Chapter 4: The cornerstones of Functional Analysis

4.1 Hahn-Banach Theorem

Recall:

Definition Let M be a set, and $D \subseteq M \times M$. Let $\prec$ ("sub") be the associated binary relation: $x \prec y :\Leftrightarrow (x,y) \in D$.

(a) $\prec$ is a partial ordering (or, M is a partially ordered set)

$:\Leftrightarrow \forall x, y, z \in M$:

(i) $x \prec x$ (reflexive)

(ii) $(x \prec y \ \& \ y \prec x) \Rightarrow (x = y)$ (antisymmetry)

(iii) $(x \prec y \ \& \ y \prec z) \Rightarrow (x \prec z)$ (transitive)

(b) $x, y \in M$ are comparable $:\Leftrightarrow x \prec y$ or $y \prec x$

x, y are incomparable $:\Leftrightarrow x, y$ are not comparable.

(c) $\prec$ is a total ordering $:\Leftrightarrow$

(i) $\prec$ is a partial ordering

(ii) x, y are comparable for all $x, y \in M$

(d) Let $W \subseteq M$. Then $u \in M$ is an upper bound of/for W

$:\Leftrightarrow w \prec u \ \forall w \in W$.

(e) $m \in W$ is a maximal element of W $:\Leftrightarrow$ The following implication holds:

$$m \prec w \text{ for } w \in W \Rightarrow m = w.$$

(Note: A maximal element need not be an upper bound and vice versa - see examples below).

Examples (a) "$\leq$" is a total ordering on $\mathbb{R}$.

$W = [0,1)$ has no maximal element, but any $u \geq 1$ is an upper bound for W

(b) "$\subseteq$" is a partial ordering on $\mathcal{P}(X)$, but not a total ordering.

(c) $(x_1, x_2) \prec (y_1, y_2) :\Leftrightarrow x_j \leq y_j$, $j = 1, 2$, is a partial ordering (but not total) on $M = \mathbb{R}^2$.
$W := \{(0,0), (1,0), (0,1)\}$ has 2 (!) maximal elements, but none of them is an upper bound for W.

The following axiom is equivalent to the axiom of choice:

Axiom 4.1 (Zorn's Lemma) Let $M \neq \emptyset$ be a partially ordered set. Assume every totally ordered subset of M has an upper bound. Then M has a maximal element.

We now finish the

Proof of Thm. 2.53 (Remains: Every Hilbert space $X \neq \{0\}$ has an orthonormal basis)

Let $M := \{\mathcal{E} \subseteq X \mid \mathcal{E} \text{ orthonormal}\}$. Then:

(i) $M \neq \emptyset$

(ii) "$\subseteq$" is partial ordering on M

(iii) Let W be a totally ordered subset of M.

Then $\mathcal{U} := \bigcup_{\mathcal{E} \in W} \mathcal{E} \in M$:

If $x_1, x_2 \in \mathcal{U}$ then $\exists\, \mathcal{E}_j \in W$: $x_j \in \mathcal{E}_j$, $j = 1, 2$.

As W is totally ordered, $\mathcal{E}_1 \subseteq \mathcal{E}_2$ (w.log.)

Hence, $x_1, x_2 \in \mathcal{E}_2 \Rightarrow x_1 \perp x_2 \Rightarrow \mathcal{U}$ orthonormal

That is, $\mathcal{U}$ is an upper bound for W.

By Zorn: $\exists$ maximal element $\mathcal{M} \in M$.

Claim: $\mathcal{M}$ is an orthonormal basis for $\mathfrak{X}$

(i) Orthonormal clear, since $\mathcal{M} \in \mathfrak{M}$

(ii) Assume $\mathcal{M}$ is not complete, i.e., not a basis.

Then (by Def. 2.47(d)), there is some $0 \neq x \in \mathfrak{X}$ such that $x \perp m$ $\forall m \in \mathcal{M}$. Hence,

$$\mathcal{M}' := \left\{ \frac{x}{\|x\|} \right\} \cup \mathcal{M} \in \mathfrak{M} \quad (\text{since } \mathcal{M}' \text{ is orthonormal})$$

So $\mathcal{M} \subsetneq \mathcal{M}'$ which contradicts $\mathcal{M}$ being maximal ☒

$\boxed{\text{Remark 4.2}}$ Similar arguments prove Thm. 2.4 (existence of Hamel basis; see exercise)

$\boxed{\text{Theorem 4.3}}$ (Hahn-Banach (H-B)) Let $\mathfrak{X}$ be a vectorspace, let $p: \mathfrak{X} \to \mathbb{R}$ be convex, i.e. $\forall x, x' \in \mathfrak{X}$, $\forall \alpha \in [0,1]$:

$$p(\alpha x + (1-\alpha)x') \leq \alpha p(x) + (1-\alpha) p(x'). \quad (*)$$

Let Y be a subspace, and let $\lambda: Y \to \mathbb{K}$ be linear, with

$$\operatorname{Re} \lambda(y) \leq p(y) \quad \forall y \in Y \quad (\square)$$

Then there exists $\Lambda: \mathfrak{X} \to \mathbb{K}$ linear such that:

(i) $\Lambda|_Y = \lambda$ (i.e. Λ is extension of λ)

(ii) $\operatorname{Re} \Lambda(x) \leq p(x)$ $\forall x \in \mathfrak{X}$ (i.e. $(\square)$ is preserved)

If, in addition, p also satisfies

$(**)$ $\quad p(\alpha x) \leq p(x)$ $\forall x \in \mathfrak{X}$, $\forall \alpha \in \mathbb{K}$ with $|\alpha| = 1$

then we even have $|\Lambda(x)| \leq p(x)$ $\forall x \in \mathfrak{X}$.

