) Let $x_n \overset{n\to\infty}{\to} x$ in X. Let $V_0 \subseteq Y$ be a neighbourhood of $f(x)$. Then $\exists\, V \subseteq V_0$ open with $f(x) \in V$. Set $U := f^{-1}(V)$. U is open (because f is continuous and V is open), and $x \in U$, so U is a neighbourhood of x. Hence, we can apply the def. of convergence of $(x_n)_n$ to x: $\exists\, n_0 \in \mathbb{N}: \forall n \geq n_0 : x_n \in U$. But this means: $\forall n \geq n_0$: $f(x_n) \in V \subseteq V_0$. So $f(x_n) \overset{n\to\infty}{\to} f(x)$ in Y.

(b) By contradiction: Suppose f not continuous, i.e., there exists open subset $V \subseteq Y$ such that (s.t.) $U := f^{-1}(V)$ is not open, i.e.,

(*) $\exists x \in U$: $\forall$ neighbourhood N of x: $N \cap U^c \neq \emptyset$

Let $\{N_k\}_{k\in\mathbb{N}}$ be a countable neighbourhood base at x. Consider $\tilde{N}_k := \bigcap_{j=1}^{k} N_j$ for $k \in \mathbb{N}$. Then $\{\tilde{N}_k\}_{k\in\mathbb{N}}$ is a countable neighbourhood base of x, with $\tilde{N}_{k+1} \subseteq \tilde{N}_k$ $\forall k$. $\forall k \in \mathbb{N}$: $\tilde{N}_k$ is a neighbourhood of $x \overset{(*)}{\Rightarrow} \exists x_k \in \tilde{N}_k \cap U^c$.

Thus (1) $\forall k \in \mathbb{N}\ \forall l \geq k : x_l \in \tilde{N}_k$, so: $x_l \overset{l\to\infty}{\to} x$

(2) $\forall k \in \mathbb{N}$: $f(x_k) \in f(U^c) \subseteq V^c$. But $f(x) \in V$, so $\{f(x_k)\}_k$ cannot converge to $f(x)$ ↯ (contradiction) ∎

1.2. Metric spaces

Lemma 1.7 Let X be a metric space. Then, the open sets of X are precisely the ones of the metric topology, i.e., the topology generated by the base $\{B_{1/n}(x)\}_{\substack{n\in\mathbb{N}\\ x\in X}}$. Moreover, X is 1st countable and Hausdorff with respect to (w.r.t.) the metric top.

Pf: (i) Open sets coincide: Let $A \subseteq \mathbb{X}$.

Claim: A open in metric space $\Leftrightarrow$ A open in metric top. (see Handout)

Pf claim: "$\Rightarrow$" Let A be open according to Def. 8(iii) in Handout.

Then: $\forall x \in A \; \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x) \subseteq A$, so $A = \bigcup_{x \in A} B_{\varepsilon_x}(x)$.

Now, for all $x \in A$ choose $\mathbb{N} \ni u_x > \frac{1}{\varepsilon_x}$, then $A = \bigcup_{x \in A} B_{\frac{1}{u_x}}(x)$

so A is open in the metric topology.

"$\Leftarrow$": Exercise!

(ii) 1st countable follows from:

Claim: $\forall x \in \mathbb{X}$: $\{B_{1/n}(x)\}_{n \in \mathbb{N}}$ is a neighbourhood base at x.

Pf claim: (a) $B_{1/n}(x)$ is a neighbourhood of x for every $n \in \mathbb{N}$

(b) Let N be a neighbourhood of $x \Rightarrow \exists G$ open s.t. $x \in G \subseteq N$.

By (i), use Def. 8(iii) in Handout for openness: $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq G$.

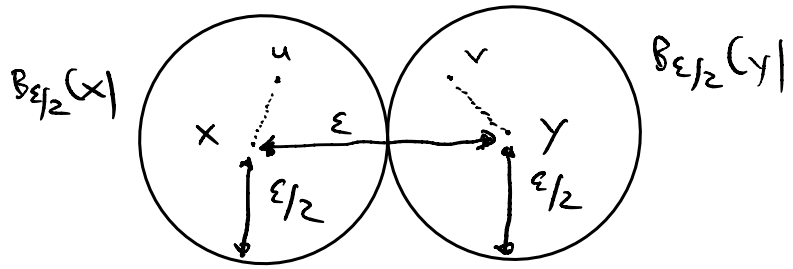
Let $\mathbb{N} \ni n > \frac{1}{\varepsilon} \Rightarrow B_{1/n}(x) \subseteq N$.

(iii) $\mathbb{X}$ is Hausdorff: Let $x, y \in \mathbb{X}$ with $x \neq y$.

Then $\varepsilon := d(x,y) > 0$, and for all $u \in B_{\varepsilon/2}(x)$, $v \in B_{\frac{\varepsilon}{2}}(y)$, the triangle inequality yields:

$$\varepsilon = d(x,y) \leq d(x,u) + d(u,v) + d(v,y) < \frac{\varepsilon}{2} + d(u,v) + \frac{\varepsilon}{2}.$$

Hence $d(u,v) > 0 \Rightarrow u \neq v \Rightarrow B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(y) = \emptyset \Rightarrow$ $\mathbb{X}$ is Hausdorff



Corollary 1.8 All topological notions are available in a metric space $\mathbb{X}$. In particular, for $(x_n)_n \subseteq \mathbb{X}$, $x \in \mathbb{X}$, $A \subseteq \mathbb{X}$:

(a) $\lim_{n\to\infty} x_n = x$ (as in Def. 1.3 wrt. metric topology)
$\Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$ (in $\mathbb{R}$, with $|\cdot|$)

(b) $x \in \overline{A} \Leftrightarrow \exists (y_n)_n \subseteq A : \lim_{n\to\infty} y_n = x$.

(c) If $\mathbb{Y}$ is a topological space, and $f : \mathbb{X} \to \mathbb{Y}$, then
f is continuous $\Leftrightarrow$ f is sequentially continuous

Pf: (a), (b): simple exercises (do!) (c): See Theorem 1.6 ∎

Theorem 1.9 Let $\mathbb{X}$ be a metric space.
Then: $\mathbb{X}$ separable $\Rightarrow$ $\mathbb{X}$ 2nd countable
(and, hence, $\Leftrightarrow$ by Theorem 1.2)

Pf: See exercises. Strategy: Let $A \subseteq \mathbb{X}$ be countable and dense. Prove that for every open subset $G \subseteq \mathbb{X}$ we have the representation $G = \bigcup_{x\in G} B_{\frac{1}{n(x)}}(a(x))$ with $n(x) \in \mathbb{N}$ and $a(x) \in A$ $\forall x \in G$, that is, $\{B_{\frac{1}{n}}(a)\}_{\substack{n\in\mathbb{N} \\ a\in A}}$ is a countable base ∎

Example 1.10 The above proof establishes the claim of the Handout (Ex. 2(2)): $\{B_{1/n}(q)\}_{\substack{n\in\mathbb{N} \\ q\in\mathbb{Q}^d}}$ is a base of the Euclidean topology on $\mathbb{R}^d$.

Example 1.11 Consider the space of all continuous (cont.) functions $f: [0,1] \to \mathbb{C}$:

$$C([0,1];\mathbb{C}) := C([0,1]) := \{ f: [0,1] \to \mathbb{C} \mid f \text{ is continuous} \}$$

with two different metrics:

(a) Supremum metric:

$$d_\infty(f,g) := \|f-g\|_\infty, \text{ where } \|f\|_\infty := \sup_{x \in [0,1]} |f(x)|.$$

The metric space $(C([0,1]), d_\infty)$ is complete and separable (see Thms. 1.31 & 1.32 later)

(b) 1-metric:

$$d_1(f,g) := \|f-g\|_1, \text{ where } \|f\|_1 := \int_0^1 |f(x)|\, dx.$$

Then: (i) d_1 defines a metric on $C([0,1])$ (see Exercise)

(ii) $(C([0,1]), d_1)$ is separable because $d_1(f,g) \le d_\infty(f,g)$ and $(C([0,1]), d_\infty)$ is separable

(iii) $(C([0,1]), d_1)$ is not complete (see Exercise)

Definition 1.12 Let (X, d_X), (Y, d_Y) be metric spaces, $T: X \to Y$ map

(i) T is called an isometry $:\Leftrightarrow \forall x, x' \in X$:

$$d_X(x,x') = d_Y(Tx, Tx').$$

(ii) X, Y are isometric $:\Leftrightarrow$ there exists a bijective isometry $T: X \to Y$ (correctly, (X, d_X) & (Y, d_Y))

Remark 1.13 Let $T: X \to Y$ be an isometry. Then

(a) T is injective and continuous

(b) X and the range of T (or image; see 1.5(c)

$$\operatorname{ran} T := T(X) = \{ y \in Y \mid \exists x \in X \text{ with } y = Tx \}$$ are isometric

Theorem 1.14 Let X be a metric space. Then there exists a complete metric space $\tilde{X}$, and an isometry $i: X \to \tilde{X}$ such that $i(X)$ is dense in $\tilde{X}$. The space $\tilde{X}$ is unique up to isometric spaces.

Pf: Consists of 4 steps:

(a) Construct $\tilde{X}$

(b) Construct the isometry i, and $W := i(X)$ is dense in $\tilde{X}$

(c) $\tilde{X}$ is complete

(d) Uniqueness

(a): Define an equivalence relation $\sim$ on the set of all Cauchy sequences $(x_n)_n \subseteq X$:

$$(x_n)_n \sim (y_n)_n :\Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$$

($\sim$ is clearly reflexive, symmetric, and transitive). Consider the equivalence class $[(x_n)_n] := \{(y_n)_n \subseteq X \text{ Cauchy} \mid (y_n)_n \sim (x_n)_n\}$ we obtain this way, and let

$$\tilde{X} := \{ [(x_n)_n] \mid (x_n)_n \subseteq X \text{ Cauchy} \}.$$

Given $\tilde{x} \in \tilde{X}$, we write $(x_n)_n \in \tilde{x}$ for a representative $(x_n)_n$ of the equivalence class $\tilde{x}$. Define a metric on $\tilde{X}$:

$$\tilde{d}(\tilde{x}, \tilde{y}) := \lim_{n\to\infty} d(x_n, y_n), \quad \tilde{x}, \tilde{y} \in \tilde{X}, (x_n)_n \in \tilde{x}, (y_n)_n \in \tilde{y}$$

We have to check that:

(1) $\tilde{d}$ is well-defined:

(i) Existence of the limit:

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

$\Rightarrow \quad d(x_n,y_n) - d(x_m,y_m) \le d(x_n,x_m) + d(y_m,y_n)$,

and the same inequality holds with n & m interchanged.

Thus we have:

$$(*) \qquad \underbrace{|d(x_n,y_n)}_{=:\alpha_n} - \underbrace{d(x_m,y_m)|}_{=:\alpha_m} \le d(x_n,x_m) + d(y_m,y_n).$$

Since $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences, we have:

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that $\forall n,m \ge N$:

$$d(x_n,x_m) < \frac{\varepsilon}{2} \quad \text{and} \quad d(y_n,y_m) < \frac{\varepsilon}{2}$$

So $|\alpha_n - \alpha_m| < \varepsilon$ for $n,m \ge N$ and so $(\alpha_n)_n \subseteq \mathbb{R}$ is a Cauchy sequence, thus convergent (because $\mathbb{R}$ is complete)

(iii) Independence of representatives: Let $(x_n)_n \sim (x_n')_n$ and $(y_n)_n \sim (y_n')_n$. We have to prove that

$$\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n',y_n')$$

Repeat the derivation of $(*)$ above, interchanging $x_n \leftrightarrow x_n'$ and $y_n \leftrightarrow y_n'$, and obtain

$$|d(x_n,y_n) - d(x_n',y_n')| \le d(x_n,x_n') + d(y_n,y_n') \xrightarrow{n\to\infty} 0$$

by definition of $\sim$.

(2) $\tilde{d}$ fulfills the axioms of a metric:

(i) $\tilde{d} \ge 0$: $\quad \tilde{d}(\tilde{x},\tilde{y}) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n,y_n) = 0$ for some representatives

$$\Leftrightarrow (x_n)_n \sim (y_n)_n \Leftrightarrow \tilde{x} = \tilde{y}.$$

(ii) Symmetry

and (iii) Triangle inequality

are clear from the properties of d (do!).

(b): Define $i: X \to \tilde{X}$

$$b \longmapsto \tilde{b} := [(b, b, b, \dots)]$$

Set $W := i(X)$. The map i is an isometry since

$$\tilde{d}(i(a), i(b)) = \tilde{d}(\tilde{a}, \tilde{b}) = \lim_{n\to\infty} d(a,b) = d(a,b).$$

It remains to show that W is dense in $\tilde{X}$ with respect to $\tilde{d}$.

Let $\tilde{x} \in \tilde{X}$ and $\varepsilon > 0$. Pick any representative $(x_n)_n \in \tilde{x}$. As $(x_n)_n$ is a Cauchy sequence (in X) we have: $\exists N \in \mathbb{N}\ \forall n, m \geq N$: $d(x_n, x_m) < \frac{\varepsilon}{2}$. Let $\tilde{b} := [(x_N, x_N, x_N, \dots)] = i(x_N) \in W$.

Then $\tilde{d}(\tilde{b}, \tilde{x}) = \lim_{n\to\infty} d(x_N, x_n) \leq \frac{\varepsilon}{2} < \varepsilon$.

So W is dense in $\tilde{X}$.

(c): Let $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ be a Cauchy seq. in $\tilde{X}$; W is dense in $\tilde{X}$: $\forall k \in \mathbb{N}\ \exists\, \tilde{z}^{(k)} := [(z_k, z_k, \dots)] \in W$, $z_k \in X$, s.t.

$$\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)}) < \frac{1}{k}$$

For every $k, \ell \in \mathbb{N}$ we get (due to the isometry property of i):

$$(**)\quad d(z_k, z_\ell) = \tilde{d}(\underbrace{i(z_k)}_{\tilde{z}^{(k)}}, \underbrace{i(z_\ell)}_{\tilde{z}^{(\ell)}}) \leq \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x}^{(k)})}_{<\frac{1}{k}} + \tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(\ell)}) + \underbrace{\tilde{d}(\tilde{x}^{(\ell)}, \tilde{z}^{(\ell)})}_{<\frac{1}{\ell}}$$

Let $\varepsilon > 0$. Since $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ is Cauchy (wrt. $\tilde{d}$), there exists $K \in \mathbb{N}$ such that $\tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(\ell)}) < \frac{\varepsilon}{3}$ for all $k, \ell \geq K$. Hence, (**) implies that $d(z_k, z_\ell) < \varepsilon$ for all $k, \ell \geq \max\{\frac{3}{\varepsilon}, K\}$, that is, $(z_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy. Let $\tilde{x} := [(z_n)_n] \in \tilde{X}$.

We prove now that

$$\lim_{k\to\infty} \tilde{d}(\tilde{x}^{(k)}, \tilde{x}) = 0.$$

Indeed, let $\varepsilon > 0$. Then, as $(z_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. $d(z_k, z_n) < \frac{\varepsilon}{2}$ for all $k, n \geq N$. Thus, for all $k > \max\{\frac{2}{\varepsilon}, N\}$,

$$\tilde{d}(\tilde{x}^{(k)}, \tilde{x}) \leq \underbrace{\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)})}_{<\frac{1}{k}} + \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x})}_{=\lim_{n\to\infty} d(z_k, z_n)} < \varepsilon \qquad \square$$

(d) See exercise.

1.3. Example: sequence space ℓ^p

Definition 1.15 (ℓ^p-spaces)

Let, for $p \in [1,\infty)$ $(= [1,\infty[)$,

$$\ell^p := \ell^p(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_p := \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} < \infty \right\}$$

and $(p = \infty)$

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

($\|\cdot\|_p$ will be a norm for every $p \in [1,\infty]$, see later)

Lemma 1.16 For every $p \in [1,\infty]$, $d_p(x,y) := \|x-y\|_p$, $x, y \in \ell^p$, defines a metric d_p on ℓ^p

Pf: All properties clear (check!), except for triangle inequality, this follows from Lemma 1.17 (b) below (do!)

Lemma 1.17 (Hölder & Minkowski)

(a) Let $p, q \in [1,\infty]$ be (Hölder) conjugated exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ (convention: "$\frac{1}{\infty} = 0$").

Dual pairing and Hölder inequality: For all $x \in \ell^p$, $y \in \ell^q$:

$\langle x, y\rangle := \sum_{n\in\mathbb{N}} x_n y_n$ is well-defined, and

$$|\langle x, y\rangle| \le \sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \, \|y\|_q$$

(b) Minkowski inequality:

For all $x, y \in \ell^p$: $\quad \|x+y\|_p \le \|x\|_p + \|y\|_p$

Pf: Both (a) & (b) follow from corresponding ineq.'son $\mathbb{C}^N$, and then passing to the limit; for (a) f.ex.

$$\sum_{n=1}^{N} |x_n y_n| \le \left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{N} |y_n|^q\right)^{1/q}$$ (see ex. Forster, vol. 1)

& carefully (!) take limit ∎

Remarks 1.18 (a) Note: alternative notation: l_p

(b) Note: $l^p(\{1,\dots,n\}) := \{x=(x_1,\dots,x_n) \mid x_i \in \mathbb{C},\ \|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\}$ $(n\in\mathbb{N})$ is nothing but $\mathbb{C}^n$ with $\|\cdot\|_p$-norm.

(c) l^p-spaces can also be defined for $p\in(0,1)$ $(=]0,1[)$. But the Minkowski inequality does not remain true in this case. Instead, one has $\|x+y\|_p^p \le \|x\|_p^p + \|y\|_p^p \ \forall p\in(0,1)$ (quasi-norm). Therefore we consider l^p-spaces for $p\ge 1$ only.

Theorem 1.19 (a) $\forall p\in[1,\infty)$: l^p is separable

(b) l^∞ is not separable

Pf: (a) We use the separability of $\mathbb{C}$. For $n\in\mathbb{N}$, let

$$M_n := \{(x_1,\dots,x_n,0,\dots) \mid x_j \in \mathbb{Q}+i\mathbb{Q},\ j=1,\dots,n\}$$

So M_n is countable and $M := \bigcup_{n\in\mathbb{N}} M_n$ is also countable.

Claim: $\overline{M} = l^p$. Let $y\in l^p$ and $\varepsilon>0$, then there exists $N\in\mathbb{N}$:

$$\sum_{j=N+1}^{\infty} |y_j|^p < \frac{\varepsilon^p}{2} \qquad \left(\|y\|_p<\infty \Rightarrow \lim_{L\to\infty}\left(\sum_{j=L}^{\infty}|y_j|^p\right)^{1/p} = 0\right),$$

and since $\mathbb{Q}+i\mathbb{Q}$ is dense in $\mathbb{C}$ there exists $x\in M_N$ such that

$$\sum_{j=1}^{N} |x_j-y_j|^p < \frac{\varepsilon^p}{2}.$$ This implies $(d_p(x,y))^p = \|x-y\|_p^p < \varepsilon^p$

(b) See exercise. ∎

Theorem 1.20 l^p is complete for every $p\in[1,\infty]$

(that is, $(l^p(\mathbb{N}), d_p)$ is a complete metric space)

Pf: (a) Case $p \in [1, \infty)$: Let $(x^{(n)})_{n \in \mathbb{N}} \subseteq \ell^p$ be a Cauchy sequence wrt $\|\cdot\|_p$, where $x^{(n)} := (x_1^{(n)}, x_2^{(n)}, \dots)$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$(*) \quad \sum_{j=J_1}^{J_2} |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j \in \mathbb{N}} |x_j^{(n)} - x_j^{(m)}|^p < \varepsilon^p \quad \forall n, m \geq N, \ \forall J_1 \leq J_2 \in \mathbb{N}.$$

Step 1: Find a candidate for the limit using the completeness of $\mathbb{C}$. Let $J_1 = J_2 = J$, then $(*)$ implies $(x_J^{(n)})_n \subseteq \mathbb{C}$ is a Cauchy seq. $(\forall J \in \mathbb{N})$ and, since $\mathbb{C}$ is complete, there is some $x_J \in \mathbb{C}$ s.t

$$(**) \quad \lim_{n \to \infty} x_J^{(n)} = x_J \quad \forall J \in \mathbb{N}.$$

so our candidate is $x := (x_1, x_2, \dots)$

Step 2: Prove that the candidate belongs to the space: Use the Minkowski inequality in $\mathbb{C}^{J_2}$, $(**)$, and set $J_1 = 1$ in $(*)$:

$$\left(\sum_{j=1}^{J_2} |x_j|^p\right)^{1/p} \leq \lim_{n \to \infty} \left(\sum_{j=1}^{J_2} |x_j^{(n)} - x_j^{(n)}|^p\right)^{1/p} + \underbrace{\left(\sum_{j=1}^{J_2} |x_j^{(n)}|^p\right)^{1/p}}_{\leq \|x^{(n)}\|_p < \infty}$$

$$\leq \varepsilon + \|x^{(n)}\|_p \quad \forall J_2 \in \mathbb{N}, \forall n \geq N$$

so we get $\|x\|_p \leq \varepsilon + \|x^{(n)}\|_p$ for every $n \geq N$, and hence $x \in \ell^p$

Step 3: Prove that the sequence converges to the candidate, in the norm of the space: Set $J_1 = 1$ in $(*)$ and $n \to \infty$, using $(**)$. Then, for every $n \geq N$,

$$\sum_{j=1}^{J_2} |x_j^{(n)} - x_j|^p \leq \varepsilon^p \quad \forall J_2 \in \mathbb{N},$$

so by sending $J_2 \to \infty$ in addition, we have

$$d_p(x^{(n)}, x) = \|x^{(n)} - x\|_p \leq \varepsilon, \text{ so } x^{(n)} \to x \text{ in } \ell^p.$$

(b) Case $p = \infty$: Replace "$\sum_{j=J_1}^{J_2}$" by "$\sup_{J_1 \leq j \leq J_2}$" and "$|\cdot|^p$" by "$|\cdot|$" ∎

1.4. Compactness

Definition 1.21 Let X be a topological space, and $A \subseteq X$

(a) A is compact $:\Leftrightarrow$ For every open cover $\bigcup_{\alpha \in I} B_\alpha \supseteq A$ with an index set $I \neq \emptyset$, and $B_\alpha \subseteq X$ open for all $\alpha \in I$, there exists a finite open subcover, i.e., $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in I$ such that $\bigcup_{n=1}^{N} B_{\alpha_n} \supseteq A$ (Heine-Borel-property).

(b) A is sequentially compact $:\Leftrightarrow$ Every sequence in A has a convergent subsequence with limit in A (Bolzano-Weierstrass property).

(c) A is relatively (seq.) compact $:\Leftrightarrow \bar{A}$ is (seq.) compact.

Remark 1.22 (a) Def. applies in particular to X. In this case we have "$=$" instead of "$\supseteq$".

(b) Some books (e.g. Bourbaki) use compactness only for Hausdorff spaces; they call our property quasi-compact.

Theorem 1.23 Let X be a topological space

(a) Assume X is 1st countable. Then:

X compact $\Rightarrow$ X sequentially compact

(b) Assume X is 2nd countable. Then:

X compact $\Leftrightarrow$ X sequentially compact.

In the proof we will use/need:

Theorem 1.24 (Lindelöf's Lemma(!)) Let X be a 2nd countable top. space, and let $\bigcup_{\alpha \in I} A_\alpha = X$ be an open cover. Then there exists $(\alpha_k)_{k \in \mathbb{N}} \subseteq I$ such that $\bigcup_{k \in \mathbb{N}} A_{\alpha_k} = X$, i.e., there exists a countable(!) open subcover

---

Pf: Let $\{B_k\}_{k \in \mathbb{N}}$ be a countable base of the topology. Let

$K := \{k \in \mathbb{N} \mid \exists \alpha = \alpha_k \in I \text{ with } B_k \subseteq A_{\alpha_k}\}$. Then $\bigcup_{k \in K} B_k \subseteq \bigcup_{\alpha \in I} A_\alpha$ (A)

Now, every A_α is a union of certain B_k's, and the corresponding k's necessarily belong to K. Thus $\bigcup_{\alpha\in I} A_\alpha \subseteq \bigcup_{k\in K} B_k$ (Δ1)

(Δ1) & ($\Delta\Delta$1) imply $X = \bigcup_{\alpha\in I} A_\alpha = \bigcup_{k\in K} B_k = \bigcup_{k\in K} A_{\alpha_k}$ ■

Pf. of Thm 1.23: (a) By contradiction: Assume X is compact, but that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ without convergent subsequence.

Claim: $\forall x \in X$ there exists a neighbourhood $U(x)$ such that $x_n \in U(x)$ for at most finitely many n.

Pf. of claim: Suppose claim false. Then $\exists x \in X$ and a countable neighbourhood base $\{V_k\}_{k\in\mathbb{N}}$ of x with $V_k \supseteq V_{k+1}$ $\forall k \in \mathbb{N}$ (holds w.l.o.g. (o.B.d.A), see pf. Thm. 1.6(b)) such that

(*) $\qquad \forall k \in \mathbb{N}: x_n \in V_k$ for infinitely many n

So, for all $k \in \mathbb{N}$, define $n_k \in \mathbb{N}$ such that $x_{n_k} \in V_k$. Since (*) holds for infinitely many n, we can choose the n_k s.t. $n_k < n_{k+1}$ $\forall k \in \mathbb{N}$

But then $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and $\lim_{k\to\infty} x_{n_k} = x$ ↯

Wlog., the neighbourhoods $U(x)$ from the claim can be assumed to be open. (Otherwise shrink $U(x)$ to the open set contained in it which itself contains x). Now, $X = \bigcup_{y\in X} U(y) = \bigcup_{j=1}^{n} U(y_j)$ for some $n \in \mathbb{N}$ and some $y_1, \dots, y_n \in X$ because X is compact. The claim implies that each $U(y_j)$ contains at most finitely many members of the sequence $(x_n)_{n\in\mathbb{N}}$, so X contains at most finitely many members of the sequence $(x_n)_{n\in\mathbb{N}}$ ↯

(b) "⇒" follows from Theorem 1.2 and (a). We prove "⇐" by contradiction: Assume every sequence has a convergent subsequence, but there exists an open cover of X without finite subcover. As X is 2nd countable, there exists a countable subcover $X = \bigcup_{j\in\mathbb{N}} C_j$ of this cover (by Thm. 1.24). For every $n \in \mathbb{N}$ pick a point

(**) $\qquad x_n \in X \setminus \left(\bigcup_{j=1}^{n} C_j\right)$

(possible $\forall n \in \mathbb{N}$, since there exists no finite subcover)

By hypothesis, $(x_n)_n \subseteq \overline{X}$ has a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x \in \overline{X}$. There exists $N \in \mathbb{N}$ such that $x \in C'_N$, so C'_N is a neighbourhood of x. Now, $(x_{n_k})_{k\in\mathbb{N}}$ being convergent means $x_{n_k} \in C_N$ for finally all k (i.e. $\exists K : k \geq K \Rightarrow x_{n_k} \in C_N$). On the other hand, $n_k \geq N$ for finally all k, hence $x_{n_k} \notin C_N$ for finally all k by (**) ↯ ∎

Theorem 1.25 Let $\overline{X}$ be a topological space and $A \subseteq \overline{X}$. Then:

(a) $\overline{X}$ compact and A closed $\Rightarrow$ A compact

(b) $\overline{X}$ Hausdorff and A compact $\Rightarrow$ A closed.

Pf. (a) Let $\bigcup_{\alpha\in I} U_\alpha \supseteq A$ be an open cover. Since A is closed, A^c is open, and

$$\overline{X} = A^c \cup \left(\bigcup_{\alpha\in I} U_\alpha\right)$$ is an open cover.

Since $\overline{X}$ is compact, there exists $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in I$ such that

$$\overline{X} = A^c \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

and so $\bigcup_{i=1}^{n} U_{\alpha_i} \supseteq A$ is a finite subcover.

(b) See Exercise.

Warning 1.26 Bounded and closed do NOT imply compact in general!

Example: ℓ^p, $p \in [1, \infty]$, and (the closed unit ball)

$$\overline{B}_1(0) := \{x \in \ell^p \mid \|x\|_p \leq 1\} = \overline{\{x \in \ell^p \mid \|x\|_p < 1\}} = \overline{B_1(0)}$$

is bounded and closed but: consider $e^{(n)} := (\dots, 0, 1, 0, \dots) \in \overline{B}_1(0)$ (with 1 at the nth position), $n \in \mathbb{N}$. Then

$$d_p(e^{(n)}, e^{(m)}) = \|e^{(n)} - e^{(m)}\|_p = \begin{cases} 2^{1/p}, & p < \infty \\ 1, & p = \infty \end{cases} \quad \forall n, m \in \mathbb{N},\ n \neq m,$$

so there exists no convergent subsequence, and $\overline{B}_1(0)$ is not seq. compact. Hence, by Thm. 1.23 (a), $\overline{B}_1(0)$ is not compact.

Theorem 1.27 Let $\overline{X}$ be a metric space. Then

$$\begin{array}{ccc} \overline{X} \text{ is compact} & \overset{(a)}{\Longleftrightarrow} & \overline{X} \text{ is sequentially compact} \\ & & \Downarrow (c) \\ \overline{X} \text{ is } 2^{\text{nd}} \text{ countable} & \underset{(b)}{\Longleftarrow} & \overline{X} \text{ is separable} \end{array}$$

(Recall that $\overline{X}$ is 1^{st} countable & Hausdorff (1.21))

Pf: (b) was proved in Thm. 1.9.

(a) "$\Rightarrow$": This is Thm. 1.23 (a) (since $\overline{X}$ is 1^{st} countable).

"$\Leftarrow$": Follows from pf. of (c) (see below), "$\Leftarrow$" of (b), and Thm. 1.23 (b).

(c): We prove that sequential compactness implies separability by constructing a countable set M with $\overline{M} = \overline{X}$. Fix $n \in \mathbb{N}$ and use the following algorithm to define points $x_k^{(n)}$:

(i) Choose an arbitrary $x_1^{(n)} \in \overline{X}$; set $k := 1$

(ii) WHILE $R_k^{(n)} := \overline{X} \setminus \left(\bigcup_{j=1}^{k} B_{\frac{1}{n}}(x_j^{(n)}) \right) \neq \emptyset$ DO

$$\{ \text{pick } x_{k+1}^{(n)} \in R_k^{(n)} \text{ and increment } k \to k+1 \}$$

Claim: This algorithm stops after finitely many steps.

True, because for $k \neq l$, we have $d(x_k^{(n)}, x_l^{(n)}) \geq \frac{1}{n}$. Hence, if the algorithm did not stop after finitely many steps, we would have an infinite sequence $(x_k^{(n)})_{k \in \mathbb{N}} \subseteq \overline{X}$ without a convergent subseq., in contradiction to $\overline{X}$ being sequentially compact. ✓

The claim now implies that: $\forall n \in \mathbb{N}\ \exists K_n \in \mathbb{N}$ such that

$$(*) \qquad \overline{X} = \bigcup_{j=1}^{K_n} B_{\frac{1}{n}}(x_j^{(n)})$$

Set $M_n := \{ x_j^{(n)} \mid j = 1, \dots, K_n \}$ and $M := \bigcup_{n \in \mathbb{N}} M_n$. Then M is countable.

To prove denseness ($\overline{M} = \overline{X}$), let $x \in \overline{X}$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$.

Then $(*) \Rightarrow \exists l \in \{1, \dots, K_n\}$ such that $x \in B_{\frac{1}{n}}(x_l^{(n)})$. Hence,

$$d(x, M) \leq d(x, x_l^{(n)}) < \frac{1}{n} < \varepsilon, \text{ so } \overline{M} = \overline{X} \quad \blacksquare$$

Theorem 1.28 (Tychonoff's Theorem - or Tikhonov)

Let $J \neq \emptyset$ be an index set, and X_α a compact topological space for all $\alpha \in J$. Then

$$\prod_{\alpha \in J} X_\alpha = \left\{ f : J \to \bigcup_{\alpha \in J} X_\alpha \mid f(\alpha) \in X_\alpha \right\}$$

is compact in the product topology.

Pf: See any textbook on topology (Kelley, Munkres, v. Querenburg f.ex.)

Definition 1.29 Let X, Y be topological spaces. Define

(i) $C(X,Y) := \{ f : X \to Y \mid f \text{ is continuous} \}$

In particular, for $Y = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, set $C(X) := C(X, \mathbb{K})$

(ii) $C_b(X) := \{ f \in C(X) \mid \|f\|_\infty < \infty \}$ (bounded continuous functions)

where $\|f\|_\infty := \sup_{x \in X} |f(x)| = \sup \{ |f(x)| \mid x \in X \}$

Theorem 1.30 Let X, Y be topological spaces, X compact, $f \in C(X,Y)$.

Then: (a) $f(X)$ is compact

(b) Assume: Y is Hausdorff and f is a bijection.

Then f is a homeomorphism (i.e., f^{-1} is continuous)

(c) If X, Y are metric spaces, then f is uniformly continuous:

$$\forall \varepsilon > 0 \; \exists \delta = \delta_\varepsilon > 0 : \forall x \in X : f(B_\delta(x; d_X)) \subseteq B_\varepsilon(f(x); d_Y)$$

(equivalently: $\forall \varepsilon > 0 \; \exists \delta > 0 : d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$)

(d) Assume $Y = \mathbb{R}$ (i.e. $f : X \to \mathbb{R}$). Then f takes on its maximum and minimum: $\exists x_1, x_2 \in X$:

$$\forall x \in X : f(x_1) \leq f(x) \leq f(x_2)$$

Pf: (a) Let $\bigcup_{\alpha \in J} V_\alpha \supseteq f(X)$ be an open cover. Then

(*) $$X \subseteq f^{-1}\left(\bigcup_{\alpha \in J} V_\alpha \right) = \bigcup_{\alpha \in J} f^{-1}(V_\alpha) \quad (\text{"}=\text{"} \text{ in fact})$$

Because f is continuous & V_α open, $f^{-1}(V_\alpha)$ is open $\forall \alpha \in J$, hence (*1) gives an open cover of X – which is compact. Hence, $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N$ s.t. $X \subseteq \bigcup_{n=1}^{N} f^{-1}(V_\alpha)$, and therefore,

$$f(X) \subseteq \bigcup_{n=1}^{N} V_{\alpha_n}$$

(b), (c), (d): See Exercises.

1.5. Example: Spaces of continuous functions

General assumptions in this section:

(i) X is a compact Hausdorff space

(ii) $C(X)$ is equipped with the uniform (supremum) metric:

$$d_\infty(f,g) := \|f-g\|_\infty := \sup_{x \in X} |f(x)-g(x)|$$

(Note: sup = max is finite $\forall f, g \in C(X)$ by Thm. 1.30 (d))

Theorem 1.31 $C(X)$ is complete.

Pf: Follows from completeness of $C_b(X)$ (bounded cont. functions; see exercise), and that $C(X) = C_b(X)$, which follows from compactness of X, and Thm. 1.30 (d). ∎

Theorem 1.32 X is metrisable $\iff$ $C(X)$ is separable

(A topological space $(X, \mathcal{T})$ is metrisable $:\iff \exists$ metric d on X that generates the topology $\mathcal{T}$)

Pf: For "$\Leftarrow$": See (f.ex) Bourbaki, "Elements of Mathematics, General Topology", Part 2, Sect. X 3.3, Thm. 1-1.

Here, we only prove:

"$\Rightarrow$": Fix any metric that is compatible with the topology.

For $m, n \in \mathbb{N}$, define

$$G_{m,n} := \left\{ f \in C(X) \;\middle|\; f\left(B_{\frac{1}{m}}(x)\right) \subseteq B_{\frac{1}{n}}(f(x)) \quad \forall x \in X \right\}$$

By compactness of X we get

(i) Any $f \in C(X)$ is even uniformly continuous, by Thm. 1.30(c).

Choose $\varepsilon := \frac{1}{n}$ there, and let m large enough s.t. $\frac{1}{m} \leq \delta$. Then $f \in G_{mn}$, so $C(X) = \bigcup_{m \in \mathbb{N}} G_{mn}$ $\forall n \in \mathbb{N}$ (i)

(ii) For any $n \in \mathbb{N}$ we can find $k_n \in \mathbb{N}$ and $a_1, \dots, a_{k_n} \in X$ such that X can be written as a union of open balls of radius $\frac{1}{n}$: $X = \bigcup_{k=1}^{k_n} B_{\frac{1}{n}}(a_k)$ (2) (X is compact)

Now, $\mathbb{K}$ is separable, i.e. there exists countable set $\{x_\nu \in \mathbb{K} \mid \nu \in \mathbb{N}\}$ which is dense in $\mathbb{K}$.

