Chapter 5: Bounded operators

5.1 Topologies on the space of bounded linear operators

Definition 5.1 Let $\mathfrak{X}, Y$ be normed spaces.

(a) The *uniform (operator) topology* on $BL(\mathfrak{X}, Y) :=$ norm topology wrt. to $\|\cdot\|_{\mathfrak{X}\to Y}$

(b) The *strong (operator) topology* on $BL(\mathfrak{X}, Y) :=$ locally convex topology induced by the seminorms $\{T \mapsto \|Tx\|_Y\}_{x\in\mathfrak{X}}$

(c) The *weak (operator) topology* on $BL(\mathfrak{X}, Y) :=$ locally convex topology induced by the seminorms $\{T \mapsto |\ell(Tx)|\}_{\substack{x\in\mathfrak{X}\\ \ell\in Y^*}}$

Remark 5.2 (a) The families of seminorms in 5.1(a) & 5.2(b) are separating (check!) $\Rightarrow$ Hausdorff topologies.

(b) Strong and weak operator topologies are not 1^{st} countable if $\dim \mathfrak{X} = \infty$.

Lemma 5.3 (a) The strong topology is the coarsest vector-space topology on $BL(X, Y)$ such that all maps

$$M_x : BL(X,Y) \to Y, \quad T \mapsto Tx \qquad , x \in \mathfrak{X}$$

are continuous.

The weak (op.) topology is the coarsest vector-space topology on $BL(X,Y)$ such that all maps

$$M_{\ell,x}: BL(X,Y) \to \mathbb{K}, \quad T \mapsto \ell(Tx), \qquad x \in X,\ \ell \in Y^*$$

are continuous.

(b) coarser ⟵⟶ finer

weak op. top. $\subseteq$ weak topology on $BL(X,Y)$ ⟵ less important

weak op. top. $\subseteq$ strong op. top $\subseteq$ uniform op. top.

(c) Let $T, T_n \in BL(X,Y)$, $n \in \mathbb{N}$. Then

(i) $(T_n)_n$ converges to T in the strong operator topology, written $T_n \xrightarrow{s} T$ (strongly)

$$\iff \lim_{n\to\infty} T_n x = Tx \quad \forall x \in X$$

(ii) $(T_n)_n$ converges to T in the weak operator topology, written $T_n \xrightarrow{w} T$ (weakly) (!)

$$\iff \lim_{n\to\infty} \ell(T_n x) = \ell(Tx) \quad \forall x \in X,\ \forall \ell \in Y^*$$

Pf: (a) and (c) from Lemma 4.24

(b) Upper branch: Follows from $M_{\ell,x} \in (BL(X,Y))^*$ $\forall x \in X$, $\forall \ell \in Y^*$, since

$$|M_{\ell,x} T| = |\ell(Tx)| \le \|\ell\|_{Y^*} \underbrace{\|Tx\|_Y}_{\le \|T\| \cdot \|x\|}$$

$$\Rightarrow \|M_{\ell,x}\|_{(BL(X,Y))^*} \le \|\ell\|_{Y^*} \|x\| < \infty.$$

Hence, the weak topology on $BL(X,Y)$ is inducted by a family of seminorms that containg the inducing family of the weak operator topology. Hence, the latter is coarser.

Lower branch:

(i) $M_x : BL(X,Y) \to Y$ is linear $\forall x \in X$.

It is also bounded wrt. the uniform topology:

$$\|M_x T\|_Y = \|Tx\|_Y \leq \|T\| \cdot \|x\| \Rightarrow \sup_{0 \neq T \in BL(X,Y)} \frac{\|M_x T\|}{\|T\|} \leq \|x\|$$

i.e. M_x is continuous wrt. the uniform topology on $BL(X,Y)$ for all x. Also, the uniform topology is a vector-space topology (because it is a norm topology). So from (a): strong operator topology is coarser than the uniform topology.

(ii) $M_{\ell,x} = \ell \circ M_x$, and M_x is continuous wrt. strong operator topology on $BL(X,Y)$. Hence, $M_{\ell,x}$ is continuous wrt. strong operator topology. So from (a): the strong operator topology must be finer than the weak operator topology

Lemma 5.4 Let $\mathcal{H}$ be a Hilbert space, and $(T_n)_n \subseteq BL(\mathcal{H})$

(a) If $(T_n x)_n$ is Cauchy in $\mathcal{H}$ $\forall x \in \mathcal{H}$ then there exists $T \in BL(\mathcal{H})$ such that $T_n \xrightarrow{s} T$

(b) If $(\langle y, T_n x\rangle)_n$ is Cauchy in $\mathbb{K}$ $\forall x, y \in \mathcal{H}$ then there exists $T \in BL(\mathcal{H})$ such that $T_n \xrightarrow{w} T$.