$\boxed{\text{Remark 4.4}}$ (a) Condition $(**)$ is equivalent to:

$$p(\alpha x) = p(x) \quad \forall x \in \mathbb{K} \text{ with } |\alpha| = 1 \quad (\text{check!})$$

(b) If p is a (semi-)norm on $\mathfrak{X}$, then p satisfies $(*)$ & $(**)$.

Pf (of 4.3): 4 Steps: Steps 1 & 2 prove the main part for $\mathbb{R}$-vectorspaces, step 3 for $\mathbb{C}$-vector-spaces, and step 4 proves the addendum under add. condition ~~(*~~*).

Wlog: $Y \subsetneq X$ (otherwise trivial).

Step 1: Case $\mathbb{K}=\mathbb{R}$. Extend by one dimension — a preparation for Step 2.

By assumption: $\exists\, z \in X \setminus Y$ (so $z \neq 0$). Let $\tilde{Y} := \operatorname{span}(Y, \{z\})$.

For every $\tilde{y} \in \tilde{Y}$, $\exists!$ decomposition $\tilde{y} = y + \alpha z$, $y \in Y$, $\alpha \in \mathbb{R}$

Candidate for extension $\tilde{\lambda}$ of λ to $\tilde{Y}$:

$$\tilde{\lambda}(\tilde{y}) := \lambda(y) + \alpha \xi \qquad \text{for some } \xi \in \mathbb{R}$$

(to be chosen)

Interpretation: $\xi = \tilde{\lambda}(z)$. Clearly, $\tilde{\lambda}$ is ($\mathbb{R}$-) linear on $\tilde{Y}$, and $\tilde{\lambda}|_Y = \lambda$. Will choose ξ s.t. $\tilde{\lambda} \leq p$ on $\tilde{Y}$ (recall (ii)):

Let $\beta_1, \beta_2 > 0$, $y_1, y_2 \in Y$, then

$$\beta_1 \lambda(y_1) + \beta_2 \lambda(y_2) = (\beta_1+\beta_2)\, \lambda\underbrace{\left(\frac{\beta_1}{\beta_1+\beta_2} y_1 + \frac{\beta_2}{\beta_1+\beta_2} y_2\right)}\quad (\nabla)$$

$$\overset{\text{by } (\square)}{\leq} p\underbrace{\left(\frac{\beta_1}{\beta_1+\beta_2} y_1 + \frac{\beta_2}{\beta_1+\beta_2} y_2\right)}$$

$$= \frac{\beta_1}{\beta_1+\beta_2}(y - \beta_2 z) + \frac{\beta_2}{\beta_1+\beta_2}(y_2 + \beta_1 z)$$

Hence,

$$(\nabla) \overset{p \text{ convex}}{\leq} \beta_1 p(y_1 - \beta_2 z) + \beta_2 p(y_2 + \beta_1 z).$$

Re-arranging, we get

$$\frac{1}{\beta_2}\left(\lambda(y_1) - p(y_1 - \beta_2 z)\right) \leq \frac{1}{\beta_1}\left(p(y_2 + \beta_1 z) - \lambda(y_2)\right)$$

$$\forall \beta_1, \beta_2 > 0, \ \forall y_1, y_2 \in Y$$

Hence, there exists $a \in \mathbb{R}$:

$$(\nabla\nabla) \quad \sup_{\substack{\beta_2 > 0 \\ y_1 \in Y}} \left[\frac{1}{\beta_2} (\lambda(y_1) - p(y_1 - \beta_2 z)) \right] \le a \le \inf_{\substack{\beta_1 > 0 \\ y_2 \in Y}} \left[\frac{1}{\beta_1} (p(y_2 + \beta_1 z) - \lambda(y_2)) \right]$$

Set $\xi = \tilde{\lambda}(z) := a$. Then, $\forall \tilde{y} = y + \alpha z \in \tilde{Y}$ with $\alpha > 0$, the right inequality in $(\nabla\nabla)$ implies

$$\lambda(\tilde{z}) \le \frac{1}{\alpha} (p(\tilde{y}) - \lambda(y)), \text{ so } \tilde{\lambda}(\tilde{y}) \le p(\tilde{y}).$$

If $\tilde{y} = y - \alpha z$ with $\alpha > 0$, use the left ineq. in $(\nabla\nabla)$ instead.

Therefore, $\tilde{\lambda} \le p$ on $\tilde{Y}$.

Step 2: Case $\mathbb{K} = \mathbb{R}$. Idea: Use Zorn to construct the extension.

Let $M := \{ (\mathbb{R}\text{-}) \text{ linear extensions } e \text{ of } \lambda \text{ with } e \le p \text{ on } \mathrm{dom}(e) \}$

We have

(i) $M \neq \emptyset$ since $\lambda \in M$

(ii) Define partial ordering $\prec$ on M:

$$e_1 \prec e_2 :\Leftrightarrow \mathrm{dom}(e_1) \subseteq \mathrm{dom}(e_2) \wedge e_2|_{\mathrm{dom}(e_1)} = e_1.$$

(iii) Let $W \subseteq M$ be totally ordered.