For given $n \in \mathbb{N}$ and any $\varphi : \{1, \dots, k_n\} \to \mathbb{N}$, i.e. $\varphi = (\varphi(1), \dots, \varphi(k_n)) \in \mathbb{N}^{k_n}$, define

$$G_{mn}^{\varphi} := \left\{ g \in G_{mn} \mid |g(a_k) - x_{\varphi(k)}| < \frac{1}{n} \ \forall k = 1, \dots, k_m \right\}$$

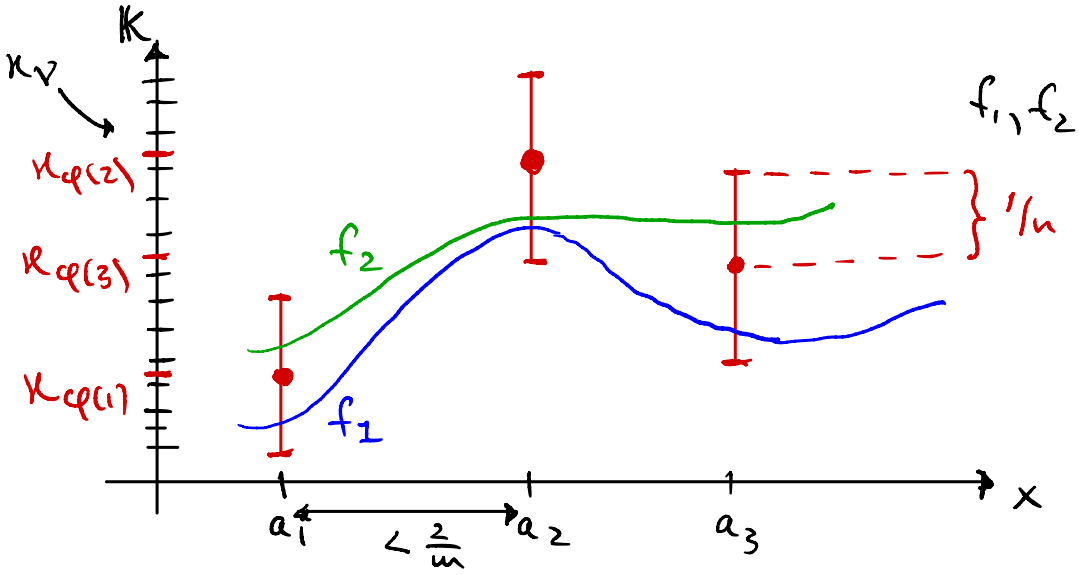
We only want to consider those φ with $G_{mn}^{\varphi} \neq \emptyset$, so let

$$\Phi_{mn} := \left\{ \varphi \in \mathbb{N}^{k_m} \mid G_{mn}^{\varphi} \neq \emptyset \right\}$$

Φ_{mn} is not empty (since $\varphi_\nu := (\nu, \dots, \nu) \in \Phi_{mn}$, because the constant function $f \equiv x_\nu \in G_{mn}^{\varphi_\nu}$).

For every $\varphi \in \Phi_{mn}$ pick some $g_\varphi \in G_{mn}^{\varphi}$. Now define

$$L_{mn} := \{ g_\varphi \mid \varphi \in \Phi_{mn} \} \quad \text{and} \quad L := \bigcup_{m,n \in \mathbb{N}} L_{mn}$$



$f_1, f_2 \in G_{mn}^{\varphi}$

Notions in proof of Thm. 1.32

Note: L_{mn} is countable because $\Phi_{mn} \subseteq \mathbb{N}^{k_m}$, hence L is countable. The theorem now follows from:

Claim: $\overline{L}^{d_\infty} = C(X)$

Pf: Fix $f \in C(X)$, $n \in \mathbb{N}$.

(1) $\Rightarrow \exists m \in \mathbb{N} : f \in G_{mn}$. Also, $\exists \varphi_f \in \Phi_{mn}$ s.t. $f \in G_{mn}^{\varphi_f}$: Simply choose $\varphi_f(k)$ such that $|f(a_k) - \kappa_{\varphi_f(k)}| < \frac{1}{n}$, which is possible by the denseness of $\{\kappa_\nu \mid \nu \in \mathbb{N}\}$ in K.

(2) $\Rightarrow \forall x \in X \; \exists k_x \in \{1, \dots, k_m\} : x \in B_{\frac{1}{m}}(a_{k_x})$

Take $g_{\varphi_f} \in L_{mn} \subseteq L$ as approximant. Since $f, g_{\varphi_f} \in G_{mn}^{\varphi_f}$ we have, for all $x \in X$,

$$|f(x) - g_{\varphi_f}(x)| \leq \underbrace{|f(x) - f(a_{k_x})|}_{< \frac{1}{n} \text{ by def. of } G_{mn}} + \underbrace{|f(a_{k_x}) - \kappa_{\varphi_f(k_x)}|}_{< \frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} + \underbrace{|\kappa_{\varphi_f(k_x)} - g_{\varphi_f}(a_{k_x})|}_{< \frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} + \underbrace{|g_{\varphi_f}(a_{k_x}) - g_{\varphi_f}(x)|}_{< \frac{1}{n} \text{ by def. of } G_{mn}}$$

$$< \frac{4}{n}.$$

Since f and n were arbitrary, the claim follows ∎

An alternative approach to separability:

Definition 1.33 (a) A K-vectorspace A is a K-algebra $:\Leftrightarrow$ there exists a multiplication $A \times A \to A$ which satisfies:

$$(a+b)c = ac + bc \qquad \forall a, b, c \in A$$

$$c(a+b) = ca + cb \qquad \forall a, b, c \in A$$

$$\lambda(ac) = (\lambda a)c = a(\lambda c) \qquad \forall a, c \in A, \lambda \in K$$

(Example: $C(X)$ is a (commutative!) K-algebra)

(b) A subspace $B \subseteq A$ is a subalgebra $:\Leftrightarrow$ B is closed under multiplication

(c) A subset $B \subseteq C(X)$ separates points in X $:\Leftrightarrow$ for all distinct points $x, y \in X$, $x \neq y$, there exists $f \in B$: $f(x) \neq f(y)$

Theorem 1.34 (Stone-Weierstraß) Let $B \subseteq C(X)$ be a subalgebra with the following properties:

(i) $1 \in B$ (where $1 : X \to \mathbb{K}$, $x \mapsto 1$ is the constant function) — B is unital

(ii) B separates points.

(iii) If $\mathbb{K} = \mathbb{C}$: Assume further that B is closed under complex conjugation (i.e. $f \in B \Rightarrow \bar{f} \in B$)

Then: $\overline{B}^{d_\infty} = C(X)$ (that is, B is dense in $C(X)$)

Pf: Not here; see e.g. Reed-Simon, Vol. 1, App. to Sect. IV.3

Corollary 1.35 Let $X \subseteq \mathbb{R}^d$ be compact, $d \in \mathbb{N}$. Then the set of polynomials is dense (wrt. d_∞) in $C(X)$. Furthermore, $C(X)$ is separable.

Pf: Let B be the $\mathbb{K}$-subalgebra of the $\mathbb{K}$-algebra $C(X)$ generated by the monomials

(*) $M_{n,\alpha} : X \ni x = (x_1, \dots, x_d) \mapsto x_\alpha^n \in \mathbb{R}$, $n \in \mathbb{N}_0, \alpha \in \{1, \dots, d\}$

Then Stone-Weierstraß gives $\overline{B} = C(X)$ since, clearly,

(1) $1 \in B$ and, if $\mathbb{K} = \mathbb{C}$, then B is closed under complex conjugation, since $x \in X \subseteq \mathbb{R}^d$ and B is a $\mathbb{C}$-subspace.

(2) If $x, y \in X$, $x \neq y$, then $\exists \alpha \in \{1, \dots, d\} : x_\alpha \neq y_\alpha$, so $M_{1,\alpha}(x) \neq M_{1,\alpha}(y)$, and so B separates points.

We prove separability: Let $\mathbb{K}_0 := \begin{cases} \mathbb{Q}, & \mathbb{K} = \mathbb{R} \\ \mathbb{Q}+i\mathbb{Q}, & \mathbb{K} = \mathbb{C} \end{cases}$

and let B_0 be the $\mathbb{K}_0$-algebra generated by the monomials in (*).

(Note: This is not a subalgebra of $C(X)$ - why?!)

(1) B_0 is countable (it consists of polynomials with rational coeff.)

(2) Since X is bounded:

$$\forall \varepsilon > 0 \; \forall n \in \mathbb{N}_0 \; \forall \alpha \in \{1,\dots,d\} \; \forall c \in \mathbb{K} \; \exists q \in \mathbb{K}_0 : d_\infty(c M_{n,\alpha}, q M_{n,\alpha}) < \varepsilon$$

$$\Rightarrow \quad \overline{B_0} = \overline{B} = C(X) \qquad \blacksquare$$

Definition 1.36 Let X be a metric space (not necessarily compact)

Let $\mathcal{F} \subseteq C(X)$.

(i) $\mathcal{F}$ is <u>equicontinuous</u> $:\Leftrightarrow \forall \varepsilon > 0 \; \forall x \in X \; \exists \delta > 0 : \forall f \in \mathcal{F}$:

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

(i.e. $d_X(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}$; $\delta = \delta(\varepsilon, x)$)

(ii) $\mathcal{F}$ is <u>uniformly equicontinuous</u> $:\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in X \; \forall f \in \mathcal{F}$

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

(i.e., $\delta = \delta(\varepsilon)$, but uniformly in $f \in \mathcal{F}$ & $x, y \in X$)

Remark 1.37 (a) If X is even compact (by Thm. 1.30(c)):

equicontinuous $\Leftrightarrow$ uniformly equicontinuous.

(b) Examples:

(i) Any <u>finite</u> collection of continuous functions is equicontinuous

(ii) $X = [0,1]$: $\{x \mapsto \cos(\frac{x}{n}) \mid n \in \mathbb{N}\}$ is uniformly equicontinuous

(iii) $X = [0,1]$: $\{x \mapsto x^{1/n} \mid n \in \mathbb{N}\}$ is <u>not</u> equicontinuous

(iv) $X = (0,\infty)$: $\{x \mapsto \arctan(nx) \mid n \in \mathbb{N}\}$ is equicontinuous but not uniformly equicont. However, each function <u>is</u> uniformly continuous.

Theorem 1.38 Let X be a metric space (not necessarily cpt.) and $(f_n)_{n\in\mathbb{N}} \subseteq C(X)$ an equicontinuous sequence. Assume there exists dense subset $D \subseteq X$ such that

$$\forall x \in D: \quad \lim_{n\to\infty} f_n(x) \quad \text{exists.}$$

Then $\lim_{n\to\infty} f_n(x) =: f(x)$ exists $\forall x \in X$, and the limit function

$$f \in C(X).$$

Pf: We first prove convergence for all $x \in X$:

Let $x \in X$ (fixed), $\varepsilon > 0$. By assumption there exists $\delta > 0$ s.t.

$$(*) \qquad \forall n \in \mathbb{N}\ \forall x' \in B_\delta(x): \ |f_n(x) - f_n(x')| < \varepsilon.$$

Since $\overline{D} = X$, there exists $y \in B_\delta(x) \cap D$. At y, the seq. $(f_n(y))_{n\in\mathbb{N}}$ converges (by assumption), hence, it is Cauchy. So: $\exists N \in \mathbb{N}\ \forall n, m \geq N$:

$$(**) \qquad |f_n(y) - f_m(y)| < \varepsilon$$

and so, $\forall n, m \geq N$

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(y)|}_{<\varepsilon \text{ by } (*)} + \underbrace{|f_n(y) - f_m(y)|}_{<\varepsilon \text{ by } (**)} + \underbrace{|f_m(y) - f_m(x)|}_{<\varepsilon \text{ by } (*)}$$

$$< 3\varepsilon$$

So $\lim_{n\to\infty} f_n(x) =: f(x)$ exists for every $x \in X$ (since $(f_n(x))_{n\in\mathbb{N}}$ Cauchy & $\mathbb{K}$ complete)

We now prove continuity of f:

Note that $(*)$ holds for all $n \in \mathbb{N}$, and ε & δ there are indep. of n.

Hence, we can take the limit $n \to \infty$ in $(*)$ to get

$$\forall \varepsilon > 0\ \exists \delta > 0: \forall x' \in B_\delta(x): \ |f(x) - f(x')| \leq \varepsilon \qquad \blacksquare$$

Corollary 1.39 Assume – in addition to the assumptions in Thm. 1.38 – that X is compact. Then in fact

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0$$

(i.e., the convergence is uniform on X)

Pf: Let $\varepsilon > 0$. As X is compact,

(i) $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous (see Rmk 1.37(a)), so:

$\exists \delta > 0$: $\forall x \in X \ \forall n \in \mathbb{N} \ \forall x' \in X$ with $d(x,x') < \delta$:

$$|f_n(x) - f_n(x')| < \varepsilon \qquad (*)$$

(ii) there exists $K \in \mathbb{N}$ and $a_1, \dots, a_K \in X$ such that

$$X = \bigcup_{k=1}^{K} B_\delta(a_k). \qquad (**)$$

Using (**), for every $x \in X$ we can choose $k_x \in \{1, \dots, K\}$ s.t. $x \in B_\delta(a_{k_x})$.

Then, $\forall n \in \mathbb{N}$,

$$|f_n(x) - f(x)| \le \underbrace{|f_n(x) - f_n(a_{k_x})|}_{=:T_1} + \underbrace{|f_n(a_{k_x}) - f(a_{k_x})|}_{=:T_2} + \underbrace{|f(a_{k_x}) - f(x)|}_{=:T_3}.$$

(1) $T_1 < \varepsilon$ for every $n \in \mathbb{N}$ by the definition of a_{k_x} and (*)

(2) $\exists N \in \mathbb{N}$: $\forall n \ge N \ \forall x \in X$: $T_2 < \varepsilon$ by simultaneous convergence of $(f_n(a_k))_n$ for $k = 1, \dots, K$ (finitely many!) by Thm. 1.38

(3) $T_3 \le \varepsilon$ by the definition of a_{k_x}, (*), and the limit $n \to \infty$ in (*)

Hence, we have

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \ge N \ \forall x \in X : |f_n(x) - f(x)| < 3\varepsilon \qquad \blacksquare$$

Theorem 1.40 (Arzelà-Ascoli) Let X be a compact metric space, and $(f_n)_{n\in\mathbb{N}} \subseteq C(X)$ an equicontinuous and pointwise bounded sequence, i.e. $\forall x \in X : \sup_{n\in\mathbb{N}} |f_n(x)| < \infty$

Then there exists a uniformly convergent subsequence $(f_{n_j})_{j\in\mathbb{N}}$.

Equivalently: Every equicontinuous and pointwise bounded subset $\mathcal{F} \subseteq C(X)$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact).

Pf. The equivalence of the 2 statements follows from:
Z metric space, $A \subseteq Z$: A relatively compact $\Leftrightarrow$ every sequence in A has a convergent subsequence (with limit not necessarily in A but only in $\bar{A}$) (see exercise).

We prove the "sequence version":
Since X is compact, X is also separable (by Thm. 1.27), so there exists a dense subset $\{a_\ell \in X \mid \ell \in \mathbb{N}\} \subseteq X$. Pointwise boundedness gives

$$\sup_{n \in \mathbb{N}} |f_n(a_\ell)| < \infty \quad \forall \ell \in \mathbb{N}, \qquad (*)$$

so, by the Bolzano-Weierstraß Thm. (in $\mathbb{K}$), for every ℓ there exists a subsequence $(n_j^{(\ell)})_{j \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{j \to \infty} f_{n_j^{(\ell)}}(a_\ell) \quad \text{exists.} \qquad (**)$$

But: We need simultaneous convergence on the set of points $\{a_\ell \in X \mid \ell \in \mathbb{N}\}$ for a single subsequence. Use diagonal sequence trick: W.l.o.g., assume $(n_j^{(\ell+1)})_{j \in \mathbb{N}} \subseteq (n_j^{(\ell)})_{j \in \mathbb{N}}$

[Indeed, $\sup_{j \in \mathbb{N}} |f_{n_j^{(1)}}(a_2)| < \infty$ by (*). Thus, (**) holds, with $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$. Now proceed by induction].

Define $\nu_j := n_j^{(j)}$ for $j \in \mathbb{N}$ ("diagonal sequence"). Then $(\nu_j)_{j \geq \ell} \subseteq (n_j^{(\ell)})_j$ and $\lim_{j \to \infty} f_{\nu_j}(a_\ell)$ exists for all $\ell \in \mathbb{N}$.
Since the set of the a_ℓ's is dense, Cor. 1.39 gives the claim ∎

Remark 1.41 Both assumptions – compactness of X and equicontinuity – are essential for Thm 1.40 to turn pointwise boundedness of a sequence into (uniform) convergence of a subsequence.

1.6 Baire's Theorem

– the basis of 3 (out of 4) fundamental theorems of FA.

Remark 1.42 Let X be a metric space, $A_1, A_2 \subseteq X$ both dense, and A_1 also open. Then $A_1 \cap A_2$ is also dense

Completeness gives more:

Theorem 1.43 (Baire) Let X be a complete metric space, and, for all $n \in \mathbb{N}$, let $A_n \subseteq X$ be open and dense.
Then $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.

Remark 1.44 (a) Completeness is essential: Consider $\mathbb{Q}$ (with metric from $\mathbb{R}$). Let $\{q_n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Define $A_n := \mathbb{Q} \setminus \{q_n\}$. This is open and dense in $\mathbb{Q}$ but $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ is not dense in $\mathbb{Q}$.

(b) Openness of $\bigcap_{n\in\mathbb{N}} A_n$ is false in general. Consider $A_n := \mathbb{R} \setminus \{q_n\}$.

(c) Baire's Theorem also holds if X is a locally compact Hausdorff space.

Pf (of 1.43): Define $D := \bigcap_{n\in\mathbb{N}} A_n$. Let $x_0 \in X$ be arbitrary and fix $\varepsilon > 0$.
To prove the denseness of D in X we have to prove that $D \cap B_\varepsilon(x_0) \neq \emptyset$.
We do this by constructing a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to some $x \in D \cap B_\varepsilon(x_0)$ (see drawing)
That A_1 is dense implies $A_1 \cap B_\varepsilon(x_0) \neq \emptyset$, so pick $x_1 \in A_1 \cap B_\varepsilon(x_0)$ and $\varepsilon_1 \in (0, \frac{\varepsilon}{2})$ s.t.

$$\overline{B_{\varepsilon_1}(x_1)} \subseteq A_1 \cap B_\varepsilon(x_0) \quad (\leftarrow \text{open})$$

Proceeding similarly with A_2 and $B_{\varepsilon_1}(x_1)$ we pick $x_2 \in A_2 \cap B_{\varepsilon_1}(x_1)$ and $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2})$ s.t.

$$\overline{B_{\varepsilon_2}(x_2)} \subseteq A_2 \cap B_{\varepsilon_1}(x_1) \subseteq A_1 \cap A_2 \cap B_\varepsilon(x_0)$$

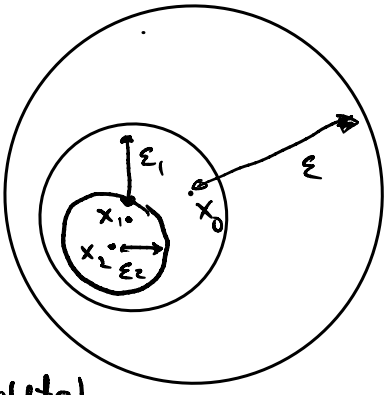
We proceed inductively in the same way, to get 2 sequences

(a) $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n < \frac{\varepsilon}{2^n}$ $\forall n\in\mathbb{N}$

(b) $(x_n)_n \subseteq X$ with, $\forall n\in\mathbb{N}$,

$$\overline{B_{\varepsilon_n}(x_n)} \subseteq A_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \subseteq A_n \cap \dots \cap A_1 \cap B_\varepsilon(x_0)$$

- hence, $\forall n\in\mathbb{N}\ \forall m\geq n$: $x_m \in B_{\varepsilon_n}(x_n)$. (*)



Now (*) & (a) implies $(x_n)_n$ is Cauchy,

and so: $\exists x\in X : x_n \to x$, $n\to\infty$ (since X complete)

But by (*) we have $x \in \overline{B_{\varepsilon_n}(x_n)}$ $\forall n\in\mathbb{N}$, and (b) gives

$$x \in \bigcap_{n\in\mathbb{N}} \overline{B_{\varepsilon_n}(x_n)} \subseteq D \cap B_\varepsilon(x_0), \text{ so } D\cap B_\varepsilon(x_0) \neq \emptyset \quad \blacksquare$$

Definition 1.45 Let X be a topological space and $A\subseteq X$.

(i) A is a G_δ(-set) $:\Leftrightarrow$ A is a countable intersection of open sets.

(ii) A is nowhere dense $:\Leftrightarrow$ $\overline{A}$ has no interior points.

(iii) A is meagre (or of 1st category) $:\Leftrightarrow$ A is a countable union of nowhere dense sets

(iv) A is non-meagre (or of 2nd category) $:\Leftrightarrow$ A is not meagre

Example 1.46 $\mathbb{Q}$ is meagre in $\mathbb{R}$ $\left(\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}\right)$

Lemma 1.47 Let X be a topological space and $A\subseteq X$. Then

(a) A is nowhere dense $\Leftrightarrow$ $(\overline{A})^c$ dense

(b) A is meagre and $B\subseteq A$ $\Rightarrow$ B is meagre

(c) $A_n \subseteq X$ meagre $\forall n\in\mathbb{N}$ $\Rightarrow$ $\bigcup_{n\in\mathbb{N}} A_n$ meagre.

Pf: (b) & (c) are clear by the very definitions (do!)

Statement (a) follows from equivalence

$$B \text{ has no interior points} \Leftrightarrow B^c \text{ dense}$$

This is equivalent to

$$\exists \text{ interior point of } B \Leftrightarrow B^c \text{ is } \underline{\text{not}} \text{ dense}$$

which is clearly true $\blacksquare$

We now prove 3 equivalent reformulations of Baire's Thm:

Lemma 1.48 Let X be a topological space. Then the following 4 statements are equivalent:

(i) $A_n \subseteq X$ open & dense $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ dense

(ii) $A_n \subseteq X$ is a dense G_δ $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ

(iii) $A \subseteq X$ has an interior point $\Rightarrow$ A non-meagre

(iv) $A \subseteq X$ meagre $\Rightarrow$ A^c dense

If one (hence, all) of the above holds, X is called a Baire space.

Corollary 1.49 A complete metric space X is a Baire space.

In particular: If $X \neq \emptyset$, then X is non-meagre.

Pf (of 1.48): (i) $\Leftrightarrow$ (ii) by definition of G_δ

(i) $\Rightarrow$ (iii): Suppose A is meagre, that is, $A = \bigcup_{n\in\mathbb{N}} A_n$ with A_n nowhere dense $\forall n \in \mathbb{N}$. By 1.47(a) this is equivalent to $(\overline{A_n})^c$ dense & open $\forall n \in \mathbb{N}$, and (i) implies that the intersection $\bigcap_{n\in\mathbb{N}} (\overline{A_n})^c =: B$ is also dense. But then $B^c = \bigcup_{n\in\mathbb{N}} \overline{A_n} \supseteq A$ has no interior points, hence A has no interior points ↯

(iii) $\Rightarrow$ (iv): See exercise

(iv) $\Rightarrow$ (i): See exercise

Chapter 2: Banach and Hilbert Spaces

2.1 Vector spaces

General assumption: $X \neq \{0\}$ is a $\mathbb{K}$-vectorspace, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1 Let $\emptyset \neq M \subseteq X$.

(i) M is linearly independent iff all non-empty finite (!) subsets $F \subseteq M$ are linearly independent, i.e., the following implication holds:

$$\sum_{f \in F} \underbrace{\alpha_f}_{\in \mathbb{K}} f = 0 \implies \alpha_f = 0 \quad \forall f \in F$$

(ii) M is linearly dependent iff M is not linearly independent.

(iii) $B \subseteq X$ is a Hamel basis (or algebraic basis) iff

(1) B is linearly independent

(2) Every $x \in X$ can be represented as a (finite!) linear combination of elements in B (B spans X).

(iv) X has finite dimension iff there exists a Hamel basis with $|B| < \infty$. Then $\dim X := |B|$ is called the dimension of X

(v) X has infinite dimension iff X does not have finite dimension

Remark 2.2 The dimension is well-defined: $|B|$ is the same for every Hamel basis in a given space (Pf: see linear algebra (LA)).

Example 2.3 (a) Consider

$$c_c := \{ x = (x_j)_{j\in\mathbb{N}} \mid x_j \in \mathbb{C}\ \forall j\in\mathbb{N},\ \text{and } x_j \neq 0 \text{ for only finitely many } j\text{'s} \}$$

(see also exercise; the index 'c' stands for "compact support"; also: c_{00})

Let $e_m := (\dots, 0, 1, 0, \dots)$ with a 1 at the m'th position.

Claim: $B := \{ e_n \mid n\in\mathbb{N} \}$ is a Hamel basis for c_c

(b) Even though ℓ^1 is separable, there exists no countable Hamel basis for ℓ^1 (see exercise)

Theorem 2.4 Every vector space $X \neq \{0\}$ has a Hamel basis

Pf.: Uses Zorn's Lemma; see later.

Corollary 2.5 X has infinite dimension iff For every $n\in\mathbb{N}$ there exists $M_n \subseteq X$ such that $|M_n| = n$ and M_n is linearly indep.

Pf: Existence of Hamel basis with $|B| = \infty$ ∎

Example 2.6 Infinite dimensional vector spaces:

c_c, ℓ^p, $C(X)$ (where $\emptyset \neq X \subseteq \mathbb{R}^d$ open)

2.2 Banach Spaces

Definition 2.7 Let X be a vectorspace. A map $X \to [0,\infty)$, $x \mapsto \|x\|$ is a norm :$\Leftrightarrow$

(1) $\|x\| > 0 \quad \forall\, 0 \neq x \in X$

(2) $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda\in\mathbb{K}\ \forall x \in X$ $\quad (\Rightarrow \|x\| = 0)$

(3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

$(X, \|\cdot\|)$ is called a normed space.

If only (2) and (3) hold, $\|\cdot\|$ is called a semi-norm

Remark 2.8 Let X be a normed space. Then $d(x,y) := \|x-y\|$ is a metric on X. Thus all topological notions and results from the theory of metric spaces are available.

A base of the norm topology on X:

$$\{ B_{\frac{1}{k}}(x) \mid x \in X, k \in \mathbb{N} \} = \{ x + B_{\frac{1}{k}}(0) \mid x \in X, k \in \mathbb{N} \}$$

[Minkowski sum of sets A, B: $A + B := \{a+b \mid a \in A, b \in B\}$

and $a + B := \{a\} + B$]

Warning: Not every metric (on vector spaces) comes from a norm.

Example 2.9 (a) ℓ^p is a normed space with $\|\cdot\| = \|\cdot\|_p$ $\forall p \in [1, \infty]$

(b) $C(X; \mathbb{K})$ (where X is a compact Hausdorff space) is a normed space with $\|f\| := \|f\|_\infty := \sup_{x \in X} |f(x)|$

(c) $\mathbb{R}^d, \mathbb{C}^d$ are normed spaces for $d \in \mathbb{N}$ (wrt. p-norm)

Lemma 2.10 Let X be normed space. Then the following maps are continuous:

(a) Norm: $X \to [0, \infty)$, $x \mapsto \|x\|$

(b) Addition: $X \times X \to X$, $(x, y) \mapsto x + y$

(c) Scalar multiplication: $\mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$

Pf: (a) Let $(x_k)_{k \in \mathbb{N}} \subseteq X$ with $x_k \to x$, $k \to \infty$, i.e., $\|x_k - x\| \to 0$, $k \to \infty$ (see Cor. 1.8). Hence, the claim follows from reverse triangle inequality:

$$\big| \|x\| - \|y\| \big| \leq \|x + y\|$$

(b), (c) see exercise ∎

Definition 2.11 A complete normed space $(X, \|\cdot\|)$ is called a Banach space

Examples 2.12 (a) All spaces in Ex. 2.9 are Banach spaces

(b) Let $X := C([0,1])$ with L^1-norm $\|f\|_1 := \int_0^1 |f(t)|\,dt$. Then $(X, \|\cdot\|_1)$ is not a Banach space (recall Ex. 1.11 (b1)).

Theorem 2.13 Every normed space X can be completed, so that X is isometric to a dense linear subspace W of a Banach space $\widehat{X}$, which is unique up to isometric isomorphisms.

Pf: Analogous to the proof of Theorem 1.14. Note: The isometry is even a linear bijection (hence, an isomorphism) in this case (see Section 2.3 below).

Definition 2.14 Let X be a normed space. A sequence $(e_n)_n \subseteq X$ is called a (Schauder) basis in X :$\Leftrightarrow$ For all $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} x_n e_n \Big\| = 0$$

Notation: $x = \sum_{n\in\mathbb{N}} x_n e_n$, infinite linear combination, convergent series

Note: Linear independence not required for Schauder basis

Example 2.15 Let $p \in [1,\infty)$. Then $(e_n)_{n\in\mathbb{N}}$ with $e_n := (0,\dots,0,1,0,\dots)$ (the 1 in the n'th position) is a (Schauder) basis of ℓ^p:

For $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ we have

$$\Big\| x - \sum_{n=1}^{N} x_n e_n \Big\|^p = \sum_{n=N+1}^{\infty} |x_n|^p \xrightarrow{N\to\infty} 0.$$

Note: This construction fails for ℓ^∞!

Lemma 2.16 Let X be a normed space. Then

$$X \text{ has a (Schauder) basis} \Rightarrow X \text{ is separable}$$

Pf: Let $\mathbb{K}_0 := \mathbb{Q}$, if $\mathbb{K} = \mathbb{R}$, resp. $\mathbb{K}_0 := \mathbb{Q} + i\mathbb{Q}$ for $\mathbb{K} = \mathbb{C}$. Define

$$A_N := \left\{ \sum_{n=1}^{N} x_n e_n \mid x_n \in \mathbb{K}_0 \text{ for } n \in \{1, \dots, N\} \right\}$$

Then the union $A := \bigcup_{N \in \mathbb{N}} A_N$ is dense in X, and countable ∎

Remark 2.17 The implication "$\Leftarrow$" in Lemma 2.16 does not hold (Enflo, 1973)

Remark 2.18 All norms in finite-dimensional spaces are equivalent. That is, for norms $\|\cdot\|$ and $|||\cdot|||$ on $\mathbb{K}^n$ there exists constants $c, \tilde{c} > 0$ such that

$$c\|x\| \le |||x||| \le \tilde{c}\|x\| \qquad \forall x \in \mathbb{K}^n.$$

(see exercise)

Theorem 2.19 Let X be a normed space, and $F \subseteq X$ a finite-dim. subspace. Then F is complete and closed.

Pf: Let $n := \dim F < \infty$. Fix a basis $\{e_1, \dots, e_n\}$ in F. For every $x \in F$ there exists unique $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ s.t. $x = \sum_{j=1}^{n} \alpha_j e_j$.

Let $$|||\alpha||| := \left\| \sum_{j=1}^{n} \alpha_j e_j \right\| \qquad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n.$$

Then the normed spaces $(F, \|\cdot\|)$ and $(\mathbb{K}^n, |||\cdot|||)$ are isometrically isomorphic via $x \mapsto \alpha$. Now, $\mathbb{K}^n$ is closed and complete wrt. the Euclidean norm, and all norms in $\mathbb{K}^n$ are equivalent (by 2.18). Hence, $(\mathbb{K}^n, |||\cdot|||)$ is closed and complete, and, because of the isometry, so is $(F, \|\cdot\|)$. ∎

As a preparation for Theorem 2.21 we prove the following Lemma:

Lemma 2.20 (Riesz' Lemma, 1916) Let X be a normed space, and $U \subsetneq X$ a closed (!) subspace. Then, for all $\lambda \in (0,1)$, there exists $x_\lambda \in X \setminus U$ such that

$$\|x_\lambda\| = 1 \quad \text{and} \quad \|x_\lambda - u\| \geq \lambda \quad \forall u \in U$$

Pf: Let $x \in X \setminus U$ (open !). Then $\exists \varepsilon_x > 0 : B_{\varepsilon_x}(x) \subseteq X \setminus U$. Hence

$$d := \operatorname{dist}(x, U) = \inf_{u \in U} \|x - u\| \geq \varepsilon_x > 0 \text{ for all } x \in X \setminus U.$$

Since $0 < \lambda < 1$ there exists $u_\lambda \in U$ s.t. $d \leq \|x - u_\lambda\| \leq \frac{d}{\lambda}$.

Hence, $\gamma := \frac{1}{\|x - u_\lambda\|} \geq \frac{\lambda}{d}$. Define $x_\lambda := \gamma(x - u_\lambda) \in X \setminus U$ (!)

By definition of γ, we have $\|x_\lambda\| = \gamma \cdot \|x - u_\lambda\| = 1$, and

$$\|x_\lambda - u\| = \|\gamma(x - u_\lambda) - u\| = \|\gamma x - (u + \gamma u_\lambda)\| = \gamma \cdot \|x - \underbrace{(u_\lambda - \tfrac{u}{\gamma})}_{\in U}\|$$

$$\geq \gamma d \geq \lambda \quad \forall u \in U \quad \blacksquare$$

Warning 1.26 illustrates the following general result:

Theorem 2.21 Let X be a normed space. Then

$$\overline{B_1(0)} = \{x \in X \mid \|x\| \leq 1\} \text{ compact} \iff \dim X < \infty.$$

Pf: "$\Leftarrow$": Let $n := \dim X < \infty$. From (the proof of) Thm 2.19: X is isometric to $(\mathbb{K}^n, |||\cdot|||)$. The statement follows from Heine-Borel for $(\mathbb{K}^n, |\cdot|)$ and the equivalence of norms.

"$\Rightarrow$": Assume (for contradiction) that $\dim X = \infty$. Will prove: this implies $\overline{B_1(0)}$ is not seq. compact, by constructing a sequence $(x_n)_{n \in \mathbb{N}}$ in $\overline{B_1(0)}$ with no convergent subsequences:

(i) Let $x_1 \in X$ be arbitrary, s.t. $\|x_1\| = 1$. Let $U_1 := \operatorname{span}\{x_1\} \subsetneq X$ be the closed (!) subspace spanned by x_1. Riesz' Lemma, applied with $\lambda = \frac{1}{2}$, gives existence of $x_2 \in X \setminus U_1$ s.t. $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$. Let $U_2 := \operatorname{span}\{x_1, x_2\} \subsetneq X$.

(iii) Let $U_n := \operatorname{span}\{x_1, \dots, x_n\} \subsetneq X$ be the closed subspace of the vectors constructed before. Again, by Riesz' Lemma, there exists $x_{n+1} \in X \setminus U_n$ s.t. $\|x_{n+1}\| = 1$ and $\operatorname{dist}(x_{n+1}, U_n) \geq \frac{1}{2}$.

By assumption, $\dim X = \infty$, hence this procedure does not stop. We get a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \overline{B_1(0)}$ s.t. $\|x_n - x_m\| \geq \frac{1}{2}$ $\forall n \neq m$. Clearly, $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequence - contradicting the (seq-) compactness of $\overline{B_1(0)}$ ↯ ∎

2.3 Linear operators

Definition 2.22 Let X, Y be vector spaces (over the same field $\mathbb{K}$), $X_0 \subseteq X$ a (linear) subspace, and $T : X_0 \to Y$.

(i) T is a (linear) operator $:\Leftrightarrow$
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \alpha, \beta \in \mathbb{K}, \forall x, y \in X_0$$

(ii) $\operatorname{dom}(T) := D(T) := X_0$ is the domain of T.

(iii) $\operatorname{ran}(T) := R(T) := T(X_0)$ is the range of T

(iv) $\ker(T) := N(T) := \{x \in X_0 \mid Tx = 0\}$ is the kernel (or nullspace) of T

(v) $U \subseteq \operatorname{dom}(T)$ a subspace,
$$T|_U : \begin{array}{ccc} U & \to & Y \\ x & \mapsto & Tx \end{array}$$ is the restriction of T to U

(vi) $W \supseteq \operatorname{dom}(T)$ a vector space, $\tilde{T} : W \to Y$ linear with $\tilde{T}|_{\operatorname{dom}(T)} = T$ is called a (!) linear extension of T to W.

Examples 2.23 Some linear operators:

(a) Identity operator on vector space X:
$$\mathbb{1} := \mathbb{1}_X : \begin{array}{ccc} X & \to & X \\ x & \mapsto & x \end{array}$$

(b) $X = Y = C([0,1])$

(i) Differentiation operator ($X_0 = C^1([0,1])$):

$$\frac{d}{dx} : C^1([0,1]) \to C([0,1]), \quad f \mapsto f'$$

(ii) Anti-derivative operator ($X_0 = X$):

$$T : C([0,1]) \to C([0,1]), \quad f \mapsto Tf$$

with $(Tf)(x) := \int_0^x f(t)\,dt \quad \forall x \in [0,1]$

(iii) Multiplication operator by argument: As above (in (ii)), with $(Tf)(x) := x f(x) \quad \forall x \in [0,1]$

Lemma 2.24 Let $T : X \supseteq \mathrm{dom}(T) \to Y$ be a linear operator. Then

(a) $\mathrm{ran}(T)$ and $\ker(T)$ are vector subspaces (linear subspaces)

(b) $\dim \mathrm{ran}(T) \le \dim \mathrm{dom}(T)$

(c) $\ker(T) = \{0\} \iff T$ injective

$\iff \exists$ inverse T^{-1} of T s.t. $T^{-1} : \mathrm{ran}(T) \to \mathrm{dom}(T)$, $T^{-1}T = \mathbb{1}|_{\mathrm{dom}(T)}$, $TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$

Pf: Copy from Linear Algebra ∎

Remark 2.25 (a) If T^{-1} exists, it is linear.

