

5.5 Fredholm alternative and the spectral theorem for compact operators

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General assumption: H is a separable Hilbert space over $\mathbb{C}$.

(Motivation 5.32)

(i) Let M be an $N \times N$ -matrix (over $\mathbb{C}$) for $N \in \mathbb{N}$, and let $z \in \mathbb{C}$.

Then: Either: $M\psi = z\psi$ has a solution $0 \neq \psi \in \mathbb{C}^N$

Or: $(M - z)^{-1}$ exists

(ii) Let $H = L^2([0,1])$ and $M \in BL(L^2([0,1]))$ be defined by

$$(M\psi)(x) = x\psi(x) \quad \text{for a.e. } x \in [0,1] \quad \forall \psi \in L^2([0,1]).$$

(a) $M\psi = z\psi$ has no solutions $0 \neq \psi \in L^2$ since:

$$x\psi(x) = z\psi(x) \text{ a.e. } x \in [0,1] \Rightarrow \psi(x) = 0 \text{ a.e. } x \in [0,1]$$

(b) $(M - z)$ is not surjective $\forall z \in [0,1]$: (so, $(M - z)^{-1}$ does not exist).

$$- ((M - z)\psi)(x) = (x - z)\psi(x)$$

- Any element $\varphi \in L^2([0,1])$ which has a representative which is (1) continuous at $x = z$ (2) with $\varphi(z) \neq 0$ satisfies $\varphi \notin \text{ran}(M - z)$. Assume $\varphi \in \text{ran}(M - z)$.

Then there exists $\psi \in L^2([0,1])$ s.t. $(M - z)\psi = \varphi$,

hence $\psi(x) = \frac{1}{x - z} \varphi(x)$ (a.e. x) - which is not an element of $L^2([0,1])$ ∇ .

(c) For compact operators, the above either/or-statement in (a) still holds (see Cor. 5.34).

Theorem 5.33 (Analytic Fredholm theorem)

Let $D \subseteq \mathbb{C}$ be open and connected, let $f: D \rightarrow BL(H)$ be (norm) analytic, and $f(z)$ compact $\forall z \in D$ ($f(z) \in K(H)$).

Then: Either (A1): $(f(z) - \pi)^{-1}$ does not exist for any $z \in D$

or (A2): There exists a discrete subset $S \subseteq D$ without accumulation points in D , such that the map

$$D \setminus S \rightarrow BL(H) \quad \text{is well-defined and analytic.}$$

$$z \mapsto (f(z) - \pi)^{-1}$$

Moreover: $z \in S$ iff $f(z)\varphi = \varphi$ has a solution $0 \neq \varphi \in H$.

Corollary 5.34 Let $T \in BL(H)$ be compact (i.e. $T \in K(H)$)

and $z \in \mathbb{C} \setminus \{0\}$. Then

Either: $T\varphi = z\varphi$ has a solution $0 \neq \varphi \in H$

or: $(T - zI)^{-1} \in BL(H)$ exists.

Pf (5.34): Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z} T = \frac{T}{z}$.

Alternative (A1) does not hold in Thm. 5.33, since $\frac{T}{z} - \pi$ has an inverse in $BL(H)$ $\forall |z| > \|T\|$ (Neumann series!)

Hence, Alternative (A2) holds. $\square$

Pf (5.33): It suffices to prove: $\forall z_0 \in D \exists$ neighbourhood $\mathcal{N}(z_0)$ s.t.

either (A1) or (A2) holds on $\mathcal{N}(z_0)$. [Indeed: $\forall z_0 \in D$, colour

$\mathcal{N}(z_0)$ red or blue, depending on which alternative holds;

D is connected, i.e. if both colours appear, then there

exists $\tilde{z}_0$ s.t. $\mathcal{N}(\tilde{z}_0)$ cannot be uniquely coloured

- contradiction]