Define $u: \bigcup_{e \in W} \mathrm{dom}(e) \to \mathbb{R}$, $x \mapsto \tilde{e}(x)$ where $\tilde{e}$ is any element of W such that $x \in \mathrm{dom}(\tilde{e})$.

u is well-defined: (i.e. indep. of the chosen $\tilde{e}$ among allowed elements):

Let $x \in \mathrm{dom}(\tilde{e}_1) \cap \mathrm{dom}(\tilde{e}_2)$. Since W is totally ordered, we have (wlog.) $\tilde{e}_2$ is an extension of $\tilde{e}_1$. Hence, $\tilde{e}_2(x) = \tilde{e}_1(x)$.

u is $(\mathbb{R}\text{-})$ linear: Let $x_1, x_2 \in \bigcup_{e \in W} \mathrm{dom}(e)$, $\gamma \in \mathbb{R}$. Hence, $\exists \tilde{e}_j \in W : x_j \in \mathrm{dom}(\tilde{e}_j)$, $j = 1, 2$. Since W is totally ordered, (wlog.) $\tilde{e}_2$ is extension of $\tilde{e}_1$. Then $x_1, x_2, x_1 + \gamma x_2 \in \mathrm{dom}(\tilde{e}_2)$ & linearity of u follows from that of $\tilde{e}_2$.

Lastly: $u(x) = \tilde{e}(x) \le p(x) \quad \forall x \in \bigcup_{e \in W} \mathrm{dom}(e)$.

Hence: $u \in M$, and (check!) u is an upper bound for W.

By Zorn's Lemma: M has a maximal element Λ.

Claim: $\mathrm{dom}(\Lambda) = X$: Suppose $\mathrm{dom}(\Lambda) \subsetneq X$. Then there is some $0 \ne z \in X \setminus \mathrm{dom}(\Lambda)$. Let $\tilde{Y} := \mathrm{span}(\mathrm{dom}(\Lambda), \{z\})$.

By Step 1, Λ has some ($\mathbb{R}$-) linear extension $\tilde{\Lambda} \in M$ to $\tilde{Y}$, which contradicts M being maximal.

Hence, the main part of the thm. follows for $\mathbb{K} = \mathbb{R}$.

Step 3: Case $\mathbb{K} = \mathbb{C}$. Reduce to the real case.

Define $\ell(y) := \mathrm{Re}\,\lambda(y) \quad \forall y \in Y$. Then $\ell : X \to \mathbb{R}$ is $\mathbb{R}$-linear, and $\ell \le p$ on Y. So, Step 2 implies: There exists $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ with $L|_Y = \ell$ and $L \le p$ on X.

Note: $\lambda(y) = \ell(y) + i\,\ell(-iy) \quad \forall y \in Y$ since:

$$\ell(-iy) = \mathrm{Re}\,\lambda(-iy) \overset{\lambda\ \mathbb{C}\text{-lin}}{=} \mathrm{Re}[-i\lambda(y)] = \mathrm{Im}\,\lambda(y)$$

Define $\Lambda(x) := L(x) + i\,L(-ix)$, $x \in X$.

Then, by Step 2,

(i) Λ is $\mathbb{R}$-linear on X

(ii) $\Lambda|_Y = \lambda$

(iii) $\mathrm{Re}\,\Lambda \overset{L(-ix) \in \mathbb{R}}{=} L \le p$ on X

Also, Λ is $\underline{\mathbb{C}}$-linear, since it is $\mathbb{R}$-linear and

$$\Lambda(ix) = L(ix) + i\,L(x) \overset{L\ \mathbb{R}\text{-lin}}{=} i\,(L(x) + i\,L(-ix)) = i\,\Lambda(x),$$

proving the main part for $\mathbb{K} = \mathbb{C}$.

Step 4: Addendum: We fix $\in X$ and use the polar repr.

$$\Lambda(x) = |\Lambda(x)|\,e^{i\theta(x)} \quad [\text{If } \mathbb{K} = \mathbb{R} \text{ then } e^{i\theta(x)} \in \{-1, 1\}]$$

Then $|\Lambda(x)| = e^{-i\theta(x)}\Lambda(x) \overset{\mathbb{K}\text{-lin}}{=} \Lambda(e^{-i\theta(x)}x) \overset{\text{LHS} \in \mathbb{R}}{=} \mathrm{Re}\,\Lambda(e^{-i\theta(x)}x)$

$$\overset{\text{main part}}{\le} p(e^{-i\theta(x)}x) \overset{(**)}{\le} p(x)$$

■

Corollary 4.5 Let X be a normed space, $Y \subseteq X$ a subspace, and $\varphi \in Y^*$. Then there exists $f \in X^*$ with $f|_Y = \varphi$ and $\|f\|_{X^*} = \|\varphi\|_{Y^*}$.

Pf: Apply Thm. 4.3 with $p(x) := \|\varphi\|_{Y^*} \cdot \|x\|$ for $x \in X$ (fulfills assumptions - check!) to $\lambda = \varphi$. Then $\exists f : X \to \mathbb{K}$ linear with $|f(x)| \leq \|\varphi\|_{Y^*} \|x\|$, hence $\|f\|_{X^*} \leq \|\varphi\|_{Y^*}$.
But
$$\|f\|_{X^*} = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \geq \sup_{0 \neq x \in Y} \frac{|\varphi(x)|}{\|x\|} = \|\varphi\|_{Y^*} \quad \blacksquare$$

Corollary 4.6 Let X be a normed space and let $0 \neq x_0 \in X$. Then there exists $f \in X^*$ with $f(x_0) = \|x_0\|$ and $\|f\|_{X^*} = 1$.