Pf: $\alpha x = T^{-1}T(\alpha x) = T^{-1}(\alpha Tx) \quad \forall x \in \mathrm{dom}(T), \forall \alpha \in \mathbb{K}$. Hence, $\alpha T^{-1}y = T^{-1}(\alpha y) \quad \forall y \in \mathrm{ran}(T)$. Similarly for addition.

(b) Even if $T : X \to X$ with $\ker(T) = \{0\}$, then not necessarily $\mathrm{ran}(T) = X$, nor $TT^{-1} = \mathbb{1}$ (unlike in the case $\dim X < \infty$!) but only $TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$.

Example 2.26 (Illustrating Rmk 2.25)

Let $X = l^\infty$.

Right-shift operator: $R: \ell^\infty \longrightarrow \ell^\infty$
$(x_1, x_2, x_3, \dots) \longmapsto (0, x_1, x_2, \dots)$

Clearly: $\ker(R) = \{0\}$, but $\operatorname{ran}(R) \subsetneq \ell^\infty$.

Left-shift operator: $L: \ell^\infty \longrightarrow \ell^\infty$
$(x_1, x_2, x_3, \dots) \longmapsto (x_2, x_3, x_4, \dots)$

Then $R^{-1} = L|_{\operatorname{ran}(R)}$ satisfies

$$R^{-1}R = \mathbb{1} \quad \text{and} \quad RR^{-1} = \mathbb{1}|_{\operatorname{ran}(R)}.$$

Definition 2.27 Let X, Y be normed spaces, $T: X \supseteq \operatorname{dom}(T) \to Y$.

T is bounded (operator) $:\Leftrightarrow$ its operator norm is finite, i.e.,

$$\|T\|_{\operatorname{dom}(T)\to Y} := \|T\| := \sup_{\substack{x \in \operatorname{dom}(T) \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in \operatorname{dom}(T) \\ \|x\| = 1}} \|Tx\| < \infty.$$

Examples 2.28 (Recall Ex. 2.23)

(a) $\mathbb{1}: (X, \|\cdot\|) \to (X, \|\cdot\|)$, $x \mapsto x$, is bounded with $\|\mathbb{1}\| = 1$.

(b) Let $(X, \|\cdot\|) = (C([0,1]), \|\cdot\|_\infty)$.

(i) Differentiation $\frac{d}{dx}: C^1([0,1]) \to C([0,1])$, $f \mapsto f'$

is unbounded (i.e., not bounded); see exercise.

(ii) Anti-derivative $T: C([0,1]) \to C([0,1])$:

$$\|Tf\|_\infty = \sup_{x \in [0,1]} \left| \int_0^x f(t)\,dt \right| \leq \|f\|_\infty \quad \forall f \in C([0,1])$$

Because of $\frac{\|Tf\|}{\|f\|} \leq 1$ $(f \neq 0)$, we have $\|T\| \leq 1$.

Also, $\|T1\|_\infty = \sup_{x \in [0,1]} \int_0^x dt = \sup_{x \in [0,1]} x = 1$,

and, since $\|1\|_\infty = 1$, we obtain $\|T\| = 1$.

(iii) Multiplication operator $T: C([0,1]) \to C([0,1])$ by argument has $\|T\| = 1$, because

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |x f(x)| \leq \|f\|_\infty, \text{ with equality for } f \equiv 1.$$

Theorem 2.29 Let X, Y be normed spaces, $T: X \supseteq \mathrm{dom}(T) \to Y$ a linear operator. Then the following statements are equivalent:

(i) T is continuous

(ii) T is continuous at some $x_0 \in \mathrm{dom}(T)$.

(iii) There exists $c \in (0,\infty)$ s.t.

$$\|Tx\| \leq c\|x\| \quad \forall x \in \mathrm{dom}(T).$$

(iv) T is bounded

Pf: (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (iii): Step 1: Prove T is continuous at $x = 0$.

For this, let $(x_n)_n \subseteq \mathrm{dom}(T)$ be a sequence s.t. $x_n \to 0$, $n \to \infty$.

Then (by 2.10(b)), $(x_n + x_0)_n \to x_0$, $n \to \infty$, and by linearity and continuity at x_0, we get

$$Tx_n + Tx_0 = T(x_n + x_0) \xrightarrow{n\to\infty} Tx_0, \text{ so } Tx_n \xrightarrow{n\to\infty} 0 \text{ (again, by 2.10(b))}.$$

Step 2: From Step 1 we get (ε-δ criterion with $\varepsilon = 1$)

$$\exists \delta > 0: \forall x \in \mathrm{dom}(T) \text{ with } \|x\| < \delta : \quad \|Tx\| < 1 \qquad (*)$$

Hence, for all $0 \neq x \in \mathrm{dom}(T)$, by linearity:

$$\|Tx\| = \frac{2\|x\|}{\delta} \left\| T\Big(\underbrace{\frac{\delta}{2\|x\|}x}_{\text{has norm} = \frac{\delta}{2} < \delta}\Big)\right\| \overset{(*)}{<} \frac{2}{\delta}\|x\|.$$

(iii) $\Rightarrow$ (iv): We know that $\|Tx\| \leq c\|x\|$, so

$$\sup_{\substack{x\in\mathrm{dom}(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \leq c < \infty.$$

(iv) $\Rightarrow$ (i): Let $(x_n)_n \subseteq \mathrm{dom}(T)$ with $x_n \xrightarrow{n\to\infty} x$, then

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \cdot \|x_n - x\| \xrightarrow{n\to\infty} 0 \qquad \blacksquare$$

Lemma 2.30 Let T be linear and $\dim \mathrm{dom}(T) < \infty$. Then T is bounded.

Pf: T is continuous: Fix (finite!) basis of $\mathrm{dom}(T)$, expand $x \in \mathrm{dom}(T)$ wrt. this basis, and use linearity to deduce (sequential) continuity. The claim follows from Thm. 2.29 ∎

Theorem 2.31 (Bounded linear extension) Let $\mathbb{X}$ be a normed space, and $\mathbb{Y}$ a Banach (!) space. Let $T: \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. Let $\widetilde{\mathbb{X}}$ be the completion of $\mathbb{X}$. Then there exists a bounded linear extension $\widetilde{T}: \widetilde{\mathbb{X}} \to \mathbb{Y}$ of T, which is unique if $\mathbb{X}$ is identified with a (dense) subspace of its completion $\widetilde{\mathbb{X}}$, i.e. $\mathbb{X} = \mathbb{W}$ in Theorem 2.13. Moreover, we have $\|\widetilde{T}\| = \|T\|$.

Corollary 2.32 Let $\mathbb{X}$ be a normed space and $\mathbb{Y}$ a Banach space. Let $T: \mathbb{X} \supseteq \mathrm{dom}(T) \to \mathbb{Y}$ be a bounded linear operator with $\mathrm{dom}(T) \subseteq \mathbb{X}$ a dense linear subspace. Then there exists a unique bounded linear extension $\widehat{T}: \mathbb{X} \to \mathbb{Y}$ of T. Moreover, we have $\|\widehat{T}\| = \|T\|$.

Pf (of 2.32): Because of denseness, the completion $\mathbb{Z}$ of $\mathrm{dom}(T)$ and $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ are isometrically isomorphic (Check!). Wlog, we identify $\mathbb{Z} = \widetilde{\mathbb{X}}$. Also identify $\mathbb{X}$ with a dense linear subspace of $\widetilde{\mathbb{X}}$, so that $\mathrm{dom}(T) \subseteq \mathbb{X} \subseteq \widetilde{\mathbb{X}}$. The claim follows from Theorem 2.31 with $\widehat{T} := \widetilde{T}|_{\mathbb{X}}$. ∎

Pf (of Thm. 2.31): We identify $\mathbb{X}$ with a dense subspace of $\widetilde{\mathbb{X}}$. Let $x \in \widetilde{\mathbb{X}}$. Then there exists a sequence $(x_n)_n \subseteq \mathbb{X}$ with $x_n \xrightarrow{n\to\infty} x$ in $\widetilde{\mathbb{X}}$. Since $(x_n)_n$ is a Cauchy sequence, $(Tx_n)_n$ is a Cauchy sequence in $\mathbb{Y}$ because

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \cdot \|x_n - x_m\|.$$

Using that Y is a Banach space, there exists $y \in Y$ s.t. $Tx_n \xrightarrow{n\to\infty} y$ in Y. Define $\tilde{T}x := y$ (then $\tilde{T}|_X = T$: take constant sequences!). We have to check several things:

(i) Well-definedness, i.e., independence of approximating seq.: Let $x_n' \xrightarrow{n\to\infty} x$ in $\tilde{X}$. As above, $\exists\, y' \in Y$ s.t. $Tx_n' \xrightarrow{n\to\infty} y'$.

Then we have

$$\|Tx_n - Tx_n'\| \leq \|T\| \cdot \|x_n - x_n'\|.$$

In the limit $n, m \to \infty$ we get $\|y - y'\| = 0$, so $y = y'$.

(ii) Linearity: Let $x_n \xrightarrow{n\to\infty} x$, $x_n' \xrightarrow{n\to\infty} x'$. Then (by 2.10 (b) & (c)), $\alpha x_n + \alpha' x_n' \xrightarrow{n\to\infty} \alpha x + \alpha' x'$ $\forall \alpha, \alpha' \in \mathbb{K}$, and so (by def. of $\tilde{T}$)

$$\tilde{T}(\alpha x + \alpha' x') = \lim_{n\to\infty} T(\alpha x_n + \alpha' x_n') = \lim_{n\to\infty} \alpha Tx_n + \lim_{n\to\infty} \alpha' Tx_n'$$
$$= \alpha \tilde{T}x + \alpha \tilde{T}x'.$$

(iii) Norm: As $\tilde{T}$ is an extension, we have $\|\tilde{T}\| \geq \|T\|$, since

$$\|\tilde{T}\| = \sup_{\substack{x \in \tilde{X} \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} \geq \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|.$$

On the other hand, since $\|\cdot\|$ is continuous (Thm. 2.10 (a)),

$$\|\tilde{T}x\| = \lim_{\substack{n\to\infty \\ x_n \in X}} \|\tilde{T}x_n\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\| \cdot \|x_n\|$$
$$= \|T\| \cdot \lim_{n\to\infty} \|x_n\| = \|T\| \cdot \|x\|.$$

So, $\frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\|$ $\forall x \in \tilde{X}$, hence $\|\tilde{T}\| \leq \|T\|$, so $\|\tilde{T}\| = \|T\|$.

(iv) Uniqueness: The fact that we defined $\tilde{T}x := \lim_{n\to\infty} Tx_n$ is necessary to ensure continuity at x. ∎

Definition 2.33 Let X, Y be normed spaces. Define

$$BL(X,Y) := \{T : X \to Y \mid T \text{ is linear and bounded}\}$$

and set $BL(X) := BL(X,X)$.

Warning: Notation varies - other choices: $B(X,Y)$, $L(X,Y)$, ...

Theorem 2.34 $(BL(X,Y), \|\cdot\|_{X\to Y})$ is a normed space.
If Y is complete, then so is $BL(X,Y)$.

Pf: (i) First of all, $BL(X,Y)$ is a $\mathbb{K}$-vectorspace with zero element $0 := 0 : X \to Y$, $x \mapsto 0$

For $T_1, T_2 \in BL(X,Y)$ and $\alpha \in \mathbb{K}$, define T_1+T_2 and αT_1 by

$$(T_1+T_2)x := T_1x + T_2x \qquad \forall x \in X.$$
$$(\alpha T_1)x := \alpha T_1 x$$

Then clearly $T_1+T_2, \alpha T_1 : X \to Y$ are linear.

(ii) $\|\cdot\|_{X\to Y}$ is a norm on $BL(X,Y)$:

(a) $\|T\|_{X\to Y} \geq 0$ ✓ and if $\|T\|_{X\to Y} = 0$, then $\|Tx\| = 0$ $\forall x \in X$, so $Tx = 0$ $\forall x \in X$, and so $T = 0$ $(= 0)$.

(b) We have

$$\|\alpha T\|_{X\to Y} = \sup_{0\neq x\in X} \frac{\|(\alpha T)x\|}{\|x\|} = \sup_{0\neq x\in X} \frac{\|\alpha Tx\|}{\|x\|}$$
$$= \sup_{0\neq x\in X} |\alpha| \cdot \frac{\|Tx\|}{\|x\|} = |\alpha| \cdot \|T\|$$

Also, $\|(T_1+T_2)x\| = \|T_1x + T_2x\| \leq \|T_1x\| + \|T_2x\|$, so

$$\|T_1+T_2\| \leq \sup_{0\neq x\in X} \left(\frac{\|T_1x\|}{\|x\|} + \frac{\|T_2x\|}{\|x\|} \right) \leq \|T_1\| + \|T_2\|.$$

Hence, if $T_1, T_2 \in BL(X,Y)$, then αT_1 and T_1+T_2 are bounded, so $T_1+T_2, \alpha T_1 \in BL(X,Y)$ (vectorspace!), and $\|\cdot\|_{X\to Y}$ is a norm.

(iii) Assume Y is complete. We prove completeness of $BL(X,Y)$. Let $(T_k)_{k\in\mathbb{N}} \subseteq BL(X,Y)$ be a Cauchy sequence (wrt. $\|\cdot\|_{X\to Y}$).

For every $x \in X$, and $k,l \in \mathbb{N}$, we have $\|T_k x - T_l x\| \le \|T_k - T_l\| \cdot \|x\|$.

So,

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall k,l \ge N \; \forall x \in X : \|T_k x - T_l x\| \le \varepsilon \|x\| \quad (*)$$

This implies that $(T_k x)_{k\in\mathbb{N}} \subseteq Y$ is a Cauchy seq. (in Y) for every $x \in X$, and, since Y is complete, there is some limit

$$\lim_{k\to\infty} T_k x =: Tx \in Y.$$

This defines a map $T : X \to Y$, $x \mapsto Tx := \lim_{k\to\infty} T_k x$

(a) T is linear, since: Let $\alpha, \alpha' \in \mathbb{K}$, $x, x' \in X$, then

$$T(\alpha x + \alpha' x') = \lim_{k\to\infty} T_k(\alpha x + \alpha' x') = \lim_{k\to\infty} (\alpha T_k x + \alpha' T_k x')$$
$$= \alpha Tx + \alpha' Tx'$$

(b) T is bounded, and the norm limit of the T_k's:

Taking the limit $l \to \infty$ in $(*)$ above gives

$\|T_k x - Tx\| \le \varepsilon \|x\|$ and so

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall k \ge N : \|T_k - T\| \le \varepsilon \quad (**)$$

This gives two things:

(1) T is bounded: $T_k \in BL(X,Y)$, $T_k - T \overset{\text{by } (**)}{\in} BL(X,Y)$ and $BL(X,Y)$ is a vector space, so

$$T = T_k - (T_k - T) \in BL(X,Y).$$

(2) $T_k \overset{k\to\infty}{\to} T$ in $\|\cdot\|_{X\to Y}$ ∎

Note: Compare also with the proofs of Thm 1.31 (a exercise!) and Thm. 1.20 (for $p = \infty$).

2.4. Linear functionals and dual space

Definition 2.35 Let $\mathbb{X}$ be a normed space. A linear functional (on $\mathbb{X}$) is a linear operator $\ell : \mathbb{X} \supseteq \mathrm{dom}(\ell) \to \mathbb{K}$.

The dual space of $\mathbb{X}$ is $\mathbb{X}^* := BL(\mathbb{X}, \mathbb{K})$.

Notations for the norm on $\mathbb{X}^*$: $\|\cdot\|_{\mathbb{X}\to\mathbb{K}} =: \|\cdot\|_{\mathbb{X}^*} =: \|\cdot\|_* =: \|\cdot\|$.

Note: $\mathbb{X}^*$ consists of the bounded (equivalently, continuous) linear functionals; $\mathbb{X}^*$ is therefore also called the topological dual - not to be confused with the algebraic dual of all linear functionals, $\mathbb{X}' := \{ f : \mathbb{X} \to \mathbb{K} \mid f \text{ linear} \} = \mathrm{Hom}_{\mathbb{K}}(\mathbb{X}, \mathbb{K})$, common in Linear Algebra!

Corollary 2.36 (to Thm. 2.34). $(\mathbb{X}^*, \|\cdot\|_{\mathbb{X}^*})$ is a Banach space (whether $\mathbb{X}$ is complete or not).

Examples 2.37 Let $\mathbb{X} = C([a,b])$ (with $a < b \in \mathbb{R}$) equipped with $\|\cdot\|_\infty$

(a) For $f \in \mathbb{X}$, let
$$\ell(f) := \int_a^b f(x)\,dx \in \mathbb{K}$$
This is clearly a linear functional $\ell : \mathbb{X} \to \mathbb{K}$ with
$$|\ell(f)| \le \int_a^b \underbrace{|f(x)|}_{\le \|f\|_\infty} dx \le \|f\|_\infty (b-a) \quad (*)$$
so $\|\ell\|_{\mathbb{X}^*} \le (b-a)$, and $f \equiv 1$ gives equality in $(*)$, so $\|\ell\|_{\mathbb{X}^*} = b-a$.

(b) For $f \in \mathbb{X}$ and $t_0 \in [a,b]$, let $\delta_{t_0}(f) := f(t_0) \in \mathbb{K}$.
This gives a linear functional $\delta_{t_0} : \mathbb{X} \to \mathbb{K}$, called the Dirac δ-functional (at t_0), with $|\delta_{t_0}(f)| = |f(t_0)| \le \|f\|_\infty$, and again equality for $f \equiv 1$. So $\|\delta_{t_0}\|_{\mathbb{X}^*} = 1$.

NB! Boundedness (i.e., continuity) of δ_{t_0} depends on the norm on $\mathbb{X}$. For ex., δ_{t_0} is not bounded, if $\mathbb{X}$ is equipped with the L^1-norm ($\|f\|_1 = \int_0^1 |f(x)|\,dx$).

Notation: Let X, Y be normed spaces. We write $X \cong Y$ (or $X \simeq Y$) iff X is isometrically isomorphic to Y.

Theorem 2.38 Let $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$.

More precisely: The map $\ell^q \ni y = (y_n)_n \longmapsto f_y \in (\ell^p)^*$,

$$f_y(x) := \sum_{n \in \mathbb{N}} x_n y_n \quad , \quad x \in \ell^p$$

is a bijective isometry

Pf: (a) Case $1 < p < \infty$: Let $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$, $f \in (\ell^p)^*$.

We work with the canonical (Schauder) basis $(e_n)_{n \in \mathbb{N}}$ for ℓ^p (see 2.15)

Then x can be written as a $\|\cdot\|_p$-convergent series $x = \sum_{n \in \mathbb{N}} x_n e_n$

Since f is continuous (wrt. $\|\cdot\|_p$) and linear,

$$f(x) = f\left(\lim_{N \to \infty} \sum_{n=1}^{N} x_n e_n\right) = \sum_{n \in \mathbb{N}} f(x_n e_n) = \sum_{n \in \mathbb{N}} x_n f(e_n). \qquad (*1)$$

For $N \in \mathbb{N}$ (fixed), define $\tilde{x} := (\tilde{x}_n)_{n \in \mathbb{N}} \in \ell^p$ by

$$\tilde{x}_n := \begin{cases} \dfrac{|f(e_n)|^q}{f(e_n)} & \text{if } n \le N \text{ and } f(e_n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Apply $(*1)$ to $\tilde{x}$:

$$f(\tilde{x}) = \sum_{n=1}^{N} |f(e_n)|^q \qquad (**)$$

so that $0 \le f(\tilde{x}) \le \|f\|_{(\ell^p)^*} \cdot \|\tilde{x}\|_p$, where (compute!)

$$\|\tilde{x}\|_p = \left(\sum_{n=1}^{N} |f(e_n)|^{(q-1)p} \right)^{1/p}.$$

Since $\frac{1}{p} = 1 - \frac{1}{q} \Leftrightarrow p = \frac{q}{q-1}$ (take note!), we have $p(q-1) = q$, so

$$(**) \Rightarrow \sum_{n=1}^{N} |f(e_n)|^q \le \|f\|_{(\ell^p)^*} \left(\sum_{n=1}^{N} |f(e_n)|^q \right)^{1/p}$$

$$\Rightarrow \left(\sum_{n=1}^{N} |f(e_n)|^q \right)^{1/q} \le \|f\|_{(\ell^p)^*} \quad \forall N \in \mathbb{N} \qquad (***)$$

Hence, the map $J: (\ell^p)^* \to \ell^q$, $f \mapsto (f(e_n))_{n\in\mathbb{N}}$

is well-defined, and

(i) J is linear

(ii) $\|Jf\|_q = \|(f(e_n))_{n\in\mathbb{N}}\|_q \le \|f\|_{(\ell^p)^*}$ by (**)

(iii) J is onto/surjective: If $y = (y_n)_n \in \ell^q$, define

$$f_y: \ell^p \to \mathbb{K},\quad x=(x_n)_n \mapsto f_y(x) := \sum_{n\in\mathbb{N}} x_n y_n$$

Clearly, f_y is linear. It is well-defined: $\sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$ by Hölder's inequality, so

$$\|f_y\|_{(\ell^p)^*} \le \|y\|_q$$

and so $f_y \in (\ell^p)^*$. But $f_y(e_n) = y_n$ $\forall n\in\mathbb{N}$, so $J(f_y) = y$.

(iv) (*) & Hölder imply:

$$\forall f\in(\ell^p)^*\ \forall x\in\ell^p: |f(x)| \le \|x\|_p \cdot \underbrace{\|(f(e_n))_{n\in\mathbb{N}}\|_q}_{Jf}$$

$$\Rightarrow \|f\|_{(\ell^p)^*} \le \|Jf\|_q$$

So $\|Jf\|_q = \|f\|_{(\ell^p)^*}$, and J is an isometric isomorphism. (The map in the statement of the thm. is J^{-1}).

(b) The case $p=1$ is analogous, but instead of defining $\tilde{x}_n$, use

$$|f(e_n)| \le \|f\|_{(\ell^1)^*} \cdot \underbrace{\|e_n\|_1}_{=1}$$

This implies $\|(f(e_n))_n\|_\infty \le \|f\|_{(\ell^1)^*}$, which replaces (***), and the properties of J follow as above. ∎

[Remark 2.39] (a) $(c_0)^* \cong \ell^1$ (see exercise)

(b) The map $\ell^1 \to (\ell^\infty)^*$, $y \mapsto f_y$, defined (as in Thm. 2.38) by

$$f_y(x) = \sum_{n\in\mathbb{N}} x_n y_n,\quad x\in\ell^\infty,\ y\in\ell^1$$

is well-defined (Hölder!), linear, isometric, but <u>not</u> onto!

In other words: $(\ell^\infty)^*$ is strictly "larger" than ℓ^1! (see later)

2.5 Hilbert spaces

The main new feature is the "geometry" from the scalar product

Definition 2.40 Let $\mathbb{X}$ be a ($\mathbb{K}$-) vectorspace.

A map $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{K}$ is a scalar product (or inner product) $:\Leftrightarrow$

(i) $\langle x, x \rangle \geq 0 \quad \forall x \in \mathbb{X}$, and $\langle x, x \rangle = 0 \Rightarrow x = 0$
(positive definit) (non-degenerate)

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle \quad \forall \alpha, \beta \in \mathbb{K}, \forall x, y, z \in \mathbb{X}$
(linearity)

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in \mathbb{X}$

$(\mathbb{X}, \langle \cdot, \cdot \rangle)$ is an inner product space (or pre-Hilbert space)

Note: If $\mathbb{K} = \mathbb{R}$, (ii) & (iii) imply that $\langle \cdot, \cdot \rangle$ is bi-linear

If, however, $\mathbb{K} = \mathbb{C}$, (ii) & (iii) give that

$$\langle \alpha x, y \rangle = \bar{\alpha} \langle x, y \rangle \quad \forall \alpha \in \mathbb{C}, x, y \in \mathbb{X} \quad (\text{conjugate linear})$$

Then $\langle \cdot, \cdot \rangle$ is called sesqui-linear ("$1\frac{1}{2}$-linear")

Warning: In literature also: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ & $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$

Lemma 2.41 (Cauchy-Schwarz(-Bunyakovsky) inequality)

Let $\mathbb{X}$ be an inner product space. Then

(C-S) $$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \quad \forall x, y \in \mathbb{X}$$

with equality iff x & y are linearly dependent

Proof: See Linear Algebra (and/or Ana 1-3). ∎

Lemma 2.42 Let $\mathbb{X}$ be an inner product space.
Then $\mathbb{X}$ is a normed space with norm $\|x\| := \langle x,x\rangle^{1/2}$.
All notions & results from topological, metric, and normed spaces are available. In particular, the scalar product
$\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{K}$ is continuous.

Pf: (i) $\|\cdot\|$ fulfills all axioms of a norm, see LA (or do!)

(ii) Continuity of $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{K}$:

Note that $\mathbb{X} \times \mathbb{X}$ is a normed space (f. ex. with norm $\|(x,y)\|_1 := \|x\| + \|y\|$), hence seq. cont. enough.

So let $x_n \overset{n\to\infty}{\to} x$, $y_n \overset{n\to\infty}{\to} y$ be two conv. seq.'s in $\mathbb{X}$.

Then

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \le |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle|$$

$$= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\overset{\text{C-S}}{\le} \|x_n\| \cdot \underbrace{\|y_n - y\|}_{\to 0} + \underbrace{\|y\|}_{<\infty} \cdot \underbrace{\|x_n - x\|}_{\to 0} \overset{n\to\infty}{\to} 0$$

since $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ (as $(x_n)_n$ converges & $\|\cdot\|$ cont.). ∎

Definition 2.43 A complete inner product space is called a Hilbert space

Note: Hence, any Hilbert space is also a Banach space (for the other direction, see 2.46 below).

Theorem 2.44 Let X be an inner product space. Then there exists a Hilbert space $\tilde{X}$, a dense subspace $W \subseteq \tilde{X}$, and a unitary map $U : X \to W$.

(U is unitary $:\Leftrightarrow$ U is an isomorphism with $\langle x, y \rangle_X = \langle Ux, Uy \rangle_{\tilde{X}} \quad \forall x, y \in X$)

Idea of pf: See the pf's of Thm's 2.13 & 1.14.
Additional aspect here: Define a scalar product on
$\tilde{X} := \{$ equivalence classes $\tilde{x}$ of Cauchy sequences in $X \}$:

$$\langle \tilde{x}, \tilde{y} \rangle_{\tilde{X}} := \lim_{n \to \infty} \langle x_n, y_n \rangle_X$$

with $(x_n)_n, (y_n)_n \subseteq X$ being representatives of the equivalence classes $\tilde{x}$ and $\tilde{y}$, and $U : \begin{array}{ccc} X & \to & W \\ x & \longmapsto & \tilde{x} \end{array}$
where $\tilde{x}$ is the equivalence class of
the constant representative $(x, x, x, \ldots)$ ∎

Examples 2.45 (a) $\ell^2 = \ell^2(\mathbb{N})$ is a Hilbert space with scalar product $\langle x, y \rangle := \sum_{n \in \mathbb{N}} \overline{x_n} y_n$, where $x = (x_n)_n$, $y = (y_n)_n$.
Of course, $\ell^2(\{1, \ldots, N\}) \equiv \mathbb{C}^N$ is also a Hilbert space.

(b) $C([0,1])$ is an inner product space with the scalar product $\langle f, g \rangle := \int_0^1 \overline{f(x)} g(x) \, dx$

but not a Hilbert space (proof ambiguous to Ex. 1.11 (b)).
(Applying Thm. 2.44 to this example gives $L^2([0,1])$, see next chapter).

(c) ℓ^p for $p \neq 2$ is not an inner product space, because of the following theorem:

Theorem 2.46 (Jordan-von Neumann) Let $(X, \|\cdot\|)$ be a normed space. Then X is an inner product space with $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ iff $\|\cdot\|$ satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in X.$$

Pf: "$\Rightarrow$": Elementary computation (do!)

"$\Leftarrow$": Define inner product (!) by polarisation:

$$\langle x, y \rangle := \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) - \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right) & \mathbb{K} = \mathbb{C} \end{cases}$$

We verify that this definition satisfies axioms of an inner product:

(i) Symmetry and definiteness are obvious (check!)

(ii) Bi-/sesqui-linearity follows from parallelogram identity (here only $\mathbb{K} = \mathbb{R}$)

$$\langle x, y \rangle + \langle x, z \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2\right)$$
$$= \frac{1}{2}\left(\|x + \tfrac{y+z}{2}\|^2 - \|x - \tfrac{y+z}{2}\|^2\right)$$
$$= 2\left\langle x, \tfrac{y+z}{2} \right\rangle \quad (1)$$

$z = 0$ in (1) $\Rightarrow$ $\langle x, y \rangle = 2\langle x, \frac{y}{2} \rangle$ (2)

(1) & (2) $\Rightarrow$ $\langle x, y \rangle + \langle x, z \rangle = \langle x, y+z \rangle$ (3)

(2) with $2y$ instead of y: $\langle x, 2y \rangle = 2\langle x, y \rangle$ (4)

By induction from (3) & (4): $\langle x, ny \rangle = n\langle x, y \rangle \quad \forall n \in \mathbb{N}_0$ (5)

$z = -y$ in (3) $\rightarrow$ $\langle x, y \rangle = -\langle x, -y \rangle \Rightarrow$ (5) holds $\forall n \in \mathbb{Z}$

Let $m \in \mathbb{Z} \setminus \{0\}$ and use (5) with $\frac{y}{m}$ instead of y (writing $\lambda := \frac{n}{m}$):

$$\langle x, \lambda y \rangle = n\langle x, \tfrac{y}{m} \rangle = \lambda m \langle x, \tfrac{y}{m} \rangle = \lambda \langle x, y \rangle.$$

This holds for all $\lambda \in \mathbb{Q}$, and thus for all $\lambda \in \mathbb{R}$ by continuity (2.10)

∎

Definition 2.47 Let $\mathbb{X}$ be an inner product space, and $A \subseteq \mathbb{X}$ (not necessarily subspace)

(a) $x, y \in \mathbb{X}$ are orthogonal $:\Leftrightarrow \langle x, y \rangle = 0$. In symbols: $x \perp y$

(b) Orthogonal complement of A: $A^\perp := \{x \in \mathbb{X} \mid x \perp a \quad \forall a \in A\}$

(c) Let $J \neq \emptyset$ be an index set, and for all $\alpha \in J$, let $e_\alpha \in \mathbb{X}$.
We call $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set (or family) $:\Leftrightarrow$

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} \quad \forall \alpha, \beta \in J$$

(and orthogonal set if $\langle e_\alpha, e_\beta \rangle = 0 \ \forall \alpha \neq \beta$)

(d) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal basis (ONB) or complete orthonormal set $:\Leftrightarrow$

(i) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal set.

(ii) "Completeness": If $\langle e_\alpha, x \rangle = 0$ for some $x \in \mathbb{X}$ and all $\alpha \in J$, then $x = 0$ (I.e.: The zero vector is the only vector orthogonal to all e_α's).

Lemma 2.48 Let $\mathbb{X}$ be an inner product space.

(a) Let $A \subseteq \mathbb{X}$ be a subset (not necessarily subspace). Then $A^\perp$ is a closed (linear) subspace in $\mathbb{X}$.

(b) Every orthonormal set $\{e_\alpha\}_{\alpha \in J}$ is linearly independent.

Pf: (a) See exercises.

(b) Assume $\sum_{j=1}^{n} \lambda_j e_{\alpha_j} = 0$ for some $n \in \mathbb{N}$, some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, and some $\alpha_1, \dots, \alpha_n \in J$ with $\alpha_j \neq \alpha_k \ \forall j \neq k$. Then we have

$$0 = \langle e_{\alpha_k}, 0 \rangle = \langle e_{\alpha_k}, \sum_{j=1}^{n} \lambda_j e_{\alpha_j} \rangle = \sum_{j=1}^{n} \lambda_j \underbrace{\langle e_{\alpha_k}, e_{\alpha_j} \rangle}_{\delta_{kj}} = \lambda_k \quad \forall k = 1, \dots, n \ \square$$

Next we prove two lemmas/lemmata as preparation for Theorem 2.51:

Lemma 2.49 (Uncountable series) Let $\emptyset \neq J$ be an index set and $0 \leq c_\alpha < \infty$ $\forall \alpha \in J$. Then

$$\sum_{\alpha \in J} c_\alpha := \sup_{\substack{J_0 \subseteq J \\ |J_0| < \infty}} \sum_{\alpha \in J_0} c_\alpha < \infty$$

implies $c_\alpha = 0$ for all but countably many α's.

Note: The definition agrees with the usual one, if J is countable.

Proof: Define the sets $S_0 := \{\alpha \in J \mid c_\alpha \geq 1\}$

$$S_n := \left\{\alpha \in J \mid \tfrac{1}{n} > c_\alpha \geq \tfrac{1}{n+1}\right\}, \quad n \in \mathbb{N}$$

Claim: $|S_n| < \infty$ $\forall n \in \mathbb{N}_0$

True, since, if this was violated for some $n \in \mathbb{N}$, then $\sum_{\alpha \in J} c_\alpha \geq \sum_{\alpha \in S_n} c_\alpha = \infty$. But then $\{\alpha \in J \mid c_\alpha > 0\} = \bigcup_{n \in \mathbb{N}_0} S_n$ is a countable union of finite sets, hence countable ∎

Lemma 2.50 (Series in Banach & Hilbert spaces)

(a) Let X be a Banach space and consider a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$.

Then $\sum_{n \in \mathbb{N}} \|x_n\| < \infty \Rightarrow \sum_{n \in \mathbb{N}} x_n := \lim_{N \to \infty} \left(\sum_{n=1}^{N} x_n\right)$ exists in X.

(b) Let X be a Hilbert space, and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence of pairwise orthogonal vectors (i.e. an orthogonal family).

Then $\sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty \iff \sum_{n \in \mathbb{N}} x_n$ exists in X

(Note: Recall that $\ell^1 \subsetneq \ell^2$)

Pf: Let $S_n := \sum_{j=1}^{n} x_j$ for $n \in \mathbb{N}$. Since X is complete, we have:

$\sum_{n \in \mathbb{N}} x_n$ exists $\iff (S_n)_n$ is Cauchy (in X).

(a) Let $m \geq n$. Then

$$\|S_m - S_n\| = \left\|\sum_{j=n+1}^{m} x_j\right\| \leq \sum_{j=n+1}^{m} \|x_j\| = \sigma_m - \sigma_n$$

with $\sigma_n := \sum_{j=1}^{n} \|x_j\|$. But by the assumption, $(\sigma_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges in $\mathbb{R}$, and hence is Cauchy.

(b) Consider again the sequence of partial sums $(s_n)_n$. Let $m \geq n$, then

$$\|s_m - s_n\|^2 = \Big\| \sum_{j=n+1}^{m} x_j \Big\|^2 = \sum_{j,k=n+1}^{m} \langle x_j, x_k \rangle = \sum_{j=n+1}^{m} \|x_j\|^2 = T_m - T_n$$

with $T_n := \sum_{j=1}^{n} \|x_j\|^2$. Hence, $(s_n)_n$ is Cauchy (in $\mathbb{X}$) iff. $(T_n)_{n\in\mathbb{N}}$ is Cauchy (in $\mathbb{R}$) ∎

Theorem 2.51 Let $\mathbb{X}$ be an inner product space, $x \in \mathbb{X}$, and $\{e_\alpha\}_{\alpha\in J}$ an orthonormal set. Then:

$$\|x\|^2 \geq \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Bessel's inequality)}.$$

If $\mathbb{X}$ is even a Hilbert space, and if $\{e_\alpha\}_{\alpha\in J}$ is even an ONB, then

$$x = \sum_{\alpha\in J} \langle e_\alpha, x\rangle e_\alpha$$

where at most countably many terms are $\neq 0$, and

$$\|x\|^2 = \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Parseval's equality)}.$$

Pf: Let $J_0 \subseteq J$ be any finite subset. Then we can write x as follows:

$$x = \underbrace{\sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha}_{=:u} + \underbrace{\Big(x - \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha\Big)}_{=:v}$$

Using that $\{e_\alpha\}_{\alpha\in J_0}$ is an orthonormal set, we get:

$$\langle u, u\rangle = \Big\langle \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha, \sum_{\beta\in J_0} \langle e_\beta, x\rangle e_\beta \Big\rangle = \sum_{\alpha,\beta\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\beta, x\rangle \langle e_\alpha, e_\beta\rangle$$

$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, x\rangle = \sum_{\alpha\in J_0} |\langle e_\alpha, x\rangle|^2 ,$$

$$\langle u, v\rangle = \langle u, x-u\rangle = \langle u, x\rangle - \langle u, u\rangle$$
$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha, x\rangle}\langle e_\alpha, x\rangle - \langle u, u\rangle = 0.$$

Hence, we can estimate $\|x\|^2$:

$$\|x\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \underbrace{\langle v, v\rangle}_{\geq 0} \geq \sum_{\alpha\in J_0} |\langle e_\alpha, x\rangle|^2 \quad \forall\, J_0 \subseteq J,\ |J_0| < \infty$$

Taking sup over all finite subsets $J_0 \subseteq J$ gives Bessel's inequality.