Fix $z_0 \in D$. As f continuous: $\exists r > 0: \|f(z) - f(z_0)\| < \frac{1}{2}$

for all $z \in D_r := B_r(z_0)$. By Thm. 5-31 (separability!),

$\exists F \in BL(H)$, finite rank, s.t. $\|f(z_0) - F\| < \frac{1}{2}$. Hence

$\|f(z) - F\| < 1$, and so (by 5-18 (Neumann series))

(1) $D_r \rightarrow BL(H)$ is well-defined and analytic.
 $z \mapsto (\mathbb{I} - f(z) + F)^{-1}$

Let $g(z) := F(\mathbb{I} - f(z) + F)^{-1}$, let $N := \text{rank}(F) < \infty$,

and let $\varphi_1, \dots, \varphi_N \in H$ be an ONB of $\text{ran}(F)$.

Then $\text{ran}(g(z)) \subseteq \text{ran}(F)$, so: $\forall \varphi \in H$ (expand $g(z)\varphi$ in ONB)

(2) $g(z)\varphi = \sum_{n=1}^N \langle \varphi_n, g(z)\varphi \rangle \varphi_n$

Note also that

(3) $f(z) - \mathbb{I} = (g(z) - \mathbb{I})(\mathbb{I} - f(z) + F)$

For all $z \in D_r$ we have the following equivalences:

$$f(z) - \mathbb{I} \text{ not invertible} \stackrel{(1), (3)}{\iff} g(z) - \mathbb{I} \text{ not invertible}$$

$$\Updownarrow (*) \text{ (Pf. below!)}$$

$$\exists \text{ sol. } 0 \neq \varphi \in H \text{ of } f(z)\varphi = \varphi \iff \exists \text{ sol. } 0 \neq \varphi \in H \text{ of } g(z)\varphi = \varphi$$

$$\Updownarrow (2) \varphi = \sum_{n=1}^N \beta_n \varphi_n$$

$$\exists \beta_1, \dots, \beta_N \in \mathbb{C} \text{ (not all } = 0): \beta_n = \sum_{m=1}^N \underbrace{\langle \varphi_n, g(z)\varphi_m \rangle}_{=: A_{nm}(z)} \beta_m$$

$$\Updownarrow$$

$$d(z) := \det(A(z) - \mathbb{I}_{N \times N}) = 0, \text{ where } A := (A_{nm})_{1 \leq n, m \leq N}$$

But: $d: D_r \rightarrow \mathbb{C}$ is analytic, because, by (1), A_{nm} is analytic

By the identity thm (Complex Analysis): $\forall n, m$

Either $S_r := \{z \in D_r \mid d(z) = 0\}$ has no accum.-point in D_r

i.e. $(*)$ holds on D_r

or $S_r = D_r$, i.e. (A) holds on D_r ; $\Rightarrow$ claim! (128)

It remains to prove (*): The pb. is reduced to the finite rank case.

$\Uparrow$: Clear.

$\Downarrow$: By contradiction. Assume $d(z) \neq 0$. Then:

$\forall \xi \in \mathbb{C}^N: (A(z) - \Pi_{N \times N})\beta = \xi$ has a solution $\beta \in \mathbb{C}^N$.

For an arbitrary (fixed) $\varphi \in H$, choose $\xi_n = \xi_n^\varphi = \langle \varphi_n, g(z)\varphi \rangle$ for $n = 1, \dots, N$, and let $\beta_n^\varphi \in \mathbb{C}^N$ be a corresponding solution.

Then: $\tilde{\varphi} := -\varphi + \sum_{n=1}^N \beta_n^\varphi \varphi_n$ solves $(g(z) - \Pi)\tilde{\varphi} = \varphi$,

since:

$$\begin{aligned} (g(z) - \Pi)\tilde{\varphi} &= \varphi - g(z)\varphi + (g(z) - \Pi) \sum_{n=1}^N \beta_n^\varphi \varphi_n \\ &\stackrel{(2)}{=} \varphi + \underbrace{\sum_{n=1}^N \left[-\underbrace{\langle \varphi_n, g(z)\varphi \rangle}_{\xi_n^\varphi} + \sum_{m=1}^N (A_{nm}(z) - \delta_{nm}) \beta_m^\varphi \right] \varphi_n}_{=0} \end{aligned}$$

$\Rightarrow g(z) - \Pi$ is invertible ∇

Corollary 5.35 (Riesz-Schauder) Let $T \in BL(H)$ be compact.