Pf: Let $Y = \operatorname{span}\{x_0\}$. If $y \in Y$ then $y = \alpha x_0$ for some unique $\alpha \in \mathbb{K}$. Define φ on Y by $\varphi(y) := \alpha \cdot \|x_0\|$.
This implies $\varphi(x_0) = \|x_0\|$ and $\varphi \in Y^*$ with $\|\varphi\|_{Y^*} = 1$, since $|\varphi(y)| = \|y\|$. By Cor. 4.5, $\exists f : X \to \mathbb{K}$ linear, with $f|_Y = \varphi$ (in particular, $f(x_0) = \|x_0\|$) and $\|f\|_{X^*} = \|\varphi\|_{Y^*} = 1$ $\blacksquare$

Corollary 4.7 Let X be a normed space, $Z \subseteq X$ a closed subspace, and $x_0 \in X \setminus Z$ with $0 < \operatorname{dist}(x_0, Z) =: d$. Then there exists $f \in X^*$ with $f|_Z = 0$, $f(x_0) = d$ and $\|f\|_{X^*} = 1$.

Proof: See exercise.

4.2 Three consequences of Baire's theorem

$\boxed{\text{Theorem 4.8}}$ (Banach-Steinhaus, Uniform Boundedness Principle)

Let X be a Banach space, Y a normed (!) space, and $\mathcal{F} \subseteq BL(X,Y)$.

If
$$\sup_{T\in\mathcal{F}} \|Tx\| < \infty \qquad \forall x \in X$$
then
$$\sup_{T\in\mathcal{F}} \|T\| < \infty .$$

Pf: For $n \in \mathbb{N}$, define

$$A_n := \{x\in X \mid \|Tx\| \le n \ \forall T\in\mathcal{F}\} = \bigcap_{T\in\mathcal{F}} T^{-1}(\overline{B_n^Y(0)}) \quad \left(\begin{array}{c}\text{closed, since}\\ T \text{ cont.}\end{array}\right)$$

Then, by hypothesis, $X = \bigcup_{n\in\mathbb{N}} A_n$. By Cor 1.49 (conseq. Baire) there exists $n_0 \in \mathbb{N}$ s.t. A_{n_0} is not no-where dense.

Since A_{n_0} is also closed: $\exists\, x_0 \in A_{n_0}$ & $r>0$: $B_r(x_0) \subseteq A_{n_0}$.

Now let $0 \ne x \in X$ and $T \in \mathcal{F}$, then

$$\frac{r}{2\|x\|}\|Tx\| \le \|T(\underbrace{\tfrac{r}{2\|x\|}x + x_0}_{\in B_r(x_0)\subseteq A_{n_0}})\| + \|Tx_0\| \le n_0 + \|Tx_0\|,$$

hence,

$\|T\| \le \frac{2}{r}(n_0 + \|Tx_0\|)$. Taking the supremum over all $T\in\mathcal{F}$ proves the claim. ∎

$\boxed{\text{Theorem 4.9}}$ (Open mapping theorem) Let X, Y be Banach spaces. Let $T \in BL(X,Y)$ be onto (surjective). Then T is open (an open map), i.e.:

$$A \subseteq X \text{ open} \Rightarrow T(A) \subseteq Y \text{ open}$$

Pf: The thm. follows from 3 claims:

Claim 1: $\exists r>0$: $T(B_r^X(0))$ has non-empty interior $\Rightarrow$ T is open (i.e. claim is "$\Rightarrow$" holds).

Pf: Assume $T(B_r^X(0))$ has non-empty interior.

Let $y := Tx$ for some $x \in B_r^X(0)$ be an interior point of $T(B_r^X(0))$, i.e., $\exists r_y$: $B_{r_y}^Y(y) \subset T(B_r^X(0))$.

Note: $B_r^X(0) \subseteq B_\delta^X(x)$ for some $\delta > 0$ (large enough, e.g. $3r$)

Hence, $B_{r_y}^Y(y) \subseteq T(B_\delta^X(x))$. $(*)$

By scaling, translation and linearity, we get $\forall r'>0$:

$$T(B_{r'}^X(x)) = T(B_{r'}^X(0)+x) = T\left(\frac{r'}{\delta} B_\delta^X(0)+x\right)$$
$$= \frac{r'}{\delta} T(B_\delta^X(0)) + Tx = \frac{r'}{\delta} T(B_\delta^X(x)-x) + Tx$$
$$\overset{(*)}{\supseteq} \frac{r'}{\delta}\Big(B_{r_y}^Y(y) - \underbrace{Tx}_{=y}\Big) + Tx = \frac{r'}{\delta} B_{r_y}^Y(0) + Tx$$
$$= B_{\frac{r' r_y}{\delta}}^Y(0) + Tx = B_{\frac{r' r_y}{\delta}}^Y(y). \quad (**)$$