Now, Lemma 2.49 $\Rightarrow \exists\, N \in \mathbb{N}\cup\{\infty\}$ and $\alpha_n \in J$ $\forall n \in \{1,\dots,N\}$ s.t.

(*) $\sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 = \sum_{n=1}^{N} |\langle e_{\alpha_n}, x\rangle|^2 \leq \|x\|^2 < \infty$ (using Bessel's ineq.)

Then (*) and Lemma 2.50(b) (which requires completeness!) imply that

$$x' := \sum_{n=1}^{N} \langle e_{\alpha_n}, x\rangle e_{\alpha_n} \text{ exists in } X.$$

We need to prove that $x = x'$:

We have $x - x' \perp e_{\alpha_m}$ $\forall m \in \{1,\dots,N\}$:

$$\langle e_{\alpha_m}, x-x'\rangle = \langle e_{\alpha_m}, x\rangle - \langle e_{\alpha_m}, x'\rangle = \langle e_{\alpha_m}, x\rangle - \sum_{n=1}^{N}\langle e_{\alpha_n}, x\rangle \underbrace{\langle e_{\alpha_m}, e_{\alpha_n}\rangle}_{\delta_{mn}}$$
$$= 0 \quad \forall m \in \{1,\dots,N\}$$

If $\alpha \notin \{\alpha_n \mid n \in \{1,\dots,N\}\}$ then we have:

$$\langle e_\alpha, x-x'\rangle = \langle e_\alpha, x\rangle - \underbrace{\sum_{n=1}^{N}\langle e_{\alpha_n}, x\rangle \underbrace{\langle e_\alpha, e_{\alpha_n}\rangle}_{=0}}_{=0} = \langle e_\alpha, x\rangle \overset{(*)}{=} 0$$

Taken together, we have $x - x' \perp e_\alpha$ $\forall \alpha \in J$ $\overset{\text{ONB}}{\Rightarrow}$ $x = x'$

Moreover (if $N < \infty$, no limits are needed):

$$\|x\|^2 = \Big\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j} \Big\|^2 \overset{\|\cdot\| \text{ cont.}}{=} \lim_{n\to\infty} \Big\| \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j} \Big\|^2$$
$$= \lim_{n\to\infty} \Big\langle \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j}, \sum_{k=1}^{n} \langle e_{\alpha_k}, x\rangle e_{\alpha_k} \Big\rangle$$
$$= \lim \sum_{j=1}^{n}\sum_{k=1}^{n} \overline{\langle e_{\alpha_j}, x\rangle}\langle e_{\alpha_k}, x\rangle \underbrace{\langle e_{\alpha_j}, e_{\alpha_k}\rangle}_{\delta_{jk}} = \sum_{j=1}^{N} |\langle e_{\alpha_j}, x\rangle|^2$$

(*) $= \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2$ ∎

Lemma 2.50 (b) and Theorem 2.51 gives

Corollary 2.52 Let $\mathcal{X}$ be a Hilbert space and $\{e_\alpha\}_{\alpha\in J}$ an orthonormal set in $\mathcal{X}$

(a) If $(c_\alpha)_{\alpha\in J}\subseteq \mathbb{K}$ with $\sum_{\alpha\in J}|c_\alpha|^2<\infty$, then $\sum_{\alpha\in J} c_\alpha e_\alpha$ is well-defined in $\mathcal{X}$

(b) If J is countable, and $\{e_\alpha \mid \alpha\in J\}$ is even an orthonormal basis (ONB), then $\{e_\alpha\}_{\alpha\in J}$ is a Schauder basis.

Theorem 2.53 Every Hilbert space $\mathcal{X}\neq\{0\}$ has an ONB. Moreover,

$\mathcal{X}$ separable $\Leftrightarrow$ there exists a countable ONB.

Pf: The existence of an ONB in the non-separable case will be proven later (when Zorn's Lemma is available).

"$\Leftarrow$": Suppose there exists a countable ONB. By Cor. 2.52(b), this is a Schauder basis. Thus, $\mathcal{X}$ is separable by Lemma 2.16. ✓

"$\Rightarrow$": Let $\{x_n \mid n\in\mathbb{N}\}$ be dense in $\mathcal{X}$. We construct an ONB using the Gram-Schmidt procedure:

(i) If $x_1\neq 0$, define $e_1 := \frac{x_1}{\|x_1\|}$ (otherwise, use x_{n_0} with $n_0 := \min\{k\in\mathbb{N} \mid x_k\neq 0\}$)

(ii) Throw away all x_n's that are linearly dependent of e_1.
Let $n_1 := \min\{k\in\mathbb{N} \mid x_k\notin \operatorname{span}\{e_1\}\} > n_0$ be the smallest index of the remaining elements. Set

$$\tilde{e}_2 := x_{n_1} - \langle e_1, x_{n_1}\rangle e_1 \quad\text{and}\quad e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|}.$$

(iii) Throw away all x_n's that are linearly dependent of $\{e_1, e_2\}$. Let $n_2 := \min\{k\in\mathbb{N} \mid x_k\notin \operatorname{span}\{e_1,e_2\}\} > n_1$ be the smallest index of the remaining elements. Set

$$\tilde{e}_3 := x_{n_2} - \langle e_1, x_{n_2}\rangle e_1 - \langle e_2, x_{n_2}\rangle e_2 \quad\text{and}\quad e_3 := \frac{\tilde{e}_3}{\|\tilde{e}_3\|}.$$

(iv) Continue this procedure.

This terminates iff $\dim\mathcal{X}<\infty$. It is clear, that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set. Also, $\operatorname{span}\{e_n \mid n\in\mathbb{N}\}$ is dense in $\mathcal{X}$ because

each x_n is a finite linear combination of e_n's. It remains to prove that this is a basis: Assume there is $y \in \mathbb{X}$ such that $y \perp e_n$ $\forall n$. By denseness, $\exists (y_k)_k \subseteq \operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ such that $y_k \xrightarrow{k\to\infty} y$ in $\mathbb{X}$. Thus $\langle y, y_k \rangle = 0$ $\forall k$ and

$$\|y\|^2 = \langle y, y \rangle = \lim_{k\to\infty} \underbrace{\langle y, y_k \rangle}_{=0} = 0 \qquad \blacksquare$$

Definition Let $\mathbb{X}$, $\mathbb{X}'$ be inner product spaces.

$\mathbb{X}, \mathbb{X}'$ are unitarily equivalent $:\Leftrightarrow \exists$ isometric isomorphism $U: \mathbb{X} \to \mathbb{X}'$

Note: U isometric & linear $\Rightarrow$ U preserves scalar products:

$$\langle x, y \rangle_{\mathbb{X}} = \langle Ux, Uy \rangle_{\mathbb{X}'} \quad \forall x, y \in \mathbb{X}$$

(follows from polarisation, see proof of Thm. 2.46)

Theorem 2.54 Let $\mathbb{X}$ be a Hilbert space. If $n := \dim \mathbb{X} < \infty$ then $\mathbb{X}$ is unitarily equivalent to $\mathbb{K}^n$ (with Euclidean scalar product).
If $\dim \mathbb{X} = \infty$ and $\mathbb{X}$ is separable, then $\mathbb{X}$ is unit. equiv. to ℓ^2

Pf: 2nd case: Fix a countable ONB $\{e_n\}_{n\in\mathbb{N}}$ (exists by Thm. 2.53 b/c $\mathbb{X}$ is separable) and consider the map

$$U: \mathbb{X} \to \ell^2, \quad x \mapsto (\langle e_n, x \rangle)_{n\in\mathbb{N}}$$

(i) U is well-defined by Bessel's inequality, and linear.
(ii) U is surjective by Cor. 2.52 (a)
(iii) U is isometric (→ injective) by Parseval's equality.
1st case: Analogous ∎

Theorem 2.55 (Projection theorem) Let $\mathbb{X}$ be a Hilbert space and $A \subseteq \mathbb{X}$ a closed subspace. Then: For all $x \in \mathbb{X}$ there exists a unique $a \in A$ and a unique $w \in A^\perp$ such that $x = a + w$

The proof relies on:

Lemma 2.56 Let X, A be as in Thm. 2.55.
For every $x \in X$ there exists a unique $a \in A$ such that
$\operatorname{dist}(x, A) = \|x - a\|$. That is, a is the "closest element" to x in A.

Pf(2.56): Existence: Let $d := d(x, A) = \inf_{y \in A} \|x - y\|$.
There exists a sequence $(y_n)_{n \in \omega} \subseteq A$ s.t.

(*) $d = \lim_{n \to \infty} \underbrace{\|x - y_n\|}_{\geq d}$ (by def. of "inf"; $(y_n)_n$ is a minimizing sequence)

Using the parallelogram identity
$\|u + v\|^2 = 2(\|u\|^2 + \|v\|^2) - \|u - v\|^2$ (Thm. 2.46) we get

$$\|y_n - y_m\|^2 = \|(y_n - x) + (x - y_m)\|^2$$
$$= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \|2x - y_m - y_n\|^2$$
$$= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - 4\,\Big\|x - \underbrace{\frac{y_m + y_n}{2}}_{\in A \text{ (subspace)}}\Big\|^2 \quad (**)$$
$$\leq 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4d^2 \qquad \underbrace{\geq 4d^2}$$
$$\longrightarrow 0, \quad m, n \to \infty \quad \text{(by (*))}.$$

Hence, $(y_n)_n$ is Cauchy, so (X complete) $\exists a \in X : y_n \to a, n \to \infty$
But $(y_n)_n \subseteq A$ and A is closed, so $a \in A$, and $d = \|x - a\|$ by (*) and continuity of the norm

Uniqueness: Assume there exists $(y_n')_n \subseteq A$ with

$$\lim_{n \to \infty} \|y_n' - x\| = d \quad \text{and} \quad a' = \lim_{n \to \infty} y_n'$$

Replacing y_m by y_m' in (**) gives

$$\|a - a'\|^2 = \lim_{n \to \infty} \lim_{m \to \infty} \|y_n - y_m'\|^2$$
$$\leq \lim_{n \to \infty} \lim_{m \to \infty} 2(\|y_n - x\|^2 + \|y_m' - x\|^2) - 4d^2 = 0$$
$$\Rightarrow a = a' \quad \blacksquare$$

Remark: Note that the proof works if A is closed and convex (not necessarily a subspace)

Pf (of 2.55): Existence: For given $x \in \mathbb{X}$, let $a \in A$ be defined by Lemma 2.56, and set $w := x - a$. Need to prove: $w \in A^\perp$

Let $\xi \in \mathbb{K}$ and $y \in A$. Then

$$\|w\|^2 \leq \|x - \underbrace{(a + \xi y)}_{\in A}\|^2 = \|\underbrace{w - \xi y}_{}\|^2 = \|w\|^2 + \|\xi y\|^2 - 2\,\mathrm{Re}\langle w, \xi y\rangle.$$

So for $0 \neq y \in A$ we have $|\xi|^2 - \dfrac{2\,\mathrm{Re}\langle w, \xi y\rangle}{\|y\|^2} \geq 0$

(i) For $\xi = t \in \mathbb{R}$ this gives

$$t^2 - \frac{2\,\mathrm{Re}\langle w, y\rangle}{\|y\|^2}\, t \geq 0 \quad \forall t \in \mathbb{R}$$

This implies $\mathrm{Re}\langle w, y\rangle = 0$. If $\mathbb{K} = \mathbb{R}$ we are done, since $y \in A$ was arbitrary.

For $\mathbb{K} = \mathbb{C}$ we also consider the following case:

(ii) For $\xi = it$, $t \in \mathbb{R}$:

$$t^2 + \frac{2\,\mathrm{Im}\langle w, y\rangle}{\|y\|^2}\, t \geq 0 \quad \forall t \in \mathbb{R}$$

So $\mathrm{Im}\langle w, y\rangle = 0$ for every $y \in A$. Together with (i) this gives $\langle w, y\rangle = 0$ and so $w \in A^\perp$.

Uniqueness: Assume there are $a, a' \in A$ and $w, w' \in A^\perp$ with $a + w = x = a' + w'$. Then $\underbrace{a - a'}_{\in A} = \underbrace{w' - w}_{\in A^\perp}$. Then $a = a'$ and $w = w'$ as $A \cap A^\perp = \{0\}$. ∎

Theorem 2.57 (Riesz' Representation Thm.) Let $\mathbb{X}$ be a Hilbert space, and let $\ell \in \mathbb{X}^*$. Then there exists a unique $y_\ell \in \mathbb{X}$ such that

$$\ell(x) = \langle y_\ell, x\rangle \quad \forall x \in \mathbb{X} \qquad (*)$$

and

$$\|\ell\|_* = \|y_\ell\|.$$

Pf: If $\ker \ell = \mathcal{X} \Rightarrow \ell = 0$ and thm. true with $y_\ell = 0$.

Assume true for $\ker \ell \subsetneq \mathcal{X}$. Note that $\ker \ell$ is closed.

Existence of y_ℓ: From Thm. 2.55 we know $(\ker \ell)^\perp \neq \{0\}$ and this allows to choose $0 \neq x_0 \in (\ker \ell)^\perp$. Define

$$y_\ell := \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0 \in \mathcal{X}$$

(i) So (*) holds for every $x \in \ker \ell$ ($x_0 \perp \ker \ell$)

(ii) Let $x = \alpha x_0$, $\alpha \in \mathbb{K}$. Then $\ell(x) = \alpha \ell(x_0)$ and

$$\langle y_\ell, x \rangle = \left\langle \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0, \alpha x_0 \right\rangle = \frac{\ell(x_0)}{\|x_0\|^2} \alpha \langle x_0, x_0 \rangle = \alpha \ell(x_0),$$

so ℓ and $\langle y_\ell, \cdot \rangle$ agree on $\operatorname{span}\{\ker \ell, x_0\}$.

(iii) But $\operatorname{span}\{\ker \ell, x_0\} = \mathcal{X}$ because, for all $x \in \mathcal{X}$,

$$x = \underbrace{\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0\right)}_{\in \ker \ell} + \underbrace{\frac{\ell(x)}{\ell(x_0)} \cdot x_0}_{\in \operatorname{span}\{x_0\}}$$

since

$$\ell\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0\right) = \ell(x) - \frac{\ell(x)}{\ell(x_0)} \ell(x_0) = 0.$$

So $\ell = \langle y_\ell, \cdot \rangle$ on $\mathcal{X}$ by (i), (ii), and the linearity of ℓ and $\langle y_\ell, \cdot \rangle$.

Uniqueness of y_ℓ: Assume there exists $y_\ell' \in \mathcal{X}$ with $\ell = \langle y_\ell', \cdot \rangle$. Then, for every $x \in \mathcal{X}$,

$$0 = \underbrace{\ell(x)}_{\langle y_\ell, x \rangle} - \underbrace{\ell(x)}_{\langle y_\ell', x \rangle} = \langle y_\ell - y_\ell', x \rangle$$

Choose $x = y_\ell - y_\ell'$, so that $0 = \|y_\ell - y_\ell'\|^2$, so $y_\ell = y_\ell'$.

Norm: We have

$$\|\ell\|_0 = \sup_{0 \neq x \in \mathcal{X}} \underbrace{\frac{|\ell(x)|}{\|x\|}}_{\frac{|\langle y_\ell, x \rangle|}{\|x\|}} \overset{\text{C-S}}{\leq} \|y_\ell\|$$

Note that $y_\ell \neq 0$ since $\ker \ell \subsetneq X$. Hence, we may choose $x = y_\ell$ in sup above:

$$\|\ell\|_* \geq \frac{|\ell(y_\ell)|}{\|y_\ell\|} = \frac{|\langle y_\ell, y_\ell \rangle|}{\|y_\ell\|} = \|y_\ell\|$$

hence, $\|\ell\|_* = \|y_\ell\|$ ∎

Corollary 2.58 Let X be a Hilbert space. Then the map

$$J : X^* \to X$$
$$\ell \mapsto y_\ell$$

defined by Thm. 2.57 is a semi-linear, isometric bijection (linear for $\mathbb{K} = \mathbb{R}$, conjugate linear for $\mathbb{K} = \mathbb{C}$).

Pf: Let $\alpha, \beta \in \mathbb{K}$, and $\ell_1, \ell_2 \in X^*$ with $\ell_j = \langle y_j, \cdot \rangle$, i.e. $J(\ell_j) = y_j$ for $j = 1, 2$. Then

$$\alpha \ell_1 + \beta \ell_2 = \alpha \langle y_1, \cdot \rangle + \beta \langle y_2, \cdot \rangle = \langle \bar{\alpha} y_1 + \bar{\beta} y_2, \cdot \rangle$$

So $J(\alpha \ell_1 + \beta \ell_2) = \bar{\alpha} J(\ell_1) + \bar{\beta} J(\ell_2)$ (by uniqueness in 2.57)

Hence, J is semi-linear. That J is an isometry was proved in 2.57 ($\Rightarrow$ J injective). Also, J is onto / surjective:

For $y \in X$, $\ell_y := \langle y, \cdot \rangle \in X^*$ (use C-S, or Lem. 2.42, to prove continuity) and $J(\ell_y) = y$ (by uniqueness in 2.57) ∎

Chapter 3: L^p-spaces

3.1 Completeness and dual space

Notation: In this section: $(X, \mathcal{A}, \mu)$ a measure space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $f: X \to \mathbb{K}$ measurable (see also Handout!)

Definition 3.1 (L^p-spaces) Let $p \in (0, \infty)$.

(a)
$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} \qquad (\text{possibly } \infty)$$

$$\|f\|_\infty := \inf \{ \alpha > 0 \mid \mu(\{x \in X \mid |f(x)| > \alpha\}) = 0 \}$$

$$= \inf_{\substack{N \in \mathcal{A} \\ \mu(N)=0}} \sup_{x \in X \setminus N} |f(x)| =: \operatorname{ess\,sup}_{x \in X} |f(x)|$$

(μ-) essential supremum

(b) Vectorspace of p-integrable functions (wrt. μ):

$$\mathcal{L}^p := \mathcal{L}^p(\mu) := \mathcal{L}^p(X) := \mathcal{L}^p(X, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) :=$$

$$:= \{ f: X \to \mathbb{K} \mid f \text{ measurable and } \|f\|_p < \infty \} = \{ f: X \to \mathbb{K} \mid |f|^p \ \mu\text{-integrable} \}.$$

(c) Equivalence relation $\sim$ on $\mathcal{L}^p$:

$$f \sim g :\Leftrightarrow f = g \ \mu\text{-a.e.}$$

Vectorspace of equivalence classes of p-integrable fct.'s wrt μ:

$$L^p := \mathcal{L}^p / \sim .$$

Warning! Notation does not distinguish between equivalence classes of functions, and their representatives!

Lemma 3.2 (Hölder & Minkowski)

(a) $\forall r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$: $\forall f, g: X \to \mathbb{K}$ meas.

$$\|fg\|_r \leq \|f\|_p \cdot \|g\|_q \quad (\text{generalised Hölder's ineq.})$$

(b) Let $(X_i, \mathcal{A}_i, \mu_i)$, $i=1,2$, be σ-finite measure spaces, and $f: X_1 \times X_2 \to \mathbb{C}$ measurable. Then, for $p \in [1,\infty)$,

$$(*) \quad \left\{ \int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \int_{X_1} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

(Minkowski's Integral Inequality)

That is,

$$\left\| \int_{X_1} f(x,\cdot)\, d\mu_1(x) \right\|_p \le \int_{X_1} \|f(x,\cdot)\|_p \, d\mu_1(x)$$

(c) $\forall\, p \in [1,\infty]$:

$$(**) \quad \|f+g\|_p \le \|f\|_p + \|g\|_p \quad \text{(Minkowski's inequality)}.$$

Pf: (a) Ana III / Handout / see exercises.

(b) Note that

$$\left\{ \int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \left\{ \int_{X_2} \left(\int_{X_1} |f(x,y)|\, d\mu_1(x) \right)^p d\mu_2(y) \right\}^{1/p},$$

hence, it suffices to consider $f \ge 0$.

Next, if $p=1$, $(*)$ follows from Tonelli's Thm.

If $1 < p < \infty$, then

$$\int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \overset{p=(p-1)+1}{=} \int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^{p-1} \cdot \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right| d\mu_2(y)$$

$$\le \int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \left[\int_{X_1} |f(x,y)|\, d\mu_1(x) \right] d\mu_2(y)$$

$$= \int_{X_2} \left(\int_{X_1} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_1(x) \right) d\mu_2(y) \quad \left(\text{linearity } \int_{X_1} d\mu_1(x) \right)$$

$$\overset{\text{Tonelli}}{=} \int_{X_1} \left(\int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_2(y) \right) d\mu_1(x) \qquad (\square)$$

Now, with $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $q = \frac{p}{p-1}$, Hölder gives

$$\int_{X_2}\Big|\int_{X_1} f(t,y)\,d\mu_1(t)\Big|^{p-1}\cdot|f(x,y)|\,d\mu_2(x)$$

$$\le\Big(\int_{X_2}\Big|\int_{X_1} f(t,y)\,d\mu_1(t)\Big|^{q(p-1)}d\mu_2(y)\Big)^{1/q}\Big(\int_{X_2}|f(x,y)|^p d\mu_2(y)\Big)^{1/p}$$

$$=\Big(\int_{X_2}\Big|\int_{X_1} f(t,y)\,d\mu_1(t)\Big|^p d\mu_2(y)\Big)^{\frac{p-1}{p}}\Big(\int_{X_2}|f(x,y)|^p d\mu_2(y)\Big)^{1/p}$$

Inserting this in (□) gives

$$\int_{X_2}\Big|\int_{X_1} f(x,y)\,d\mu_1(x)\Big|^p d\mu_2(y)$$

$$\le\int_{X_1}\underbrace{\Big(\int_{X_2}\Big|\int_{X_1} f(t,y)\,d\mu_1(t)\Big|^p d\mu_2(y)\Big)^{\frac{p-1}{p}}}_{\text{indep. of } x}\cdot\Big(\int_{X_2}|f(x,y)|^p d\mu_2(y)\Big)^{1/p}d\mu_1(x)$$

$$=\underbrace{\Big(\int_{X_2}\Big|\int_{X_1} f(\underset{x}{t},y)\,d\mu_1(\underset{x}{t})\Big|^p d\mu_2(y)\Big)^{\frac{p-1}{p}}}_{\text{divide (!) : } 1-\frac{p-1}{p}=\frac{1}{p}}\cdot\int_{X_2}\Big(\int_{X_1}|f(x,y)|^p d\mu_2(y)\Big)^{1/p}d\mu_1(x)$$

$$\Rightarrow\Big\{\int_{X_2}\Big|\int_{X_1} f(x,y)\,d\mu_1(x)\Big|^p d\mu_2(y)\Big\}^{1/p}\le\int_{X_1}\Big(\int_{X_2}|f(x,y)|^p d\mu_2(y)\Big)^{1/p}d\mu_1(x)$$

To divide: we used integral to be finite; is the case if $\mu_i(X_i)<\infty$, $i=1,2$ and f bounded. For general case: Let $X_i^k \nearrow X_i$, $k\to\infty$, $\mu_i(X_i^k)<\infty$ (X_i σ-finite), consider the above for

$$\tilde f_k(x,y) := \mathbb{1}_{X_1^k}(x)\,\mathbb{1}_{X_2^k}(y)\,\mathbb{1}_{\{f\le k\}}\cdot f(x,y)$$ & take limit $k\to\infty$

(c) Take $X_1=\{1,2\}$, μ_1 = counting measure, $\int_{X_1}\cdots d\mu_1(x)=\sum_{i=1}^{2}$,

then (*) becomes, with $f(x,y)=f_i(y)$,

(**) $\|f_1+f_2\|_p\le\|f_1\|_p+\|f_2\|_p$, $f_i\in L^p(X_2,\mu_2)$ ■

Remark (a) Take $X_2:=\mathbb{N}$, μ_2 = counting measure, $\int_{X_2}d\mu_2(y)=\sum_{i=1}^{\infty}$,

then (**) becomes

(***) $\|x+y\|_p\le\|x\|_p+\|y\|_p$, $x,y\in\underline{\ell^p}$

(b) Take $X_2 := \{1, \dots, N\}$, $\mu_2 =$ counting measure, $\int_{X_2} d\mu_2 = \sum_{i=1}^{N}$

then (**) becomes

(***) $\|x+y\|_p \le \|x\|_p + \|y\|_p$, $x, y \in \mathbb{K}^N$

(c) The proof above is "the same" as the standard proof of (**), (***), or (****) (check!)

Theorem 3.3 (Riesz–Fischer) Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $(L^p, \|\cdot\|_p)$ is a Banach space for all $p \in [1, \infty]$. Moreover, L^2 is a Hilbert space with scalar product

$$\langle f, g \rangle := \int_X \overline{f(x)}\, g(x)\, d\mu(x) \qquad \forall f, g \in L^2$$

Pf: Ana III. (triangle inequality needs $p \ge 1$!)

Remark (a) Take $X = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, $\mu =$ counting measure, $\int_X d\mu = \sum_{i=1}^{\infty}$, then 3.3 says: $(\ell^p, \|\cdot\|_p)$ is Banach.

(b) Take $X = \{1, \dots, N\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu =$ counting measure, $\int_X d\mu = \sum_{i=1}^{N}$, then 3.3 says: $(\mathbb{K}^N, \|\cdot\|_p)$ is Banach.

Lemma 3.4 If $\mu(X) < \infty$, $1 \le r \le p \le \infty$, then L^p ($\subseteq L^r$) is dense in L^r (wrt. $\|\cdot\|_r$), and

$$\|f\|_r \le [\mu(X)]^{\frac{1}{r} - \frac{1}{p}} \|f\|_p \qquad \forall f \in L^p$$

Pf: Ana III.

The next theorem is about the geometry of L^p, but also serves as preparation to compute its dual:

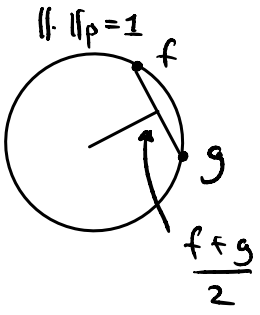
Theorem 3.5 Let $p \in (1, \infty)$. Then:

For all $\varepsilon > 0, r > 0$ there exists $\delta > 0$ s.t.:

For all $f, g \in L^p$:

$\|f\|_p, \|g\|_p \leq r$ and $\|f - g\|_p \geq \varepsilon$

$$\Rightarrow \left\| \frac{f+g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2} - \delta.$$



In particular, L^p is uniformly convex (case (see drawing) $r = \|f\|_p = \|g\|_p = 1$)

Pf (after Naoki Shioji)

Define the map $\alpha : \mathbb{K}^2 \to \mathbb{R}$

$$z = (z_1, z_2) \longmapsto \frac{|z_1|^p + |z_2|^p}{2} - \left| \frac{z_1 + z_2}{2} \right|^p$$

Claim: $\forall z \in \mathbb{K}^2 : \alpha(z) \geq 0$, with "=" iff $z_1 = z_2$ (*1)

Pf Claim: $\left| \frac{z_1 + z_2}{2} \right|^p \overset{\Delta\text{-ineq. } \mathbb{K}}{\leq} \left(\frac{|z_1| + |z_2|}{2} \right)^p \overset{(\cdot)^p \text{ strictly convex } (p>1)}{\leq} \frac{|z_1|^p + |z_2|^p}{2}$

Δ-ineq. is equality $\Leftrightarrow z_1 = 0$ or $z_2 = \lambda z_1$ for some $\lambda > 0$

Strict conv. ineq. is equality $\Leftrightarrow |z_1| = |z_2|$

$\Rightarrow$ (equality $\Leftrightarrow z_1 = z_2$) ✓ claim

Now, fix $\varepsilon, r > 0$, and choose $\eta \in (0, \frac{\varepsilon^p}{2r^p})$ small enough (f.ex. $< 2^{-p}$) such that

$$D := \{ z \in \mathbb{K}^2 \mid |z_1|^p + |z_2|^p = 1 \text{ and } |z_1 - z_2|^p \geq \eta \} \neq \emptyset$$

Then, by compactness (!) of D and continuity of α,

$$\theta := \inf \{ \alpha(z) \mid z \in D \} = \min \{ \alpha(z) \mid z \in D \} > 0 \quad \text{(by (*1))}$$

Claim: $|z_1|^p + |z_2|^p \leq \frac{\alpha(z)}{\theta}$ (**)

$\forall z \in F := \{ \tilde{z} \in \mathbb{K}^2 \mid |\tilde{z}_1 - \tilde{z}_2|^p \geq \eta (|\tilde{z}_1|^p + |\tilde{z}_2|^p) \}$

Pf claim: W.l.o.g., $z \neq (0,0)$ ((***) holds for $(0,0)$)

Let $z \in F$, define $t := (|z_1|^p + |z_2|^p)^{1/p} > 0$ (since $z \neq 0$)

Note: (i) $\left|\frac{z_1}{t}\right|^p + \left|\frac{z_2}{t}\right|^p = 1$ (compute!)

(ii) $\left|\frac{z_1}{t} - \frac{z_2}{t}\right|^p \geq \eta$ (since $z \in F$)

so $\frac{z}{t} \in D$. Hence, by def. of Θ, we have $\Theta \leq \alpha\left(\frac{z}{t}\right) = \frac{\alpha(z)}{t^p}$ ✓ (claim)

Now, let $f, g \in L^p$ with $\|f\|_p, \|g\|_p \leq r$ and $\|f-g\|_p \geq \varepsilon$.

Define $M := \{x \in X \mid (f(x), g(x)) \in F\}$ ($X = M \cup M^c$)

and estimate:

$$\varepsilon^p \leq \int_X |f-g|^p d\mu = \int_{M^c} |f-g|^p d\mu + 2^p \int_M \left|\frac{f-g}{2}\right|^p d\mu =: I_1 + 2^p I_2 \quad (***)$$

By definition of M^c (& F), $M^c \subseteq X$

$$I_1 \leq \eta \int_{M^c} (|f|^p + |g|^p) d\mu \leq \eta \left(\|f\|_p^p + \|g\|_p^p\right) \leq 2\eta r^p$$

Since $\alpha \geq 0$, we also get

$$I_2 \leq \frac{1}{2} \int_M (|f|^p + |g|^p) d\mu \overset{(**)}{\leq} \frac{1}{2\theta} \int_M \alpha(f(x), g(x)) d\mu(x)$$

$$\overset{M \subseteq X}{\leq} \frac{1}{2\theta} \int_X \left(\frac{|f|^p + |g|^p}{2} - \left|\frac{f+g}{2}\right|^p \right) d\mu.$$

Inserting the inequalities for I_1 & I_2 in (***) gives

$$\left\| \frac{f+g}{2} \right\|_p^p \leq \underbrace{\frac{\theta(2\eta r^p - \varepsilon^p)}{2^{p-1}}}_{=: -\delta} + \frac{\|f\|_p^p + \|g\|_p^p}{2}$$

with $\delta > 0$ by the definition of η ∎

Theorem 3.6 (Riesz' Representation Thm. for $(L^p)^*$)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$, assume in addition that μ is σ-finite. Then the map

$$J : L^q \to (L^p)^*, \quad f \mapsto l_f, \qquad l_f : L^p \to \mathbb{K}, \; g \mapsto \int_X f g \, d\mu \tag{*}$$

is an isometric isomorphism. Hence:

$$L^q \simeq (L^p)^*, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \in [1, \infty)$$

Remark 3.7 (a) The map J is also well-defined and isometric for $p = \infty$, but fails to be surjective if $\mathrm{supp}(\mu)$ has infinitely many elements (see exercise). I.e., $(L^\infty)^*$ is strictly larger than L^1 in this case (compare 2.38 and 2.39(b))

(b) The case $p = 1$ in 3.6 would be wrong without additional hypotheses on the measure space. σ-finiteness of μ is sufficient, but not necessary, because of:

Thm The map $J : L^\infty \to (L^1)^*$ is

(a) Injective $\Leftrightarrow$ isometric $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is semi-finite

(b) Bijective $\Leftrightarrow$ isometric $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is localisable & bijective

Note: $(X, \mathcal{A}, \mu)$ σ-finite $\Rightarrow$ localisable $\Rightarrow$ semi-finite

(For required def.'s & pf., see Salamon, "Measure and Integration", EMS, (2016) Sect. 4.5)

Pf (of 3.6): By linearity of the integral, the map J is linear.

Let $f \in L^q$. Then l_f (see (*)) is linear (in g), $fg \in L^1 \; \forall g \in L^p$ by Hölder, and $|l_f(g)| \le \|f\|_q \|g\|_p$. Hence, $l_f \in (L^p)^*$,

and $\|l_f\|_{(L^p)^*} \le \|f\|_q$ (**)

Hence, J in (*) is well-defined, linear, and bounded.

Claim: $\|l_f\|_{(L^p)^*} \overset{(***)}{=} \|f\|_q$ ($\Rightarrow J$ is isometric, hence inj.)

Pf claim: Wlog, $f \neq 0$ ((***) clear for $f=0$).

(i) Case $p \in (1,\infty)$: Let

$$g(x) := \begin{cases} \overline{f(x)}\,|f(x)|^{q-2}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

Then

$$|g|^p = |f|^{(q-1)p} = |f|^q \in L^1 \quad \text{since } (q-1)p = q\left(1-\tfrac{1}{q}\right)p = q$$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

So $g \in L^p$, and $\|g\|_p = \|f\|_q^{q/p}$. Also

$$\frac{l_f(g)}{\|g\|_p} = \frac{\int |f|^q}{\|g\|_p} = \frac{\|f\|_q^q}{\|f\|_q^{q/p}} = \|f\|_q^{q(1-\frac{1}{p})} = \|f\|_q^1, \text{ hence}$$

$\|l_f\|_{(L^p)^*} \ge \|f\|_q$. This, and (**), imply (***) (the claim).

(ii) Case $p=1$: Since μ is σ-finite, we have:

$$X = \bigcup_{n\in\mathbb{N}} X_n, \quad X_n \subseteq X_{n+1}, \quad \mu(X_n) < \infty \quad \forall n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and, for $f \in L^\infty$, define

$$M := \{x \in X \mid |f(x)| \ge \|f\|_\infty - \varepsilon\}.$$

By definition of "ess sup", $\mu(M) > 0$. Then $M_n := M \cap X_n$ satisfies $0 < \mu(M_n) < \infty$ for (all) n sufficiently large.

Let

$$g(x) := \begin{cases} \dfrac{\overline{f(x)}}{|f(x)|} \dfrac{1}{\mu(M_n)} \mathbb{1}_{M_n}(x), & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

Then $g \in L^1$ with $\|g\|_1 \le 1$, and

$$l_f(g) = \int_X fg\,d\mu = \frac{1}{\mu(M_n)} \int_{M_n} |f|\,d\mu \ge \|f\|_\infty - \varepsilon. \text{ As this holds } \forall \varepsilon > 0,$$

we get $\|l_f\|_{(L^1)^*} \ge \|f\|_\infty$. This, and (**), imply the claim (***).

It remains to prove that J is surjective.

Let $\xi \in (L^p)^*$. Need to prove: $\exists f_\xi \in L^q$ s.t. $\xi = \ell_{f_\xi}$ (see (*1))

Case $p \in (1,\infty)$: (We follow again N. Shioji)

Define the non-linear functional $F: L^p \to \mathbb{R}$

$$g \mapsto F(g) := \frac{1}{p}\|g\|_p^p - \mathrm{Re}\,\xi(g)$$

Since $p > 1$ and

$$F(g) \geq \frac{1}{p}\|g\|_p^p - \|\xi\|_* \|g\|_p ,$$

F is bounded from below:

$$m := \inf\{F(g) \mid g \in L^p\} > -\infty$$

Claim: $\exists!\, f \in L^p : F(f) = m$ (i.e. exists unique minimiser of F)

[This is where we need/use uniform convexity!]

Pf claim: For $n \in \mathbb{N}$, let $C_n := \{g \in L^p \mid F(g) \leq m + \frac{1}{n}\}$.