Then $\sigma(T)$ has no accumulation point in $\mathbb{C}$, except possibly at 0. Moreover, any $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of finite multiplicity (i.e., the corresponding eigenspace is finite-dim).

PF: Choose $D := \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z}$ in Thm. 5.33.

Then $\mathbb{C} \setminus (\{0\} \cup S) \subseteq \rho(T)$ so $\sigma(T) \subseteq \{0\} \cup S$; also, $S \subseteq \sigma_p(T) \subseteq \sigma(T) \subseteq \{0\} \cup S$. Remove 0 from the sets in this chain of inclusions and note that $S = S \setminus \{0\}$.

Hence $S \subseteq \sigma_p(T) \setminus \{0\} \subseteq \sigma(T) \setminus \{0\} \subseteq S$, and so

$$S = \sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

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Also, 0 is the only possible accumulation point of S by 5.33.

Finite multiplicity: Suppose not (only possible for $\dim(H) = \infty$)

Then there exists an infinite ONB $\{\varphi_n\}_{n \in \mathbb{N}}$ of the eigenspace corresponding to some eigenvalue $\lambda \neq 0$, and $T\varphi_n = \lambda\varphi_n$ for all n (eigenspace closed!). Compactness of T implies that $(T\varphi_n)_n = (\lambda\varphi_n)_n$ has a convergent subsequence, which is a contradiction, since $\lambda \neq 0$ and $\{\varphi_n\}_n$ ONB. $\blacksquare$

Remark 5.36 (a) If $\dim H = \infty$, then $0 \in \sigma(T)$

Pf: If not, then $0 \in \rho(T)$, i.e. $T^{-1} \in BL(H)$ and $\mathbb{I} = TT^{-1}$.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an ONB of H . Then $\varphi_n = T(T^{-1}\varphi_n)$ $\forall n$.

Since $\{T^{-1}\varphi_n\}_n \subseteq H$ is bounded, and T compact, $\{\varphi_n\}_n$ has a convergent subsequence $\nmid$ (since $\{\varphi_n\}_n$ ONB). $\blacksquare$

(b) Corollaries 5.34 & 5.35 have generalisations to Banach spaces.

In the following we use the convention $n = 1, \dots, \infty : \Leftrightarrow n \in \mathbb{N}$ (to have unified notation).

Theorem 5.37 (Singular-value decomposition for compact operators)

Let $T \in BL(H)$ be compact. Then there exists $N \in \mathbb{N}_0 \cup \{\infty\}$, orthonormal sets $\{\varphi_n\}_{n=1, \dots, N}$, $\{\psi_n\}_{n=1, \dots, N} \subseteq H$ (which are not necessarily ONB's) and singular values $(\lambda_n)_{n=1, \dots, N} \subseteq (0, \infty)$ s.t.

$$T = \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \psi_n. \quad (*)$$

If $N = \infty$, then $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, and the sum in (*) converges in the operator norm.

The proof relies on

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Theorem 5-38 (Hilbert-Schmidt theorem; Spectral theorem
for self-adjoint and compact op's on Hilbert space)

Let $T \in BL(H)$ be self-adjoint and compact. Then there exists an ONB $\{\varphi_n\}_{n=1, \dots, \dim H} \subseteq H$ of eigenvectors, and eigenvalues $(\lambda_n)_{n=1, \dots, \dim H} \subseteq \mathbb{R}$ (listed according to their multiplicities) such that

$$T\varphi_n = \lambda_n \varphi_n \quad \forall n = 1, \dots, \dim H.$$

Moreover,

$$T = \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n. \quad (*)$$

If $\dim H = \infty$, then $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, and the sum in $(*)$ converges in the operator norm.