Now, let $A \subseteq X$ be open, let $y' \in T(A)$ be arbitrary, and choose $x' \in A$ s.t. $y' = Tx'$. Since A is open, $\exists r'>0$: $B_{r'}^X(x') \subseteq A$, hence

$$T(A) \supseteq T(B_{r'}^X(x')) = T(B_{r'}^X(x)+x'-x) \overset{(**)}{\supseteq} B_{\frac{r' r_y}{\delta}}^Y(y)+y'-y = B_{\frac{r' r_y}{\delta}}^Y(y').$$

i.e. $T(A)$ is open, so Claim 1 holds. ✓

From now on: center of all balls is 0 (unless otherwise noted) and we drop the center from the notation

Claim 2: $\exists \varepsilon>0$:
$$B_\varepsilon^Y \subseteq \overline{T(B_1^X)}. \quad (***)$$

Pf: $Y \overset{\text{onto}}{=} T(X) = T\left(\bigcup_{n\in\mathbb{N}} B_n^X\right) = \bigcup_{n\in\mathbb{N}} T(B_n^X)$.

Now, Y is complete, so we can apply Baire's Thm (in form of Cor. 1.49): There exists $n \in \mathbb{N}$: $T(B_n^X)$ is not no-where dense, i.e. $\exists\, y \in \overline{T(B_n^X)}$ and $\varepsilon > 0$: $B_\varepsilon^Y(y) \subseteq \overline{T(B_n^X)}$, or, equivalently, $B_\varepsilon^Y \subseteq \overline{T(B_n^X)} - y$.

We have:

(i) $\exists\, (x_k)_{k\in\mathbb{N}} \subseteq B_n^X$ s.t. $y = \lim_{k\to\infty} Tx_k$

(ii) $\forall k \in \mathbb{N}$: $\overline{T(B_n^X)} - Tx_k = \overline{T(B_n^X - x_k)} \subseteq \overline{T(B_{2n}^X)}$

hence $B_\varepsilon^Y \subseteq \overline{T(B_{2n}^X)}$, so $B_{\frac{\varepsilon}{2n}}^Y = \frac{1}{2n} B_\varepsilon^Y \subseteq \frac{1}{2n}\overline{T(B_{2n}^X)} = \overline{T(\frac{1}{2n}B_{2n}^X)}$
$= \overline{T(B_1^X)}$. ✓

<u>Claim 3</u>: $\overline{T(B_1^X)} \subseteq T(B_2^X)$

<u>Pf</u>: Let ε be as in Claim 2. Let $y \in \overline{T(B_1^X)}$.

Then there exists $x_1 \in B_1^X$ such that

$$y - Tx_1 \in B_{\varepsilon/2}^Y \overset{(***)}{\subseteq} \overline{T(B_{1/2}^X)}.$$

Similarly, there exists $x_2 \in B_{1/2}^X$ such that

$$y - Tx_1 - Tx_2 = (y - Tx_1) - Tx_2 \in B_{\varepsilon/4}^Y \overset{(***)}{\subseteq} \overline{T(B_{1/4}^X)}$$

Inductively we get: $\forall n \in \mathbb{N}\ \exists\, x_n \in B_{\frac{1}{2^{n-1}}}^X$ s.t.

$$y - \sum_{j=1}^n Tx_j \in B_{\varepsilon/2^n}^Y \qquad (****)$$

Since $\sum_{n\in\mathbb{N}} \underbrace{\|x_n\|}_{< 2^{-(n-1)}} < 2 < \infty$, and X is a Banach space,

$\sum_{j\in\mathbb{N}} x_j =: x \in X$ exists by Lemma 2.50(a). As T is <u>cont.</u>, we have $Tx = \sum_{j\in\mathbb{N}} Tx_j$. Using $(****)$ and the continuity of $\|\cdot\|$,

$$\|y - Tx\| = \lim_{n\to\infty} \|y - \sum_{j=1}^n Tx_j\| = 0, \text{ hence } y = Tx,\ \|x\| < 2$$

i.e. $y \in T(B_2^X)$ ∎

Corollary 4.10 (Inverse mapping theorem)
Let X, Y be Banach spaces, and $T \in BL(X,Y)$ a bijection. Then $T^{-1} \in BL(Y,X)$

Pf: Clearly, T^{-1} exists and is linear (recall 2.25).
By Thm. 4.9, T is open, that is, T^{-1} is continuous ☒

Definition 4.11 Let X, Y be normed spaces, and
$T: X \supseteq \mathrm{dom}(T) \to Y$ a linear operator.

(a) Graph of T:

$$G(T) := \{(x, Tx) \subseteq X \times Y \mid x \in \mathrm{dom}(T)\}$$

We equip $X \times Y$ with the norm

$$\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$$

Then: X, Y complete $\Rightarrow$ $X \times Y$ complete.

(b) T is closed (operator) $:\Leftrightarrow$ $G(T) \subseteq X \times Y$ is closed
(in $X \times Y, \|\cdot\|_{X\times Y}$)

Remark 4.12

(a) T is closed if and only if the following implication holds:
$(x_n)_n \subseteq \mathrm{dom}(T)$ with: $x_n \xrightarrow{n\to\infty} x \in X$ & $Tx_n \xrightarrow{n\to\infty} y \in Y$
$\Longrightarrow$ $x \in \mathrm{dom}(T)$ and $y = Tx$

(b) Compare (a) with the definition of T (seq.) cont.,
where convergence of $(Tx_n)_n$ must be proved.
Here, it is given / assumed!