Then: (i) $C_n \neq \emptyset$ $\forall n \in \mathbb{N}$ (by definition of the infimum)

(ii) C_n closed $\forall n \in \mathbb{N}$ (since F cont. and $(-\infty, m+\frac{1}{n}]$ closed)

(iii) $C_n \supseteq C_{n+1}$ $\forall n \in \mathbb{N}$

(iv) $\mathrm{diam}(C_n) \xrightarrow{n\to\infty} 0$ (proof below)

From (i)-(iv) & completeness of L^p follows (see exercise!)

that $\bigcap_{n\in\mathbb{N}} C_n = \{f\}$ for some (unique) $f \in L^p$.

This implies the claim, since $\bigcap_{n\in\mathbb{N}} C_n = \{\varphi \in L^p \mid F(\varphi) = m\}$.

It remains to do the

Proof of (iv): Note that (a) $C_1 \neq \emptyset$, (b) $\forall g \in C_1$: $\frac{1}{p}\|g\|_p^p - \|\xi\|_*\|g\|_p \leq m+1$

and (c) $p > 1$ – so $r := \sup\{\|g\|_p \mid g \in C_1\} \in (0,\infty)$

Now let $\varepsilon > 0$, and let $g, h \in C_1$ (so $\|g\|_p, \|h\|_p \leq r$)

be arbitrary but fixed.

Assume (for contradiction!): $\|g-h\|_p \geq \varepsilon$ (□)

Then, by Theorem 3.5, $\exists \delta > 0$ (indep. of g, h) s.t.

$$\left\| \frac{g+h}{2} \right\|_p^p \leq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta. \quad (□□)$$

On the other hand: $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \frac{p}{n} < \delta$.

Now restrict choice of g, h to: $g, h \in C_n$, $n \geq n_0$.

Then
$$m \overset{\text{def of } m}{\leq} F\left(\frac{g+h}{2}\right) = \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2}\mathrm{Re}\,\xi(g) - \frac{1}{2}\mathrm{Re}\,\xi(h)$$

$$= \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2p}\left(\|g\|_p^p + \|h\|_p^p\right) + \underbrace{\frac{1}{2}\left(F(g)+F(h)\right)}_{\leq m + \frac{1}{n} \text{ (by def of } C_n)}$$

Hence,
$$\left\| \frac{g+h}{2} \right\|_p^p \geq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \underbrace{\frac{p}{n}}_{<\delta} > \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta.$$

This contradicts (□□), hence (□) is wrong, so:

$\forall g, h \in C_n \ \forall n \geq n_0 : \|g-h\|_p < \varepsilon$. This proves (iv), hence the claim ✓

So, let $f \in L^p$ be the unique minimiser from the claim, i.e.

$F(f) = m$, and let $g \in L^p$ be arbitrary.

Define
$$\sigma_g : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto \sigma_g(t) := F(f+tg) = \frac{1}{p}\int_X |f+tg|^p \, d\mu - t\,\mathrm{Re}\,\xi(g) - \mathrm{Re}\,\xi(f)$$

Goal: Compute $\sigma_g'(0)$.

Lebesgue's Thm. on differentiability of parameter-dependent integrals (see Handout!) applies, since, for $I \subseteq \mathbb{R}$ open & bounded, with $0 \in I$, we have:

(i) $|f+tg|^p \in L^1 \quad \forall t \in I$

(ii) $\varphi(t,\cdot) := \frac{\partial}{\partial t}|f+tg|^p \in L^1 \quad \forall t \in I$, since:

$$\varphi(t,\cdot) = \frac{\partial}{\partial t}\left[(f+tg)\overline{(f+tg)}\right]^{p/2}$$

$$= \frac{p}{2}|f+tg|^{p-2}\left[g\overline{(f+tg)} + (f+tg)\bar{g}\right]$$

$$= p\,|f+tg|^{p-2}\,\mathrm{Re}\left[g\overline{(f+tg)}\right]$$

(in the case $p \in (1,2)$ with the convention that

$$\varphi(t,x) := 0 \text{ if } |f(x)+tg(x)| = 0\ )$$

(iii) $\exists\, \Phi \in L^1 : |\varphi(t,\cdot)| \le \Phi \quad \forall t \in I.$

Hence, since f minimises F (so σ_g has minimum at $t=0$),

$$0 = \sigma_g'(0) = \int_X |f|^{p-2}\,\mathrm{Re}(\bar{f}g)\,d\mu - \mathrm{Re}\,\xi(g)$$

$$\Rightarrow \quad \mathrm{Re}\,\xi(g) = \mathrm{Re}\left(\int_X f_\xi g\,d\mu\right), \quad f_\xi := |f|^{p-2}\bar{f} \in L^q \text{ (check!)}$$

If $\mathbb{K} = \mathbb{R}$, the claim follows

If $\mathbb{K} = \mathbb{C}$, consider also $\tau_g : \mathbb{R} \ni t \mapsto \tau_g(t) := F(f - itg)$

By similar arguments, we get

$$0 = \tau_g'(0) = \int_X |f|^{p-2}\,\mathrm{Im}(\bar{f}g)\,d\mu - \mathrm{Im}(\xi g)$$

$$\Rightarrow \quad \mathrm{Im}\,\xi(g) = \mathrm{Im}\left(\int_X f_\xi g\,d\mu\right) \quad \checkmark \quad (p \in (1,\infty) \text{ done})$$

Case $p=1$: Idea: Step 1: Reduce to the above case $p \in (1,\infty)$, if μ is finite. Step 2: Generalise to σ-finite μ.

Step 1: Assume $\mu(X) < \infty$. Let $r > 1$. Recall from Lemma 3.4: $L^\infty \subseteq L^r \subseteq L^1$ (the inclusion being dense).

Then:

$$|\xi(g)| \le \|\xi\|_{(L^1)^*}\|g\|_1 \le \underbrace{\|\xi\|_{(L^1)^*}\left[\mu(X)\right]^{1-\frac{1}{r}}}_{=:\,\kappa_r}\|g\|_r \qquad \forall g \in L^r$$

Hence, $\xi_r := \xi|_{L^r} \in (L^r)^*$. From the case $p \in (1,\infty)$ proved above (on r) we get: $\exists! f_r \in L^q$ $(\frac{1}{r}+\frac{1}{q}=1)$: $\xi_r = \ell_{f_r}$ (see (*1)), and

$\|f_r\|_q = \|\xi_r\|_{(L^r)^*} \le M_r$. Hence: $\forall r_1, r_2 \ge 1$, $\forall h \in L^\infty$ $(\subseteq L^{r_i}, i=1,2)$:

$\ell_{f_{r_1}}(h) = \xi_{r_1}(h) = \xi(h) = \xi_{r_2}(h) = \ell_{f_{r_2}}(h)$ i.e. $\int_X (f_{r_1} - f_{r_2})\, h \, d\mu = 0$

Choose $h := \dfrac{\overline{(f_{r_1} - f_{r_2})}}{|f_{r_1} - f_{r_2}|} \in L^\infty$, then $f_{r_1} = f_{r_2} =: f$ is independent of the chosen/used r. Hence, $f \in \bigcap_{q \in (1,\infty]} L^q$ and

$$\limsup_{q\to\infty} \|f\|_q \underbrace{\le}_{\le M_r} \limsup_{q\to\infty} \left\{ \|\xi\|_{(L^1)^*} [\mu(X)]^{1/q} \right\} = \|\xi\|_{(L^1)^*} < \infty$$

$\leftarrow 1 - \frac{1}{r} = \frac{1}{q}$

Then (see exercise!) $f \in L^\infty$, and so $\ell_f \in (L^1)^*$ (see (*))

Since L^∞ dense in L^1, we get: ξ and ℓ_f agree on a dense subspace, so (by Thm. 2.31) $\xi = \ell_f$ on L^1.

Step 2: Generalise to μ being σ-finite. See exercise ∎

Remark (a) Compare with Thm 2.38 (for ℓ^p). Recall that

$$\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$$ with μ counting measure.

(b) Compare with Thm. 2.57 & 2.58 (for Hilb. space) Recall that L^2 is a Hilbert space, with $\langle f, g\rangle := \int_X \bar{f} g \, d\mu$. Note the complex conjugate (also in 2.57 & 2.58) $(\frac{1}{2}+\frac{1}{2}=1)$.
(Recall also 2.54 & see next section)

(c) Another proof uses Radon-Nikodym Thm. to prove surjectivity of the map J; see Werner, "Funktionalanalysis", 8. Aufl., Satz II.2.4 p. 65.

(d) A more abstract proof uses the uniform convexity & Milman-Pettis Thm, to conclude L^p, $p \in (1,\infty)$, is reflexive, to identify $(L^p)^*$; see 4.17 & 4.18 later.

3.2 Separability

We have the following general abstract result:

Theorem 3.8 Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space which is countably generated (i.e., $\exists\, \mathcal{C} \subseteq \mathcal{P}(X)$ countable such that $\mathcal{A} = \sigma(\mathcal{C})$). Let $p \in [1, \infty)$. Then L^p is separable.

Pf: See Behrends, "Maß- und Integrationstheorie" (1987) Thm IV.1.5 p. 160.

Remark 3.9 The theorem does not hold for $p = \infty$ (in general).

We turn to the concrete case of $L^p(\Omega, d\lambda^d)$ with $\Omega \subseteq \mathbb{R}^d$ open (non-empty) and λ^d Lebesgue measure, for which we will prove the result in 3.8.

Notation: (i) For $f: \Omega \to \mathbb{K}$, the (topological) support of f is

$$\operatorname{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$$

(closure wrt. rel. top. on Ω induced by Euclidean top.)

(ii) For $k \in \mathbb{N}$: Space of k-times cont. diff. fct's

$$C^k(\Omega) := \left\{ f \in C(\Omega) \mid \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f \in C(\Omega) \ \forall \alpha_1, \ldots, \alpha_d \in \mathbb{N}_0,\ \sum_{j=1}^{d} \alpha_j \leq k \right\}$$

Also, $C^0(\Omega) := C(\Omega)$ and

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega)$$

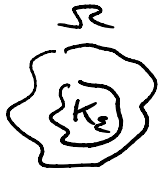
((infinitely) smooth functions)

(iii) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k-times cont. diff. fct's with compact support:

$$C_c^k(\Omega) := \{ f \in C^k(\Omega) \mid \operatorname{supp}(f) \subseteq \Omega \text{ is compact} \}$$

(iv) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k-times cont. diff. fct.'s vanishing at ∞ (!):

$$C_0^k(\Omega) := \{ f \in C^k(\Omega) \mid \forall \varepsilon > 0 \ \exists K_\varepsilon \subseteq \Omega \text{ compact s.t. } |f(x)| \leq \varepsilon \ \forall x \in \Omega \setminus K_\varepsilon \}$$



Rest this section:

Consider $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{B}(\Omega), \lambda^d)$, with $\mathcal{B}(\Omega)$ Borel-σ-algebra on Ω (trace-σ-algebra to Ω of Borel-σ-algeb. on $\mathbb{R}^d$) and λ^d Lebesgue-Borel measure on $\mathbb{R}^d$ (see Handout).

Main result is for $L^p(\Omega) := L^p(\Omega, \lambda^d)$:

Theorem 3.10 Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open and let $p \in [1, \infty)$.

Then $$\overline{C_c^\infty(\Omega)}^{\|\cdot\|_p} = L^p(\Omega)$$

(i.e. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ wrt. $\|\cdot\|_p$-norm).

Remark 3.11 (See exercises)

(a) Theorem 3.10 does not hold for $p = \infty$. This follows from:

$$\overline{C_c(\Omega)}^{\|\cdot\|_\infty} = C_0(\Omega).$$

(b) $C_c(\Omega)$ is separable wrt. $\|\cdot\|_\infty$.

Corollary 3.12 Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open, and let $p \in [1, \infty)$.

Then $L^p(\Omega)$ is separable.

Pf: From Rmk. 3.11(b) & Thm. 3.10 (see exercise for details). ∎

The proof of Thm. 3.10 requires some preparations:

Proposition 3.13 (Young's Inequality (for convolutions))

Let $p, q \in [1, \infty]$ with $1 \leq \frac{1}{p} + \frac{1}{q} \leq 2$, and let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$.

Then the convolution $f * g$ of f and g,

$$\mathbb{R}^d \ni x \longmapsto (f*g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) \, dy \qquad (*)$$

(where $dx := d\lambda^d(x)$, integration wrt. Lebesgue measure on $\mathbb{R}^d$)

exists for a.e. x, is commutative: $f*g = g*f$, and $f*g \in L^r(\mathbb{R}^d)$ with, for $\frac{1}{r} \overset{(**)}{:=} -1 + \frac{1}{p} + \frac{1}{q}$ (i.e. $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$),

$$(***) \qquad \|f*g\|_r \le \|f\|_p \|g\|_q \qquad \text{(Young's Ineq.)}.$$

Corollary 3.14 Let $p,q,r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$ (□)
Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then

$$I := I(f,g,h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\, g(x-y)\, h(y)\, dx\, dy \right|$$

$$\le \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)| \cdot |g(x-y)| \cdot |h(y)|\, dx\, dy \le \|f\|_p \cdot \|g\|_q \cdot \|h\|_r \qquad (□□)$$

Pf (of 3.14): Hölder & Young's ineq.: By 3.13, $f*g \in L^s$, $\frac{1}{s} = -1 + \frac{1}{p} + \frac{1}{q}$.
Hence, $\frac{1}{r} + \frac{1}{s} = 1$ by (□), and so (□□) follows from Hölder's ineq.

Remark 3.14 (a) Note that, by scaling, the r in (**) is the only one for which (***) could hold: For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$. Then (change of variables!), $\|f_\lambda\|_p = \lambda^{-1/p} \|f\|_p$ & $(f*g)_\lambda(x) = (\lambda f_\lambda * g_\lambda)(x)$.
Hence, if (***) holds for some $p,q,r \in [1,\infty]$, then also, $\forall \lambda > 0$

$$\|f*g\|_r = \lambda^{1/r} \|(f*g)_\lambda\|_r \le \lambda^{1/r} \lambda \|f_\lambda\| \cdot \|g_\lambda\|_q = \lambda^{\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p \|g\|_q$$

This is only possible if (*) holds (let $\lambda \searrow 0$ and $\lambda \nearrow \infty$ above).

(b) Often (in applications) one encounters

$$J_\alpha(f,h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\, |x-y|^{-\alpha}\, h(y)\, dx\, dy \right|, \text{ that is, (□□) with } g(z) = |z|^{-\alpha}.$$

Now, $g \notin L^q(\mathbb{R}^d)$ $\forall q$. The replacement for (□□) (or, for (***) in 3.13) is called the Hardy-Littlewood-Sobolev Ineq. (HLS):

$$J_\alpha(f,h) \le C_1(p,\alpha,d) \|f\|_p \|h\|_r, \quad \frac{1}{p}+\frac{1}{r}+\frac{\alpha}{d} = 2 \ \&\ 0 < \alpha < d$$

Its proof requires much more work.

Pf (of 3.13): Commutativity (if integral exists):

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \overset{y := x-z}{=} \int_{\mathbb{R}^d} f(z)\, g(x-z)\, dz = (g*f)(x)$$

The cases $p=1<q=r$ and $q=1<p=r$ follow directly from Minkowski's Ineq. (3.2(b)) (& commutativity, for $q=1<p=r$): ($r<\infty$!)

$$\left(\int_{\mathbb{R}^d}|(f*g)(x)|^r dx\right)^{1/r} = \left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(y)g(x-y)\,dy\right|^r dx\right)^{1/r}$$

$$\overset{3.2(b)}{\leq} \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)g(x-y)|^r dx\right)^{1/r} dy = \int_{\mathbb{R}^d}|f(y)|\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^r dx\right)^{1/r}}_{=\|g\|_r\ \forall x} dy = \|f\|_1\|g\|_r \quad \checkmark$$

Remains the most general case:

$1<p<r$ and $1<q<r$, $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. (**)

Strategy: write $|f(y)g(x-y)|$ as product of 3 factors & use (gen.) Hölder:

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{s_1}}|g(x-y)|^{\frac{q}{s_2}}|f(y)|^{\frac{p}{s_3}} \quad (\nabla)$$

We must have (check!)

$$p\left(\frac{1}{s_1}+\frac{1}{s_3}\right)=1\ ,\quad q\left(\frac{1}{s_1}+\frac{1}{s_2}\right)=1$$

As we want to estimate the L^r-norm of $f*g$, we want $s_1=r$.

Hence, $\begin{pmatrix}1&0&0\\ q&q&0\\ p&0&p\end{pmatrix}\begin{pmatrix}s_1^{-1}\\ s_2^{-1}\\ s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\ 1\\ 1\end{pmatrix}$ check! $\Leftrightarrow$ ($pq\neq 0$) $\begin{pmatrix}s_1^{-1}\\ s_2^{-1}\\ s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\ (1-\frac{q}{r})\frac{1}{q}\\ (1-\frac{p}{r})\frac{1}{p}\end{pmatrix}$ $(\nabla\nabla)$

We see that $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\overset{(\Delta)}{=}1$, and (∇) becomes

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{r}}|g(x-y)|^{1-\frac{q}{r}}|f(y)|^{1-\frac{p}{r}} \quad (\nabla\nabla\nabla)$$

and so, by gen. Hölder (Tut T26; 3 factors; use (Δ) & $(\nabla\nabla)$)

$$|(f*g)(x)| \leq \left(\int_{\mathbb{R}^d}|f(y)|^p\cdot|g(x-y)|^q dy\right)^{\frac{1}{r}}\|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}} \quad (\nabla\nabla\nabla\nabla)$$

Hence (raise to power r & integrate wrt x),

$$\|f*g\|_r \leq \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)|^p|g(x-y)|^q dy\right)dx\right)^{1/r}$$

$$\overset{\text{Tonelli}}{=} \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}|f(y)|^p\cdot\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^q dx\right)}_{\equiv\|g\|_q^q\ \forall x}dy\right)^{1/r}$$

$$= \|f\|_p\|g\|_q \quad \checkmark$$

If $r=\infty$, then (**) follows from (standard) Hölder, as $\frac{1}{p}+\frac{1}{q}=1$ ∎

Next, we will study **mollification**: L^p-functions are made smooth (C^∞) by convolution with a mollifier:

Let $$L^p_c(\Omega) := \{ f \in L^p(\Omega) \mid \exists K \subseteq \Omega \text{ compact s.t. } f|_{\Omega \setminus K} = 0 \text{ a.e.} \}$$
for the space of L^p-functions with compact support in Ω.
We identify $f \in L^p_c(\Omega)$ with its extension by 0 to all of $\mathbb{R}^d$.

Lemma 3.15 Let $p \in [1, \infty)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open, $f \in L^p_c(\Omega)$.
Let $j \in C^\infty_c(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} j(x)\,dx = 1$, and for $\varepsilon > 0$ define the **mollifier**
$$j_\varepsilon : \mathbb{R}^d \to [0, \infty), \quad x \mapsto \varepsilon^{-d} j\left(\frac{x}{\varepsilon}\right).$$
Then, for all $\varepsilon > 0$ sufficiently small: $f_\varepsilon := j_\varepsilon * f \in C^\infty_c(\Omega)$ and $\|f_\varepsilon\|_p \leq \|f\|_p$.

Remark Note that j_ε is a pointwise defined function (and not just an equiv. class of integrable fct's, defined a.e.), the convolution $j_\varepsilon * f$ has the canonical representative
$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y)\,dy \qquad (*)$$
which is pointwise well-defined $\forall x \in \mathbb{R}^d$, and independent of the choice of the representative for $f \in L^p$. It is the representative $(*)$ of the convolution $j_\varepsilon * f$ we denote by f_ε

Pf: **Norm**: Use Thm 3.13 with $r = p$. Then $q = 1$ then, and so $f_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j_\varepsilon\|_1 = \|f\|_p$.

Support: By assumption, $\exists K \subseteq \Omega$ compact with $f|_{\Omega \setminus K} = 0$.
Then (by T1o(iii)) $\delta := \operatorname{dist}(K, \partial\Omega) > 0$ (since K compact & $\partial\Omega$ closed)
By def. of the mollifier, $\exists \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:

$\operatorname{supp} j_\varepsilon \subseteq B_{\delta/2}(0)$. From (*1) we conclude that

$$\operatorname{supp} f_\varepsilon \subseteq \bigcup_{y\in K} B_{\delta/2}(y) \subseteq \Omega.$$

In particular, $\operatorname{supp} f_\varepsilon$ is bounded (!) and (since closed by def.), also compact.

Derivatives: Let $e\in\mathbb{R}^d$, $|e|=1$, be a unit vector, and $\lambda\in\mathbb{R}\setminus\{0\}$.

Then, $\forall x\in\mathbb{R}^d$,

$$F_\lambda(x) := \frac{1}{\lambda}\left(f_\varepsilon(x+\lambda e) - f_\varepsilon(x)\right) = \frac{1}{\lambda}\int_{\mathbb{R}^d}\left(j_\varepsilon(x-y+\lambda e) - j_\varepsilon(x-y)\right) f(y)\,dy$$

$$\overset{\text{HDI}}{=} \frac{1}{\lambda}\int_{\mathbb{R}^d}\left(\int_0^1 \underbrace{\frac{d}{dt}\left[j_\varepsilon(x-y+t\lambda e)\right]}_{\lambda\, e\cdot(\nabla j_\varepsilon)(x-y+t\lambda e)}\,dt\right) f(y)\,dy$$

$$= \int_{\mathbb{R}^d}\underbrace{\left(\int_0^1 (e\cdot\nabla j_\varepsilon)(x-y+t\lambda e)\,dt\right)}_{=:D} f(y)\,dy$$

Since $|D| \le \|\,|\nabla j_\varepsilon|\,\|_\infty < \infty$ and $f\in L^1$ (because $f\in L^p$ & of compact support), using Dominated Convergence (twice, in dt & dy integral) gives that

$$e\cdot(\nabla f_\varepsilon)(x) = \lim_{\lambda\to 0} F_\lambda(x) = \int_{\mathbb{R}^d} e\cdot(\nabla j_\varepsilon)(x-y) f(y)\,dy = \left((e\cdot\nabla j_\varepsilon) * f\right)(x)$$

exists $\forall x\in\mathbb{R}^d$. The existence of higher derivatives follows by induction in the same way. ☒

Lemma 3.16 Let $p\in[1,\infty)$, let $A\in\mathcal{B}(\mathbb{R}^d)$ be bounded.

Then $$\lim_{\varepsilon\searrow 0} \|\mathbb{1}_A - j_\varepsilon * \mathbb{1}_A\|_{L^p(\mathbb{R}^d)} = 0.$$

Proof: Step 1: It suffices to prove the lemma for $p=1$, since:

Let $g_\varepsilon := \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \in L^\infty_c(\mathbb{R}^d)$; then $g_\varepsilon\in L^p(\mathbb{R}^d)\ \forall p\in[1,\infty)$ and

$$\|g_\varepsilon\|_p^p \le \underbrace{\left(\|g_\varepsilon\|_\infty\right)^{p-1}}_{\le 2}\cdot \|g_\varepsilon\|_1.$$

Step 2: The case $p=1$. Since Lebesgue measure is (outer) regular (see Handout):

$$\forall \delta>0 \ \exists B \supseteq A, B \text{ open and } \int_{B\setminus A} dx < \delta.$$

So, fix $\delta>0$, and let B be as above (so, also bounded). Note that

$$\| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_1 \le \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 + \underbrace{\| \mathbb{1}_{B/A} \|_1}_{<\delta} + \| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1$$

with $\| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_\varepsilon(x-y) \mathbb{1}_{B\setminus A}(y)\, dy\, dx$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y)\, dx \right)}_{=1 \ \forall y} \mathbb{1}_{B\setminus A}(y)\, dy < \delta.$$

Hence, it suffices to prove that $\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 \to 0$, $\varepsilon \searrow 0$.

We have $0 \le j_\varepsilon * \mathbb{1}_B \le 1$, hence

$$\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 = \int_B (1 - (j_\varepsilon * \mathbb{1}_B)(x))\, dx + \int_{\mathbb{R}^d\setminus B} (j_\varepsilon * \mathbb{1}_B)(x)\, dx$$

$$= 2|B| - 2 \int_B \int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x - \varepsilon z)\, dz\, dx.$$

Now, since B is open: $\forall x \in B \ \forall z \in \mathbb{R}^d : \mathbb{1}_B(x-\varepsilon z) \to 1$, $\varepsilon \searrow 0$ (think!)

Also, B is bounded and $j \in L^1(\mathbb{R}^d)$, so, using Lebesgue's Thm. on Dominated Convergence twice

$$\lim_{\varepsilon \searrow 0} \int_B \left(\int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x-\varepsilon z)\, dz \right) dx = \int_B \left(\int_{\mathbb{R}^d} j(z)\, dz \right) dx = |B| \quad \blacksquare$$

Proof of Thm. 3.16: (Here $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$)

Step 1: Let $\varepsilon>0$, $f \in L^p(\Omega)$. For $n \in \mathbb{N}$, define

$$K_n := \left\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge \tfrac{1}{n} \text{ and } |x| \le n \right\} \subseteq \Omega$$

Now, Ω is open, so $\Omega = \bigcup_{n\in\mathbb{N}} K_n$, and $K_n \subseteq K_{n+1}$ $\forall n \in \mathbb{N}$.

By dominated convergence, ($\lim_{n\to\infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_\Omega(x)$)

$$\lim_{n\to\infty} \|f - f\mathbb{1}_{K_n}\|_p = \lim_{n\to\infty} \|f(1-\mathbb{1}_{K_n})\|_p = 0.$$

Hence: $\exists n \in \mathbb{N}$: $\|f - f\mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$.

Step 2: Recall (Handout & exercise) that step functions are dense in L^p : $\exists L \in \mathbb{N}, c_1, \dots, c_L \in \mathbb{K}$ and $A_1, \dots, A_L \in \mathcal{B}(\mathbb{R}^d)$ with $A_\ell \subseteq K_n$ (!) $\forall \ell = 1, \dots, L$ such that: For $g := \sum_{\ell=1}^{L} c_\ell \mathbb{1}_{A_\ell}$ we have

$$g \in L^p_c(\Omega) \quad \text{and} \quad \|g - f\mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$$

Step 3: Mollify step function: $\exists \delta > 0$ s.t.

(i) $j_\delta * g \in C_c^\infty(\Omega)$ (by Lemma 3.15)

(ii) $\|g - j_\delta * g\|_p \le \sum_{\ell=1}^{L} |c_\ell| \cdot \|\mathbb{1}_{A_\ell} - j_\delta * \mathbb{1}_{A_\ell}\|_p < \frac{\varepsilon}{3}$

(by Lemma 3.16)

Hence,

$$\|f - \underbrace{j_\delta * g}_{\in C_c^\infty(\Omega)}\|_p < \varepsilon \qquad \square$$

Chapter 4: The cornerstones of Functional Analysis

4.1 Hahn-Banach Theorem

Recall:

Definition Let M be a set, and $D \subseteq M \times M$. Let $\prec$ ("sub") be the associated binary relation: $x \prec y :\Leftrightarrow (x,y) \in D$.

(a) $\prec$ is a partial ordering (or, M is a partially ordered set)
$:\Leftrightarrow \forall x,y,z \in M$:
(i) $x \prec x$ (reflexive)
(ii) $(x \prec y \ \& \ y \prec x) \Rightarrow (x = y)$ (antisymmetry)
(iii) $(x \prec y \ \& \ y \prec z) \Rightarrow (x \prec z)$ (transitive)

(b) $x, y \in M$ are comparable $:\Leftrightarrow x \prec y$ or $y \prec x$
x, y are incomparable $:\Leftrightarrow x, y$ are not comparable.

(c) $\prec$ is a total ordering $:\Leftrightarrow$
(i) $\prec$ is a partial ordering
(ii) x, y are comparable for all $x, y \in M$

(d) Let $W \subseteq M$. Then $u \in M$ is an upper bound of/for W
$:\Leftrightarrow w \prec u \ \forall w \in W$.

(e) $u \in W$ is a maximal element of W : $\Leftrightarrow$ The following implication holds:
$$u \prec w \text{ for } w \in W \Rightarrow u = w.$$

(Note: A maximal element need not be an upper bound and vice versa - see examples below).

Examples (a) "$\leq$" is a total ordering on $\mathbb{R}$.
$W = [0,1)$ has no maximal element, but any $u \geq 1$ is an upper bound for W

(b) "$\subseteq$" is a partial ordering on $\mathcal{P}(X)$, but not a total ordering.

(c) $(x_1, x_2) \prec (y_1, y_2) :\Leftrightarrow x_j \leq y_j$, $j = 1,2$,
is a partial ordering (but not total) on $M = \mathbb{R}^2$.
$W := \{(0,0), (1,0), (0,1)\}$ has 2 (!) maximal elements,
but none of them is an upper bound for W.

The following axiom is equivalent to the axiom of choice:

Axiom 4.1 (Zorn's Lemma) Let $M \neq \emptyset$ be a partially ordered set. Assume every totally ordered subset of M has an upper bound. Then M has a maximal element.

We now finish the

Proof of Thm. 2.53 (Remains: Every Hilbert space $X \neq \{0\}$ has an orthonormal basis)

Let $M := \{E \subseteq X \mid E \text{ orthonormal}\}$. Then:

(i) $M \neq \emptyset$

(ii) "$\subseteq$" is partial ordering on M

(iii) Let W be a totally ordered subset of M.

Then $U := \bigcup_{E \in W} E \in M$:

If $x_1, x_2 \in U$ then $\exists E_j \in W : x_j \in E_j$, $j = 1,2$.

As W is totally ordered, $E_1 \subseteq E_2$ (w.l.o.g.)

Hence, $x_1, x_2 \in E_2 \Rightarrow x_1 \perp x_2 \Rightarrow U$ orthonormal

That is, U is an upper bound for W.

By Zorn: $\exists$ maximal element $N \in M$.

Claim: $\mathcal{M}$ is an orthonormal basis for $\overline{X}$

(i) Orthonormal clear, since $\mathcal{M} \in M$

(ii) Assume $\mathcal{M}$ is not complete, i.e., not a basis.

Then (by Def. 2.47(d)), there is some $0 \neq x \in \overline{X}$ such that $x \perp u$ $\forall u \in \mathcal{M}$. Hence,

$$\mathcal{M}' := \left\{ \frac{x}{\|x\|} \right\} \cup \mathcal{M} \in M \quad \text{(since } \mathcal{M}' \text{ is orthonormal)}$$

so $\mathcal{M} \subsetneq \mathcal{M}'$ which contradicts $\mathcal{M}$ being maximal ⊠

Remark 4.2 Similar arguments prove Thm. 2.4 (existence of Hamel basis; see exercise)

Theorem 4.3 (Hahn-Banach (H-B)) Let X be a vectorspace, let $p: X \to \mathbb{R}$ be convex, i.e. $\forall x, x' \in X$, $\forall \alpha \in [0,1]$:

$$p(\alpha x + (1-\alpha)x') \leq \alpha p(x) + (1-\alpha) p(x') \quad (*)$$

Let Y be a subspace, and let $\lambda: Y \to \mathbb{K}$ be linear, with

$$\operatorname{Re} \lambda(y) \leq p(y) \quad \forall y \in Y \quad (\Pi)$$

Then there exists $\Lambda: X \to \mathbb{K}$ linear such that:

(i) $\Lambda|_Y = \lambda$ (i.e. Λ is extension of λ)

(ii) $\operatorname{Re} \Lambda(x) \leq p(x)$ $\forall x \in X$ (i.e. (Π) is preserved)

If, in addition, p also satisfies

$(**)$ $\quad p(\alpha x) \leq p(x)$ $\forall x \in X$, $\forall \alpha \in \mathbb{K}$ with $|\alpha| = 1$

then we even have $|\Lambda(x)| \leq p(x)$ $\forall x \in X$.

Remark 4.4 (a) Condition $(**)$ is equivalent to:

$$p(\alpha x) = p(x) \quad \forall x \in \mathbb{K} \text{ with } |\alpha| = 1 \quad \text{(check!)}$$

(b) If p is a (semi-) norm on X, then p satisfies $(*)$ & $(**)$.

Pf (of 4.3): 4 Steps: Steps 1 & 2 prove the main part for $\mathbb{R}$-vectorspaces, step 3 for $\mathbb{C}$-vector-spaces, and step 4 proves the addendum under add. condition (**).
Wlog: $Y \subsetneq X$ (otherwise trivial).

Step 1: Case $\mathbb{K}=\mathbb{R}$. Extend by one dimension – a preparation for Step 2.
By assumption: $\exists\, z \in X \setminus Y$ (so $z \neq 0$). Let $\tilde{Y} := \operatorname{span}(Y, \{z\})$.
For every $\tilde{y} \in \tilde{Y}$, $\exists!$ decomposition $\tilde{y} = y + \alpha z$, $y \in Y$, $\alpha \in \mathbb{R}$
Candidate for extension $\tilde{\lambda}$ of λ to $\tilde{Y}$:

$$\tilde{\lambda}(\tilde{y}) := \lambda(y) + \alpha\xi \qquad \text{for some } \xi \in \mathbb{R} \text{ (to be chosen)}$$

Interpretation: $\xi = \tilde{\lambda}(z)$. Clearly, $\tilde{\lambda}$ is ($\mathbb{R}$-) linear on $\tilde{Y}$, and $\tilde{\lambda}|_Y = \lambda$. Will choose ξ s.t. $\tilde{\lambda} \leq p$ on $\tilde{Y}$ (recall (ii)):
Let $\beta_1, \beta_2 > 0$, $y_1, y_2 \in Y$, then

$$\beta_1\lambda(y_1) + \beta_2\lambda(y_2) = (\beta_1+\beta_2)\,\lambda\Big(\underbrace{\tfrac{\beta_1}{\beta_1+\beta_2}y_1 + \tfrac{\beta_2}{\beta_1+\beta_2}y_2}\Big) \quad (\nabla)$$

$$\overset{\text{by (II)}}{\leq} p\Big(\underbrace{\tfrac{\beta_1}{\beta_1+\beta_2}y_1 + \tfrac{\beta_2}{\beta_1+\beta_2}y_2}\Big)$$

$$= \tfrac{\beta_1}{\beta_1+\beta_2}(y - \beta_2 z) + \tfrac{\beta_2}{\beta_1+\beta_2}(y_2 + \beta_1 z)$$

Hence,

$$(\nabla) \overset{p \text{ convex}}{\leq} \beta_1 p(y_1 - \beta_2 z) + \beta_2 p(y_2 + \beta_1 z).$$

Re-arranging, we get

$$\frac{1}{\beta_2}\big(\lambda(y_1) - p(y_1 - \beta_2 z)\big) \leq \frac{1}{\beta_1}\big(p(y_2 + \beta_1 z) - \lambda(y_2)\big)$$

$\forall\, \beta_1, \beta_2 > 0$, $\forall\, y_1, y_2 \in Y$

Hence, there exists $a \in \mathbb{R}$:

$$(\nabla\nabla) \quad \sup_{\substack{\beta_2>0 \\ \gamma_1 \in Y}} \left[\frac{1}{\beta_2} (\lambda(\gamma_1) - p(\gamma_1 - \beta_2 z)) \right] \le a \le \inf_{\substack{\beta_1>0 \\ \gamma_2 \in Y}} \left[\frac{1}{\beta_1} (p(\gamma_2 + \beta_1 z) - \lambda(\gamma_2)) \right]$$

Set $\tilde{\lambda}(z) := a$. Then, $\forall \tilde{\gamma} = \gamma + \alpha z \in \tilde{Y}$ with $\alpha > 0$, the right inequality in $(\nabla\nabla)$ implies

$$\tilde{\lambda}(z) \le \frac{1}{\alpha}(p(\tilde{\gamma}) - \lambda(\gamma)), \text{ so } \tilde{\lambda}(\tilde{\gamma}) \le p(\tilde{\gamma}).$$

If $\tilde{\gamma} = \gamma - \alpha z$ with $\alpha > 0$, use the left ineq. in $(\nabla\nabla)$ instead. Therefore, $\tilde{\lambda} \le p$ on $\tilde{Y}$.

Step 2: Case $\mathbb{K} = \mathbb{R}$. Idea: Use Zorn to construct the extension.

Let $M := \{(\mathbb{R}\text{-})$ linear extensions e of λ with $e \le p$ on $\mathrm{dom}(e)\}$

We have

(i) $M \neq \emptyset$ since $\lambda \in M$

(ii) Define partial ordering $\prec$ on M:

$$e_1 \prec e_2 :\Leftrightarrow \mathrm{dom}(e_1) \subseteq \mathrm{dom}(e_2) \wedge e_2|_{\mathrm{dom}(e_1)} = e_1.$$

(iii) Let $W \subseteq M$ be totally ordered.