Pf of 5.38: (i) Let $0 \neq \mu_j, j = 1, 2, \dots$ be the pairwise distinct non-zero (real!) eigenvalues of T , i.e., $\mu_j \neq \mu_k$ for $j \neq k$. For every j , choose an ONB B_j in the finite-dim. (Cor 5-35) eigenspace of T corresponding to μ_j . Then $B := \bigcup_j B_j$ is an orthonormal set in H , since eigenvectors corresp. to different eigenvalues of normal operators are orthogonal (check!).

Let $H_0 := \overline{\text{span}(B)}$. Then $T(H_0) \subseteq H_0$, and as $T = T^*$ this gives $T(H_0^\perp) \subseteq H_0^\perp$:

$$\forall x_\perp \in H_0^\perp \forall x_0 \in H_0: \langle x_0, \overset{T=T^*}{Tx_\perp} \rangle = \underbrace{\langle Tx_0, x_\perp \rangle}_{\in H_0} = 0 \Rightarrow Tx_\perp \perp H_0$$

Let $T^\perp := T|_{H_0^\perp} : H_0^\perp \rightarrow H_0^\perp$. Then $T^\perp$ is selfadjoint & compact (131)

But $\sigma(T^\perp) = \{0\} \Rightarrow$ spectral radius $r(T^\perp) = 0$.

So, by 5.25(b): $\|T^\perp\| = 0$, hence $T^\perp = 0$, and so

$H_0^\perp \subseteq \ker(T)$ (in fact " $\supseteq$ "). So: if $B^\perp$ is any ONB of $H_0^\perp$, then $B \cup B^\perp =: \{\varphi_n\}_{n=1, \dots, \dim H}$ is ONB of H consisting of eigenvectors of T .

(ii) Let λ_n be the eigenvalue corresponding to φ_n , $n=1, \dots, \dim H$. Then $(\lambda_n)_{n=1, \dots, \dim H}$ is sequence of eigenvalues enumerated according to multiplicities and (if $\dim H = \infty$), a null sequence by Cor. 5.35

(iii) $T = \sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$ holds because, if

$\dim H < \infty$, both sides obviously agree on ONB $\{\varphi_n\}_{n=1, \dots, \dim H}$:

$$T\varphi_m = \lambda_m \varphi_m \quad \forall m \in \mathbb{N} \text{ and (finite sum!)} \quad \dim H < \infty$$

$$\sum_{n=1}^{\dim H} \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n = \lambda_m \varphi_m \quad (\text{ONB!})$$

If $\dim H = \infty$, we first prove that the RHS in (*) is a Cauchy seq. wrt. the operator norm. Indeed, let $\varepsilon > 0$. Since $(\lambda_n)_{n \in \mathbb{N}}$ is a null sequence, $\exists M_1 \in \mathbb{N}$ s.t. $\sup_{n \geq M_1} |\lambda_n| < \varepsilon$. Hence, $\forall M_2 \geq M_1$:

$$\sup_{\|\varphi\|=1} \left\| \sum_{n=M_1+1}^{M_2} \lambda_n \langle \varphi_n, \varphi \rangle \varphi_n \right\|^2 = \sup_{\|\varphi\|=1} \left[\sum_{n=M_1+1}^{M_2} |\lambda_n|^2 \langle \varphi_n, \varphi \rangle^2 \right]$$

$$\leq \sup_{\|\varphi\|=1} \left[\underbrace{\left(\sup_{n \geq M_1} |\lambda_n|^2 \right)}_{< \varepsilon^2} \underbrace{\sum_{n=M_1+1}^{M_2} |\langle \varphi_n, \varphi \rangle|^2}_{\leq \|\varphi\|^2 \text{ (Bessel)}} \right] < \varepsilon^2 \quad (132)$$

Since $BL(H)$ is complete, $\exists R \in BL(H)$ s.t.

$R = \sum_{n \in \mathbb{N}} \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$. Since both T & R are bounded,

it suffices to prove that they agree on a dense subset of H (Bounded Linear Extension Thm. 2.31).