Theorem 4.13 (Closed graph theorem)

Let $\mathfrak{X}, Y$ be Banach spaces and $T : \mathfrak{X} \supseteq \mathrm{dom}\, T \to Y$ a closed linear operator. Then

$$\mathrm{dom}(T) \text{ closed} \iff T \text{ bounded}.$$

Pf: "$\Leftarrow$": Let $(x_n)_n \subseteq \mathrm{dom}\, T$ with $x_n \xrightarrow{n\to\infty} x \in \mathfrak{X}$. Then $(x_n)_n$ is Cauchy in $\mathfrak{X}$. By hypothesis, T is bounded, so $(Tx_n)_n$ is Cauchy in Y. But Y is complete, so $\exists\, y \in Y : Tx_n \xrightarrow{n\to\infty} y$ and, since T is closed, it follows that $x = \lim_{n\to\infty} x_n \in \mathrm{dom}(T)$ (and $Tx = y$).

"$\Leftarrow$": Define the projection $P_1 : \begin{array}{l} G(T) \to \mathrm{dom}(T) \\ (x, Tx) \mapsto x \end{array}$

(i) $G(T)$ is closed in $X \times Y$, hence $G(T)$ is a Banach space.

(ii) $\mathrm{dom}(T)$ is closed in $\mathfrak{X}$ (by hypothesis), hence $\mathrm{dom}(T)$ is a Banach space.

(iii) P_1 is a bijection

(iv) P_1 is bounded: Let $z = (x, Tx) \in G(T)$, then

$$\|P_1 z\|_{\mathfrak{X}} = \|x\|_{\mathfrak{X}} \le \|x\|_{\mathfrak{X}} + \|Tx\|_Y = \|z\|_{X\times Y}.$$

So $\|P_1\| \le 1$

By the inverse mapping theorem $P_1^{-1} : \begin{array}{l} \mathrm{dom}(T) \to G(T) \\ x \mapsto (x, Tx) \end{array}$ is also bounded, hence:

$\exists\, c < \infty : \|P_1^{-1} x\|_{X\times Y} \le c \|x\|_X$. Since $\|P_1^{-1} x\|_{X\times Y} = \|x\|_X + \|Tx\|_Y$ this implies

$$\|Tx\|_Y \le (c+1)\|x\|_{\mathfrak{X}}$$

So T is bounded $\blacksquare$

Example 4.14 Let $\mathfrak{X} = \mathcal{Y} = C_0(\mathbb{R})$ with supremum norm (Banach!)

Consider $\quad T := \frac{d}{dx} : \operatorname{dom}(T) \to C_0(\mathbb{R})$
$$f \mapsto f'$$
with $\operatorname{dom}(T) = \{ f \in C_0(\mathbb{R}) \mid f \in C^1(\mathbb{R}) \text{ and } f' \in C_0(\mathbb{R}) \} \subseteq C_0(\mathbb{R})$.

Claim: T is closed (a closed operator).

Pf: Let $(f_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}(T)$ be a sequence such that:

(i) $\exists g \in C_0(\mathbb{R}) : \ \|f_n - g\|_\infty \xrightarrow{n\to\infty} 0$

(ii) $\exists h \in C_0(\mathbb{R}) : \ \|f_n' - h\|_\infty \xrightarrow{n\to\infty} 0$

i.e., $(f_n, f_n')_n \in G(T)$, $(f_n, f_n')_n \xrightarrow{n\to\infty} (g, h)$ in $\mathfrak{X} \times \mathcal{Y}$; we need to prove that $(g,h) \in G(T)$, equivalently, $g \in \operatorname{dom}(T)$ and $h = g'$.

By uniform convergence, we can exchange limits, so, $\forall x \in \mathbb{R}$,
$$\int_0^x h(t)\,dt = \int_0^x \lim_{n\to\infty} f_n'(t)\,dt = \lim_{n\to\infty} \underbrace{\int_0^x f_n'(t)\,dt}_{f_n(x) - f_n(0)} = g(x) - g(0)$$
(since $f_n \to g$ pointwise, by (i)).

Hence, $\quad g(x) = g(0) + \int_0^x h(t)\,dt \quad \forall x \in \mathbb{R}$

So, by the Fundamental Theorem of Calculus (HDI): $g \in C^1(\mathbb{R})$ with $g' = h \in C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$, we have $g \in \operatorname{dom}(T)$, and so $(g,h) = (g, g') \in G(T)$. So T is closed.

Also: Since T is unbounded (cf. 2.28) $\overset{\text{Thm. 4.13}}{\Longrightarrow}$ $\operatorname{dom}(T)$ is not a closed subspace of $C_0(\mathbb{R})$ wrt. $\|\cdot\|_\infty$.

4.3 (Bi-) Dual spaces and weak topologies

Theorem 4.15 Let $\mathfrak{X}$ be a normed space. Then, for every $x \in \mathfrak{X}$,
$$\|x\| = \sup_{0 \neq f \in \mathfrak{X}^*} \frac{|f(x)|}{\|f\|_*}$$

Pf: See exercise.

Definition 4.16 Let X be a normed space. We call

$X^{**} := (X^*)^*$ the bidual space of X (always Banach!)

More generally, introduce the n-fold dual space

$$X^{\overbrace{*\cdots*}^{n \text{ times}}} := \left(X^{\overbrace{*\cdots*}^{n-1 \text{ times}}}\right)^* \quad \text{for } n \in \mathbb{N}, \text{ recursively.}$$

Theorem (& definition) 4.17 Let X be a normed space.

