

Chapter 5: Bounded operators

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5.1 Topologies on the space of bounded linear operators

Definition 5.1 Let X, Y be normed spaces.

- (a) The uniform (operator) topology on $BL(X, Y) :=$
norm topology wrt. to $\|\cdot\|_{X \rightarrow Y}$
- (b) The strong (operator) topology on $BL(X, Y) :=$
locally convex topology induced by the seminorms
 $\{T \mapsto \|Tx\|_Y\}_{x \in X}$
- (c) The weak (operator) topology on $BL(X, Y) :=$
locally convex topology induced by the seminorms
 $\{T \mapsto |\ell(Tx)|\}_{\substack{x \in X \\ \ell \in Y^*}}$

Remark 5.2 (a) The families of seminorms in 5.1(a) & 5.2(b) are separating (check!) $\Rightarrow$ Hausdorff topologies.
(b) Strong and weak operator topologies are not 1st countable if $\dim X = \infty$.

Lemma 5.3 (a) The strong topology is the coarsest vector-space topology on $BL(X, Y)$ such that all maps

$$M_x: BL(X, Y) \rightarrow Y, \quad x \in X \\ T \mapsto Tx$$

are continuous.

The weak^(op.) topology is the coarsest vector-space topology on $B(L(X, Y))$ such that all maps

$$\Pi_{\ell, x}: B(L(X, Y)) \rightarrow \mathbb{K} \quad , \quad x \in X, \ell \in Y^*$$

$$T \mapsto \ell(Tx)$$

are continuous.

(b) $\xleftarrow{\text{coarser}} \quad \xrightarrow{\text{finer}}$

$$\text{weak op. top.} \subseteq \left\{ \begin{array}{c} \text{weak topology} \\ \text{on } B(L(X, Y)) \end{array} \right\} \leftarrow \text{less important}$$

$$\subseteq \text{strong op. top.} \subseteq \text{uniform op. top.}$$

(c) Let $T, T_n \in B(L(X, Y))$, $n \in \mathbb{N}$. Then

(i) $(T_n)_n$ converges to T in the strong operator topology,
written $T_n \xrightarrow{s} T$ (strongly)

$$\Leftrightarrow \lim_{n \rightarrow \infty} T_n x = T x \quad \forall x \in X$$

(ii) $(T_n)_n$ converges to T in the weak operator topology,
written $T_n \xrightarrow{w} T$ (weakly) (!)

$$\Leftrightarrow \lim_{n \rightarrow \infty} \ell(T_n x) = \ell(T x) \quad \forall x \in X, \forall \ell \in Y^*$$

Pf: (a) and (c) from Lemma 4.24

(b) Upper branch: Follows from $\Pi_{\ell, x} \in (B(L(X, Y)))^*$
 $\forall x \in X, \forall \ell \in Y^*$, since

$$|\Pi_{\ell, x} T| = |\ell(Tx)| \leq \|\ell\|_{Y^*} \underbrace{\|Tx\|_Y}_{\leq \|T\| \cdot \|x\|} \leq \|T\| \cdot \|x\|$$

$$\Rightarrow \|\Pi_{\ell, x}\|_{(B(L(X, Y)))^*} \leq \|\ell\|_{Y^*} \|x\| < \infty.$$

Hence, the weak topology on $BL(X, Y)$ is induced by a family of seminorms that contains the inducing family of the weak operator topology. Hence, the latter is coarser. (108)

Lower branch:

(i) $M_x: BL(X, Y) \rightarrow Y$ is linear $\forall x \in X$.

It is also bounded w.r.t. the uniform topology:

$$\|M_x T\|_Y = \|Tx\|_Y \leq \|T\| \cdot \|x\| \Rightarrow \sup_{0 \neq T \in BL(X, Y)} \frac{\|M_x T\|}{\|T\|} \leq \|x\|$$

i.e. M_x is continuous w.r.t. the uniform topology on $BL(X, Y)$ for all x . Also, the uniform topology is a vector-space topology (because it is a norm topology). So from (a): strong operator topology is coarser than the uniform topology.

(ii) $M_{e, x} = I \circ M_x$, and M_x is continuous w.r.t. strong operator topology on $BL(X, Y)$. Hence, $M_{e, x}$ is continuous w.r.t. strong operator topology. So from (a): the strong operator topology must be finer than the weak operator topology. ■

[Lemma 5.4] Let H be a Hilbert space, and $(T_n)_n \subseteq BL(H)$

(a) If $(T_n x)_n$ is Cauchy in H ($\forall x \in H$) then there exists

$$T \in BL(H) \text{ such that } T_n \xrightarrow{s} T$$

(b) If $(\langle y, T_n x \rangle)_n$ is Cauchy in $\mathbb{K}$ $\forall x, y \in H$ then there

$$\text{exists } T \in BL(H) \text{ such that } T_n \xrightarrow{w} T.$$

Pf: (a) Define

$$T: H \rightarrow H$$

$$x \mapsto \lim_{n \rightarrow \infty} T_n x$$

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The map T is

(i) well-defined (limit exists?)

(ii) linear

(iii) bounded: We have $\sup_n \|T_n x\| < \infty \forall x \in H$ (due to convergence). Hence, by the uniform boundedness principle: $\sup_n \|T_n\| < \infty$. So, for all $x \in H$, $\|x\| = 1$:

$$\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \left(\sup_{n \in \mathbb{N}} \|T_n\| \right) \|x\| = \sup_{n \in \mathbb{N}} \|T_n\|,$$

hence,

$$\|T\| = \sup_{x \in H, \|x\|=1} \|Tx\| \leq \sup_{n \in \mathbb{N}} \|T_n\| < \infty$$

Hence $T \in BL(H)$ and $T_n \xrightarrow{s} T$ (by def.).

(b) See exercise.

Example 5.5 Let $(T_n)_n \subseteq BL(\ell^2)$.

(a) For $x = (x_1, x_2, \dots) \in \ell^2$ and $n \in \mathbb{N}$ let

$$T_n x := \left(\frac{1}{n} x_1, \frac{1}{n} x_2, \dots \right) = \frac{1}{n} x.$$

Then $\|T_n\| = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$, i.e. uniform convergence to zero (operator).

(b) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_{n+1}, x_{n+2}, \dots)$

(i) $T_n \xrightarrow{s} 0$ because $\|T_n x\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \xrightarrow{n \rightarrow \infty} 0 \quad x \in \ell^2$.

(ii) $(T_n)_n$ does not converge uniformly to 0, because

$$T_n e_{n+1} = e_{n+1}, \text{ so } \|T_n\| \geq 1.$$

(c) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_1, x_2, \dots)$

i.e. "n times iterated right shift" (see 2.26)

(i) $T_n \xrightarrow{w} 0$, since for $y \in \ell^2$, by Cauchy-Schwarz, (110)

$$|\langle y, T_n x \rangle| = \left| \sum_{j=1}^{\infty} \overline{y_{n+j}} x_j \right| \leq \|x\| \cdot \underbrace{\left(\sum_{j=1}^{\infty} |y_{n+j}|^2 \right)^{1/2}}_{\xrightarrow{n \rightarrow \infty} 0 \text{ since } y \in \ell^2}$$

(ii) $(T_n)_n$ does not converge strongly to 0, because $\|T_n x\| = \|x\| \quad \forall x \in \ell^2$.

5.2 Adjoint operators

Definition 5.6 (a) Let X, Y be normed spaces, and $T \in BL(X, Y)$.

Then $T^*: Y^* \rightarrow X^*$ is a linear operator, the
 $\ell \mapsto \ell \circ T$ adjoint (operator) of T

(b) Let H_1, H_2 be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then

$T^*: H_2 \rightarrow H_1$ where (by Riesz!) $T^* y$ is the unique $z \in H_1$,
 $y \mapsto T^* y$ such that, $\forall x \in H_1$, $\langle y, Tx \rangle_{H_2} = \langle z, x \rangle_{H_1}$ (*)

is a linear operator, the (Hilbert space) adjoint (operator) of T .

Note that, with $\ell_y := \langle y, \cdot \rangle_{H_2}$, the linear functional

$T^* \ell_y: H_1 \rightarrow \mathbb{K}$ belongs to H_1^*
 $x \mapsto \langle y, Tx \rangle_{H_2}$

Remarks 5.7 (a) In the Hilbert space case we have that

$T^* = J_1 T^* J_2^{-1}$, with $J_j: H_j^* \rightarrow H_j$ the antilinear, isometric and bijective identification (Cor. 2-58).

