(i) $T_n \xrightarrow{w} 0$, since for $y \in \ell^2$, by Cauchy-Schwarz,

$$|\langle y, T_n x\rangle| = \left|\sum_{j=1}^{\infty} \overline{y_{n+j}}\, x_j\right| \le \|x\| \cdot \underbrace{\left(\sum_{j=1}^{\infty} |y_{n+j}|^2\right)^{1/2}}_{\xrightarrow{n\to\infty} 0 \text{ since } y \in \ell^2}$$

(ii) $(T_n)_n$ does not converge strongly to 0, because $\|T_n x\| = \|x\| \;\forall x \in \ell^2$.

5.2 Adjoint operators

Definition 5.6 (a) Let X, Y be normed spaces, and $T \in BL(X, Y)$.

Then $T^\times : Y^* \to X^*$, $\ell \mapsto \ell \circ T$ is a linear operator, the adjoint (operator) of T.

(b) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then

$T^* : H_2 \to H_1$, $y \mapsto T^* y$ where (by Riesz!) T^*y is the unique $z \in H_1$ such that, $\forall x \in H_1$, $\langle y, Tx\rangle_{H_2} = \langle z, x\rangle_{H_1}$ (*)

is a linear operator, the (Hilbert space) adjoint (operator) of T.

Note that, with $\ell_y := \langle y, \cdot\rangle_{H_2}$, the linear functional

$T^\times \ell_y : H_1 \to \mathbb{K}$, $x \mapsto \langle y, Tx\rangle_{H_2}$ belongs to H_1^*

Remarks 5.7 (a) In the Hilbert space case we have that $T^* = J_1 T^\times J_2^{-1}$, with $J_j : H_j^* \to H_j$ the antilinear, isometric and bijective identification (Cor. 2.58).

Indeed, from (*): LHS $= (J_2^{-1} y)(Tx) = (T^\times (J_2^{-1} y))(x)$,

RHS $= (J_1^{-1}(T^* y))(x)$.

(b) T^* is linear (!) by part (a) of this remark.

(c) Notation varies. Some denote the (Banach space) adjoint $T^\times$ by T', or even T^*.

Examples 5.8 (a) Let $X = Y = \ell^1$, and $Rx := (0, x_1, x_2, \dots)$ the right-shift operator. Then $R \in BL(\ell^1)$. Hence $R^\times : (\ell^1)^* \to (\ell^1)^*$ (Riesz representation: $(\ell^1)^* \ni f_\xi \overset{1\text{ to }1}{\longleftrightarrow} \xi \in \ell^\infty$ with $f_\xi(x) = \sum_{n\in\mathbb{N}} \xi_n x_n$ for every $x \in \ell^1$). So:

$$(R^\times f_\xi)(x) = f_\xi(Rx) = \sum_{n\in\mathbb{N}} \xi_n (Rx)_n = \sum_{n\in\mathbb{N}} \xi_{n+1} x_n = \sum_{n\in\mathbb{N}} (L\xi)_n x_n = f_{L\xi}(x)$$

when $L : \ell^\infty \to \ell^\infty$, $\xi \mapsto (\xi_2, \xi_3, \xi_4, \dots)$ is the left shift operator.

Hence, $R^\times f_\xi = f_{L\xi}$.

(b) For $H_1 = H_2 = H$ and $\varphi, \psi \in H$, let $P := \langle \varphi, \cdot \rangle \psi$, i.e.,

$$Px = \langle \varphi, x \rangle \psi \quad \forall x \in H \qquad (\operatorname{ran}(P) = \operatorname{span}\{\psi\}).$$

Then $P^* = \langle \psi, \cdot \rangle \varphi$ since, for all $x, y \in H$,

$$\langle y, Px \rangle = \langle y, \psi \rangle \langle \varphi, x \rangle = \langle \langle \psi, y \rangle \varphi, x \rangle = \langle P^* y, x \rangle.$$

As an application of adjoint operators, we prove that bounded operators preserve weak convergence:

Lemma 5.9 Let X, Y be normed spaces, and $T \in BL(X, Y)$. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$, $x \in X$. Then

$$x_n \xrightarrow{w} x \quad \Rightarrow \quad Tx_n \xrightarrow{w} Tx, \quad n \to \infty.$$

Pf: Let $\ell \in Y^*$. Then, $\ell(Tx_n) - \ell(Tx) = \underbrace{(T^\times \ell)}_{\in X^*}(x_n - x) \xrightarrow{n\to\infty} 0$ ∎

Theorem 5.10 Let X, Y be normed spaces and $T \in BL(X, Y)$. Then

$$\|T^\times\|_{Y^* \to X^*} = \|T\|_{X \to Y}$$

Pf: By definition and Thm. 4.15 we have

$$\|T\|_{X\to Y} = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\|_Y \overset{4.15}{=} \sup_{\substack{x\in X\\ \|x\|=1}} \sup_{\substack{\ell\in Y^*\\ \|\ell\|_{Y^*}=1}} |\ell(Tx)| \underset{\text{check}}{\overset{(!)}{=}} \sup_{\substack{\ell\in Y^*\\ \|\ell\|_{Y^*}=1}} \underbrace{\sup_{\substack{x\in X\\ \|x\|=1}} |(T^\times \ell)(x)|}_{\|T^\times \ell\|_{X^*}}$$

$$= \|T^\times\|_{Y^* \to X^*} \qquad ∎$$

Corollary 5.11 Let H_1, H_2 be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then $\|T^*\|_{H_2 \to H_1} = \|T\|_{H_1 \to H_2}$

Pf: Follows from $T^* = J_1 T^\times J_2^{-1}$ and Thm. 5.10, since $J_j : H_j^* \to H_j$, $j = 1,2$, are isometric & bijective. ∎

Theorem 5.12 Let H be a Hilbert space, and $T, S \in BL(H)$. Then

(a) The map $(\cdot)^* : BL(H) \to BL(H)$, $T \mapsto T^*$ is anti-linear, isometric and bijective, with $\mathbb{1}^* = \mathbb{1}$.