Define $u : \bigcup_{e \in W} \mathrm{dom}(e) \to \mathbb{R}$, $x \mapsto \tilde{e}(x)$ where $\tilde{e}$ is any element of W such that $x \in \mathrm{dom}(\tilde{e})$.

u is well-defined: (i.e. indep. of the chosen $\tilde{e}$ among allowed elements):

Let $x \in \mathrm{dom}(\tilde{e}_1) \cap \mathrm{dom}(\tilde{e}_2)$. Since W is totally ordered, we have (wlog.) $\tilde{e}_2$ is an extension of $\tilde{e}_1$. Hence, $\tilde{e}_2(x) = \tilde{e}_1(x)$.

u is $(\mathbb{R}\text{-})$ linear: Let $x_1, x_2 \in \bigcup_{e \in W} \mathrm{dom}(e)$, $\gamma \in \mathbb{R}$. Hence, $\exists \tilde{e}_j \in W : x_j \in \mathrm{dom}(\tilde{e}_j)$, $j = 1, 2$. Since W is totally ordered, (wlog.) $\tilde{e}_2$ is extension of $\tilde{e}_1$. Then $x_1, x_2, x_1 + \gamma x_2 \in \mathrm{dom}(\tilde{e}_2)$ & linearity of u follows from that of $\tilde{e}_2$.

Lastly: $u(x) = \widetilde{e}(x) \leq p(x) \quad \forall x \in \bigcup_{e \in W} \operatorname{dom}(e)$.

Hence: $u \in M$, and (check!) u is an upper bound for W.

By Zorn's Lemma: M has a maximal element Λ.

Claim: $\operatorname{dom}(\Lambda) = X$: Suppose $\operatorname{dom}(\Lambda) \subsetneq X$. Then there is some $0 \neq z \in X \setminus \operatorname{dom}(\Lambda)$. Let $\widetilde{Y} := \operatorname{span}(\operatorname{dom}(\Lambda), \{z\})$.

By Step 1, Λ has some ($\mathbb{R}$-) linear extension $\widetilde{\Lambda} \in M$ to $\widetilde{Y}$, which contradicts M being maximal.

Hence, the main part of the thm. follows for $\mathbb{K} = \mathbb{R}$.

Step 3: Case $\mathbb{K} = \mathbb{C}$. Reduce to the real case.

Define $\ell(y) := \operatorname{Re} \lambda(y) \quad \forall y \in Y$. Then $\ell : X \to \mathbb{R}$ is $\mathbb{R}$-linear, and $\ell \leq p$ on Y. So, Step 2 implies: There exists $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ with $L|_Y = \ell$ and $L \leq p$ on X.

Note: $\lambda(y) = \ell(y) + i\ell(-iy) \quad \forall y \in Y$ since:

$$\ell(-iy) = \operatorname{Re} \lambda(-iy) \overset{\lambda \mathbb{C}\text{-lin}}{=} \operatorname{Re}[-i\lambda(y)] = \operatorname{Im} \lambda(y)$$

Define $\Lambda(x) := L(x) + iL(-ix)$, $x \in X$.

Then, by Step 2,

(i) Λ is $\mathbb{R}$-linear on X

(ii) $\Lambda|_Y = \lambda$

(iii) $\operatorname{Re} \Lambda \overset{L(-ix) \in \mathbb{R}}{=} L \leq p$ on X

Also, Λ is $\mathbb{C}$-linear, since if is $\mathbb{R}$-linear and

$$\Lambda(ix) = L(ix) + iL(x) \overset{L \mathbb{R}\text{-lin}}{=} i(L(x) + iL(-ix)) = i\Lambda(x),$$

proving the main part for $\mathbb{K} = \mathbb{C}$.

Step 4: Addendum: We fix $x \in X$ and use the polar repr.

$$\Lambda(x) = |\Lambda(x)| e^{i\theta(x)} \quad [\text{If } \mathbb{K} = \mathbb{R} \text{ then } e^{i\theta(x)} \in \{-1, 1\}]$$

Then $|\Lambda(x)| = e^{-i\theta(x)} \Lambda(x) \overset{\mathbb{K}\text{-lin}}{=} \Lambda(e^{-i\theta(x)} x) \overset{\text{LHS} \in \mathbb{R}}{=} \operatorname{Re} \Lambda(e^{-i\theta(x)} x)$

$\overset{\text{main part}}{\leq} p(e^{-i\theta(x)} x) \overset{(**)}{\leq} p(x)$ ∎

Corollary 4.5 Let X be a normed space, $Y \subseteq X$ a subspace, and $\varphi \in Y^*$. Then there exists $f \in X^*$ with $f|_Y = \varphi$ and $\|f\|_{X^*} = \|\varphi\|_{Y^*}$.

Pf: Apply Thm. 4.3 with $p(x) := \|\varphi\|_{Y^*} \cdot \|x\|$ for $x \in X$ (fulfills assumptions - check!) to $\lambda = \varphi$. Then $\exists f: X \to \mathbb{K}$ linear with $|f(x)| \leq \|\varphi\|_{Y^*}\|x\|$, hence $\|f\|_{X^*} \leq \|\varphi\|_{Y^*}$.

But $$\|f\|_{X^*} = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \geq \sup_{0 \neq x \in Y} \frac{|\varphi(x)|}{\|x\|} = \|\varphi\|_{Y^*}$$ ∎

Corollary 4.6 Let X be a normed space and let $0 \neq x_0 \in X$. Then there exists $f \in X^*$ with $f(x_0) = \|x_0\|$ and $\|f\|_{X^*} = 1$.

Pf: Let $Y = \operatorname{span}\{x_0\}$. If $y \in Y$ then $y = \alpha x_0$ for some unique $\alpha \in \mathbb{K}$. Define φ on Y by $\varphi(y) := \alpha \cdot \|x_0\|$.

This implies $\varphi(x_0) = \|x_0\|$ and $\varphi \in Y^*$ with $\|\varphi\|_{Y^*} = 1$, since $|\varphi(y)| = \|y\|$. By Cor. 4.5, $\exists f: X \to \mathbb{K}$ linear, with $f|_Y = \varphi$ (in particular, $f(x_0) = \|x_0\|$) and $\|f\|_{X^*} = \|\varphi\|_{Y^*} = 1$ ∎

Corollary 4.7 Let X be a normed space, $Z \subseteq X$ a closed subspace, and $x_0 \in X \setminus Z$ with $0 < \operatorname{dist}(x_0, Z) =: d$. Then there exists $f \in X^*$ with $f|_Z = 0$, $f(x_0) = d$ and $\|f\|_{X^*} = 1$.

Proof: See exercise.

4.2 Three consequences of Baire's theorem

Theorem 4.8 (Banach-Steinhaus; Uniform Boundedness Principle)

Let X be a Banach space, Y a normed (!) space, and $\mathcal{F} \subseteq BL(X,Y)$.

If $$\sup_{T\in\mathcal{F}} \|Tx\| < \infty \quad \forall x \in X$$

then $$\sup_{T\in\mathcal{F}} \|T\| < \infty .$$

Pf: For $n \in \mathbb{N}$, define

$$A_n := \{x \in X \mid \|Tx\| \le n \ \forall T \in \mathcal{F}\} = \bigcap_{T\in\mathcal{F}} T^{-1}(\overline{B_n^Y(0)}) \quad \left(\begin{array}{c}\text{closed, since}\\ T \text{ cont.}\end{array}\right)$$

Then, by hypothesis, $X = \bigcup_{n\in\mathbb{N}} A_n$. By Cor 1.49 (consequ. Baire) there exists $n_0 \in \mathbb{N}$ s.t. A_{n_0} is not no-where dense.

Since A_{n_0} is also closed: $\exists\, x_0 \in A_{n_0}$ & $r > 0$: $B_r(x_0) \subseteq A_{n_0}$.

Now let $0 \neq x \in X$ and $T \in \mathcal{F}$, then

$$\frac{r}{2\|x\|}\|Tx\| \le \|T(\underbrace{\tfrac{r}{2\|x\|}x + x_0}_{\in B_r(x_0)\subseteq A_{n_0}})\| + \|Tx_0\| \le n_0 + \|Tx_0\|,$$

hence,

$$\|T\| \le \frac{2}{r}(n_0 + \|Tx_0\|).$$ Taking the supremum over all $T \in \mathcal{F}$ proves the claim. ∎

Theorem 4.9 (Open mapping theorem) Let X, Y be Banach spaces. Let $T \in BL(X,Y)$ be onto (surjective). Then T is open (an open map), i.e.:

$$A \subseteq X \text{ open} \Rightarrow T(A) \subseteq Y \text{ open}$$

Pf: The thm. follows from 3 claims:

Claim 1: $\exists r>0 : T(B_r^X(0))$ has non-empty interior
$\Rightarrow T$ is open (i.e. claim is "$\Rightarrow$" holds)

Pf: Assume $T(B_r^X(0))$ has non-empty interior.

Let $y := Tx$ for some $x \in B_r^X(0)$ be an interior point of $T(B_r^X(0))$, i.e., $\exists r_y : B_{r_y}^Y(y) \subseteq T(B_r^X(0))$.

Note: $B_r^X(0) \subseteq B_\delta^X(x)$ for some $\delta > 0$ (large enough, e.g. $3r$)

Hence, $B_{r_y}^Y(y) \subseteq T(B_\delta^X(x))$. (*)

By scaling, translation and linearity, we get $\forall r' > 0$:

$$T(B_{r'}^X(x)) = T(B_{r'}^X(0)+x) = T\left(\frac{r'}{\delta} B_\delta^X(0)+x\right)$$

$$= \frac{r'}{\delta} T(B_\delta^X(0)) + Tx = \frac{r'}{\delta} T(B_\delta^X(x)-x) + Tx$$

$$(*) \geq \frac{r'}{\delta}\left(B_{r_y}^Y(y) - \underbrace{Tx}_{=y}\right) + Tx = \frac{r'}{\delta} B_{r_y}^Y(0) + Tx$$

$$= B_{\frac{r' r_y}{\delta}}^Y(0) + Tx = B_{\frac{r' r_y}{\delta}}^Y(y). \quad (**)$$

Now, let $A \subseteq X$ be open, let $y' \in T(A)$ be arbitrary, and choose $x' \in A$ s.t. $y' = Tx'$. Since A is open, $\exists r' > 0$: $B_{r'}^X(x') \subseteq A$, hence

$$T(A) \supseteq T(B_{r'}^X(x')) = T(B_{r'}^X(x) + x' - x) \overset{(**)}{\supseteq} B_{\frac{r' r_y}{\delta}}^Y(y) + y' - y = B_{\frac{r' r_y}{\delta}}^Y(y').$$

i.e. $T(A)$ is open, so Claim 1 holds ✓

From now on: center of all balls is 0 (unless otherwise noted) and we drop the center from the notation

Claim 2: $\exists \varepsilon > 0$:

$$B_\varepsilon^Y \subseteq \overline{T(B_1^X)} \quad (***)$$

Pf: $Y \overset{\text{onto}}{=} T(X) = T\left(\bigcup_{n\in\mathbb{N}} B_n^X\right) = \bigcup_{n\in\mathbb{N}} T(B_n^X)$.

Now, Y is complete, so we can apply Baire's Thm (in form of Cor. 1.49): Then exists $n\in\mathbb{N}$: $T(B_n^X)$ is not no-where dense, i.e., $\exists\, y \in \overline{T(B_n^X)}$ and $\varepsilon > 0$: $B_\varepsilon^Y(y) \subseteq \overline{T(B_n^X)}$, or, equivalently, $B_\varepsilon^Y \subseteq \overline{T(B_n^X)} - y$.

We have:

(i) $\exists\, (x_k)_{k\in\mathbb{N}} \subseteq B_n^X$ s.t. $y = \lim_{k\to\infty} Tx_k$

(ii) $\forall k\in\mathbb{N}$: $\overline{T(B_n^X)} - Tx_k = \overline{T(B_n^X - x_k)} \subseteq \overline{T(B_{2n}^X)}$

Hence $B_\varepsilon^Y \subseteq \overline{T(B_{2n}^X)}$, so $B_{\frac{\varepsilon}{2n}}^Y = \frac{1}{2n} B_\varepsilon^Y \subseteq \frac{1}{2n}\overline{T(B_{2n}^X)} = \overline{T\left(\frac{1}{2n} B_{2n}^X\right)}$

$= \overline{T(B_1^X)}$. ✓

Claim 3: $\overline{T(B_1^X)} \subseteq T(B_2^X)$

Pf: Let ε be as in Claim 2. Let $y \in \overline{T(B_1^X)}$.

Then there exists $x_1 \in B_1^X$ such that

$$y - Tx_1 \in B_{\varepsilon/2}^Y \overset{(***)}{\subseteq} \overline{T(B_{1/2}^X)}.$$

Similarly, there exists $x_2 \in B_{1/2}^X$ such that

$$y - Tx_1 - Tx_2 = (y - Tx_1) - Tx_2 \in B_{\varepsilon/4}^Y \overset{(***)}{\subseteq} \overline{T(B_{1/4}^X)}$$

Inductively we get: $\forall n\in\mathbb{N}\ \exists\, x_n \in B_{\frac{1}{2^{n-1}}}^X$ s.t.

$$y - \sum_{j=1}^{n} Tx_j \in B_{\varepsilon/2^n}^Y \qquad (****)$$

Since $\sum_{n\in\mathbb{N}} \underbrace{\|x_n\|}_{<2^{-(n-1)}} < 2 < \infty$, and X is a Banach space,

$\sum_{j\in\mathbb{N}} x_j := x \in X$ exists by Lemma 2.50(a). As T is cont., we have $Tx = \sum_{j\in\mathbb{N}} Tx_j$. Using $(****)$ and the continuity of $\|\cdot\|$,

$$\|y - Tx\| = \lim_{n\to\infty} \|y - \sum_{j=1}^{n} Tx_j\| = 0, \text{ hence } y = Tx,\ \|x\| < 2$$

i.e. $y \in T(B_2^X)$ ∎

Corollary 4.10 (Inverse mapping theorem)

Let X, Y be Banach spaces, and $T \in BL(X,Y)$ a bijection. Then $T^{-1} \in BL(Y,X)$

Pf: Clearly, T^{-1} exists and is linear (recall 2.25). By Thm. 4.9, T is open, that is, T^{-1} is continuous ∎

Definition 4.11 Let X, Y be normed spaces, and $T : X \supseteq \mathrm{dom}(T) \to Y$ a linear operator.

(a) Graph of T:

$$G(T) := \{(x, Tx) \subseteq X \times Y \mid x \in \mathrm{dom}(T)\}$$

We equip $X \times Y$ with the norm

$$\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$$

Then: X, Y complete $\Rightarrow$ $X \times Y$ complete.

(b) T is closed (operator) $:\Leftrightarrow$ $G(T) \subseteq X \times Y$ is closed (in $X \times Y, \|\cdot\|_{X\times Y}$)

Remark 4.12

(a) T is closed if and only if the following implication holds:

$$(x_n)_n \subseteq \mathrm{dom}(T) \text{ with } x_n \xrightarrow{n\to\infty} x \in X \;\&\; Tx_n \xrightarrow{n\to\infty} y \in Y$$
$$\Rightarrow \quad x \in \mathrm{dom}(T) \text{ and } y = Tx$$

(b) Compare (a) with the definition of T (seq.) cont., where convergence of $(Tx_n)_n$ must be proved. Here, it is given / assumed!

Theorem 4.13 (Closed graph theorem)

Let X, Y be Banach spaces and $T: X \supseteq \mathrm{dom}\, T \to Y$ a closed linear operator. Then

$$\mathrm{dom}(T) \text{ closed} \iff T \text{ bounded}.$$

Pf: "$\Leftarrow$": Let $(x_n)_n \subseteq \mathrm{dom}\, T$ with $x_n \xrightarrow{n\to\infty} x \in X$. Then $(x_n)_n$ is Cauchy in X. By hypothesis, T is bounded, so $(Tx_n)_n$ is Cauchy in Y. But Y is complete, so $\exists y \in Y: Tx_n \xrightarrow{n\to\infty} y$ and, since T is closed, it follows that $x = \lim_{n\to\infty} x_n \in \mathrm{dom}(T)$ (and $Tx = y$).

"$\Rightarrow$": Define the projection $P_1: G(T) \to \mathrm{dom}(T)$, $(x, Tx) \mapsto x$

(i) $G(T)$ is closed in $X \times Y$, hence $G(T)$ is a Banach space.

(ii) $\mathrm{dom}(T)$ is closed in X (by hypothesis), hence $\mathrm{dom}(T)$ is a Banach space.

(iii) P_1 is a bijection

(iv) P_1 is bounded: Let $z = (x, Tx) \in G(T)$, then

$$\|P_1 z\|_X = \|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|z\|_{X\times Y}.$$

So $\|P_1\| \leq 1$

By the inverse mapping theorem $P_1^{-1}: \mathrm{dom}(T) \to G(T)$, $x \mapsto (x, Tx)$ is also bounded, hence:

$\exists c < \infty: \|P_1^{-1} x\|_{X\times Y} \leq c\|x\|_X$. Since $\|P_1^{-1}x\|_{X\times Y} = \|x\|_X + \|Tx\|_Y$ this implies

$$\|Tx\|_Y \leq (c+1)\|x\|_X$$

So T is bounded ∎

Example 4.14 Let $X = Y = C_0(\mathbb{R})$ with supremum norm (Banach!)

Consider $T := \frac{d}{dx} : \operatorname{dom}(T) \to C_0(\mathbb{R})$, $f \mapsto f'$

with $\operatorname{dom}(T) = \{ f \in C_0(\mathbb{R}) \mid f \in C^1(\mathbb{R}) \text{ and } f' \in C_0(\mathbb{R}) \} \subsetneq C_0(\mathbb{R})$.

Claim: T is closed (a closed operator).

Pf: Let $(f_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}(T)$ be a sequence such that:

(i) $\exists g \in C_0(\mathbb{R}) : \|f_n - g\|_\infty \xrightarrow{n\to\infty} 0$

(ii) $\exists h \in C_0(\mathbb{R}) : \|f_n' - h\|_\infty \xrightarrow{n\to\infty} 0$

i.e., $(f_n, f_n')_n \in G(T)$, $(f_n, f_n')_n \xrightarrow{n\to\infty} (g, h)$ in $X \times Y$; we need to prove that $(g,h) \in G(T)$, equivalently, $g \in \operatorname{dom}(T)$ and $h = g'$.

By uniform convergence, we can exchange limits, so, $\forall x \in \mathbb{R}$,

$$\int_0^x h(t)\,dt = \int_0^x \lim_{n\to\infty} f_n'(t)\,dt = \lim_{n\to\infty} \underbrace{\int_0^x f_n'(t)\,dt}_{f_n(x) - f_n(0)} = g(x) - g(0)$$

(since $f_n \to g$ pointwise, by (i)).

Hence, $g(x) = g(0) + \int_0^x h(t)\,dt \quad \forall x \in \mathbb{R}$

So, by the Fundamental Theorem of Calculus (HDI): $g \in C^1(\mathbb{R})$ with $g' = h \in C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$, we have $g \in \operatorname{dom}(T)$, and so $(g,h) = (g,g') \in G(T)$. So T is closed.

Also: Since T is unbounded (cf. 2.28) $\overset{\text{Thm. 4.13}}{\Rightarrow}$ $\operatorname{dom}(T)$ is not a closed subspace of $C_0(\mathbb{R})$ wrt. $\|\cdot\|_\infty$.

4.3 (Bi-) Dual spaces and weak topologies

Theorem 4.15 Let X be a normed space. Then, for every $x \in X$,

$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*}$$

Pf: See exercise.

Definition 4.16 Let X be a normed space. We call

$X^{**} := (X^*)^*$ the bidual space of X (always Banach!)

More generally, introduce the n-fold dual space

$X^{\overbrace{*\cdots*}^{n \text{ times}}} := (X^{\overbrace{*\cdots*}^{n-1 \text{ times}}})^*$ for $n \in \mathbb{N}$, recursively.

Theorem (& definition) 4.17 Let X be a normed space.

The canonical embedding $J : \begin{matrix} X \to X^{**} \\ x \mapsto Jx \end{matrix}$ with $Jx : \begin{matrix} X^* \to \mathbb{K} \\ f \mapsto f(x) \end{matrix}$

is well-defined, linear, and isometric.

If J is surjective, we call X reflexive.

Remarks 4.18 (a) Hilbert spaces are reflexive (Riesz!)
(b) Every finite dimensional normed space is reflexive (use the dual basis)
(c) ℓ^p, L^p are reflexive for $p \in (1,\infty)$; c_0 & ℓ^1 are not refl.
(d) Milman-Pettis Thm: X uniformly convex & Banach $\Rightarrow X$ reflexive
(e) X reflexive $\Rightarrow X$ complete.

Pf (4.17): J is well-defined (i.e. $Jx \in X^{**}\ \forall x \in X$):

Let $\alpha, \beta \in \mathbb{K}$, $f, g \in X^*$, then

$$(Jx)(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha (Jx)(f) + \beta (Jx)(g)$$

So, Jx is linear. Also, Jx is bounded, since

$$\|Jx\|_{**} = \sup_{0 \neq f \in X^*} \frac{|(Jx)(f)|}{\|f\|_*} = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*} \overset{\text{Thm. 4.15}}{=} \|x\| < \infty.$$

Hence, $Jx \in X^{**}$, so J is well-defined – and, an isometry.

Also, J is linear: Let $\alpha, \beta \in \mathbb{K}$, $x, y \in X$, then, for all $f \in X^*$,

$$(J(\alpha x + \beta y))(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha (Jx)(f) + \beta (Jy)(f),$$

and so $J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$ ∎

Theorem 4.19 Let X be a Banach space. Then

$$X \text{ reflexive} \iff X^* \text{ reflexive}$$

Pf. "$\Rightarrow$": $(X^*)^{**} = ((X^*)^*)^* = (X^{**})^* = X^*$

"$\Leftarrow$": See f. ex. Werner, "Funktionalanalysis", 8. Aufl., Kap. III 3.4.

Theorem 4.20 Let X be a normed space. Then

$$X^* \text{ separable} \Rightarrow X \text{ separable.}$$

Remark 4.21 $X = \ell^1$ shows that "$\Leftarrow$" does not hold, since $\ell^\infty \simeq (\ell^1)^*$, which is not separable.

Pf (4.20): Let $A := \{f_n \in X^* \mid n \in \mathbb{N}\}$ be dense (exists by hyp.).

For $n \in \mathbb{N}$ choose $x_n \in X$ s.t. $\|x_n\| = 1$ & $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|_*$.

Let $D := \operatorname{span}_{\mathbb{K}}\{x_n \mid n \in \mathbb{N}\} \subseteq X$.

Claim: D is dense in X (Note: D is not countable, but a $\mathbb{K}$-linear subspace).

Pf (claim): Assume (for contradiction) D not dense.

Then there exists $z \in X$ s.t. $\operatorname{dist}(z, D) > 0$. By Cor. 4.7 (H-B!)

$\exists f \in X^*: f|_D = 0$, $f(z) > 0$. Now:

(i) A dense in X^* implies: $\exists$ seq. $(f_{n_k})_k \subseteq A$:

$$\|f_{n_k} - f\|_* \xrightarrow{k\to\infty} 0 \quad (\boxplus)$$

(ii) We have

$$\|f_{n_k} - f\|_* \geq |f_{n_k}(x_{n_k}) - f(x_{n_k})| \overset{f|_D = 0}{=} |f_{n_k}(x_{n_k})| \geq \frac{1}{2}\|f_{n_k}\|_*$$

Then $\|f_{n_k}\|_* \xrightarrow{k\to\infty} 0$, so $f_{n_k} \xrightarrow{k\to\infty} 0$ in X^*, hence $f = 0$ by $(\boxplus)$ ↯.

So, D is dense.

Define $\widetilde{\mathbb{K}} := \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$, resp. $\widetilde{\mathbb{K}} := \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$, and set

$\tilde{D} := \operatorname{span}_{\mathbb{K}}\{x_n \mid n \in \mathbb{N}\}$. Then $\tilde{D}$ is countable & dense in X ∎

Definition 4.22 Let X be a $\mathbb{K}$-vector space, let $\{p_\alpha\}_{\alpha \in I}$ be a family of seminorms on X

(a) $\{p_\alpha\}_{\alpha \in I}$ is separating $:\Leftrightarrow \forall 0 \neq x \in X : \exists \alpha \in I : p_\alpha(x) > 0$

(b) For given $\alpha \in I$, $r > 0$, let

$$U_{\alpha,r} := \{y \in X \mid p_\alpha(y) < r\} \ni 0.$$

Also, for $x \in X$, let

$$U_{\alpha,r}(x) := x + U_{\alpha,r} = \{y \in X \mid p_\alpha(y-x) < r\} \ni x.$$

Let

$$\mathcal{N}_x := \left\{ \bigcap_{j=1}^{n} U_{\alpha_j, r_j}(x) \;\middle|\; n \in \mathbb{N}, \alpha_j \in I, r_j > 0 \text{ for } j = 1, \dots, n \right\}$$

(family of finite intersections of $U_{\alpha,r}(x)$'s)

Define the locally convex topology (l.c.t.) on X induced by $\{p_\alpha\}_{\alpha \in I}$:

$Y \subseteq X$ open in the l.c.t. $:\Leftrightarrow \forall x \in Y \; \exists V_x \in \mathcal{U}_x : V_x \subseteq Y$

$$\Leftrightarrow \forall x \in Y \; \exists V_x \in \mathcal{N}_x : Y = \bigcup_{x \in Y} V_x$$

Remark 4.23 (a) $U_{\alpha,r}(x)$ is open in l.c.t., and $\mathcal{N}_x$ is a neighbourhood base of the l.c.t. at x.

(b) If $\{p_\alpha\}_{\alpha \in I}$ is separating, then the l.c.t. is Hausdorff (see exercise).

(c) The elements of the neighbourhood base are convex sets (hence the name): For $y_1, y_2 \in U_{\alpha,r}$ and $\lambda \in [0,1]$, we have $\lambda y_1 + (1-\lambda) y_2 \in U_{\alpha,r}$, since:

$$p_\alpha(\lambda y_1 + (1-\lambda) y_2) \leq p_\alpha(\lambda y_1) + p_\alpha((1-\lambda) y_2)$$

$$= \lambda \underbrace{p_\alpha(y_1)}_{<r} + (1-\lambda) \underbrace{p_\alpha(y_2)}_{<r} < r.$$

(d) X is a topological vector space wrt. a locally convex topology, i.e., addition of vectors and mult. of a vector by a scalar are continuous wrt. a l.c.t. This relies on $U_{\alpha,r}(x) = x + U_{\alpha,r}$ & $\lambda U_{\alpha,r}(x) = U_{\alpha,|\lambda| r}(\lambda x)$ $\forall x \in X, \lambda \in \mathbb{K}, \alpha \in I, r > 0$.

Notation: vector space topology $:\Leftrightarrow$ a topology making X into a topological vector space.

The composition of a linear map with a norm gives a seminorm (check!). The following abstract result will be applied several times with different choices of S_α and Z:

Lemma 4.24 Let X be a vector space, $(Z, \|\cdot\|_Z)$ a normed space, and $S_\alpha : X \to Z$ a linear map for every index $\alpha \in I$. Let $\mathcal{T}_{l.c.t.}$ be the Locally convex topology on X induced by the family of seminorms $\{x \mapsto \|S_\alpha x\|_Z\}_{\alpha \in I}$. Then:

(a) $\mathcal{T}_{l.c.t.}$ is the coarsest vector space topology on X s.t. For every $\alpha \in I$ the map S_α is continuous.

(b) $(x_k)_{k \in \mathbb{N}} \subseteq X$ converges to $x \in X$ wrt. $\mathcal{T}_{l.c.t.} \Leftrightarrow$

$$\|S_\alpha x_k - S_\alpha x\|_Z \xrightarrow{k \to \infty} 0 \quad \forall \alpha \in I$$

Pf: See exercise.

A very important example is when $S_\alpha \in X^*$ and $Z = \mathbb{K}$:

Definition 4.25 Let X be a normed space. The weak topology on X is the locally convex topology (l.c.t.) on X induced by the family of seminorms $\{x \mapsto |f(x)|\}_{f \in X^*}$.

Lemma 4.26 Let X be a normed space. Then the weak topology on X is

(a) Hausdorff.

(b) the coarsest vector-space topology on X such that every $f \in X^*$ is continuous. In particular, the weak topology is coarser than the norm (=: strong) topology.

(c) identical to the norm topology, if $\dim X < \infty$,

(d) such that $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to x in the weak topology (or, weakly) iff

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X^*$$

Notation: $x_n \overset{w}{\to} x$, or $x_n \overset{n\to\infty}{\rightharpoonup} x$.

Pf: (a) By Rmk 4.23(b), it suffices to check that the family of seminorms is separating. This follows from Corollary 4.6 (H-B!): For every $0 \neq x \in X$ $\exists f_x \in X^*: f_x(x) = \|x\|$.

(b) Lemma 4.24(a) (c) See exercise (d) Lemma 4.24(b). ∎

Remark 4.27 (a) Weak limits are unique by Lemma 4.26(a)

(b) If $\dim X = \infty$, then the weak topology is not 1st countable (hence not metrisable!) (for pf., see exercise)

(c) Strong convergence $\overset{4.26(b)}{\Rightarrow}$ weak convergence (but in general not vice versa!)

(d) In a Hilbert space X (by Lem. 4.26(d) & Riesz):

$$x_n \overset{w}{\to} x \iff \langle y, x_n\rangle \overset{n\to\infty}{\to} \langle y, x\rangle \quad \forall y \in X$$

(e) In ℓ^1: $x_n \overset{w}{\to} x \iff \|x_n - x\|_1 \overset{n\to\infty}{\to} 0$

(Schur, 1921; See also Conway, "A course in FA", 1990, Prop. 5.2)

Example 4.28 Let $X = \ell^p$, $p \in [1,\infty]$, $e_n := (\delta_{nk})_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$.

Then (i) $(e_n)_n$ has no $\|\cdot\|_p$-convergent (i.e., strongly conv.) subsequence.

(ii) If $p = 1$, then $(e_n)_n$ is not weakly convergent:
Recall $(\ell^1)^* \cong \ell^\infty$, and choose $f \leftrightarrow (-1,1,-1,1,-1,\dots) \in \ell^\infty$
then $f(e_n) = (-1)^n \ \forall n \in \mathbb{N}$ which is not convergent as $n \to \infty$ (This is consistent with Rmk 4.27(e)).

(iii) If $1 < p \le \infty$, then $(e_n)_n$ is weakly convergent to 0:
In case $1 < p < \infty$, we have $(\ell^p)^* \cong \ell^q$ (Thm. 2.38) with $1 < q < \infty$. Hence, for $f \leftrightarrow y = (y_k)_k \in \ell^q$:
$f(e_n) = y_n \xrightarrow{n\to\infty} 0$ (since $\sum_{k\in\mathbb{N}} |y_k|^q < \infty$).
In the case $p = \infty$, use also exercises E6.1 & E6.3.

Lemma 4.29 Let X be a normed space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, $x \in X$, and $x_n \overset{w}{\to} x$. Then

(a) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$, (b) $\|x\| \le \liminf_{n\to\infty} \|x_n\|$

Pf (a): Let $x_n \overset{w}{\to} x$, so $f(x_n) \overset{n\to\infty}{\to} f(x)$ $\forall f \in X^*$, hence

$$\forall f \in X^* \text{ (fixed)}: \quad \sup_{n\in\mathbb{N}} \underbrace{|f(x_n)|}_{(Jx_n)(f)} < \infty,$$

with the canonical embedding $J : X \to X^{**}$. Since X^* is Banach, the Uniform Boundedness Principle (Thm. 4.9) gives

$$\sup_{n\in\mathbb{N}} \underbrace{\|Jx_n\|_{**}}_{=\|x_n\|} < \infty. \quad \checkmark$$

(b) By Cor. 4.6 (H-B!):

$$\exists f_x \in X^* : \|f_x\|_* = 1 \text{ and } f_x(x) = \|x\|$$

Then

$$\|x\| = |f_x(x)| = \lim_{n\to\infty} |f_x(x_n)| = \liminf_{n\to\infty} |f_x(x_n)| \le \liminf_{n\to\infty} \underbrace{\|f_x\|_*}_{=1} \|x_n\| \quad \blacksquare$$

Theorem 4.30 Let X be a normed space. Then $x_n \overset{w}{\to} x$ iff the following two statements hold:

(i) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$

(ii) $\exists F \subseteq X^*$ with $\operatorname{span}(F)$ dense (wrt. $\|\cdot\|_*$) in X^* s.t.

$$\forall f \in F: \quad \lim_{n\to\infty} f(x_n) = f(x). \qquad (*)$$

Pf: "$\Rightarrow$": From Lem. 4.29 and by definition of weak convergence

"$\Leftarrow$": Use an $\frac{\varepsilon}{3}$-argument: Let $\varepsilon > 0$ and $g \in X^*$. Let

$K := \frac{1}{2}(\|x\| + \sup_{n\in\mathbb{N}} \|x_n\|) < \infty$. Since $\operatorname{span}(F)$ is dense in X^*, there exists $f \in \operatorname{Span}(F)$: $\|f - g\|_* < \frac{\varepsilon}{3K}$. Also, for $f \in \operatorname{span}(F)$, $\exists N \in \mathbb{N}$: $\forall n \ge N$: $|f(x_n) - f(x)| < \frac{\varepsilon}{3}$

(Note: $(*)$ holds also for $f \in \operatorname{span}(F)$ as finite lin. comb.)

Hence, $\forall n \ge N$:

$$|g(x) - g(x_n)| \le |g(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - g(x_n)|$$

$$\le \underbrace{\|g - f\|_*}_{< \frac{\varepsilon}{3K}} \underbrace{(\|x\| + \|x_n\|)}_{\le 2K} + \underbrace{|f(x) - f(x_n)|}_{< \frac{\varepsilon}{3}} < \varepsilon \quad \blacksquare$$

Theorem 4.31 (Eberlein-Šmulian) Let X be a Banach space and $A \subseteq X$. Then A weakly compact $\Leftrightarrow$ A weakly sequentially compact.

Pf. Not here; see Whitley, Math. Ann. 172, 116-118 (1967). $\blacksquare$

Definition 4.32 Let X be a normed space. Then the weak* topology on X^* ("weak-star") is the locally convex topology on X^* induced by the family of seminorms $\{f \mapsto |f(x)|\}_{x\in X}$

Lemma 4.33 Let X be a normed space. Then the weak* topology is (a) Hausdorff

(b) The coarsest vector-space topology on X^* s.t. $\forall x \in X$, the map

$$X^* \to \mathbb{K}, \quad f \mapsto f(x) \quad \text{is continuous}$$

(c) coarser than the weak topology on X^*, and the two coincide iff X is reflexive.

(d) such that $(f_n)_n \subseteq X^*$ converges to $f \in X^*$ in the weak*-top. iff $f_n(x) \xrightarrow{n\to\infty} f(x)$ $\forall x \in X$

Notation: $f_n \xrightarrow{w^*} f$.

Pf: (a) $\{f \mapsto |f(x)|\}_{x\in X}$ is a separating family of seminorms:

For $f \neq 0$ there exists $x \in X$ with $f(x) \neq 0$.

(b) Lemma 4.24(a) with $Z = \mathbb{K}$ and $S_x = Jx$ ($J: X \to X^{**}$ canonical embedding)

(c) $J(X) \subseteq X^{**}$ with equality iff X is reflexive.

(d) Lemma 4.24 (b).

The next theorem is the analogue of Lem. 4.29 & Thm. 4.30. However: Note that here, X must be complete in order to apply the Uniform Boundedness Principle in (a) (i).

Theorem 4.34 Let X be a normed space, $f \in X^*$, $(f_n)_n \subseteq X^*$

(a) If $f_n \xrightarrow{w^*} f$, then

(i) If X is even a Banach space, then $\sup_{n\in\mathbb{N}} \|f_n\|_* < \infty$

(ii) $\|f\|_* \leq \liminf_{n\to\infty} \|f_n\|_*$

(b) If (i) $\sup_{n\in\mathbb{N}} \|f_n\|_* < \infty$

(ii) $\exists A \subseteq X$ with $\operatorname{span}(A)$ dense in X (wrt. $\|\cdot\|$) s.t.

$$\forall x \in A: \quad f_n(x) \xrightarrow{n\to\infty} f(x)$$

then $f_n \xrightarrow{w^*} f$. (If X Banach, the reverse holds by (a)(i))

Pf: See exercise ☒

Theorem 4.35 (Banach-Alaoglu) Let $\mathbb{X}$ be a normed space. Then the closed unit ball in $\mathbb{X}^*$, $\overline{B}_1^* := \{f \in \mathbb{X}^* \mid \|f\|_* \leq 1\}$ is compact in the weak* topology.