Indeed, from (*) : LHS = $(J_2^{-1} y)(Tx) = (T^*(J_2^{-1} y))(x)$,

RHS = $(J_1^{-1}(T^* y))(x)$.

(b) T^* is linear (!) by part (a) of this remark.

(c) Notation varies. Some denote the (Banach space) adjoint T^* by T' , or even T^* .

Examples 5.8 (a) Let $X = Y = \ell^2$, and $Rx := (0, x_1, x_2, \dots)$ (iii)
 the right-shift operator. Then $R \in BL(\ell^2)$. Hence $R^*: (\ell^2)^* \rightarrow (\ell^2)^*$
 (Riesz representation: $(\ell^2)^* \ni f_z \xrightarrow{z \mapsto z} z \in \ell^\infty$ with $f_z(x) = \sum_{n \in \mathbb{N}} z_n x_n$
 for every $x \in \ell^2$). So:

$$(R^* f_z)(x) = f_z(Rx) = \sum_{n \in \mathbb{N}} z_n (Rx)_n = \sum_{n \in \mathbb{N}} z_{n+1} x_n = \sum_{n \in \mathbb{N}} (Lz)_n x_n = f_{Lz}(x)$$

where $L: \ell^\infty \rightarrow \ell^\infty, z \mapsto (z_2, z_3, z_4, \dots)$ is the left shift operator.

Hence, $R^* f_z = f_{Lz}$.

(b) For $H_1 = H_2 = H$ and $\varphi, \psi \in H$, let $P := \langle \varphi, \cdot \rangle \psi$, i.e.,

$$Px = \langle \varphi, x \rangle \psi \quad \forall x \in H \quad (\text{ran } P = \text{span}\{\psi\}).$$

Then $P^* = \langle \psi, \cdot \rangle \varphi$ since, for all $x, y \in H$,

$$\langle y, Px \rangle = \langle y, \psi \rangle \langle \varphi, x \rangle = \langle \langle \varphi, y \rangle \varphi, x \rangle = \langle P^* y, x \rangle.$$

As an application of adjoint operators, we prove that bounded operators preserve weak convergence:

Lemma 5.9 Let X, Y be normed spaces, and $T \in BL(X, Y)$.

Let $(x_n)_{n \in \mathbb{N}} \subseteq X, x \in X$. Then

$$x_n \xrightarrow{w} x \quad \Rightarrow \quad Tx_n \xrightarrow{w} Tx, \quad n \rightarrow \infty.$$

Pf: Let $\ell \in Y^*$. Then, $\ell(Tx_n) - \ell(Tx) = \underbrace{(\ell \circ T)}_{\in X^*} (x_n - x) \xrightarrow{n \rightarrow \infty} 0$ $\square$

Theorem 5.10 Let X, Y be normed spaces and $T \in BL(X, Y)$.

Then $\|T^*\|_{Y^* \rightarrow X^*} = \|T\|_{X \rightarrow Y}$

Pf: By definition and Thm. 4.15 we have

$$\begin{aligned} \|T\|_{X \rightarrow Y} &= \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|_Y \stackrel{4.15}{=} \sup_{\substack{x \in X \\ \|x\|=1}} \sup_{\substack{\ell \in Y^* \\ \|\ell\|_{Y^*}=1}} |\ell(Tx)| \stackrel{(!)}{=} \sup_{\substack{\ell \in Y^* \\ \|\ell\|_{Y^*}=1}} \sup_{\substack{x \in X \\ \|x\|=1}} |(\ell \circ T)(x)| \\ &= \|T^*\|_{Y^* \rightarrow X^*} \quad \square \end{aligned}$$

Corollary 5.11 Let H_1, H_2 be Hilbert spaces, and

$$T \in BL(H_1, H_2). \text{ Then } \|T^*\|_{H_2 \rightarrow H_1} = \|T\|_{H_1 \rightarrow H_2}$$

Pf: Follows from $T^* = J_1 T^* J_2^{-1}$ and Thm. 5.10, since $J_j: H_j^* \rightarrow H_j, j=1,2$, are isometric & bijective. $\square$

Theorem 5.12 Let H be a Hilbert space, and $T, S \in BL(H)$. Then

(a) The map $(\cdot)^*: BL(H) \rightarrow BL(H)$ is anti-linear, isometric and bijective, with $\mathbb{I}^* = \mathbb{I}$.
 $T \mapsto T^*$

$$(b) (TS)^* = S^* T^* \quad (c) (T^*)^* = T$$

(d) If T has inverse $T^{-1} \in BL(H)$, then $(T^*)^{-1} = (T^{-1})^*$.

In particular, $(T^*)^{-1} \in BL(H)$.

$$(e) \|TT^*\| = \|T\|^2 \quad (\underline{C^*}\text{-property}).$$

Pf: (a): Anti-linear by definition of T^* ; isometric by Cor. 5.11; surjective by part (c) of this theorem.

(b), (c): See exercise.

(d): $T^{-1}T = \mathbb{I} = TT^{-1}$ together with (b) gives

$$T^*(T^{-1})^* = \mathbb{I}^* = \mathbb{I} = (T^{-1})^* T^*. \quad \checkmark$$

(e) On the one hand, we have (by Cor. 5.11) $\|TT^*\| \leq \|T\| \cdot \|T^*\| = \|T\|^2$.

On the other hand,

$$\|TT^*\| \stackrel{4.15}{=} \sup_{0 \neq x, y \in H} \frac{|\langle y, TT^*x \rangle|}{\|x\| \|y\|} \stackrel{y=x}{\geq} \sup_{0 \neq x \in H} \frac{\|T^*x\|^2}{\|x\|^2} = \|T^*\|^2 \stackrel{5.11}{=} \|T\|^2 \quad \square$$

Definition 5.13 Let H be a Hilbert space, and $T \in BL(H)$.

(i) T is normal : $\Leftrightarrow TT^* = T^*T$ (they commute!)

(ii) T is self-adjoint : $\Leftrightarrow T = T^*$

(iii) T is unitary : $\Leftrightarrow T$ is bijective and $T^{-1} = T^*$

$$(\text{i.e., } TT^* = \mathbb{I} = T^*T).$$

Remark 5.14 (a) self-adjoint or unitary $\Rightarrow$ normal.

(b) T normal $\Leftrightarrow \langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle \quad \forall x, y \in H$.

Pf: LHS = $\langle T^*Tx, y \rangle$, RHS = $\langle (T^*)^*T^*x, y \rangle = \langle TT^*x, y \rangle$.

Hence: T normal $\Rightarrow \|Tx\| = \|T^*x\| \quad \forall x \in H$.

In particular: $\ker(T) = \ker(T^*)$. (if T normal!)

(c) T self-adjoint $\Leftrightarrow \langle x, Ty \rangle = \langle Tx, y \rangle \quad \forall x, y \in H$ (*)

Setting $x = y$ gives: $\langle x, Tx \rangle \in \mathbb{R} \quad \forall x \in H$.

[The property (*) is called symmetry (T is symmetric). The notions "symmetric" and "selfadjoint" agree for bounded linear operators, but their generalisations to unbounded operators do NOT!]

(d) T unitary $\Leftrightarrow \langle Tx, Ty \rangle = \langle x, y \rangle = \langle T^*x, T^*y \rangle \quad \forall x, y \in H$.

Pf: $\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle$ and $\langle T^*x, T^*y \rangle = \langle (T^*)^*T^*x, y \rangle = \langle TT^*x, y \rangle$

Example 5.15 (a) $T = z\mathbb{I}$, $z \in \mathbb{K}$. Then $T^* = \bar{z}\mathbb{I}$. So, T is self-adjoint iff. $z \in \mathbb{R}$.

(b) Let $H = L^2([0, 1])$, $k \in C([0, 1]^2)$ and

$$(Tf)(x) := \int_0^1 k(x, y)f(y) dy \quad \forall f \in H, \forall x \in [0, 1]$$

Then $T \in BL(H)$ and, if $k(x, y) = \overline{k(y, x)} \quad \forall x, y \in [0, 1]$, then T is selfadjoint

5.3 The spectrum

In this subsection: $\mathbb{X}$ Banach over $\mathbb{C}$
(i.e. $\mathbb{K} = \mathbb{C}$)

Definition 5.16 | Let $T \in BL(\mathbb{X})$

(i) The resolvent set (of T) is $\rho(T) := \{z \in \mathbb{C} \mid T - z\mathbb{I} \text{ is bijective}\}$

(ii) The resolvent of T is $R_z := R(z, T) := (T - z\mathbb{I})^{-1} =: (T - z)^{-1}$

for which ever $z \in \mathbb{C}$ this exists as a (possibly unbounded) operator.