(b) $(TS)^* = S^* T^*$ (c) $(T^*)^* = T$

(d) If T has inverse $T^{-1} \in BL(H)$, then $(T^*)^{-1} = (T^{-1})^*$.
In particular, $(T^*)^{-1} \in BL(H)$.

(e) $\|TT^*\| = \|T\|^2$ (C*-property).

Pf: (a): Anti-linear by definition of T^*; isometric by Cor. 5.11; surjective by part (c) of this theorem.

(b),(c): See exercise.

(d): $T^{-1}T = \mathbb{1} = TT^{-1}$ together with (b) gives

$$T^*(T^{-1})^* = \mathbb{1}^* = \mathbb{1} = (T^{-1})^* T^* . \checkmark$$

(e) On the one hand, we have (by Cor. 5.11) $\|TT^*\| \le \|T\| \cdot \|T^*\| = \|T\|^2$.

On the other hand,

$$\|TT^*\| \overset{4.15}{=} \sup_{0 \ne x, y \in H} \frac{|\langle y, TT^*x\rangle|}{\|x\|\,\|y\|} \overset{y=x}{\geq} \sup_{0 \ne x \in H} \frac{\|T^*x\|^2}{\|x\|^2} = \|T^*\|^2 \overset{T.11}{=} \|T\|^2 \quad ∎$$

Definition 5.13 Let H be a Hilbert space, and $T \in BL(H)$.

(i) T is normal $:\Longleftrightarrow TT^* = T^*T$ (they commute!)

(ii) T is self-adjoint $:\Longleftrightarrow T = T^*$

(iii) T is unitary $:\Longleftrightarrow T$ is bijective and $T^{-1} = T^*$

(i.e., $TT^* = \mathbb{1} = T^*T$).

Remark 5.14 (a) self-adjoint or unitary $\Rightarrow$ normal.

(b) T normal $\Leftrightarrow \langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle \quad \forall x,y \in H$.

<u>Pf</u>: LHS $= \langle T^*Tx, y\rangle$, RHS $= \langle (T^*)^*T^*x, y\rangle = \langle TT^*x, y\rangle$.

Hence: T normal $\Rightarrow \|Tx\| = \|T^*x\| \quad \forall x \in H$.

In particular: $\ker(T) = \ker(T^*)$. (if T normal!)

(c) T self-adjoint $\Leftrightarrow \quad \langle x, Ty\rangle = \langle Tx, y\rangle \quad \forall x,y \in H \qquad (*)$.

Setting $x = y$ gives: $\quad \langle x, Tx\rangle \in \mathbb{R} \quad \forall x \in H$.

[The property $(*)$ is called symmetry (T is symmetric). The notions "symmetric" and "selfadjoint" agree for bounded linear operators, but their generalisations to <u>un</u>bounded operators do NOT!]

(d) T unitary $\Leftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle = \langle T^*x, T^*y\rangle \quad \forall x,y \in H$.

<u>Pf</u>: $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$ and $\langle T^*x, T^*y\rangle = \langle (T^*)^*T^*x, y\rangle = \langle TT^*x, y\rangle$

Example 5.15 (a) $T = z\mathbb{1}$, $z \in \mathbb{K}$. Then $T^* = \bar{z}\mathbb{1}$. So, T is self-adjoint iff. $z \in \mathbb{R}$.

(b) Let $H = L^2([0,1])$, $k \in C([0,1]^2)$ and

$$(Tf)(x) := \int_0^1 k(x,y) f(y)\, dy \quad \forall f \in H, \ \forall x \in [0,1]$$

Then $T \in BL(H)$ and, if $k(x,y) = \overline{k(y,x)} \ \forall x,y \in [0,1]$, then T is selfadjoint

---

5.3 The spectrum

In this subsection: X Banach over $\mathbb{C}$ (i.e. $\mathbb{K} = \mathbb{C}$)

Definition 5.16 Let $T \in BL(X)$

(i) The resolvent set (of T) is $\quad \rho(T) := \{ z \in \mathbb{C} \mid T - z\cdot\mathbb{1} \text{ is bijective} \}$

(ii) The resolvent of T is $\quad R_z := R(z, T) := (T - z\cdot\mathbb{1})^{-1} =: (T - z)^{-1}$

for which ever $z \in \mathbb{C}$ this exists as a (possibly unbounded) operator.

(a) If $z \in \rho(T)$, then $R_z \in BL(X)$ (bounded inverse thm!)

(b) R_z need <u>not</u> exist for $z \notin \rho(T)$.

(iii) The spectrum of T is $\quad \operatorname{spec}(T) := \sigma(T) := \mathbb{C} \setminus \rho(T)$

(iv) If there exists $0 \neq x \in X$ s.t. $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then λ is an eigenvalue and x the corresponding eigenvector.