Pf: (a) Define $T : \mathcal{H} \to \mathcal{H}$

$$x \mapsto \lim_{n\to\infty} T_n x$$

The map T is

(i) well-defined (limit exists!)

(ii) linear

(iii) bounded: We have $\sup_n \|T_n x\| < \infty$ $\forall x \in H$ (due to convergence). Hence, by the uniform boundedness principle: $\sup_n \|T_n\| < \infty$. So, for all $x \in H$, $\|x\| = 1$:

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \left(\sup_{n\in\mathbb{N}} \|T_n\|\right) \|x\| = \sup_{n\in\mathbb{N}} \|T_n\|,$$

hence,

$$\|T\| = \sup_{x\in H,\, \|x\|=1} \|Tx\| \leq \sup_{n\in\mathbb{N}} \|T_n\| < \infty$$

Hence $T \in BL(H)$ and $T_n \xrightarrow{s} T$ (by def.). ∎

(b) See exercise.

$\boxed{\text{Example 5.5}}$ Let $(T_n)_n \subseteq BL(\ell^2)$.

(a) For $x = (x_1, x_2, \dots) \in \ell^2$ and $n \in \mathbb{N}$ let

$$T_n x := \left(\tfrac{1}{n} x_1, \tfrac{1}{n} x_2, \dots\right) = \tfrac{1}{n} x.$$

Then $\|T_n\| = \frac{1}{n} \xrightarrow{n\to\infty} 0$, i.e. uniform convergence to zero (operator)

(b) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_{n+1}, x_{n+2}, \dots)$

(i) $T_n \xrightarrow{s} 0$ because $\|T_n x\|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \xrightarrow[x\in\ell^2]{n\to\infty} 0$.

(ii) $(T_n)_n$ does <u>not</u> converge uniformly to 0, because

$$T_n e_{n+1} = e_{n+1}, \quad \text{so } \|T_n\| \geq 1.$$

(c) Let $T_n x := (\underbrace{0, \dots, 0}_{n \text{ times}}, x_1, x_2, \dots)$

i.e. "n times iterated right shift" (see 2.26)

(i) $T_n \xrightarrow{w} 0$, since for $y \in \ell^2$, by Cauchy-Schwarz,

$$|\langle y, T_n x\rangle| = \left|\sum_{j=1}^{\infty} \overline{y_{n+j}}\, x_j\right| \le \|x\| \cdot \underbrace{\left(\sum_{j=1}^{\infty} |y_{n+j}|^2\right)^{1/2}}_{\xrightarrow{n\to\infty} 0 \text{ since } y \in \ell^2}$$

(ii) $(T_n)_n$ does not converge strongly to 0, because $\|T_n x\| = \|x\|$ $\forall x \in \ell^2$.

5.2 Adjoint operators

Definition 5.6 (a) Let X, Y be normed spaces, and $T \in BL(X,Y)$.

Then $T^\times : Y^* \to X^*$, $\ell \mapsto \ell \circ T$ is a linear operator, the adjoint (operator) of T

(b) Let H_1, H_2 be Hilbert spaces, and $T \in BL(H_1, H_2)$. Then

$T^* : H_2 \to H_1$, $y \mapsto T^* y$ where (by Riesz!) $T^* y$ is the unique $z \in H_1$ such that, $\forall x \in H_1$, $\langle y, Tx\rangle_{H_2} = \langle z, x\rangle_{H_1}$ (*)

is a linear operator, the (Hilbert space) adjoint (operator) of T.

Note that, with $\ell_y := \langle y, \cdot\rangle_{H_2}$, the linear functional

$T^\times \ell_y : H_1 \to \mathbb{K}$, $x \mapsto \langle y, Tx\rangle_{H_2}$ belongs to H_1^*

Remarks 5.7 (a) In the Hilbert space case we have that $T^* = J_1 T^\times J_2^{-1}$, with $J_j : H_j^* \to H_j$ the antilinear, isometric and bijective identification (Cor. 2.58).

Indeed, from (*): LHS $= (J_2^{-1} y)(Tx) = (T^\times (J_2^{-1} y))(x)$,

RHS $= (J_1^{-1}(T^* y))(x)$.

(b) T^* is linear (!) by part (a) of this remark.

(c) Notation varies. Some denote the (Banach space) adjoint $T^\times$ by T', or even T^*.