(a) If $z \in \rho(T)$, then $R_z \in BL(\mathbb{X})$ (bounded inverse theorem!)

(b) R_z need not exist for $z \notin \rho(T)$.

(iii) The spectrum of T is $\text{spec}(T) := \sigma(T) := \mathbb{C} \setminus \rho(T)$

(iv) If there exists $0 \neq x \in \mathbb{X}$ s.t. $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then λ is an eigenvalue and x the corresponding eigenvector.

(v) The point spectrum of T is
 $\text{spec}_p(T) := \sigma_p(T) := \{z \in \mathbb{C} \mid T - z \text{ not injective}\} = \{\text{eigenvalues of } T\}$ (114)

(vi) The continuous spectrum of T is

$$\text{spec}_c(T) := \sigma_c(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \text{ran}(T - z) \neq \mathbb{X}, \text{ but dense}\}$$

(vii) The residual spectrum of T is

$$\text{spec}_r(T) := \sigma_r(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \text{ran}(T - z) \text{ not dense}\}.$$

[$\sigma_r(T) = \emptyset$ for most T of interest in applications]

Lemma 5.17 Let $T \in BL(\mathbb{X})$. Then

(a) $\mathbb{C} = \sigma(T) \dot{\cup} \rho(T)$ (disjoint union)

(b) $\sigma(T) = \sigma_p(T) \dot{\cup} \sigma_c(T) \dot{\cup} \sigma_r(T)$

(c) If $\dim \mathbb{X} < \infty$, then $\sigma_c(T) = \emptyset = \sigma_r(T)$ (so $\sigma(T) = \sigma_p(T)$).

Pf: (a), (b) clear by definition.

(c) follows from Linear Algebra: $T - z$ injective $\Rightarrow \dim \ker(T - z) = 0$

$$\stackrel{\dim \mathbb{X} < \infty}{\Rightarrow} \dim \text{ran}(T - z) = \dim(\mathbb{X}) \Rightarrow \text{ran}(T - z) = \mathbb{X}$$

Lemma 5.18 (Neumann series). Let $T \in BL(\mathbb{X})$, with $\|T\| < 1$.

Then $(\mathbb{I} - T)^{-1} \in BL(\mathbb{X})$ and $(\mathbb{I} - T)^{-1} = \sum_{j=0}^{\infty} T^j$.

Pf: See exercise (T6.3 Sheet 6). Here details for completeness:

From $\|T^j\| \leq \|T\|^j$, $\|T\| < 1$, and Lemma 2.50 (a) follows: $S := \sum_{j=0}^{\infty} T^j \in BL(\mathbb{X})$

For $N \in \mathbb{N}$, we have

$$(\mathbb{I} - T) \sum_{j=0}^N T^j = \mathbb{I} - T^{N+1} = \left(\sum_{j=0}^N T^j \right) (\mathbb{I} - T).$$

Now, $\sum_{j=0}^N T^j \xrightarrow{N \rightarrow \infty} S$ and $T^{N+1} \xrightarrow{N \rightarrow \infty} 0$ (since $\|T^{N+1}\| \leq \|T\|^{N+1} \xrightarrow{N \rightarrow \infty} 0$). Hence,

$$(\mathbb{I} - T)S = \mathbb{I} = S(\mathbb{I} - T) \text{ so } S = (\mathbb{I} - T)^{-1}$$

Lemma 5.19 Let $T \in BL(\mathbb{X})$. Then $\sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|T\|\}$

Pf: For $|z| > \|T\|$ we have $\|\frac{T}{z}\| < 1$, hence 5.18 gives existence of

$$BL(\mathbb{X}) \ni \frac{1}{z} \left(\frac{T}{z} - \mathbb{I} \right)^{-1} = (T - z\mathbb{I})^{-1} = R_z.$$

Hence, $z \in \rho(T) = \mathbb{C} \setminus \sigma(T)$

Definition 5.20 Let $D \subseteq \mathbb{C}$ be open, $(Z, \|\cdot\|)$ a Banach space over $\mathbb{C}$, and $\xi: D \rightarrow Z$
 $z \mapsto \xi(z)$

(i) The map ξ is norm-differentiable (or strongly differentiable) at $z_0 \in D$

$$: \Leftrightarrow \exists \zeta \in Z: \lim_{h \rightarrow 0} \left\| \frac{\xi(z_0+h) - \xi(z_0)}{h} - \zeta \right\| = 0$$

Notation: $\frac{d\xi}{dz}(z_0) := \zeta$.

(ii) ξ is weakly differentiable at $z_0 \in D$

$$: \Leftrightarrow z \mapsto \ell(\xi(z)) \in \mathbb{C} \text{ is complex differentiable at } z_0 \quad \forall \ell \in X^*$$

(iii) ξ is norm-analytic (resp. weakly analytic) in D

$$: \Leftrightarrow \xi \text{ is norm-differentiable (resp. weakly differentiable) at } z_0 \text{ for all } z_0 \in D.$$

Remark 5.21 (a) Z -valued norm-analytic functions have properties analogous to those of $\mathbb{C}$ -valued analytic functions (e.g. a power-series expansion converging wrt. $\|\cdot\|$, etc.).

Remark: Replace $|\cdot|$ (abs. value in $\mathbb{C}$) by $\|\cdot\|$

(For a discussion, see Hille-Phillips, "Functional analysis and semigroups", or Dunford-Schwartz, "Linear operators" vol 1, Sect. III.14.)

(b) Norm-differentiable $\Leftrightarrow$ weakly differentiable (For " $\Leftarrow$ ", see Reed-Simon Thm. VI.4)

Theorem 5.22 Let $T \in BL(X)$. Then

(a) $\rho(T) \subseteq \mathbb{C}$ is open in $\mathbb{C}$, and the map $R(\cdot, T): \rho(T) \rightarrow BL(X)$
 $z \mapsto R(z, T)$
 is norm-analytic on $\rho(T)$, with
 derivative $\frac{dR_z}{dz} = R_z^2$ (*)

[Note: $Z = BL(X)$ in Def. 5.20 in this case - i.e., complex diff. holds wrt. the operator norm]

(b) For $z_1, z_2 \in \rho(T)$, we have

$$(**) \quad R_{z_1} - R_{z_2} = (z_1 - z_2) R_{z_1} R_{z_2} \quad (\text{First resolvent identity})$$

In particular: $R_{z_1} R_{z_2} = R_{z_2} R_{z_1}$ (they commute!)

Pf: (b) Equation (**) follows from

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$$\begin{aligned} (T - z_1)(R_{z_1} - R_{z_2})(T - z_2) &= (\mathbb{I} - (T - z_1)R_{z_2})(T - z_2) \\ &= T - z_2 - (T - z_1) = z_1 - z_2 \end{aligned}$$

and from multiplication by R_{z_1} from the left and R_{z_2} from the right.

Commutativity: Exchange $z_1 \leftrightarrow z_2$ in (**) & compare with (**).

(a) Openness of $\rho(T)$: Let $z_0 \in \rho(T)$ ($\neq \emptyset$ by Lemma 5.19)

and $z \in B_{1/\|R_{z_0}\|}(z_0)$ (open ball in $\mathbb{C}$). Then

$$T - z = T - z_0 - (z - z_0) = (T - z_0)(\mathbb{I} - \underbrace{(z - z_0)R_{z_0}}_{=: V_z})$$

By definition, $\|V_z\| < 1$, hence by

Lemma 5.18: $(\mathbb{I} - V_z)^{-1} \in BL(X)$. So,

$$R_z = (T - z)^{-1} = (\mathbb{I} - V_z)^{-1} R_{z_0} \in BL(X) \quad (II)$$

and so $z \in \rho(T)$, i.e. $\rho(T)$ is open.