The canonical embedding $J: \begin{array}{l} X \to X^{**} \\ x \mapsto Jx \end{array}$ with $Jx: \begin{array}{l} X^* \to \mathbb{K} \\ f \mapsto f(x) \end{array}$

is well-defined, linear, and isometric.

If J is surjective, we call X reflexive.

Remarks 4.18 (a) Hilbert spaces are reflexive (Riesz!)

(b) Every finite dimensional normed space is reflexive (use the dual basis)

(c) ℓ^p, L^p are reflexive for $p \in (1,\infty)$; c_0 & ℓ^1 are not refl.

(d) Milman-Pettis Thm: X uniformly convex & Banach $\Rightarrow$ X reflexive

(e) X reflexive $\Rightarrow$ X complete.

Pf (4.17): J is well-defined (i.e. $Jx \in X^{**}$ $\forall x \in X$):

Let $\alpha, \beta \in \mathbb{K}$, $f, g \in X^*$, then

$$(Jx)(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha (Jx)(f) + \beta (Jx)(g)$$

So, Jx is linear. Also, Jx is bounded, since

$$\|Jx\|_{**} = \sup_{0 \neq f \in X^*} \frac{|(Jx)(f)|}{\|f\|_*} = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*} \overset{\text{Thm. 4.15}}{=} \|x\| < \infty .$$

Hence, $Jx \in X^{**}$, so J is well-defined – and an isometry.

Also, J is linear: Let $\alpha, \beta \in \mathbb{K}$, $x, y \in X$, then, for all $f \in X^*$,

$$(J(\alpha x + \beta y))(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha (Jx)(f) + \beta (Jy)(f),$$

and so $J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$ ∎

Theorem 4.19 Let X be a Banach space. Then

$$X \text{ reflexive } \iff X^* \text{ reflexive}$$

Pf. "$\Rightarrow$": $(X^*)^{**} = ((X^*)^*)^* = (X^{**})^* = X^*$

"$\Leftarrow$": See f. ex. Werner, "Funktionalanalysis", 8. Aufl., Thm. III 3.4.

Theorem 4.20 Let X be a normed space. Then

$$X^* \text{ separable} \Rightarrow X \text{ separable.}$$

Remark 4.21 $X = \ell^1$ shows that "$\Leftarrow$" does not hold, since $\ell^\infty \simeq (\ell^1)^*$, which is not separable.

Pf (4.20): Let $A := \{f_n \in X^* \mid n \in \mathbb{N}\}$ be dense (exists by hyp.).

For $n \in \mathbb{N}$ choose $x_n \in X$ s.t. $\|x_n\| = 1$ & $|f_n(x_n)| \geq \frac{1}{2} \|f_n\|_*$.

Let $D := \operatorname{span}_{\mathbb{K}} \{x_n \mid n \in \mathbb{N}\} \subseteq X$.

Claim: D is dense in X (Note: D is not countable, but a $\mathbb{K}$-linear subspace).

Pf (claim): Assume (for contradiction) D not dense.

Then there exists $z \in X$ s.t. $\operatorname{dist}(z, D) > 0$. By Cor. 4.7 (H-B!)

$\exists f \in X^*$: $f|_D = 0$, $f(z) > 0$. Now:

(i) A dense in X^* implies: $\exists$ seq. $(f_{n_k})_k \subseteq A$:

$$\|f_{n_k} - f\|_* \xrightarrow{k \to \infty} 0 \quad (\square)$$

(ii) We have

$$\|f_{n_k} - f\|_* \geq |f_{n_k}(x_{n_k}) - f(x_{n_k})| \overset{f|_D = 0}{=} |f_{n_k}(x_{n_k})| \geq \frac{1}{2} \|f_{n_k}\|_*$$

Then $\|f_{n_k}\|_* \xrightarrow{k\to\infty} 0$, so $f_{n_k} \xrightarrow{k\to\infty} 0$ in X^*, hence $f = 0$ by $(\square)$ ↯.

So, D is dense.

Define $\tilde{\mathbb{K}} := \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$, resp. $\tilde{\mathbb{K}} := \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$, and set

$\tilde{D} := \mathrm{span}_{\mathbb{K}}\{x_n \mid n \in \mathbb{N}\}$. Then $\tilde{D}$ is countable & dense in X ■

Definition 4.22 Let X be a $\mathbb{K}$-vector space, let $\{p_\alpha\}_{\alpha \in I}$ be a family of seminorms on X

(a) $\{p_\alpha\}_{\alpha \in I}$ is *separating* $:\iff \forall\, 0 \neq x \in X : \exists\, \alpha \in I : \; p_\alpha(x) > 0$

(b) For given $\alpha \in I$, $r > 0$, let

$$U_{\alpha,r} := \{ y \in X \mid p_\alpha(y) < r \} \ni 0.$$

Also, for $x \in X$, let

$$U_{\alpha,r}(x) := x + U_{\alpha,r} = \{ y \in X \mid p_\alpha(y-x) < r \} \ni x.$$

Let

$$\mathcal{N}_x := \Big\{ \bigcap_{j=1}^{n} U_{\alpha_j, r_j}(x) \;\Big|\; n \in \mathbb{N},\ \alpha_j \in I,\ r_j > 0 \text{ for } j = 1, \dots, n \Big\}$$

(family of finite intersections of $U_{\alpha,r}(x)$'s)

Define the *Locally convex topology* (l.c.t.) on X induced by $\{p_\alpha\}_{\alpha \in I}$:

$Y \subseteq X$ open in the l.c.t. $:\iff \forall x \in Y\ \exists\, V_x \in \mathcal{N}_x : V_x \subseteq Y$

$$\iff \forall x \in Y\ \exists\, V_x \in \mathcal{N}_x : Y = \bigcup_{x \in Y} V_x$$

Remark 4.23 (a) $U_{\alpha,r}(x)$ is open in l.c.t., and $\mathcal{N}_x$ is a neighbourhood base of the l.c.t. at x.