Pf: Equip the set of maps $\{f : \mathbb{X} \to \mathbb{K}\} = \mathbb{K}^{\mathbb{X}}$ with the product topology $\mathcal{T}_{prod} := \bigtimes_{x \in \mathbb{X}} \mathcal{T}_{\mathbb{K}}$ of the Euclidean topology $\mathcal{T}_{\mathbb{K}}$ on $\mathbb{K}$. Then:

(i) $\mathcal{T}_{prod}$ is the coarsest topology on $\mathbb{K}^{\mathbb{X}}$ such that the projection/evaluation map $\Pi_x : \mathbb{K}^{\mathbb{X}} \to \mathbb{K}$, $f \mapsto f(x)$ is continuous for all $x \in \mathbb{X}$ (see T2.3, Tut Sheet 2)

(ii) $\mathcal{T}_{prod}$ is a vector-space topology on $\mathbb{K}^{\mathbb{X}}$:

Addition: $\mathcal{A} : \mathbb{K}^{\mathbb{X}} \times \mathbb{K}^{\mathbb{X}} \to \mathbb{K}^{\mathbb{X}}$, $(f,g) \mapsto f+g$. It suffices to prove openness of $\mathcal{A}^{-1}(U)$ in $\mathbb{K}^{\mathbb{X}} \times \mathbb{K}^{\mathbb{X}}$ for U in a base of $\mathcal{T}_{prod}$, that is, for $U = \bigtimes_{x \in \mathbb{X}} U_x$ with $U_x \in \mathcal{T}_{\mathbb{K}}$, and $U_x = \mathbb{K}$ for all but at most finitely many $x \in \mathbb{X}$. Addition in the field $\mathbb{K}$, $\mathcal{A}_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(z,z') \mapsto z+z'$, is continuous. Hence, $\mathcal{A}^{-1}(U) = \bigtimes_{x \in \mathbb{X}} \mathcal{A}_{\mathbb{K}}^{-1}(U_x)$ with

$$\mathcal{A}_{\mathbb{K}}^{-1}(U_x) \subseteq \mathcal{T}_{\mathbb{K}} \times \mathcal{T}_{\mathbb{K}} \quad \forall x \in \mathbb{X}$$

and $\mathcal{A}_{\mathbb{K}}^{-1}(U_x) = \mathbb{K} \times \mathbb{K}$ except for at most finitely many $x \in \mathbb{X}$.

Hence, $\mathcal{A}^{-1}(U) \in \mathcal{T}_{prod} \times \mathcal{T}_{prod}$, as finite intersection of a union of product sets $V_1 \times V_2$ with each $V_j \in \mathcal{T}_{prod}$, $j = 1,2$, being a product set of factors all of which are equal to $\mathbb{K}$ except for one. ✓ The case of scalar mult. is analogous & simpler.

(iii) Given a linear subspace $S \subseteq \mathbb{K}^{\mathbb{X}}$, the subspace topology $\mathcal{T}_{prod}|_S := \{U \cap S \mid U \in \mathcal{T}_{prod}\}$ is a vector space topology on S wrt. which the restricted eval. maps $\Pi_x|_S$ are continuous $\forall x \in \mathbb{X}$

Now, choose $S = X^*$, and let $\mathcal{T}_{w*}$ denote the weak* top. on X^*. By 4.33(b), $\mathcal{T}_{w*} \subseteq \mathcal{T}_{prod}|_{X^*}$. Therefore, the map

$$\begin{aligned} \sigma : (X^*, \mathcal{T}_{prod}|_{X^*}) &\longrightarrow (X^*, \mathcal{T}_{w*}) \\ f &\longmapsto f \end{aligned}$$

is continuous, and the theorem follows if we prove: $\bar{B}_1^*$ is $\mathcal{T}_{prod}$-compact: We claim: Then it is $\mathcal{T}_{prod}|_{X^*}$-compact (and hence, by continuity of σ, $\mathcal{T}_{w*}$-compact)

Pf Claim: Consider a $\mathcal{T}_{prod}|_{X^*}$-open cover $\bigcup_\alpha V_\alpha$ of $\bar{B}_1^*$ and use that, for $V_\alpha \in \mathcal{T}_{prod}|_{X^*}$, there exists $U_\alpha \in \mathcal{T}_{prod}$ with $V_\alpha = U_\alpha \cap X^*$. The $\mathcal{T}_{prod}$-open cover $\bigcup_\alpha U_\alpha$ has a finite subcover (by assumption), and intersection with X^* gives finite subcover of $\bigcup_\alpha V_\alpha$. ✓ (claim).

To prove $\mathcal{T}_{prod}$-compactness: Define $A := \bigtimes_{x \in X} A_x$, where $A_x := \{z \in \mathbb{K} \mid |z| \le \|x\|\}$ is compact in $\mathbb{K}$. By Tychonoff (Thm. 7.28), A is $\mathcal{T}_{prod}$-compact. Note that: $f \in A \Leftrightarrow f(x) \in A_x \ \forall x \in X$

$\Leftrightarrow \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \le 1$. For $x, y \in X$, $\alpha, \beta \in \mathbb{K}$, let

$$L_{x,y,\alpha,\beta} := \{f \in \mathbb{K}^X \mid \underbrace{f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)}_{= (\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y)(f)} = 0\}$$

$$= (\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y)^{-1}(\{0\}).$$

But $\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y$ is $\mathcal{T}_{prod}$-$\mathcal{T}_{\mathbb{K}}$-continuous, and $\{0\} \subseteq \mathbb{K}$ is $\mathcal{T}_{\mathbb{K}}$-closed, hence $L_{x,y,\alpha,\beta}$ is $\mathcal{T}_{prod}$-closed in $\mathbb{K}^X$, and so also

$$L := \bigcap_{\substack{x,y \in X \\ \alpha,\beta \in \mathbb{K}}} L_{x,y,\alpha,\beta} = \{f : X \to \mathbb{K} \text{ is linear}\}$$

is closed.

Finally, $\bar{B}_1^* = A \cap L$ is $\mathcal{T}_{prod}$-compact (as closed subset of the compact A) ∎

Theorem 4.36 (Helly; version 2 of Banach-Alaoglu)

Let X be a separable normed space. Then $\overline{B}_1^*$ is weak* sequentially compact.

Pf: Let $\{x_k \mid k \in \mathbb{N}\} \subseteq X$ be (countable) dense in X. We need to prove: Any sequence $(f_n)_n \subseteq \overline{B}_1^*$ has a weak*-convergent subsequence: Fix $k \in \mathbb{N}$ arbitrary. Consider the sequence $(f_n(x_k))_{n \in \mathbb{N}} \subseteq \mathbb{K}$. This sequence is bounded, since

$$|f_n(x_k)| \leq \underbrace{\|f_n\|_*}_{\leq 1} \|x_k\| \leq \|x_k\| \quad (k \text{ fix})$$

for all k fixed, $(f_n(x_k))_n$ has a convergent subsequence (in $\mathbb{K}$).

Claim: Then exist a common subsequence $(n_j)_j \subseteq \mathbb{N}$:

$$\forall k \in \mathbb{N}: \quad (f_{n_j}(x_k))_{j \in \mathbb{N}} \text{ is convergent.}$$

Pf: Use Cantor's diagonal sequence trick:

There exists $(n_j^{(1)})_j \subseteq \mathbb{N}$ such that $(f_{n_j^{(1)}}(x_1))_j$ converges.

Then there exists $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$ s.t. $(f_{n_j^{(2)}}(x_2))_j$ converges

Continuing this procedure, there exists $(n_j^{(k+1)})_j \subseteq (n_j^{(k)})_j$ such that $(f_{n_j^{(k+1)}}(x_{k+1}))_{j \in \mathbb{N}}$ converges. The claim then holds with $n_j := n_j^{(j)}$.

Now, define $g(x) := \lim_{j \to \infty} f_{n_j}(x) \quad \forall x \in \operatorname{span}\{x_k \mid k \in \mathbb{N}\} =: \operatorname{dom}(g)$

(i) $\operatorname{dom}(g)$ is a dense subspace of X with respect to $\|\cdot\|$

(ii) $g: \operatorname{dom}(g) \to \mathbb{K}$ is linear

(iii) g is bounded: $|g(x)| = \lim_{j \to \infty} \underbrace{|f_{n_j}(x)|}_{\leq \|f_{n_j}\|_* \|x\| \leq 1} \leq \|x\|$

Since $\mathbb{K}$ is complete, we can apply the Bounded Linear Ext. Thm. 2.31:

There exists $\tilde{g}: X \to \mathbb{K}$ such that $\tilde{g}|_{\operatorname{dom}(g)} = g$ & $\|\tilde{g}\|_* = \|g\|_* \leq 1$

Hence, $\tilde{g} \in \overline{B}_1^*$. Then, by Thm. 4.34(b), $f_{n_j} \xrightarrow{w^*} \tilde{g}$ as $j \to \infty$ ∎

Theorem 4.37 (Kakutani, version 3 of Banach-Alaoglu)

Let X be a Banach space. Then

X reflexive $\iff$ $\overline{B}_1 := \{x \in X \mid \|x\| \leq 1\}$ is weakly compact

Pf: "$\Rightarrow$": By Thm 4.19, $X^* =: Y$ is reflexive. Applying 4.33(c) to Y, we get: The weak topology on $Y^* = (X^*)^* = X$ coincides with the weak* topology on $Y^* = X$, hence the claim follows from Thm. 4.35 applied to Y. ∎

"$\Leftarrow$": See f.ex. Werner, "Funktionalanalysis", 8. Aufl, Satz VIII.3.18. ∎

Example 4.38 Compactness of $\overline{B}_1$ in different spaces.

Here, $p, q \in (1,\infty)$ are Hölder conjugate

	weak	weak*	seq. weak	seq. weak*
$\ell^1 \; (\cong c_0^*)$	no	yes	no	yes
$\ell^p \; (\cong (\ell^q)^*)$	yes	yes	yes	yes
$\ell^\infty \; (\cong (\ell^1)^*)$	no	yes	no	yes
(*) L^1 (not a dual!)	no	—	no	—
$L^p \; (\cong (L^q)^*)$	yes	yes	yes	yes
$L^\infty \; (\cong (L^1)^*)$	no	yes	no	yes
Thm. used	4.37	4.35	4.31	4.36

(*) Cor. to Krein-Milman Theorem, see f.ex. Werner, Sect. VIII.4.

Chapter 5: Bounded operators

5.1 Topologies on the space of bounded linear operators

Definition 5.1 Let X, Y be normed spaces.

(a) The uniform (operator) topology on $BL(X,Y) :=$ norm topology wrt. to $\|\cdot\|_{X\to Y}$

(b) The strong (operator) topology on $BL(X,Y) :=$ locally convex topology induced by the seminorms $\{T \mapsto \|Tx\|_Y\}_{x\in X}$

(c) The weak (operator) topology on $BL(X,Y) :=$ locally convex topology induced by the seminorms $\{T \mapsto |\ell(Tx)|\}_{\substack{x\in X \\ \ell\in Y^*}}$

Remark 5.2 (a) The families of seminorms in 5.1(a) & 5.2(b) are separating (check!) $\Rightarrow$ Hausdorff topologies.

(b) Strong and weak operator topologies are not 1st countable if $\dim X = \infty$.

Lemma 5.3 (a) The strong topology is the coarsest vector-space topology on $BL(X,Y)$ such that all maps

$$M_x : \begin{array}{ccc} BL(X,Y) & \to & Y \\ T & \mapsto & Tx \end{array}, \quad x \in X$$

are continuous.

The weak (op.) topology is the coarsest vector-space topology on $BL(X,Y)$ such that all maps

$$M_{l,x} : BL(X,Y) \to \mathbb{K}, \quad T \mapsto l(Tx), \quad x \in X,\ l \in Y^*$$

are continuous.

(b) coarser ⟵ ⟶ finer

$\subseteq$ { weak topology on $BL(X,Y)$ } ← less important

weak op. top. $\subseteq$ strong op. top $\subseteq$ uniform op. top.

(c) Let $T, T_n \in BL(X,Y)$, $n \in \mathbb{N}$. Then

(i) $(T_n)_n$ converges to T in the strong operator topology, written $T_n \xrightarrow{s} T$ (strongly)

$\iff \lim_{n\to\infty} T_n x = Tx \quad \forall x \in X$

(ii) $(T_n)_n$ converges to T in the weak operator topology, written $T_n \xrightarrow{w} T$ (weakly) (!)

$\iff \lim_{n\to\infty} l(T_n x) = l(Tx) \quad \forall x \in X,\ \forall l \in Y^*$

Pf: (a) and (c) from Lemma 4.24

(b) Upper branch: Follows from $M_{l,x} \in (BL(X,Y))^*$ $\forall x \in X, \forall l \in Y^*$, since

$$|M_{l,x} T| = |l(Tx)| \le \|l\|_{Y^*} \underbrace{\|Tx\|_Y}_{\le \|T\| \cdot \|x\|}$$

$$\Rightarrow \|M_{l,x}\|_{(BL(X,Y))^*} \le \|l\|_{Y^*} \|x\| < \infty.$$

Hence, the weak topology on $BL(X,Y)$ is inducted by a family of seminorms that contains the inducing family of the weak operator topology. Hence, the latter is coarser.

Lower branch:

(i) $\Pi_x : BL(X,Y) \to Y$ is linear $\forall x \in X$.
It is also bounded wrt. the uniform topology:

$$\|\Pi_x T\|_Y = \|Tx\|_Y \leq \|T\| \cdot \|x\| \Rightarrow \frac{\|\Pi_x T\|}{\|T\|} \leq \|x\| \Rightarrow \sup_{0 \neq T \in BL(X,Y)} \frac{\|\Pi_x T\|}{\|T\|} \leq \|x\|$$

i.e. Π_x is continuous wrt. the uniform topology on $BL(X,Y)$ for all x. Also, the uniform topology is a vector-space topology (because it is a norm topology). So from (a): strong operator topology is coarser than the uniform topology.

(ii) $\Pi_{f,x} = f \circ \Pi_x$, and Π_x is continuous wrt. strong operator topology on $BL(X,Y)$. Hence, $\Pi_{f,x}$ is continuous wrt. strong operator topology. So from (a): the strong operator topology must be finer than the weak operator topology ∎

Lemma 5.4 Let $\mathcal{H}$ be a Hilbert space, and $(T_n)_n \subseteq BL(\mathcal{H})$

(a) If $(T_n x)_n$ is Cauchy in $\mathcal{H}$ $\forall x \in \mathcal{H}$ then there exists $T \in BL(\mathcal{H})$ such that $T_n \xrightarrow{s} T$

(b) If $(\langle y, T_n x \rangle)_n$ is Cauchy in $\mathbb{K}$ $\forall x, y \in \mathcal{H}$ then there exists $T \in BL(\mathcal{H})$ such that $T_n \xrightarrow{w} T$.

Pf: (a) Define $T: H \to H$, $x \mapsto \lim_{n\to\infty} T_n x$

The map T is

(i) well-defined (limit exists!)

(ii) linear

(iii) bounded: We have $\sup_n \|T_n x\| < \infty \ \forall x \in H$ (due to convergence). Hence, by the uniform boundedness principle: $\sup_n \|T_n\| < \infty$. So, for all $x \in H$, $\|x\| = 1$:

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \left(\sup_{n\in\mathbb{N}} \|T_n\|\right) \|x\| = \sup_{n\in\mathbb{N}} \|T_n\|,$$

hence,

$$\|T\| = \sup_{x\in H,\, \|x\|=1} \|Tx\| \leq \sup_{n\in\mathbb{N}} \|T_n\| < \infty$$

Hence $T \in BL(H)$ and $T_n \xrightarrow{s} T$ (by def.). ∎

(b) See exercise.

Example 5.5 Let $(T_n)_n \subseteq BL(\ell^2)$.

(a) For $x = (x_1, x_2, \dots) \in \ell^2$ and $n \in \mathbb{N}$ let

$$T_n x := \left(\tfrac{1}{n} x_1, \tfrac{1}{n} x_2, \dots\right) = \tfrac{1}{n} x.$$

Then $\|T_n\| = \frac{1}{n} \xrightarrow{n\to\infty} 0$, i.e. uniform convergence to zero (operator)

(b) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_{n+1}, x_{n+2}, \dots)$

(i) $T_n \xrightarrow{s} 0$ because $\|T_n x\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \xrightarrow[x\in\ell^2]{n\to\infty} 0$.

(ii) $(T_n)_n$ does not converge uniformly to 0, because $T_n e_{n+1} = e_{n+1}$, so $\|T_n\| \geq 1$.

(c) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_1, x_2, \dots)$

i.e. "n times iterated right shift" (see 2.26)

(i) $T_n \xrightarrow{w} 0$, since for $y \in \ell^2$, by Cauchy-Schwarz,

$$|\langle y, T_n x\rangle| = \Big|\sum_{j=1}^{\infty} \overline{y_{n+j}}\, x_j\Big| \le \|x\| \cdot \underbrace{\Big(\sum_{j=1}^{\infty} |y_{n+j}|^2\Big)^{1/2}}_{\substack{n\to\infty \\ \to 0 \text{ since } y\in\ell^2}}$$

(ii) $(T_n)_n$ does not converge strongly to 0, because $\|T_n x\| = \|x\| \ \forall x \in \ell^2$.

5.2 Adjoint operators

Definition 5.6 (a) Let X, Y be normed spaces, and $T \in BL(X,Y)$.

Then $T^\times : Y^* \to X^*$, $\ell \mapsto \ell \circ T$ is a linear operator, the adjoint (operator) of T

(b) Let H_1, H_2 be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then

$T^* : H_2 \to H_1$, $y \mapsto T^* y$ where (by Riesz!) $T^* y$ is the unique $z \in H_1$ such that, $\forall x \in H_1$, $\langle y, Tx\rangle_{H_2} = \langle z, x\rangle_{H_1}$ (*)

is a linear operator, the (Hilbert space) adjoint (operator) of T.

Note that, with $\ell_y := \langle y, \cdot\rangle_{H_2}$, the linear functional

$T^\times \ell_y : H_1 \to \mathbb{K}$, $x \mapsto \langle y, Tx\rangle_{H_2}$ belongs to H_1^*

Remarks 5.7 (a) In the Hilbert space case we have that $T^* = J_1 T^\times J_2^{-1}$, with $J_j : H_j^* \to H_j$ the antilinear, isometric and bijective identification (Cor. 2.58).

Indeed, from (*): LHS $= (J_2^{-1} y)(Tx) = (T^*(J_2^{-1} y))(x)$,

RHS $= (J_1^{-1}(T^* y))(x)$.

(b) T^* is linear (!) by part (a) of this remark.

(c) Notation varies. Some denote the (Banach space) adjoint $T^\times$ by T', or even T^*.

Examples 5.8 (a) Let $X = Y = \ell^1$, and $Rx := (0, x_1, x_2, \dots)$ the right-shift operator. Then $R \in BL(\ell^1)$. Hence $R^* : (\ell^1)^* \to (\ell^1)^*$ (Riesz representation: $(\ell^1)^* \ni f_\xi \overset{1\text{ to }1}{\longleftrightarrow} \xi \in \ell^\infty$ with $f_\xi(x) = \sum_{n\in\mathbb{N}} \xi_n x_n$ for every $x \in \ell^1$). So:

$$(R^* f_\xi)(x) = f_\xi(Rx) = \sum_{n\in\mathbb{N}} \xi_n (Rx)_n = \sum_{n\in\mathbb{N}} \xi_{n+1} x_n = \sum_{n\in\mathbb{N}} (L\xi)_n x_n = f_{L\xi}(x)$$

where $L : \ell^\infty \to \ell^\infty$, $\xi \mapsto (\xi_2, \xi_3, \xi_4, \dots)$ is the left shift operator. Hence, $R^* f_\xi = f_{L\xi}$.

(b) For $H_1 = H_2 = H$ and $\varphi, \psi \in H$, let $P := \langle \varphi, \cdot \rangle \psi$, i.e.,

$Px = \langle \varphi, x \rangle \psi \quad \forall x \in H$ ($\operatorname{ran}(P) = \operatorname{span}\{\psi\}$).

Then $P^* = \langle \psi, \cdot \rangle \varphi$ since, for all $x, y \in H$,

$$\langle y, Px \rangle = \langle y, \psi \rangle \langle \varphi, x \rangle = \langle \langle \psi, y \rangle \varphi, x \rangle = \langle P^* y, x \rangle.$$

As an application of adjoint operators, we prove that bounded operators preserve weak convergence:

Lemma 5.9 Let X, Y be normed spaces, and $T \in BL(X, Y)$. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$, $x \in X$. Then

$$x_n \overset{w}{\to} x \quad \Rightarrow \quad Tx_n \overset{w}{\to} Tx, \quad n \to \infty.$$

Pf: Let $\ell \in Y^*$. Then, $\ell(Tx_n) - \ell(Tx) = \underbrace{(T^*\ell)}_{\in X^*}(x_n - x) \overset{n\to\infty}{\longrightarrow} 0$ ∎

Theorem 5.10 Let X, Y be normed spaces and $T \in BL(X, Y)$. Then

$$\|T^*\|_{Y^* \to X^*} = \|T\|_{X \to Y}$$

Pf: By definition and Thm. 4.15 we have

$$\|T\|_{X\to Y} = \sup_{\substack{x\in X \\ \|x\|=1}} \|Tx\|_Y \overset{4.15}{=} \sup_{x\in X,\ \|x\|=1} \sup_{\ell\in Y^*,\ \|\ell\|_{Y^*}=1} |\ell(Tx)| \overset{(!)}{=} \sup_{\ell\in Y^*,\ \|\ell\|_{Y^*}=1} \sup_{x\in X,\ \|x\|=1} |(T^*\ell)(x)|$$

check

$$= \|T^*\|_{Y^* \to X^*} \qquad \text{where } \sup_{x\in X,\ \|x\|=1} |(T^*\ell)(x)| = \|T^*\ell\|_{X^*}$$

∎

Corollary 5.11 Let H_1, H_2 be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then $\|T^*\|_{H_2 \to H_1} = \|T\|_{H_1 \to H_2}$

Pf: Follows from $T^* = J_1 T^\times J_2^{-1}$ and Thm. 5.10, since $J_j : H_j^* \to H_j$, $j = 1,2$, are isometric & bijective. ∎

Theorem 5.12 Let H be a Hilbert space, and $T, S \in BL(H)$. Then

(a) The map $(\cdot)^* : BL(H) \to BL(H)$, $T \mapsto T^*$ is anti-linear, isometric and bijective, with $\mathbb{1}^* = \mathbb{1}$.

(b) $(TS)^* = S^* T^*$ (c) $(T^*)^* = T$

(d) If T has inverse $T^{-1} \in BL(H)$, then $(T^*)^{-1} = (T^{-1})^*$.
In particular, $(T^*)^{-1} \in BL(H)$.

(e) $\|TT^*\| = \|T\|^2$ (C^*-property).

Pf: (a): Anti-linear by definition of T^*; isometric by Cor. 5.11; surjective by part (c) of this theorem.

(b), (c): See exercise.

(d): $T^{-1}T = \mathbb{1} = T T^{-1}$ together with (b) gives
$T^*(T^{-1})^* = \mathbb{1}^* = \mathbb{1} = (T^{-1})^* T^*$. ✓

(e) On the one hand, we have (by Cor. 5.11) $\|TT^*\| \le \|T\| \cdot \|T^*\| = \|T\|^2$.
On the other hand,
$$\|TT^*\| \overset{4.15}{=} \sup_{0 \neq x,y \in H} \frac{|\langle y, TT^\times x\rangle|}{\|x\|\,\|y\|} \overset{y=x}{\ge} \sup_{0 \neq x \in H} \frac{\|T^\times x\|^2}{\|x\|^2} = \|T^*\|^2 \overset{T.11}{=} \|T\|^2$$ ∎

Definition 5.13 Let H be a Hilbert space, and $T \in BL(H)$.

(i) T is normal $:\Leftrightarrow TT^* = T^* T$ (they commute!)

(ii) T is self-adjoint $:\Leftrightarrow T = T^*$

(iii) T is unitary $:\Leftrightarrow T$ is bijective and $T^{-1} = T^*$
(i.e., $TT^* = \mathbb{1} = T^* T$).

Remark 5.14 (a) self-adjoint or unitary $\Rightarrow$ normal.

(b) T normal $\Leftrightarrow \langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle \quad \forall x, y \in H$.

Pf: LHS $= \langle T^*Tx, y\rangle$, RHS $= \langle (T^*)^* T^*x, y\rangle = \langle TT^*x, y\rangle$.

Hence: T normal $\Rightarrow \|Tx\| = \|T^*x\| \quad \forall x \in H$.

In particular: $\ker(T) = \ker(T^*)$. (if T normal!)

(c) T self-adjoint $\Leftrightarrow \quad \langle x, Ty\rangle = \langle Tx, y\rangle \quad \forall x, y \in H \quad (*)$.

Setting $x = y$ gives: $\langle x, Tx\rangle \in \mathbb{R} \quad \forall x \in H$.

[The property $(*)$ is called symmetry (T is symmetric). The notions "symmetric" and "selfadjoint" agree for bounded linear operators, but their generalisations to unbounded operators do NOT!]

(d) T unitary $\Leftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle = \langle T^*x, T^*y\rangle \quad \forall x, y \in H$.

Pf: $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$ and $\langle T^*x, T^*y\rangle = \langle (T^*)^* T^*x, y\rangle = \langle TT^*x, y\rangle$

Example 5.15 (a) $T = z\mathbb{1}$, $z \in \mathbb{K}$. Then $T^* = \bar{z}\mathbb{1}$. So, T is self-adjoint iff $z \in \mathbb{R}$.

(b) Let $H = L^2([0,1])$, $k \in C([0,1]^2)$ and

$$(Tf)(x) := \int_0^1 k(x,y) f(y)\, dy \quad \forall f \in H, \ \forall x \in [0,1]$$

Then $T \in BL(H)$ and, if $k(x,y) = \overline{k(y,x)} \ \forall x, y \in [0,1]$, then T is selfadjoint

5.3 The spectrum

In this subsection: X Banach over $\mathbb{C}$ (i.e. $\mathbb{K} = \mathbb{C}$)

Definition 5.16 Let $T \in BL(X)$

(i) The resolvent set (of T) is $\rho(T) := \{z \in \mathbb{C} \mid T - z\cdot\mathbb{1} \text{ is bijective}\}$

(ii) The resolvent of T is $R_z := R(z,T) := (T - z\cdot\mathbb{1})^{-1} =: (T-z)^{-1}$

for which every $z \in \mathbb{C}$ this exists as a (possibly unbounded) operator.

(a) If $z \in \rho(T)$, then $R_z \in BL(X)$ (bounded inverse thm!)

(b) R_z need not exist for $z \notin \rho(T)$.

(iii) The spectrum of T is $\operatorname{spec}(T) := \sigma(T) := \mathbb{C} \setminus \rho(T)$

(iv) If there exists $0 \neq x \in X$ s.t. $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then λ is an eigenvalue and x the corresponding eigenvector.

(v) The point spectrum of T is

$\operatorname{spec}_p(T) := \sigma_p(T) := \{z \in \mathbb{C} \mid T - z \text{ not injective}\} = \{\text{eigenvalues of } T\}$.

(vi) The continuous spectrum of T is

$\operatorname{spec}_c(T) := \sigma_c(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \operatorname{ran}(T-z) \neq X, \text{ but dense}\}$

(vii) The residual spectrum of T is

$\operatorname{spec}_r(T) := \sigma_r(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \operatorname{ran}(T-z) \text{ not dense}\}$.

[$\sigma_r(T) = \emptyset$ for most T of interest in applications]

Lemma 5.17 Let $T \in BL(X)$. Then

(a) $\mathbb{C} = \sigma(T) \dot\cup \rho(T)$ (disjoint union)

(b) $\sigma(T) = \sigma_p(T) \dot\cup \sigma_c(T) \dot\cup \sigma_r(T)$

(c) If $\dim X < \infty$, then $\sigma_c(T) = \emptyset = \sigma_r(T)$ (so $\sigma(T) = \sigma_p(T)$).

Pf: (a), (b) clear by definition.

(c) follows from Linear Algebra: $T - z$ injective $\Rightarrow \dim \ker(T-z) = 0$

$\overset{\dim X < \infty}{\Rightarrow} \dim \operatorname{ran}(T-z) = \dim(X) \Rightarrow \operatorname{ran}(T-z) = X$ ∎

Lemma 5.18 (Neumann series). Let $T \in BL(X)$, with $\|T\| < 1$.

Then $(\mathbb{1} - T)^{-1} \in BL(X)$ and $(\mathbb{1} - T)^{-1} = \sum_{j=0}^{\infty} T^j$.

Pf: See exercise CT6.3 Sheet 6). Here details for completeness:

From $\|T^j\| \leq \|T\|^j$, $\|T\| < 1$, and Lemma 2.50(a) follows: $S := \sum_{j=0}^{\infty} T^j \in BL(X)$

For $N \in \mathbb{N}$, we have

$$(\mathbb{1} - T) \sum_{j=0}^{N} T^j = \mathbb{1} - T^{N+1} = \left(\sum_{j=0}^{N} T^j\right)(\mathbb{1} - T).$$

Now, $\sum_{j=0}^{N} T^j \overset{N\to\infty}{\to} S$ and $T^{N+1} \overset{N\to\infty}{\to} 0$ (since $\|T^{N+1}\| \leq \|T\|^{N+1} \overset{N\to\infty}{\to} 0$). Hence,

$(\mathbb{1} - T)S = \mathbb{1} = S(\mathbb{1} - T)$ so $S = (\mathbb{1} - T)^{-1}$ ∎

Lemma 5.19 Let $T \in BL(X)$. Then $\sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|T\|\}$

Pf: For $|z| > \|T\|$ we have $\|\frac{T}{z}\| < 1$, hence 5.18 gives existence of

$$BL(X) \ni \frac{1}{z}\left(\frac{T}{z} - \mathbb{1}\right)^{-1} = (T - z)^{-1} = R_z.$$

Hence, $z \in \rho(T) = \mathbb{C} \setminus \sigma(T)$ ∎

Definition 5.20 Let $D \subseteq \mathbb{C}$ be open, $(Z, \|\cdot\|)$ a Banach space over $\mathbb{C}$, and $\xi : D \to Z$, $z \mapsto \xi(z)$.

(i) The map ξ is norm-differentiable (or strongly differentiable) at $z_0 \in D$
$:\Leftrightarrow \exists\, \zeta \in Z : \lim_{\mathbb{C} \ni h \to 0} \left\| \frac{\xi(z_0+h) - \xi(z_0)}{h} - \zeta \right\| = 0$

Notation: $\frac{d\xi}{dz}(z_0) := \zeta$.

(ii) ξ is weakly differentiable at $z_0 \in D$
$:\Leftrightarrow z \mapsto \ell(\xi(z)) \in \mathbb{C}$ is complex differentiable at z_0 $\forall\, \ell \in X^*$

(iii) ξ is norm-analytic (resp. weakly analytic) in D
$:\Leftrightarrow \xi$ is norm-differentiable (resp. weakly differentiable) at z_0 for all $z_0 \in D$.

Remark 5.21 (a) Z-valued norm-analytic functions have properties analogous to those of $\mathbb{C}$-valued analytic functions (f.ex. a power-series expansion converging wrt. $\|\cdot\|$, etc.).
Moral: Replace $|\cdot|$ (abs. value in $\mathbb{C}$) by $\|\cdot\|$
(For a discussion, see Hille-Phillips, "FA and semigroups", or Dunford-Schwartz, "Linear operators" vol 1, Sect. III.14.)

(b) Norm-differentiable $\Leftrightarrow$ weakly differentiable (For "$\Leftarrow$", see Reed-Simon Thm. VI.4)

Theorem 5.22 Let $T \in BL(X)$. Then

(a) $\rho(T) \subseteq \mathbb{C}$ is open in $\mathbb{C}$, and the map $R(\cdot, T) : \rho(T) \to BL(X)$, $z \mapsto R(z,T) = R_z$ is norm-analytic on $\rho(T)$, with derivative
$$\frac{dR_z}{dz} = R_z^2 \qquad (*)$$

[Note: $Z = BL(X)$ in Def. 5.20 in this case – i.e., complex diff. holds wrt. the operator norm]

(b) For $z_1, z_2 \in \rho(T)$, we have
$$(**) \qquad R_{z_1} - R_{z_2} = (z_1 - z_2) R_{z_1} R_{z_2} \quad \text{(First resolvent identity)}$$
In particular: $R_{z_1} R_{z_2} = R_{z_2} R_{z_1}$ (they commute!)

Pf: (b) Equation (**) follows from

$$(T-z_1)(R_{z_1}-R_{z_2})(T-z_2) = (\mathbb{1}-(T-z_1)R_{z_2})(T-z_2)$$
$$= T-z_2-(T-z_1) = z_1-z_2$$

and from multiplication by R_{z_1} from the left and R_{z_2} from the right.

Commutativity: Exchange $z_1 \leftrightarrow z_2$ in (**) & compare with (**).

(a) Openness of $\rho(T)$: Let $z_0 \in \rho(T)$ ($\neq \emptyset$ by Lemma 5.19) and $z \in B_{1/\|R_{z_0}\|}(z_0)$ (open ball in $\mathbb{C}$). Then

$$T-z = T-z_0-(z-z_0) = (T-z_0)(\mathbb{1}-\underbrace{(z-z_0)R_{z_0}}_{=:V_z})$$

By definition, $\|V_z\| < 1$, hence by Lemma 5.18: $(\mathbb{1}-V_z)^{-1} \in BL(X)$. So,

$$R_z = (T-z)^{-1} = (\mathbb{1}-V_z)^{-1}R_{z_0} \in BL(X) \quad (\Pi)$$

and so $z \in \rho(T)$, i.e. $\rho(T)$ is open.

Analytic: Let $z_0 \in \rho(T)$. By the above: $\forall 0 \neq h \in \mathbb{C}$, $|h| < \|R_{z_0}\|^{-1}$: $z_0+h \in \rho(T)$. Hence

$$\frac{R_{z_0+h}-R_{z_0}}{h} \overset{(**)}{=} R_{z_0+h}R_{z_0} \overset{(\Pi)\,\&\,5.18}{=} \sum_{n\in\mathbb{N}_0} V_{z_0+h}^n R_{z_0}^2 = \sum_{n\in\mathbb{N}_0} h^n R_{z_0}^{n+2}$$

and so

$$\left\| \frac{R_{z_0+h}-R_{z_0}}{h} - R_{z_0}^2 \right\| \leq \sum_{n\in\mathbb{N}} |h|^n \|R_{z_0}\|^{n+2} = \|R_{z_0}\|^2 \left\{ \frac{1}{1-|h|\|R_z\|} - 1 \right\} \xrightarrow[|h|\to 0]{} 0$$

Hence, $z \mapsto R_z \in BL(X)$ is norm-differentiable at z_0, with derivative $R_{z_0}^2$. As $z_0 \in \rho(T)$ was arbitrary, $R(\cdot,T)$ is norm-analytic in $\rho(T)$ (and (*) holds) ∎

Lemma 5.23 Let $T \in BL(X)$. Then $\sigma(T) \neq \emptyset$.

Pf: Let $|z| > \|T\|$. Then (see pf Lemma 5.19)

$$R_z = (T-z)^{-1} = \frac{1}{z}\left(\frac{T}{z}-\mathbb{1}\right)^{-1} = -\frac{1}{z}\sum_{j=0}^{\infty}\frac{T^j}{z^j}.$$

This expansion shows that

$$\|R_z\| \le \frac{|z|^{-1}}{1-\|T\|/|z|} = \frac{1}{|z|-\|T\|} \xrightarrow{|z|\to\infty} 0 \qquad (\nabla)$$

Assume $\sigma(T)=\emptyset$ (for contradiction). Then $\rho(T)=\mathbb{C}$ and, by Thm 5.22, $\mathbb{C}\to BL(X)$, $z\mapsto R_z$, is entire (i.e. analytic on all of $\mathbb{C}$). Also, $\mathbb{C}\ni z\mapsto \|R_z\|$ is bounded (using (∇) & continuity). By Liouville's Thm, (for $\mathbb{C}$-valued fct's, see any book on Complex Analysis) $z\mapsto \|R_z\|$ is constant, and by (∇), this constant has to be 0, i.e. $R_z=0$ $\forall z\in\mathbb{C}$ ↯ ∎

Definition 5.24 For $T\in BL(X)$, the spectral radius of T is

$$r(T) := \sup_{\lambda\in\sigma(T)} |\lambda| \qquad \le \|T\| \text{ (Lemma 5.18)}$$

Theorem 5.25 Let $T\in BL(X)$. Then

(a) $$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n}$$

(b) If, in addition, X is a Hilbert space, and T is normal, then: $\lim_{n\to\infty} \|T^n\|^{1/n} = \|T\|$. Hence: $r(T) = \|T\|$.

Pf: (a) We first prove: $\lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n}$:

W.l.o.g. assume $T^n\neq 0$ $\forall n\in\mathbb{N}$ (otherwise, if $\exists n_0: T^{n_0}=0$, then $T^n=0$ $\forall n\ge n_0$ and claim is clear).

Let $a_n := \ln\|T^n\|$. Then $a_{n_1+n_2}\le a_{n_1}+a_{n_2}$ $\forall n_1,n_2\in\mathbb{N}$ (check!)