Analytic: Let $z_0 \in \rho(T)$. By the above: $\forall 0 \neq h \in \mathbb{C}$, $|h| < \|R_{z_0}\|^{-1}$:

$z_0 + h \in \rho(T)$. Hence

$$\frac{R_{z_0+h} - R_{z_0}}{h} \stackrel{(**)}{=} R_{z_0+h} R_{z_0} \stackrel{(II) \& 5.18}{=} \sum_{n \in \mathbb{N}_0} V_{z_0+h}^n R_{z_0}^2 = \sum_{n \in \mathbb{N}_0} h^n R_{z_0}^{n+2}$$

and so

$$\left\| \frac{R_{z_0+h} - R_{z_0}}{h} - R_{z_0}^2 \right\| \leq \sum_{n \in \mathbb{N}} \frac{|h|^n}{|h|} \|R_{z_0}\|^{n+2} = \|R_{z_0}\|^2 \left\{ \frac{1}{1 - |h|\|R_{z_0}\|} - 1 \right\}$$

$|h| \rightarrow 0$

Hence, $z \mapsto R_z \in BL(X)$ is norm-differentiable at z_0 , with derivative $R_{z_0}^2$. As $z_0 \in \rho(T)$ was arbitrary, $R(\cdot, T)$ is norm-analytic in $\rho(T)$ (and (**) holds) ■

(Lemma 5.23) Let $T \in BL(X)$. Then $\sigma(T) \neq \emptyset$.

Pf: Let $|z| > \|T\|$. Then (see pf Lemma 5.19)

$$R_z = (T - z)^{-1} = \frac{1}{z} \left(\frac{T}{z} - \mathbb{I} \right)^{-1} = -\frac{1}{z} \sum_{j=0}^{\infty} \frac{T^j}{z^j}$$

This expansion shows that

$$\|R_z\| \leq \frac{|z|^{-1}}{1 - \|T\|/|z|} = \frac{1}{|z| - \|T\|} \xrightarrow{|z| \rightarrow \infty} 0 \quad (\nabla)$$

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Assume $\sigma(T) = \emptyset$ (for contradiction). Then $\rho(T) = \mathbb{C}$ and, by Thm 5.22, $\mathbb{C} \rightarrow BL(X)$, $z \mapsto R_z$, is entire (i.e. analytic on all of $\mathbb{C}$). Also, $\mathbb{C} \ni z \mapsto \|R_z\|$ is bounded (using (∇) & continuity). By Liouville's Thm, (for $\mathbb{C}$ -valued fct's, see any book on Complex Analysis) $z \mapsto \|R_z\|$ is constant, and by (∇) , this constant has to be 0, i.e. $R_z = 0 \ \forall z \in \mathbb{C}$ $\hookrightarrow$ $\square$

Definition 5.24 For $T \in BL(X)$, the spectral radius of T is

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\| \quad \text{Lemma 5.19}$$

Theorem 5.25 Let $T \in BL(X)$. Then

$$(a) \quad r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$$

(b) If, in addition, X is a Hilbert space, and T is normal, then: $\lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \|T\|$. Hence: $r(T) = \|T\|$.

Pf: (a) We first prove: $\lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$:

Wlog. assume $T^n \neq 0 \ \forall n \in \mathbb{N}$ (otherwise, if $\exists n_0: T^{n_0} = 0$, then $T^n = 0 \ \forall n \geq n_0$ and claim is clear).

Let $a_n := \ln \|T^n\|$. Then $a_{n_1+n_2} \leq a_{n_1} + a_{n_2} \ \forall n_1, n_2 \in \mathbb{N}$ (check!).

Fix $m \in \mathbb{N}$ (arbitrary), and let $u(q,r) := qm + r$ for $q \in \mathbb{N}$ and $r \in \{0, 1, \dots, m-1\}$. Then

$$\frac{a_{u(q,r)}}{u(q,r)} \leq \frac{qa_m + a_r}{qm + r} = \frac{a_m}{m} \frac{1}{1 + r/(qm)} + \frac{a_r}{qm + r}$$

Hence,

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \leq \limsup_{q \rightarrow \infty} \max_{r \in \{0, \dots, m-1\}} \frac{a_n(q, r)}{n(q, r)} \leq \frac{a_m}{m} \quad \text{118} \quad \forall m \in \mathbb{N}$$

So
$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \leq \inf_{m \in \mathbb{N}} \frac{a_m}{m}.$$

But (trivially) $\liminf_{n \rightarrow \infty} \frac{a_n}{n} \geq \inf_{m \in \mathbb{N}} \frac{a_m}{m}$. Taken together,

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \in \mathbb{N}} \frac{a_n}{n} \text{ exists, and } \lim_{n \rightarrow \infty} e^{a_n/n} = \inf_{n \in \mathbb{N}} e^{a_n/n}$$

because the exp. function is continuous and increasing. ✓

Now, by Thm. 5.22, $z \mapsto R_z$ is strongly analytic in $D := \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| \leq r(T)\}$, hence it has a Laurent series expansion about the origin ($\in \mathbb{C}$):

$$R_z = \sum_{j \in \mathbb{Z}} z^j A_j, \quad A_j \in B(X) \text{ for } j \in \mathbb{Z},$$

which is norm-convergent $\forall z \in D$. [For $\mathbb{C}$ -valued fct's, see Conway, "Fct.'s of one complex variable", 2nd ed. (1978), Thm V.1.11]

On the other hand, on $\{z \in \mathbb{C} \mid |z| > \|T\|\} \subseteq D$, the expansion $(*) \quad R_z = -\frac{1}{z} \sum_{j=0}^{\infty} \frac{T^j}{z^j}$ holds (see p. Lemma 5.23).

Hence, by uniqueness of the Laurent series, $(*)$ is the Laurent series of the resolvent, hence norm convergent $\forall z \in \underline{D}$.

Equivalently, the power series

$$z \mapsto \sum_{j \in \mathbb{N}_0} z^j T^j \quad (**)$$

converges on $B_{1/r(T)}(0)$.

On the other hand: For any $\varepsilon > 0$, the power series in $(**)$ is not norm-convergent on $B_{\varepsilon + \frac{1}{r(T)}}(0)$, since if, then $(*)$ would converge for some $z \in \delta(T)$ (recall: $r(T) = \sup_{\lambda \in \delta(T)} |\lambda|$). Therefore $\frac{1}{r(T)}$ is the radius of convergence of $(**)$, and as such, it is

given by Hadamard's root criterion:

(119)

$$r(T) = \limsup_{j \rightarrow \infty} \|T^j\|^{1/j} = \lim_{j \rightarrow \infty} \|T^j\|^{1/j}$$

where the last equality was proved in the first part. ✓

(b) Let C^* refer to the C^* -property $\|TT^*\| = \|T\|^2$ from Thm. 5.12(e).

Then

$$\|T^2\|^{2^{C^* \text{ for } T^2}} = \|T^2(T^2)^*\|^{T \text{ normal}} = \|(TT^*)(TT^*)^*\|^{C^* \text{ for } TT^*} = \|TT^*\|^2 = \|T\|^4,$$

so $\|T^2\| = \|T\|^2$. By induction on $k \in \mathbb{N}$:

$$\|T^{2^k}\| = \|T\|^{2^k} \quad (\forall)$$

$$\text{hence } \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \lim_{k \rightarrow \infty} \|T^{2^k}\|^{1/2^k} \stackrel{(\forall)}{=} \|T\|$$

Example 5.26 Right-shift & left-shift operator on ℓ^2 :

For $x = (x_1, x_2, \dots) \in \ell^2$, let $Rx := (0, x_1, x_2, \dots)$ & $Lx := (x_2, x_3, \dots)$

Then $\|R\| = 1 = \|L\|$, R is isometric and $R^* = L$ (check!)

Claims: (i) $\sigma_p(L) = B_1(0)$ (open ball in $\mathbb{C}$) (ii) $\sigma_c(L) = \partial B_1(0)$

(iii) $\sigma_r(L) = \emptyset$ (iv) $\sigma_p(R) = \emptyset$ (v) $\sigma_c(R) = \partial B_1(0)$

(vi) $\sigma_r(R) = B_1(0)$.