But as R is a uniform (thus strong) limit, we have

$$\begin{aligned} R\varphi_m &= \left[\lim_{N \rightarrow \infty} \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \right] \varphi_m \\ &= \lim_{N \rightarrow \infty} \left[\underbrace{\left(\sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n \right) \varphi_m}_{\sum_{n=1}^N \lambda_n \langle \varphi_n, \varphi_m \rangle \varphi_n} \right] = \lambda_m \varphi_m \quad \forall m \in \mathbb{N} \end{aligned}$$

Hence, T and R agree on ONB, so they agree on

$\text{span} \{ \varphi_m \mid m \in \mathbb{N} \} \subseteq H$, which is dense in H $\blacksquare$

Pf of 5.37: Since T is compact, so is T^*T (by 5.30(c)).

Since T^*T is also self-adjoint (check!!), by 5.38:

$\exists$ orthonormal set $\{ \varphi_n \}_{n=1, \dots, \dim H} \subseteq H$ and $(\tilde{\lambda}_n)_{n=1, \dots, \dim H} \subseteq \mathbb{R}$ such that

$$T^*T = \sum_{n=1}^{\dim H} \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n.$$

Now, since $0 \leq \|T\varphi\|^2 = \langle \varphi, T^*T\varphi \rangle = \sum_{n=1}^{\dim H} \tilde{\lambda}_n |\langle \varphi_n, \varphi \rangle|^2 \quad \forall \varphi \in H$

we have (take $\varphi = \varphi_m$) that

$\tilde{\lambda}_m \geq 0 \quad \forall m \in \mathbb{N}$. Terms with $\tilde{\lambda}_n = 0$ do not contribute to

(*) in 5.38. Re-name the eigenvalues $(\tilde{\lambda}_n)_{n=1, \dots, \dim H}$ such that

$(\tilde{\lambda}_n)_{n=1, \dots, N} \subseteq (0, \infty)$ be the collection of non-zero eigenvalues,

and $(\tilde{\lambda}_{n'} = 0)_{n'=1, \dots, N'}$ are the zero eigenvalues, where $N, N' \in \mathbb{N}_0 \cup \{\infty\}$. Let $\{\varphi_n \mid n=1, \dots, N\}$ and $\{\varphi_{n'} \mid n'=1, \dots, N'\}$ be the set of the corresponding eigenvectors, which, together, form the ONB of H .

Hence

$$T^*T = \sum_{n=1}^N \tilde{\lambda}_n \langle \varphi_n, \cdot \rangle \varphi_n \quad \text{and} \quad \ker(T) = \overline{\text{span}\{\varphi_{n'} \mid n'=1, \dots, N'\}}.$$

Set $\lambda_n := \tilde{\lambda}_n^{1/2} = \sqrt{\tilde{\lambda}_n}$ and $\varphi_n = \frac{1}{\lambda_n} T \varphi_n$ for all $n=1, \dots, N$

Then $T \varphi_n = \lambda_n \varphi_n$, and

$$\langle \varphi_m, \varphi_n \rangle = (\lambda_m \lambda_n)^{-1} \langle \varphi_m, \underbrace{T^* T \varphi_n}_{\tilde{\lambda}_n \varphi_n} \rangle = \delta_{mn} \quad \forall m, n=1, \dots, N$$

That is, $\{\varphi_n\}_{n=1, \dots, N} \subseteq H$ is an orthonormal set.

Note that if $N = \infty$ (only possible if $\dim(H) = \infty$) then $(\lambda_n)_{n \in \mathbb{N}}$ is a null-sequence by 5.38.

Finally, the same (!) argument as in the proof of 5.38 gives

$$T = \sum_{n=1}^N \lambda_n \langle \varphi_n, \cdot \rangle \varphi_n$$

with the sum being uniformly (i.e. norm) convergent

if $N = \infty$ (use the ONB $\{\varphi_1, \dots, \varphi_N\} \cup \{\varphi_{n'}, \dots, \varphi_{N'}\}$ to test the equality)