Fix $m\in\mathbb{N}$ (arbitrary), and let $n(q,r) := qm+r$ for $q\in\mathbb{N}_0$ and $r\in\{0,1,\dots,m-1\}$. Then

$$\frac{a_{n(q,r)}}{n(q,r)} \le \frac{q a_m + a_r}{qm+r} = \frac{a_m}{m}\,\frac{1}{1+r/(qm)} + \frac{a_r}{qm+r}$$

Hence,

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \limsup_{q\to\infty} \max_{r\in\{0,\dots,m-1\}} \frac{a_{n(q,r)}}{n(q,r)} \le \frac{a_m}{m} \quad \forall m\in\mathbb{N}$$

so

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \inf_{m\in\mathbb{N}} \frac{a_m}{m}.$$

But (trivially) $\liminf_{n\to\infty} \frac{a_n}{n} \ge \inf_{m\in\mathbb{N}} \frac{a_m}{m}$. Taken together,

$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$ exists, and $\lim_{n\to\infty} e^{a_n/n} = \inf_{n\in\mathbb{N}} e^{a_n/n}$ ✓

because the exp. function is continuous and increasing.

Now, by Thm. 5.22, $z\mapsto R_z$ is strongly analytic in $D := \mathbb{C}\setminus\{z\in\mathbb{C} \mid |z| \le r(T)\}$, hence it has a Laurent series expansion about the origin (in $\mathbb{C}$):

$$R_z = \sum_{j\in\mathbb{Z}} z^j A_j, \quad A_j \in \mathcal{BL}(X) \text{ for } j\in\mathbb{Z},$$

which is norm-convergent $\forall z\in D$. [For $\mathbb{C}$-valued fct's, see Conway, "Fct.'s of one complex variable", 2nd ed. (1978), Thm V.1.1]

On the other hand, on $\{z\in\mathbb{C} \mid |z| > \|T\|\} \subseteq D$, the expansion

$$(*) \qquad R_z = -\frac{1}{z}\sum_{j=0}^{\infty} \frac{T^j}{z^j} \quad \text{holds (see pf. Lemma 5.23)}$$

Hence, by uniqueness of the Laurent series, $(*)$ is the Laurent series of the resolvent, hence norm convergent $\forall z\in D$.

Equivalently, the power series

$$\xi \mapsto \sum_{j\in\mathbb{N}_0} \xi^j T^j \qquad (**)$$

converges on $B^{(0)}_{1/r(T)}$.

On the other hand: For any $\varepsilon > 0$, the power series in $(**)$ is not norm-convergent on $B^{(0)}_{\varepsilon+\frac{1}{r(T)}}$, since if, then $(*)$ would converge for some $z\in\sigma(T)$ (recall: $r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$). Therefore $\frac{1}{r(T)}$ is the radius of convergence of $(**)$, and as such, it is

given by Hadamard's root criterion:

$$r(T) = \limsup_{j\to\infty} \|T^j\|^{1/j} = \lim_{j\to\infty} \|T^j\|^{1/j}$$

where the last equality was proved in the first part. ✓

(b) Let "C*" refer to the C*-property $\|TT^*\| = \|T\|^2$ from Thm. 5.12(e).

Then

$$\|T^2\|^2 \overset{C^* \text{ for } T^2}{=} \|T^2(T^2)^*\| \overset{T \text{ normal}}{=} \|(TT^*)(TT^*)^*\| \overset{C^* \text{ for } TT^*}{=} \|TT^*\|^2 \overset{C^* \text{ for } T}{=} \|T\|^4,$$

so $\|T^2\| = \|T\|^2$. By induction on $k \in \mathbb{N}$:

$$\|T^{2^k}\| = \|T\|^{2^k} \qquad (\nabla)$$

Hence $\lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{k\to\infty} \|T^{2^k}\|^{1/2^k} \overset{(\nabla)}{=} \|T\|$ ∎

Example 5.26 Right-shift & left-shift operator on ℓ^2:

For $x = (x_1, x_2, \ldots) \in \ell^2$, let $Rx := (0, x_1, x_2, \ldots)$ & $Lx := (x_2, x_3, \ldots)$

Then $\|R\| = 1 = \|L\|$, R is isometric and $R^* = L$ (check!)

Claims: (i) $\sigma_p(L) = B_1(0)$ (open ball in $\mathbb{C}$) (ii) $\sigma_c(L) = \partial B_1(0)$

(iii) $\sigma_r(L) = \emptyset$ (iv) $\sigma_p(R) = \emptyset$ (v) $\sigma_c(R) = \partial B_1(0)$

(vi) $\sigma_r(R) = B_1(0)$.

Proofs: (i) Let $\lambda \in \mathbb{C}$, then $Lx = \lambda x \iff (x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$

$\iff x_2 = \lambda x_1, x_3 = \lambda x_2 = \lambda^2 x_1, \ldots \iff x = x_1(1, \lambda, \lambda^2, \lambda^3, \ldots)$

Hence, $x \in \ell^2$ & $Lx = \lambda x \iff \sum_{n\in\mathbb{N}} (|\lambda|^2)^n < \infty \iff |\lambda| < 1$ ✓

(iv) R isometric $\Rightarrow \sigma_p(R) \subseteq \partial B_1(0)$ (check!). Let $|\lambda| = 1$,

then $Rx = \lambda x \iff (0, x_1, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$

$\iff 0 = \lambda x_1, x_1 = \lambda x_2 \overset{\lambda \neq 0}{\iff} x_1 = 0, x_2 = 0, \ldots$ ✓

(iii) Lemma (pf: see exercise). Let X be a Hilb. sp., $T \in BL(X)$

Then (a) $z \in \sigma_r(T) \Rightarrow \bar{z} \in \sigma_p(T^*)$

(b) $z \in \sigma_p(T) \Rightarrow \bar{z} \in \sigma_p(T^*) \cup \sigma_r(T^*)$

Hence from (a) with $T = L$: (iv) $\Rightarrow$ (iii).

(ii), (v) and (vi): See exercises.

5.4 Compact operators

General assumption: X and Y are both Banach spaces

Recall: In metric spaces: compactness $\overset{1.27}{\Leftrightarrow}$ sequential cpt.ness

Hence the equivalence in

Definition 5.27 An operator $T \in BL(X,Y)$ is compact

$:\Leftrightarrow$

$\forall A \subset X$: A bounded $\Rightarrow$ $\overline{T(A)} \subset Y$ compact (i.e. $T(A)$ rel. cpt. in Y)

$\Leftrightarrow \forall (x_n)_n \subset X$: $(x_n)_n$ bounded $\Rightarrow$ $(Tx_n)_n$ has a convergent subsequence

Example 5.28 (a) For $k \in C([0,1]^2)$, let $T: L^2([0,1]) \to L^2([0,1])$,

$$(Tf)(x) := \int_0^x k(x,y) f(y)\, dy, \quad f \in L^2([0,1]).$$

Then T is compact (see Exercise E13.2)

(b) Finite-rank operators:

$T \in BL(X,Y)$ is of finite rank $:\Leftrightarrow \dim \operatorname{ran}(T) < \infty$.

Then T is compact:

$(x_n)_n \subseteq X$ bounded $\overset{T \text{ bdd.}}{\Rightarrow}$ $(Tx_n)_n \subseteq \operatorname{ran}(T)$ bounded,

hence, $(Tx_n)_n$ is a bounded seq. in finite-dim. space, so, by Bolzano–Weierstraß, it has convergent subsequence.

Theorem 5.29 Let $T \in BL(X,Y)$ be compact, and $(x_n)_n \subseteq X$ s.t. $x_n \overset{w}{\to} x \in X$, $n \to \infty$. Then $Tx_n \overset{n\to\infty}{\to} Tx$ strongly, i.e. wrt. the norm in Y ($\|Tx_n - Tx\|_Y \to 0$, $n \to \infty$)

Pf: Let $y_n := Tx_n$ and $y := Tx$. Since $x_n \overset{w}{\to} x$, we have (by Lem. 5.9): $y_n \overset{w}{\to} y$. Assume, for contradiction, that $y_n \not\overset{s}{\to} y$ (i.e. not conv. in norm).

Then $\exists\, \varepsilon > 0,\ \exists\, (y_{n_k})_k : \|y_{n_k} - y\| \geq \varepsilon \ \forall k \in \mathbb{N}$.

But $(x_{n_k})_{k\in\mathbb{N}}$ is weakly conv., so (by 4.29) bounded.

Since T is compact, $\exists (y_{n_{k_l}})_l : y_{n_{k_l}} \overset{l\to\infty}{\to} \tilde{y} \in Y,\ \tilde{y} \neq y$.

Since $y_{n_{k_l}} \overset{w}{\underset{l\to\infty}{\to}} y \neq \tilde{y}$ (4.27(c)) this is a contradiction, so $y_n \overset{n\to\infty}{\to} y$ strongly / in norm ☒

Theorem 5.30 Let $T \in BL(X,Y)$.

(a) Assume $(T_n)_{n\in\mathbb{N}} \subseteq BL(X,Y)$, T_n compact $\forall n \in \mathbb{N}$, and

$$\|T_n - T\| \overset{n\to\infty}{\to} 0, \quad T \in BL(X,Y)$$

Then T is compact.

(b) T is compact iff T^x is compact (Schauder's Theorem)

(c) Let Z be a Banach space and $S \in BL(Y,Z)$.
If S or T is compact, then $ST \in BL(X,Z)$ is compact.

(d) In particular: Denote $K(X,Y) := \{T \in BL(X,Y) \mid T \text{ compact}\}$, $K(X) := K(X,X)$. Then $K(X) \subseteq BL(X)$ is a two-sided norm-closed ideal.

Pf: (d) follows from (a) & (c) ($Y = Z = X$).

(c) Clear, since bounded linear operators preserve convergence & boundedness.

(a) Let $(x_m)_{m\in\mathbb{N}} \subseteq X$ be bounded, wlog. $\|x_m\| \leq 1\ \forall m \in \mathbb{N}$.
As T_n is compact, $(T_n x_m)_m$ has a convergent subseq (dep. on n) with limit $y_n \in Y\ \forall n \in \mathbb{N}$. By Cantor's diagonal sequence trick, $\exists$ common subseq. $(x_{m_k})_{k\in\mathbb{N}}$ (i.e. indep. of n !) s.t. $T_n x_{m_k} \overset{k\to\infty}{\to} y_n\ \forall n \in \mathbb{N}$ (check!). (*)

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|T_n - T_{n'}\| < \varepsilon \ \forall n, n' \geq N$.

Then

$$\|y_n - y_{n'}\| \leq \underbrace{\|y_n - T_n x_{m_k}\|}_{(1)} + \underbrace{\|T_n x_{m_k} - T_{n'} x_{m_k}\|}_{\leq \|T_n - T_{n'}\| < \varepsilon} + \underbrace{\|T_{n'} x_{m_k} - y_{n'}\|}_{(2)}$$

$$\overset{k \to \infty}{<} \varepsilon, \quad \text{since (1) \& (2) tend to 0 as } k \to \infty \text{ (by (*))}$$

So $(y_n)_{n \in \mathbb{N}}$ is Cauchy, hence (Y Banach) convergent, i.e.

$y_n \overset{n \to \infty}{\to} y \in Y$

Claim: $\|T x_{m_k} - y\| \overset{k \to \infty}{\longrightarrow} 0$.

Indeed, let $\varepsilon > 0$. $\exists n \in \mathbb{N}$ s.t.

$$\|T - T_n\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|y_n - y\| < \frac{\varepsilon}{3}$$

Fix this. Then there is $K \in \mathbb{N}$ s.t. $\|T_n x_{m_k} - y_n\| < \frac{\varepsilon}{3} \ \forall k \geq K$ (by (*))

Hence, for all $k \geq K$,

$$\|T x_{m_k} - y\| \leq \underbrace{\|T x_{m_k} - T_n x_{m_k}\|}_{\leq \|T - T_n\| \cdot \underbrace{\|x_{m_k}\|}_{\leq 1} < \frac{\varepsilon}{3}} + \underbrace{\|T_n x_{m_k} - y_n\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|y_n - y\|}_{< \frac{\varepsilon}{3}}$$

$$< \varepsilon$$

So, claim holds, and T is compact.

(b) "$\Rightarrow$": Let $(f_n)_{n \in \mathbb{N}} \subseteq Y^*$ be a bounded sequence.

Then $K := T(\overline{B_1^X(0)}) = \overline{T(B_1^X(0))} \subseteq Y$ is a compact metric space. The restrictions $f_n := f_n|_K \in C(K)$ (equipped with $\|\cdot\|_\infty$-norm) form a bounded and equicontinuous sequence, because

$$|f_n(y_1) - f_n(y_2)| \leq \underbrace{\left(\sup_{k \in \mathbb{N}} \|f_k\|\right)}_{< \infty} \cdot \|y_1 - y_2\|_Y \quad \forall y_1, y_2 \in K$$

By the theorem of Arzelà-Ascoli (1.40) there exists a uniformly convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$, hence Cauchy, i.e. $\forall \varepsilon>0\ \exists N\in\mathbb{N}: \forall k,l\geq N:$ $\|f_{n_k}-f_{n_l}\|_{C(K)}<\varepsilon$

So, for all $k,l\geq N$:

$$\|T^X\ell_{n_k}-T^X\ell_{n_l}\|_{X^*} = \sup_{x\in B_1^X(0)} |T^X\ell_{n_k}x - T^X\ell_{n_l}x|$$

$$= \sup_{x\in B_1^X(0)} |\ell_{n_k}(Tx)-\ell_{n_l}(Tx)| \leq \|f_{n_k}-f_{n_l}\|_{C(K)} < \varepsilon.$$

(* in fact "=", since $T(B_1^X(0))$ dense in K).

So $(T^X\ell_{n_k})_{k\in\mathbb{N}}$ converges in X^*, hence $T^X: Y^*\to X^*$ is cpt.

"$\Leftarrow$": If T^X compact, then $T^{XX}: X^{**}\to Y^{**}$ is compact by "$\Rightarrow$". So, from (c), $T^{XX}J_X$ is compact, where $J^X: X\to X^{**}$ is the canonical embedding, $(J_X x)(f)=f(x)$ $\forall x\in X, \forall f\in X^*$. Now, for $x\in X$ and $\ell\in Y^*$, we have

$$(T^{XX}J_X x)(\ell) = J_X(x)(T^X\ell) = (T^X\ell)(x) = \ell(Tx) = (J_Y(Tx))(\ell)$$

This implies $T^{XX}J_X = J_Y T$ being compact. Since Y is closed in Y^{**} (i.e. $J_Y(Y)\subseteq Y^{**}$ is closed), T is also compact. ∎

Theorem 5.31 Let H be a separable Hilbert space.

Then $K(H)=\{T\in BL(H)\mid T \text{ compact}\} = \overline{\{T\in BL(H)\mid T \text{ has finite rank}\}}^{\|\cdot\|}$

where the closure is wrt. the operator norm.

Pf: "$\supseteq$" holds because of Ex. 5.28(b) and Thm. 5.30(a)

(this does not need separability - or that H is Hilbert space (Banach ok)).

"$\subseteq$": Need to prove: Any compact $T \in BL(H)$ is the norm-limit of finite-rang operators:

If $\dim(H) < \infty$, there is nothing to prove. So let $\dim(H) = \infty$ and let $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq H$ be an ONB. For $n \in \mathbb{N}$, define the orthogonal rank-n-projection $P_n := \sum_{j=1}^{n} \langle \varphi_j, \cdot \rangle \varphi_j$.

By Thm. 2.51: $P_n \xrightarrow{s} \mathbb{1}$, hence $\mathbb{1} - P_n = \sum_{j=n+1}^{\infty} \langle \varphi_j, \cdot \rangle \varphi_j$ (where the series converges in the strong operator sense).

Let $R_n := TP_n \in BL(H)$ for $n \in \mathbb{N}$. Then R_n has rank $\leq n$, and converges to T strongly ($R_n \xrightarrow{s} T$).

Strategy: Use compactness of T to prove convergence even uniformly.

Let $\psi \in H$, $\|\psi\| = 1$. Then

$$\|(\mathbb{1} - P_n)\psi\|^2 = \sum_{j=n+1}^{\infty} |\langle \varphi_j, \psi \rangle|^2 \overset{\text{Bessel}}{\leq} \|\psi\|^2 = 1$$

so that

$$\|T - R_n\| = \sup_{\substack{\psi \in H, \\ \|\psi\|=1}} \|T(\mathbb{1} - P_n)\psi\| \leq \sup_{\substack{\psi \in H \\ \|\psi\|=1}} \underbrace{\sup_{\substack{\psi \in \operatorname{ran}(\mathbb{1}-P_n) \\ \|\psi\| \leq 1}} \|T\psi\|}_{=: \lambda_n} = \lambda_n.$$

Note: $(\lambda_n)_{n\in\mathbb{N}}$ is non-negative and decreasing, so $\lim_{n\to\infty} \lambda_n =: \lambda \geq 0$ exists. The theorem now follows from:

Claim: $\lambda = 0$: For $n \in \mathbb{N}$, $\exists\, \psi_n \in \operatorname{ran}(\mathbb{1} - P_n)$ with $\|\psi_n\| \leq 1$ and $\|T\psi_n\| \geq \frac{\lambda_n}{2} \geq \frac{\lambda}{2}$. Now, for all $\varphi \in H$,

$$|\langle \varphi, \psi_n \rangle|^2 = |\langle \varphi, (\mathbb{1} - P_n)\psi_n \rangle|^2 = |\langle (\mathbb{1} - P_n)\varphi, \psi_n \rangle|^2$$

$$\overset{\text{C.S.}}{\leq} \underbrace{\|(\mathbb{1} - P_n)\varphi\|^2}_{\sum_{j=n+1}^{\infty} |\langle \varphi_j, \varphi \rangle|^2} \cdot \underbrace{\|\psi_n\|^2}_{\leq 1} \xrightarrow{n\to\infty} 0.$$

So $\psi_n \xrightarrow{w} 0$ and (since T cpt.) $T\psi_n \xrightarrow{n\to\infty} 0$ (by S. 29). So $\lambda = 0$ ∎

5.5 Fredholm alternative and the spectral theorem for compact operators

General assumption: H is a separable Hilbert space over $\mathbb{C}$.

Motivation 5.32

(i) Let M be an $N \times N$-matrix (over $\mathbb{C}$) for $N \in \mathbb{N}$, and let $z \in \mathbb{C}$.
Then: Either: $My = zy$ has a solution $0 \neq y \in \mathbb{C}^N$
Or: $(M - z)^{-1}$ exists

(ii) Let $H = L^2([0,1])$ and $M \in BL(L^2([0,1]))$ be defined by

$$(My)(x) := x\,y(x) \quad \text{for a.e. } x \in [0,1] \quad \forall y \in L^2([0,1])$$

(a) $My = zy$ has no solutions $0 \neq y \in L^2$ since:
$x\,y(x) = z\,y(x)$ a.e. $x \in [0,1] \Rightarrow y(x) = 0$ a.e. $x \in [0,1]$

(b) $(M - z)$ is not surjective $\forall z \in [0,1]$: (so, $(M-z)^{-1}$ does not exist!)

- $\big((M-z)y\big)(x) = (x - z)\,y(x)$
- Any element $\varphi \in L^2([0,1])$ which has a representative which is (1) continuous at $x = z$ (2) with $\varphi(z) \neq 0$ satisfies $\varphi \notin \operatorname{ran}(M - z)$: Assume $\varphi \in \operatorname{ran}(M - z)$.
Then there exists $y \in L^2([0,1])$ s.t. $(M - z)y = \varphi$,
hence $y(x) = \dfrac{1}{x - z}\,\varphi(x)$ (a.e. x) – which is not an element of $L^2([0,1])$ ↯.

(c) For compact operators, the above either/or-statement in (a) still holds (see Cor. 5.34).

Theorem 5.33 (Analytic Fredholm theorem)

Let $D \subseteq \mathbb{C}$ be open and connected, let $f: D \to BL(H)$ be (norm) analytic, and $f(z)$ compact $\forall z \in D$ ($f(z) \in K(H)$).

Then: Either (A1): $(f(z) - \mathbb{1})^{-1}$ does not exist for any $z \in D$

or (A2): There exists a discrete subset $S \subseteq D$ without accumulation points in D, such that the map

$$D \setminus S \to BL(H), \quad z \mapsto (f(z) - \mathbb{1})^{-1}$$

is well-defined and analytic.

Moreover: $z \in S$ iff $f(z)\varphi = \varphi$ has a solution $0 \neq \varphi \in H$.

Corollary 5.34 Let $T \in BL(H)$ be compact (i.e. $T \in K(H)$) and $z \in \mathbb{C} \setminus \{0\}$. Then

Either: $T\varphi = z\varphi$ has a solution $0 \neq \varphi \in H$

or: $(T - z\mathbb{1})^{-1} \in BL(H)$ exists.

Pf (5.34): Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z} T = \frac{T}{z}$.

Alternative (A1) does not hold in Thm. 5.33, since $\frac{T}{z} - \mathbb{1}$ has an inverse in $BL(H)$ $\forall |z| > \|T\|$ (Neumann series!)

Hence, Alternative (A2) holds. □

Pf (5.33): It suffices to prove: $\forall z_0 \in D$ $\exists$ neighbourhood $\mathcal{N}(z_0)$ s.t. either (A1) or (A2) holds on $\mathcal{N}(z_0)$. [Indeed: $\forall z_0 \in D$, colour $\mathcal{N}(z_0)$ red or blue, depending on which alternative holds; D is connected, i.e. if both colours appear, then there exists $\tilde{z}_0$ s.t. $\mathcal{N}(\tilde{z}_0)$ cannot be uniquely coloured - contradiction]

Fix $z_0 \in D$. As f continuous: $\exists r > 0 : \|f(z) - f(z_0)\| < \frac{1}{2}$ for all $z \in D_r := B_r(z_0)$. By Thm. 5-31 (separability!), $\exists F \in BL(H)$, finite rank, s.t. $\|f(z_0) - F\| < \frac{1}{2}$. Hence $\|f(z) - F\| < 1$, and so (by 5-18 (Neumann series))

(1) $$\begin{aligned} D_r &\to BL(H) \\ z &\mapsto (\mathbb{1} - f(z) + F)^{-1} \end{aligned}$$ is well-defined and analytic.

Let $g(z) := F(\mathbb{1} - f(z) + F)^{-1}$, let $N := \operatorname{rank}(F) < \infty$, and let $\varphi_1, \dots, \varphi_N \in H$ be an ONB of $\operatorname{ran}(F)$.

Then $\operatorname{ran}(g(z)) \subseteq \operatorname{ran}(F)$, so: $\forall \varphi \in H$ (expand $g(z)\varphi$ in ONB)

(2) $$g(z)\varphi = \sum_{n=1}^{N} \langle \varphi_n, g(z)\varphi \rangle \varphi_n$$

Note also that

(3) $$f(z) - \mathbb{1} = (g(z) - \mathbb{1})(\mathbb{1} - f(z) + F)$$

For all $z \in D_r$ we have the following equivalences:

$f(z) - \mathbb{1}$ not invertible $\overset{(1),(3)}{\iff}$ $g(z) - \mathbb{1}$ not invertible

$\Updownarrow$ (*) (Pf. below!)

$\exists$ sol. $0 \neq \varphi \in H$ of $f(z)\varphi = \varphi$ $\iff$ $\exists$ sol. $0 \neq \varphi \in H$ of $g(z)\varphi = \varphi$

$\Updownarrow$ (2) $\varphi = \sum_{n=1}^{N} \beta_n \varphi_n$

$$\exists \beta_1, \dots, \beta_N \in \mathbb{C} \text{ (not all } = 0) : \beta_n = \sum_{m=1}^{N} \underbrace{\langle \varphi_n, g(z)\varphi_m \rangle}_{=: A_{nm}(z)} \beta_m$$

$\Updownarrow$

$$d(z) := \det(A(z) - \mathbb{1}_{N \times N}) = 0, \text{ where } A := (A_{nm})_{1 \le n,m \le N}$$

But: $d : D_r \to \mathbb{C}$ is analytic, because, by (1), A_{nm} is analytic $\forall n,m$

By the identity thm (Complex Analysis):

Either $S_r := \{z \in D_r \mid d(z) = 0\}$ has no accum. point in D_r

i.e. (A2) holds on D_r

or $S_r = D_r$, i.e. (A1) holds on D_r; $\Rightarrow$ claim!

It remains to prove (*): The pb. is reduced to the finite rank case.

"$\Uparrow$": Clear.

"$\Downarrow$": By contradiction. Assume $d(z) \neq 0$. Then:

$\forall \xi \in \mathbb{C}^N : \big(A(z) - \mathbb{1}_{N\times N}\big)\beta = \xi$ has a solution $\beta \in \mathbb{C}^N$.

For an arbitrary (fixed) $\varphi \in \mathcal{H}$, choose $\xi_n = \xi_n^\varphi := \langle \varphi_n, g(z)\varphi\rangle$ for $n = 1, \dots, N$, and let $\beta^\varphi \in \mathbb{C}^N$ be a corresponding solution.

Then: $\tilde{\varphi} := -\varphi + \sum_{n=1}^{N} \beta_n^\varphi \varphi_n$ solves $\big(g(z) - \mathbb{1}\big)\tilde{\varphi} = \varphi$,

since:

$$\big(g(z) - \mathbb{1}\big)\tilde{\varphi} = \varphi - g(z)\varphi + \big(g(z) - \mathbb{1}\big)\sum_{m=1}^{N} \beta_m^\varphi \varphi_m$$

$$\overset{(2)}{=} \varphi + \sum_{n=1}^{N} \Big[\underbrace{-\langle \varphi_n, g(z)\varphi\rangle}_{\xi_n^\varphi} + \sum_{m=1}^{N} \big(A_{nm}(z) - \delta_{nm}\big)\beta_m^\varphi \Big] \varphi_n$$

$$= 0$$

$\Rightarrow g(z) - \mathbb{1}$ is invertible ↯ ∎

Corollary 5.35 (Riesz–Schauder) Let $T \in BL(H)$ be compact. Then $\sigma(T)$ has no accumulation point in $\mathbb{C}$, except possibly at 0. Moreover, any $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity (i.e., the corresponding eigenspace is finite-dim.)

Pf: Choose $D := \mathbb{C}\setminus\{0\}$ and $f(z) := \frac{1}{z}$ in Thm. 5.33.

Then $\mathbb{C}\setminus(\{0\}\cup S) \subseteq \rho(T)$ so $\sigma(T) \subseteq \{0\}\cup S$; also, $S \subseteq \sigma_p(T) \subseteq \sigma(T) \subseteq \{0\}\cup S$. Remove 0 from the sets in this chain of inclusions and note that $S = S\setminus\{0\}$.

Hence $S \subseteq \sigma_p(T)\setminus\{0\} \subseteq \sigma(T)\setminus\{0\} \subseteq S$, and so

$$S = \sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

Also, 0 is the only possible accumulation point of S by 5.33.

Finite multiplicity: Suppose not (only possible for $\dim(H) = \infty$). Then there exists an infinite ONB $\{\varphi_n\}_{n\in\mathbb{N}}$ of the eigenspace corresponding to some eigenvalue $\lambda \neq 0$, and $T\varphi_n = \lambda\varphi_n$ for all n (eigenspace closed!). Compactness of T implies that $(T\varphi_n)_n = (\lambda\varphi_n)_n$ has a convergent subsequence, which is a contradiction, since $\lambda \neq 0$ and $\{\varphi_n\}_n$ ONB. ∎

Remark 5.36 (a) If $\dim H = \infty$, then $0 \in \sigma(T)$

Pf: If not, then $0 \in \rho(T)$, i.e. $T^{-1} \in BL(H)$ and $\mathbb{1} = TT^{-1}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an ONB of H. Then $\varphi_n = T(T^{-1}\varphi_n)$ $\forall n$. Since $\{T^{-1}\varphi_n\}_n \subseteq H$ is bounded, and T compact, $\{\varphi_n\}_n$ has a convergent subsequence ↯ (since $\{\varphi_n\}_n$ ONB). ∎

(b) Corollaries 5.34 & 5.35 have generalisations to Banach spaces.

In the following we use the convention $n = 1, \dots, \infty :\Leftrightarrow n \in \mathbb{N}$ (to have unified notation).

Theorem 5.37 (Singular-value decomposition for compact operators)
Let $T \in BL(H)$ be compact. Then there exists $N \in \mathbb{N}_0 \cup \{\infty\}$, orthonormal sets $\{\varphi_n\}_{n=1,\dots,N}$, $\{\psi_n\}_{n=1,\dots,N} \subseteq H$ (which are not necessarily ONB's) and singular values $(\lambda_n)_{n=1,\dots,N} \subseteq (0,\infty)$ s.t.

$$T = \sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot \rangle \psi_n \qquad (*)$$

If $N = \infty$, then $(\lambda_n)_{n\in\mathbb{N}}$ is a null sequence, and the sum in (*) converges in the operator norm.

The proof relies on

Theorem 5-38 (Hilbert-Schmidt theorem; Spectral theorem for self-adjoint and compact op's on Hilbertspace)

Let $T \in BL(H)$ be self-adjoint and compact. Then there exists an ONB $\{\varphi_n\}_{n=1,\dots,\dim H} \subseteq H$ of eigenvectors, and eigenvalues $(\lambda_n)_{n=1,\dots,\dim H} \subseteq \mathbb{R}$ (listed according to their multiplicities) such that

$$T\varphi_n = \lambda_n \varphi_n \qquad \forall n = 1, \dots, \dim H.$$

Moreover,

$$T = \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n. \qquad (*)$$

If $\dim H = \infty$, then $(\lambda_n)_{n\in\mathbb{N}}$ is a nullsequence, and the sum in $(*)$ converges in the operator norm.

Pf of 5.38: (i) Let $0 \neq \mu_j$, $j = 1, 2, \dots$ be the pairwise distinct non-zero (real!) eigenvalues of T, i.e., $\mu_j \neq \mu_k$ for $j \neq k$. For every j, choose an ONB B_j in the finite-dim. (Cor 5-35) eigenspace of T corresponding to μ_j. Then $B := \bigcup_j B_j$ is an orthonormal set in H, since eigenvectors corresp. to different eigenvalues of normal operators are orthogonal (check!).

Let $H_0 := \overline{\operatorname{span}(B)}$. Then $T(H_0) \subseteq H_0$, and as $T = T^*$ this gives $T(H_0^\perp) \subseteq H_0^\perp$:

$$\forall x_\perp \in H_0^\perp \; \forall x_0 \in H_0 : \langle x_0, Tx_\perp \rangle \overset{T=T^*}{=} \langle \underbrace{Tx_0}_{\in H_0}, x_\perp \rangle = 0$$

$$\Rightarrow Tx_\perp \perp H_0$$

Let $T^\perp := T|_{H_0^\perp} : H_0^\perp \to H_0^\perp$. Then $T^\perp$ is selfadjoint & compact

But $\sigma(T^\perp) = \{0\} \Rightarrow$ spectral radius $r(T^\perp) = 0$.

So, by 5.25(b): $\|T^\perp\| = 0$, hence $T^\perp = 0$, and so $H_0^\perp \subseteq \ker(T)$ (in fact "$=$"). So: if $B^\perp$ is any ONB of $H_0^\perp$, then $B \cup B^\perp =: \{\varphi_n\}_{n=1,\dots,\dim H}$ is ONB of H consisting of eigenvectors of T.

(ii) Let λ_n be the eigenvalue corresponding to φ_n, $n = 1, \dots, \dim H$. Then $(\lambda_n)_{n=1,\dots,\dim H}$ is sequence of eigenvalues enumerated according to multiplicities and (if $\dim H = \infty$), a null sequence by Cor. 5.35

(iii) $T \overset{(*)}{=} \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$ holds because, if $\dim H < \infty$, both sides obviously agree on ONB $\{\varphi_n\}_{n=1,\dots,\dim H}$:

$T\varphi_m = \lambda_m \varphi_m \quad \forall m \in \mathbb{N}$ and (finite sum!)

$$\sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n = \lambda_m \varphi_m \quad \text{(ONB!)}$$

If $\dim H = \infty$, we first prove that the RHS in (*) is a Cauchy seq. wrt. the operator norm. Indeed, let $\varepsilon > 0$. Since $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, $\exists M_1 \in \mathbb{N}$ s.t. $\sup_{n \geq M_1} |\lambda_n| < \varepsilon$. Hence, $\forall M_2 \geq M_1$:

$$\sup_{\|\varphi\|=1} \Big\| \sum_{n=M_1+1}^{M_2} \lambda_n \langle \varphi_n, \varphi \rangle \varphi_n \Big\|^2 = \sup_{\|\varphi\|=1} \Big[\sum_{n=M_1+1}^{M_2} |\lambda_n|^2 |\langle \varphi_n, \varphi \rangle|^2 \Big]$$

$$\leq \sup_{\|\varphi\|=1} \Big[\underbrace{\Big(\sup_{n \geq M_1} |\lambda_n|^2 \Big)}_{< \varepsilon^2} \underbrace{\sum_{n=M_1+1}^{M_2} |\langle \varphi_n, \varphi \rangle|^2}_{\leq \|\varphi\|^2 \text{ (Bessel)}} \Big] < \varepsilon^2$$

Since $BL(H)$ is complete, $\exists R \in BL(H)$ s.t. $R = \sum_{n \in \mathbb{N}} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$. Since both T & R are bounded, it suffices to prove that they agree on a dense subset of H (Bounded Linear Extension Thm. 2.31). But as R is a uniform (thus strong) limit, we have

$$R\varphi_m = \Big[\lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \Big] \varphi_m$$

$$= \lim_{N \to \infty} \Big[\underbrace{\Big(\sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \Big)}_{\sum_{n=1}^{N} \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n} \varphi_m \Big] = \lambda_m \varphi_m \qquad \forall m \in \mathbb{N}$$

Hence, T and R agree on ONB, so they agree on $\operatorname{span}\{\varphi_m \mid m \in \mathbb{N}\} \subseteq H$, which is dense in H ∎

Pf of 5.37: Since T is compact, so is T^*T (by 5.30(c1)). Since T^*T is also self-adjoint (check!), by 5.38:

$\exists$ orthonormal set $\{\varphi_n\}_{n=1,\dots,\dim H} \subseteq H$ and $(\tilde{\lambda}_n)_{n=1,\dots,\dim H} \subseteq \mathbb{R}$ such that

$$T^*T = \sum_{n=1}^{\dim H} \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n.$$

Now, since $0 \leq \|T\varphi\|^2 = \langle \varphi, T^*T\varphi \rangle = \sum_{n=1}^{\dim H} \tilde{\lambda}_n |\langle \varphi_n, \varphi \rangle|^2 \quad \forall \varphi \in H$

we have (take $\varphi = \varphi_m$) that $\tilde{\lambda}_m \geq 0 \quad \forall m \in \mathbb{N}$. Terms with $\tilde{\lambda}_n = 0$ do not contribute to (*) in 5.38. Re-name the eigenvalues $(\tilde{\lambda}_n)_{n=1,\dots,\dim H}$ such that $(\tilde{\lambda}_n)_{n=1,\dots,N} \subseteq (0,\infty)$ be the collection of non-zero eigenvalues,

and $(\tilde{\lambda}'_{n'}=0)_{n'=1,\dots,N'}$ are the zero eigenvalues, where $N, N' \in \mathbb{N}_0 \cup \{\infty\}$. Let $\{\varphi_n \mid n=1,\dots,N\}$ and $\{\varphi'_{n'} \mid n'=1,\dots,N'\}$ be the set of the corresponding eigenvectors, which, together, form the ONB of H.

Hence

$$T^*T = \sum_{n=1}^{N} \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n \quad \text{and} \quad \ker(T) = \operatorname{span}\{\varphi'_{n'} \mid n=1,\dots,N'\}.$$

Set $\lambda_n := \tilde{\lambda}_n^{1/2} = \sqrt{\tilde{\lambda}_n}$ and $\psi_n = \frac{1}{\lambda_n} T\varphi_n$ for all $n=1,\dots,N$

Then $T\varphi_n = \lambda_n \psi_n$, and

$$\langle \psi_m, \psi_n \rangle = (\lambda_m \lambda_n)^{-1} \langle \varphi_m, \underbrace{T^*T\varphi_n}_{\tilde{\lambda}_n \varphi_n} \rangle = \delta_{mn} \quad \forall m,n=1,\dots,N$$

That is, $\{\psi_n\}_{n=1,\dots,N} \subseteq H$ is an orthonormal set. Note that if $N=\infty$ (only possible if $\dim(H)=\infty$) then $(\lambda_n)_{n\in\mathbb{N}}$ is a null-sequence by 5.38. Finally, the same (!) argument as in the proof of 5.38 gives

$$T = \sum_{n=1}^{N} \lambda_n \langle \varphi_n, \cdot \rangle \psi_n$$

with the sum being uniformly (i.e. norm) convergent if $N=\infty$ (use the ONB $\{\varphi_1,\dots,\varphi_N\} \cup \{\varphi'_1,\dots,\varphi'_{N'}\}$ to test the equality) ∎