Proofs: (i) Let $\lambda \in \mathbb{C}$, then $Lx = \lambda x \Leftrightarrow (x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots)$

$$\Leftrightarrow x_2 = \lambda x_1, x_3 = \lambda x_2 = \lambda^2 x_1, \dots \Leftrightarrow x = x_1 (1, \lambda, \lambda^2, \lambda^3, \dots)$$

$$\text{Hence, } x \in \ell^2 \text{ \& } Lx = \lambda x \Leftrightarrow \sum_{n \in \mathbb{N}} (|\lambda|^2)^n < \infty \Leftrightarrow |\lambda| < 1 \vee$$

(iv) R is isometric $\Rightarrow \sigma_p(R) \subseteq \partial B_1(0)$ (check!). Let $|\lambda| = 1$,

$$\text{then } Rx = \lambda x \Leftrightarrow (0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$$

$$\Leftrightarrow 0 = \lambda x_1, x_1 = \lambda x_2 \stackrel{\lambda \neq 0}{\Leftrightarrow} x_1 = 0, x_2 = 0, \dots \quad \checkmark$$

(iii) Lemma (pf: see exercise). Let X be a Hilb. sp., $T \in BL(X)$

Then (a) $z \in \sigma_r(T) \Rightarrow \bar{z} \in \sigma_p(T^*)$

(b) $z \in \sigma_p(T) \Rightarrow \bar{z} \in \sigma_p(T^*) \cup \sigma_r(T^*)$

Hence from (a) with $T=L: (iv) \Rightarrow (iii)$.

(120)

(ii), (v) and (vi): See exercises.

5.4 Compact operators

General assumption: X and Y are both Banach spaces

Recall: In metric spaces: compactness $\stackrel{1.27}{\Leftrightarrow}$ sequential cpt.ness

Hence the equivalence in

Definition 5.27 An operator $T \in BL(X, Y)$ is compact

$\Leftrightarrow$
 $\forall A \subseteq X: A$ bounded $\Rightarrow \overline{T(A)} \subseteq Y$ compact (i.e. $T(A)$ rel. cpt. in Y)
 $\Leftrightarrow \forall (x_n)_n \subseteq X: (x_n)_n$ bounded $\Rightarrow (Tx_n)_n$ has a convergent subsequence

Example 5.28 (a) For $k \in C([0,1]^2)$, let $T: L^2([0,1]) \rightarrow L^2([0,1])$,

$$(Tf)(x) = \int_0^x k(x,y)f(y) dy, \quad f \in L^2([0,1]).$$

Then T is compact (see Exercise E13.2)

(b) Finite-rank operators:

$T \in BL(X, Y)$ is of finite rank: $\Leftrightarrow \dim \text{ran}(T) < \infty$.

Then T is compact:

$(x_n)_n \subseteq X$ bounded $\stackrel{T \text{ bdd.}}{\Rightarrow} (Tx_n)_n \subseteq \text{ran}(T)$ bounded,
hence, $(Tx_n)_n$ is a bounded seq. in finite-dim. space, so,
by Bolzano-Weierstraß, it has convergent subsequence.

Theorem 5.29 Let $T \in BL(X, Y)$ be compact, and $(x_n)_n \subseteq X$ s.t.

$x_n \xrightarrow{w} x \in X, n \rightarrow \infty$. Then $Tx_n \xrightarrow{w} Tx$ strongly, i.e.

wrt. the norm in Y $(\|Tx_n - Tx\|_Y \rightarrow 0, n \rightarrow \infty)$

Pf: Let $y_n := Tx_n$ and $y := Tx$. Since $x_n \xrightarrow{w} x$, we have (by Lem. 5.9): $y_n \xrightarrow{w} y$. Assume, for contradiction, that $y_n \not\xrightarrow{s} y$ (i.e. not conv. in norm).

Then $\exists \varepsilon > 0, \exists (y_{n_k})_k : \|y_{n_k} - y\| \geq \varepsilon \quad \forall k \in \mathbb{N}$.

But $(x_{n_k})_{k \in \mathbb{N}}$ is weakly conv., so (by 4.29) bounded.

Since T is compact, $\exists (y_{n_k})_k : y_{n_k} \xrightarrow{l \rightarrow \infty} \tilde{y} \in Y, \tilde{y} \neq y$.

Since $y_{n_k} \xrightarrow{w} y \neq \tilde{y}$ (4.27(c)) this is a contradiction,

so $y_n \xrightarrow{w \rightarrow s} y$ strongly / in norm $\square$

Theorem 5.30 | Let $T \in BL(X, Y)$.

(a) Assume $(T_n)_{n \in \mathbb{N}} \subseteq BL(X, Y)$, T_n compact $\forall n \in \mathbb{N}$, and

$$\|T_n - T\| \xrightarrow{n \rightarrow \infty} 0, \quad T \in BL(X, Y)$$

Then T is compact.

(b) T is compact iff T^* is compact (Schauder's Theorem)

(c) Let Z be a Banach space and $S \in BL(Y, Z)$.

If S or T is compact, then $ST \in BL(X, Z)$ is compact.

(d) In particular: Denote $K(X, Y) := \{T \in BL(X, Y) \mid T \text{ compact}\}$,

$K(X) := K(X, X)$. Then $K(X) \subseteq BL(X)$ is a two-sided norm-closed ideal.

Pf: (d) follows from (a) & (c) ($Y = Z = X$).

(c) Clear, since bounded linear operators preserve convergence & boundedness.

(a) Let $(x_m)_{m \in \mathbb{N}} \subseteq X$ be bounded, wlog. $\|x_m\| \leq 1 \quad \forall m \in \mathbb{N}$.

As T_n is compact, $(T_n x_m)_m$ has a convergent subseq (dep. on n)

with limit $y_n \in Y \quad \forall n \in \mathbb{N}$. By Cantor's diagonal sequence trick, $\exists$ common subseq. $(x_{m_k})_{k \in \mathbb{N}}$ (i.e. indep. of n !)

s.t. $T_n x_{m_k} \xrightarrow{k \rightarrow \infty} y_n \quad \forall n \in \mathbb{N}$ (check!). $\square$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|T_n - T_{n'}\| < \varepsilon \quad \forall n, n' \geq N$. (122)

Then

$$\|y_n - y_{n'}\| \leq \underbrace{\|y_n - T_n x_{m_k}\|}_{(1)} + \underbrace{\|T_n x_{m_k} - T_{n'} x_{m_k}\|}_{\leq \|T_n - T_{n'}\| < \varepsilon} + \underbrace{\|T_{n'} x_{m_k} - y_{n'}\|}_{(2)}$$

$\xrightarrow{k \rightarrow \infty} < \varepsilon$, since (1) & (2) tend to 0 as $k \rightarrow \infty$ (by (*))

So $(y_n)_{n \in \mathbb{N}}$ is Cauchy, hence (Y Banach) convergent, i.e.

$$y_n \xrightarrow{n \rightarrow \infty} \gamma \in Y$$

Claim: $\|T x_{m_k} - \gamma\| \xrightarrow{k \rightarrow \infty} 0$.

Indeed, let $\varepsilon > 0$. $\exists n \in \mathbb{N}$ s.t.

$$\|T - T_n\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|y_n - \gamma\| < \frac{\varepsilon}{3}$$

Fix this. Then there is $K \in \mathbb{N}$ s.t. $\|T_n x_{m_k} - y_n\| < \frac{\varepsilon}{3} \quad \forall k \geq K$

Hence, for all $k \geq K$, (by (*))

$$\begin{aligned} \|T x_{m_k} - \gamma\| &\leq \underbrace{\|T x_{m_k} - T_n x_{m_k}\|}_{\leq \|T - T_n\| \cdot \|x_{m_k}\| \leq \varepsilon} + \underbrace{\|T_n x_{m_k} - y_n\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|y_n - \gamma\|}_{< \frac{\varepsilon}{3}} \\ &< \varepsilon \end{aligned}$$

So, claim holds, and T is compact.

(b) $\Rightarrow$: Let $(f_n)_{n \in \mathbb{N}} \subseteq Y^*$ be a bounded sequence.

Then $K := T(\overline{B_1^X(0)}) = \overline{T(B_1^X(0))} \subseteq Y$ is a compact metric space. The restrictions $f_n := f_n|_K \in C(K)$ (equipped with $\|\cdot\|_\infty$ -norm) form a bounded and equicontinuous sequence, because

$$|f_n(y_1) - f_n(y_2)| \leq \underbrace{\left(\sup_{k \in \mathbb{N}} \|f_k\| \right)}_{< \infty} \cdot \|y_1 - y_2\|_Y \quad \forall y_1, y_2 \in K$$

By the theorem of Arzelà-Ascoli (1.40) then (123)
exists a uniformly convergent subsequence $(f_{n_k})_{k \in \mathbb{N}}$,
hence Cauchy, i.e. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall k, l \geq N :$

$$\|f_{n_k} - f_{n_l}\|_{C(K)} < \varepsilon$$

So, for all $k, l \geq N :$

$$\begin{aligned} \|T^x l_{n_k} - T^x l_{n_l}\|_{\mathbb{X}^*} &= \sup_{x \in B_1^{\mathbb{X}}(0)} |T^x l_{n_k}(x) - T^x l_{n_l}(x)| \\ &= \sup_{x \in B_1^{\mathbb{X}}(0)} |l_{n_k}(Tx) - l_{n_l}(Tx)| \leq \|f_{n_k} - f_{n_l}\|_{C(K)} < \varepsilon. \end{aligned}$$

($\uparrow$ in fact " $=$ ", since $T(B_1^{\mathbb{X}}(0))$ dense in K).

So $(T^x l_{n_k})_{k \in \mathbb{N}}$ converges in $\mathbb{X}^*$, hence $T^x : \mathbb{Y}^* \rightarrow \mathbb{X}^*$ is cpt.

" $\Leftarrow$ ": If T^x compact, then $T^{xx} : \mathbb{X}^{**} \rightarrow \mathbb{Y}^{**}$ is compact

by " $\Rightarrow$ ". So, from (c), $T^{xx} J_{\mathbb{X}}$ is compact, where
 $J_{\mathbb{X}} : \mathbb{X} \rightarrow \mathbb{X}^{**}$ is the canonical embedding, $(J_{\mathbb{X}} x)(f) = f(x)$
 $\forall x \in \mathbb{X}, \forall f \in \mathbb{X}^*$. Now, for $x \in \mathbb{X}$ and $l \in \mathbb{Y}^*$ we have

$$(T^{xx} J_{\mathbb{X}} x)(l) = J_{\mathbb{X}}(x)(T^x l) = (T^x l)(x) = l(Tx) = (J_{\mathbb{Y}}(Tx))(l)$$

This implies $T^{xx} J_{\mathbb{X}} = J_{\mathbb{Y}} T$ being compact. Since
 $\mathbb{Y}$ is closed in $\mathbb{Y}^{**}$ (i.e. $J_{\mathbb{Y}}(\mathbb{Y}) \subseteq \mathbb{Y}^{**}$ is closed), T is
also compact. $\blacksquare$

Theorem 5.31 Let H be a separable Hilbert space.

Then $K(H) = \{T \in BL(H) \mid T \text{ compact}\} = \overline{\{T \in BL(H) \mid T \text{ has finite rank}\}}^{\|\cdot\|}$
where the closure is wrt. the operator norm.

Pf: " $\supseteq$ " holds because of Ex. 5.28(b) and Thm. 5.30(a)

(this does not need separability - or that H is Hilbert space
(Banach ok)).

" $\subseteq$ ": Need to prove: Any compact $T \in BL(H)$ is the norm-limit of finite-rank operators: (124)

If $\dim(H) < \infty$, there is nothing to prove. So let $\dim(H) = \infty$ and let $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq H$ be an ONB. For $n \in \mathbb{N}$, define the orthogonal rank- n -projection $P_n := \sum_{j=1}^n \langle \varphi_j, \cdot \rangle \varphi_j$.

By Thm. 2.51: $P_n \xrightarrow{s} \Pi$, hence $\Pi - P_n = \sum_{j=n+1}^{\infty} \langle \varphi_j, \cdot \rangle \varphi_j$ (where the series converges in the strong operator sense).

Let $R_n := TP_n \in BL(H)$ for $n \in \mathbb{N}$. Then R_n has rank $\leq n$, and converges to T strongly ($R_n \xrightarrow{s} T$).

Strategy: Use compactness of T to prove convergence even uniformly.

Let $\varphi \in H$, $\|\varphi\| = 1$. Then

$$\|(\Pi - P_n)\varphi\|^2 = \sum_{j=n+1}^{\infty} |\langle \varphi_j, \varphi \rangle|^2 \stackrel{\text{Bessel}}{\leq} \|\varphi\|^2 = 1$$

so that

$$\|T - R_n\| = \sup_{\substack{\varphi \in H, \\ \|\varphi\|=1}} \|T(\Pi - P_n)\varphi\| \leq \sup_{\substack{\varphi \in H, \\ \|\varphi\|=1}} \sup_{\substack{\varphi \in \text{ran}(\Pi - P_n) \\ \|\varphi\| \leq 1}} \|T\varphi\| = \lambda_n.$$

$=: \lambda_n$

Note: $(\lambda_n)_{n \in \mathbb{N}}$ is non-negative and

decreasing, so $\lim_{n \rightarrow \infty} \lambda_n =: \lambda \geq 0$ exists. The theorem now

follows from:

Claim: $\lambda = 0$: For $n \in \mathbb{N}$, $\exists \varphi_n \in \text{ran}(\Pi - P_n)$ with $\|\varphi_n\| \leq 1$

and $\|T\varphi_n\| \geq \frac{\lambda_n}{2} \geq \frac{\lambda}{2}$. Now, for all $\gamma \in H$,

$$|\langle \gamma, \varphi_n \rangle|^2 = |\langle \gamma, (\Pi - P_n)\varphi_n \rangle|^2 = |\langle (\Pi - P_n)\gamma, \varphi_n \rangle|^2$$

$$\text{C.S.} \leq \underbrace{\|(\Pi - P_n)\gamma\|^2}_{\sum_{j=n+1}^{\infty} |\langle \varphi_j, \gamma \rangle|^2} \cdot \underbrace{\|\varphi_n\|^2}_{\leq 1} \xrightarrow{n \rightarrow \infty} 0.$$

So $\varphi_n \xrightarrow{w} 0$ and (since T cpl.) $T\varphi_n \xrightarrow{n \rightarrow \infty} 0$ (by S. 29). So $\lambda = 0$

5.5 Fredholm alternative and the spectral theorem for compact operators

(125)

General assumption: H is a separable Hilbert space over $\mathbb{C}$.

(Motivation 5.32)

(i) Let M be an $N \times N$ -matrix (over $\mathbb{C}$) for $N \in \mathbb{N}$, and let $z \in \mathbb{C}$.

Then: Either: $M\psi = z\psi$ has a solution $0 \neq \psi \in \mathbb{C}^N$

Or: $(M - zI)^{-1}$ exists

(ii) Let $H = L^2([0,1])$ and $M \in BL(L^2([0,1]))$ be defined by

$$(M\psi)(x) = x\psi(x) \quad \text{for a.e. } x \in [0,1] \quad \forall \psi \in L^2([0,1]).$$

(a) $M\psi = z\psi$ has no solutions $0 \neq \psi \in L^2$ since:

$$x\psi(x) = z\psi(x) \text{ a.e. } x \in [0,1] \Rightarrow \psi(x) = 0 \text{ a.e. } x \in [0,1]$$

(b) $(M - zI)$ is not surjective $\forall z \in [0,1]$: (so, $(M - zI)^{-1}$ does not exist).

$$- ((M - zI)\psi)(x) = (x - z)\psi(x)$$

- Any element $\varphi \in L^2([0,1])$ which has a representative which is (1) continuous at $x = z$ (2) with $\varphi(z) \neq 0$ satisfies $\varphi \notin \text{ran}(M - zI)$. Assume $\varphi \in \text{ran}(M - zI)$.

Then there exists $\psi \in L^2([0,1])$ s.t. $(M - zI)\psi = \varphi$,

hence $\psi(x) = \frac{1}{x - z} \varphi(x)$ (a.e. x) - which is not an element of $L^2([0,1])$ ∇ .

(c) For compact operators, the above either/or-statement in (a) still holds (see Cor. 5.34).

Theorem 5.33 (Analytic Fredholm theorem)

Let $D \subseteq \mathbb{C}$ be open and connected, let $f: D \rightarrow BL(H)$ be (norm) analytic, and $f(z)$ compact $\forall z \in D$ ($f(z) \in K(H)$).

Then: Either (A1): $(f(z) - \pi)^{-1}$ does not exist for any $z \in D$

or (A2): There exists a discrete subset $S \subseteq D$ without accumulation points in D , such that the map

$$D \setminus S \rightarrow BL(H)$$

is well-defined and analytic.

$$z \mapsto (f(z) - \pi)^{-1}$$

Moreover: $z \in S$ iff $f(z)\varphi = \varphi$ has a solution $0 \neq \varphi \in H$.

Corollary 5.34 Let $T \in BL(H)$ be compact (i.e. $T \in K(H)$)

and $z \in \mathbb{C} \setminus \{0\}$. Then

Either: $T\varphi = z\varphi$ has a solution $0 \neq \varphi \in H$

or: $(T - zI)^{-1} \in BL(H)$ exists.

Pf (S.34): Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z} T = \frac{T}{z}$.

Alternative (A1) does not hold in Thm. 5.33, since $\frac{T}{z} - \pi$ has

an inverse in $BL(H)$ $\forall |z| > \|T\|$ (Neumann series!)

Hence, Alternative (A2) holds. $\square$

Pf (S.33): It suffices to prove: $\forall z_0 \in D \exists$ neighbourhood $\mathcal{N}(z_0)$ s.t.

either (A1) or (A2) holds on $\mathcal{N}(z_0)$. [Indeed: $\forall z_0 \in D$, colour

$\mathcal{N}(z_0)$ red or blue, depending on which alternative holds;

D is connected, i.e. if both colours appear, then there

exists $\tilde{z}_0$ s.t. $\mathcal{N}(\tilde{z}_0)$ cannot be uniquely coloured

- contradiction]

Fix $z_0 \in D$. As f continuous: $\exists r > 0: \|f(z) - f(z_0)\| < \frac{1}{2}$

for all $z \in D_r := B_r(z_0)$. By Thm. 5-31 (separability!),

$\exists F \in BL(H)$, finite rank, s.t. $\|f(z_0) - F\| < \frac{1}{2}$. Hence

$\|f(z) - F\| < 1$, and so (by 5-18 (Neumann series))

(1) $D_r \rightarrow BL(H)$ is well-defined and analytic.
 $z \mapsto (\mathbb{I} - f(z) + F)^{-1}$

Let $g(z) := F(\mathbb{I} - f(z) + F)^{-1}$, let $N := \text{rank}(F) < \infty$,

and let $\varphi_1, \dots, \varphi_N \in H$ be an ONB of $\text{ran}(F)$.

Then $\text{ran}(g(z)) \subseteq \text{ran}(F)$, so: $\forall \varphi \in H$ (expand $g(z)\varphi$ in ONB)

(2) $g(z)\varphi = \sum_{n=1}^N \langle \varphi_n, g(z)\varphi \rangle \varphi_n$

Note also that

(3) $f(z) - \mathbb{I} = (g(z) - \mathbb{I})(\mathbb{I} - f(z) + F)$

For all $z \in D_r$ we have the following equivalences:

$$f(z) - \mathbb{I} \text{ not invertible} \stackrel{(1), (3)}{\iff} g(z) - \mathbb{I} \text{ not invertible}$$

$$\Updownarrow (*) \text{ (Pf. below!)} \quad \mathbb{I} \notin \text{ran}(g(z) - \mathbb{I})$$

$$\exists \text{ sol. } 0 \neq \varphi \in H \text{ of } f(z)\varphi = \varphi \iff \exists \text{ sol. } 0 \neq \varphi \in H \text{ of } g(z)\varphi = \varphi$$

$$\Updownarrow (2) \quad \varphi = \sum_{n=1}^N \beta_n \varphi_n$$

$$\exists \beta_1, \dots, \beta_N \in \mathbb{C} \text{ (not all } = 0): \beta_n = \sum_{m=1}^N \underbrace{\langle \varphi_n, g(z)\varphi_m \rangle}_{=: A_{nm}(z)} \beta_m$$

$$\Updownarrow$$

$$d(z) := \det(A(z) - \mathbb{I}_{N \times N}) = 0, \text{ where } A := (A_{nm})_{1 \leq n, m \leq N}$$

But: $d: D_r \rightarrow \mathbb{C}$ is analytic, because, by (1), A_{nm} is analytic

By the identity thm (Complex Analysis): $\forall n, m$

Either $S_r := \{z \in D_r \mid d(z) = 0\}$ has no accum.-point in D_r

i.e. $(*)$ holds on D_r

or $S_r = D_r$, i.e. (A) holds on D_r ; $\Rightarrow$ claim! (128)

It remains to prove (*): The pb. is reduced to the finite rank case.

$\Uparrow$: Clear.

$\Downarrow$: By contradiction. Assume $d(z) \neq 0$. Then:

$\forall \xi \in \mathbb{C}^N: (A(z) - \Pi_{N \times N})\beta = \xi$ has a solution $\beta \in \mathbb{C}^N$.

For an arbitrary (fixed) $\varphi \in H$, choose $\xi_n = \xi_n^\varphi = \langle \varphi_n, g(z)\varphi \rangle$ for $n = 1, \dots, N$, and let $\beta^\varphi \in \mathbb{C}^N$ be a corresponding solution.

Then: $\tilde{\varphi} := -\varphi + \sum_{n=1}^N \beta_n^\varphi \varphi_n$ solves $(g(z) - \Pi)\tilde{\varphi} = \varphi$,

since:

$$\begin{aligned} (g(z) - \Pi)\tilde{\varphi} &= \varphi - g(z)\varphi + (g(z) - \Pi) \sum_{n=1}^N \beta_n^\varphi \varphi_n \\ &\stackrel{(2)}{=} \varphi + \underbrace{\sum_{n=1}^N \left[-\underbrace{\langle \varphi_n, g(z)\varphi \rangle}_{\xi_n^\varphi} + \sum_{m=1}^N (A_{nm}(z) - \delta_{nm}) \beta_m^\varphi \right] \varphi_n}_{=0} \end{aligned}$$

$\Rightarrow g(z) - \Pi$ is invertible ∇

Corollary 5.35 (Riesz-Schauder) Let $T \in BL(H)$ be compact.

Then $\sigma(T)$ has no accumulation point in $\mathbb{C}$, except possibly at 0. Moreover, any $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity (i.e., the corresponding eigenspace is finite-dim).

PF: Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z}$ in Thm. 5.33.

Then $\mathbb{C} \setminus (\{0\} \cup S) \subseteq \rho(T)$ so $\sigma(T) \subseteq \{0\} \cup S$; also, $S \subseteq \sigma_p(T) \subseteq \sigma(T) \subseteq \{0\} \cup S$. Remove 0 from the sets in this chain of inclusions and note that $S = S \setminus \{0\}$.

Hence $S \subseteq \sigma_p(T) \setminus \{0\} \subseteq \sigma(T) \setminus \{0\} \subseteq S$, and so

$$S = \sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

(129)

Also, 0 is the only possible accumulation point of S by 5.33.

Finite multiplicity: Suppose not (only possible for $\dim(H) = \infty$)

Then there exists an infinite ONB $\{\varphi_n\}_{n \in \mathbb{N}}$ of the eigenspace corresponding to some eigenvalue $\lambda \neq 0$, and $T\varphi_n = \lambda\varphi_n$ for all n (eigenspace closed!). Compactness of T implies that $(T\varphi_n)_n = (\lambda\varphi_n)_n$ has a convergent subsequence, which is a contradiction, since $\lambda \neq 0$ and $\{\varphi_n\}_n$ ONB. $\blacksquare$

Remark 5.36 (a) If $\dim H = \infty$, then $0 \in \sigma(T)$

Pf: If not, then $0 \in \rho(T)$, i.e. $T^{-1} \in BL(H)$ and $I = TT^{-1}$.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an ONB of H . Then $\varphi_n = T(T^{-1}\varphi_n) \forall n$.

Since $\{T^{-1}\varphi_n\}_n \subseteq H$ is bounded, and T compact, $\{\varphi_n\}_n$ has a convergent subsequence $\nexists$ (since $\{\varphi_n\}_n$ ONB). $\blacksquare$

(b) Corollaries 5.34 & 5.35 have generalisations to Banach spaces.

In the following we use the convention $n = 1, \dots, \infty : \Leftrightarrow n \in \mathbb{N}$ (to have unified notation).

Theorem 5.37 (Singular-value decomposition for compact operators)

Let $T \in BL(H)$ be compact. Then there exists $N \in \mathbb{N}_0 \cup \{\infty\}$, orthonormal sets $\{\varphi_n\}_{n=1, \dots, N}$, $\{\psi_n\}_{n=1, \dots, N} \subseteq H$ (which are not necessarily ONB's) and singular values $(\lambda_n)_{n=1, \dots, N} \subseteq (0, \infty)$ s.t.

$$T = \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \psi_n. \quad (*)$$

If $N = \infty$, then $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, and the sum in (*) converges in the operator norm.

The proof relies on

(130)

Theorem 5-38 (Hilbert-Schmidt theorem; Spectral theorem
for self-adjoint and compact op's on Hilbert space)

Let $T \in BL(H)$ be self-adjoint and compact. Then there exists an ONB $\{\varphi_n\}_{n=1, \dots, \dim H} \subseteq H$ of eigenvectors, and eigenvalues $(\lambda_n)_{n=1, \dots, \dim H} \subseteq \mathbb{R}$ (listed according to their multiplicities) such that

$$T\varphi_n = \lambda_n \varphi_n \quad \forall n = 1, \dots, \dim H.$$

Moreover,

$$T = \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n. \quad (*)$$

If $\dim H = \infty$, then $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, and the sum in $(*)$ converges in the operator norm.

Pf of 5.38: (i) Let $0 \neq \mu_j, j = 1, 2, \dots$ be the pairwise distinct non-zero (real!) eigenvalues of T , i.e., $\mu_j \neq \mu_k$ for $j \neq k$. For every j , choose an ONB B_j in the finite-dim. (Cor 5-35) eigenspace of T corresponding to μ_j . Then $B := \bigcup_j B_j$ is an orthonormal set in H , since eigenvectors corresp. to different eigenvalues of normal operators are orthogonal (check!).

Let $H_0 := \overline{\text{span}(B)}$. Then $T(H_0) \subseteq H_0$, and as $T = T^*$ this gives $T(H_0^\perp) \subseteq H_0^\perp$:

$$\forall x_\perp \in H_0^\perp \forall x_0 \in H_0: \langle x_0, \overset{T=T^*}{Tx_\perp} \rangle = \underbrace{\langle Tx_0, x_\perp \rangle}_{\in H_0} = 0 \Rightarrow Tx_\perp \perp H_0$$

Let $T^\perp := T|_{H_0^\perp} : H_0^\perp \rightarrow H_0^\perp$. Then $T^\perp$ is selfadjoint & compact (131)

But $\sigma(T^\perp) = \{0\} \Rightarrow$ spectral radius $r(T^\perp) = 0$.

So, by 5.25(b): $\|T^\perp\| = 0$, hence $T^\perp = 0$, and so $H_0^\perp \subseteq \ker(T)$ (in fact " $\supseteq$ "). So: if $B^\perp$ is any ONB of $H_0^\perp$, then $B \cup B^\perp =: \{\psi_n\}_{n=1, \dots, \dim H}$ is ONB of H consisting of eigenvectors of T .

(ii) Let λ_n be the eigenvalue corresponding to ψ_n , $n=1, \dots, \dim H$. Then $(\lambda_n)_{n=1, \dots, \dim H}$ is sequence of eigenvalues enumerated according to multiplicities and (if $\dim H = \infty$), a null sequence by Cor. 5.35

(iii) $T = \sum_{n=1}^{\dim H} \lambda_n \langle \psi_n, \cdot \rangle \psi_n$ holds because, if

$\dim H < \infty$, both sides obviously agree on ONB $\{\psi_n\}_{n=1, \dots, \dim H}$:
 $T\psi_m = \lambda_m \psi_m \quad \forall m \in \mathbb{N}$ and (finite sum!)

$$\sum_{n=1}^{\dim H} \lambda_n \langle \psi_n, \psi_m \rangle \psi_n = \lambda_m \psi_m \quad (\text{ONB!})$$

If $\dim H = \infty$, we first prove that the RHS in (*) is a Cauchy seq. wrt. the operator norm. Indeed, let $\varepsilon > 0$. Since $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, $\exists M_1 \in \mathbb{N}$ s.t. $\sup_{n \geq M_1} |\lambda_n| < \varepsilon$. Hence, $\forall M_2 \geq M_1$:

$$\sup_{\|\psi\|=1} \left\| \sum_{n=M_1+1}^{M_2} \lambda_n \langle \psi_n, \psi \rangle \psi_n \right\|^2 = \sup_{\|\psi\|=1} \left[\sum_{n=M_1+1}^{M_2} |\lambda_n|^2 \langle \psi_n, \psi \rangle^2 \right]$$

$$\leq \sup_{\|\varphi\|=1} \left[\underbrace{\left(\sup_{n \geq M_1} |\lambda_n|^2 \right)}_{< \varepsilon^2} \underbrace{\sum_{n=M_1+1}^{M_2} |\langle \varphi_n, \varphi \rangle|^2}_{\leq \|\varphi\|^2 \text{ (Bessel)}} \right] < \varepsilon^2 \quad (132)$$

Since $BL(H)$ is complete, $\exists R \in BL(H)$ s.t.

$R = \sum_{n \in \mathbb{N}} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$. Since both T & R are bounded,

it suffices to prove that they agree on a dense subset of H (Bounded Linear Extension Thm. 2.31).

But as R is a uniform (thus strong) limit, we have

$$\begin{aligned} R\varphi_m &= \left[\lim_{N \rightarrow \infty} \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \right] \varphi_m \\ &= \lim_{N \rightarrow \infty} \left[\underbrace{\left(\sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \right) \varphi_m}_{\sum_{n=1}^N \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n} \right] = \lambda_m \varphi_m \quad \forall m \in \mathbb{N} \end{aligned}$$

Hence, T and R agree on ONB, so they agree on $\text{span}\{\varphi_m \mid m \in \mathbb{N}\} \subseteq H$, which is dense in H . $\blacksquare$

Pf of 5.37: Since T is compact, so is T^*T (by 5.30(c)).

Since T^*T is also self-adjoint (check!), by 5.38:

$\exists$ orthonormal set $\{\varphi_n\}_{n=1, \dots, \dim H} \subseteq H$ and $(\tilde{\lambda}_n)_{n=1, \dots, \dim H} \subseteq \mathbb{R}$ such that

$$T^*T = \sum_{n=1}^{\dim H} \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n.$$

Now, since $0 \leq \|T\varphi\|^2 = \langle \varphi, T^*T\varphi \rangle = \sum_{n=1}^{\dim H} \tilde{\lambda}_n |\langle \varphi_n, \varphi \rangle|^2 \quad \forall \varphi \in H$

we have (take $\varphi = \varphi_m$) that

$\tilde{\lambda}_m \geq 0 \quad \forall m \in \mathbb{N}$. Terms with $\tilde{\lambda}_n = 0$ do not contribute to

(*) in 5.38. Re-name the eigenvalues $(\tilde{\lambda}_n)_{n=1, \dots, \dim H}$ such that

$(\tilde{\lambda}_n)_{n=1, \dots, N} \subseteq (0, \infty)$ be the collection of non-zero eigenvalues,

and $(\tilde{\lambda}_{n'} = 0)_{n'=1, \dots, N'}$ are the zero eigenvalues, where $N, N' \in \mathbb{N}_0 \cup \{\infty\}$. Let $\{\varphi_n \mid n=1, \dots, N\}$ and $\{\varphi_{n'} \mid n'=1, \dots, N'\}$ be the set of the corresponding eigenvectors, which, together, form the ONB of H .

Hence

$$T^*T = \sum_{n=1}^N \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n \quad \text{and} \quad \ker(T) = \overline{\text{span}\{\varphi_{n'} \mid n'=1, \dots, N'\}}.$$

Set $\lambda_n := \tilde{\lambda}_n^{1/2} = \sqrt{\tilde{\lambda}_n}$ and $\varphi_n = \frac{1}{\lambda_n} T \varphi_n$ for all $n=1, \dots, N$

Then $T \varphi_n = \lambda_n \varphi_n$, and

$$\langle \varphi_m, \varphi_n \rangle = (\lambda_m \lambda_n)^{-1} \langle \varphi_m, \underbrace{T^*T}_{\tilde{\lambda}_n} \varphi_n \rangle = \delta_{mn} \quad \forall m, n=1, \dots, N$$

That is, $\{\varphi_n\}_{n=1, \dots, N} \subseteq H$ is an orthonormal set.

Note that if $N = \infty$ (only possible if $\dim(H) = \infty$) then $(\lambda_n)_{n \in \mathbb{N}}$ is a null-sequence by 5.38.

Finally, the same (!) argument as in the proof of 5.38 gives

$$T = \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$$

with the sum being uniformly (i.e. norm) convergent

if $N = \infty$ (use the ONB $\{\varphi_1, \dots, \varphi_N\} \cup \{\varphi_{n'}, \dots, \varphi_{N'}\}$ to test the equality)