(b) If $\{p_\alpha\}_{\alpha \in I}$ is separating, then the l.c.t. is Hausdorff (see exercise).

(c) The elements of the neighbourhood base are convex sets (hence the name): For $y_1, y_2 \in U_{\alpha,r}$ and $\lambda \in [0,1]$, we have $\lambda y_1 + (1-\lambda) y_2 \in U_{\alpha,r}$, since:

$$p_\alpha(\lambda y_1 + (1-\lambda) y_2) \leq p_\alpha(\lambda y_1) + p_\alpha((1-\lambda) y_2)$$

$$= \lambda \underbrace{p_\alpha(y_1)}_{<r} + (1-\lambda) \underbrace{p_\alpha(y_2)}_{<r} < r.$$

(d) X is a topological vector space w.r.t. a locally convex topology, i.e., addition of vectors and mult. of a vector by a scalar are continuous w.r.t. a l.c.t.
This relies on $U_{\alpha,r}(x) = x + U_{\alpha,r}$ & $\lambda U_{\alpha,r}(x) = U_{\alpha,|\lambda|r}(\lambda x)$
$\forall x \in X, \lambda \in \mathbb{K}, \alpha \in I, r > 0.$

Notation: vector space topology :$\Leftrightarrow$ a topology making X into a topological vector space.

The composition of a linear map with a norm gives a seminorm (check!). The following abstract result will be applied several times with different choices of S_α and Z:

Lemma 4.24 Let X be a vector space, $(Z, \|\cdot\|_Z)$ a normed space, and $S_\alpha : X \to Z$ a linear map for every index $\alpha \in I$. Let $\mathcal{T}_{l.c.t.}$ be the locally convex topology on X induced by the family of seminorms $\{x \mapsto \|S_\alpha x\|_Z\}_{\alpha \in I}$. Then:

(a) $\mathcal{T}_{l.c.t.}$ is the coarsest vector space topology on X s.t. For every $\alpha \in I$ the map S_α is continuous.

(b) $(x_k)_{k\in\mathbb{N}} \subseteq X$ converges to $x \in X$ w.r.t. $\mathcal{T}_{l.c.t.}$ $\Leftrightarrow$
$$\|S_\alpha x_k - S_\alpha x\|_Z \xrightarrow{k\to\infty} 0 \quad \forall \alpha \in I$$

Pf: See exercise.

A very important example is when $S_\alpha \in X^*$ and $Z = \mathbb{K}$:

Definition 4.25 Let X be a normed space. The weak topology on X is the locally convex topology (l.c.t.) on X induced by the family of seminorms $\{x \mapsto |f(x)|\}_{f \in X^*}$.

$\boxed{\text{Lemma 4.26}}$ Let X be a normed space. Then the weak topology on X is

(a) Hausdorff.

(b) the coarsest vector-space topology on X such that every $f \in X^*$ is continuous. In particular, the weak topology is coarser than the norm (=: strong) topology.

(c) identical to the norm topology, if $\dim X < \infty$,

(d) such that $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to x in the weak topology (or, weakly) iff

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X^*$$

Notation: $x_n \xrightarrow{w} x$, or $x_n \xrightarrow[]{n\to\infty} x$.

Pf: (a) By Rmk 4.23(b), it suffices to check that the family of seminorms is separating. This follows from Corollary 4.6 (H-B!): For every $0 \neq x \in X$ $\exists f_x \in X^*: f_x(x) = \|x\|$.

(b) Lemma 4.24 (a) (c) See exercise (d) Lemma 4.24 (b). ∎

$\boxed{\text{Remark 4.27}}$ (a) Weak limits are unique by Lemma 4.26(a)

(b) If $\dim X = \infty$, then the weak topology is not 1st countable (hence not metrisable!) (for pf., see exercise)

(c) Strong convergence $\overset{4.26(b)}{\Longrightarrow}$ weak convergence (but in general not vice versa!)

(d) In a Hilbert space X (by Lem. 4.26(d) & Riesz):

$$x_n \xrightarrow{w} x \iff \langle y, x_n\rangle \xrightarrow{n\to\infty} \langle y, x\rangle \quad \forall y \in X$$

(e) In ℓ^1: $\quad x_n \xrightarrow{w} x \iff \|x_n - x\|_1 \xrightarrow{n\to\infty} 0$

(Schur, 1921; See also Conway, "A course in FA", 1990, Prop. 5.2)

$\boxed{\text{Example 4.28}}$ Let $X = \ell^p$, $p \in [1,\infty]$, $e_n := (\delta_{nk})_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$.

Then (i) $(e_n)_n$ has no $\|\cdot\|_p$-convergent (i.e., strongly conv.) subsequence.

(ii) If $p=1$, then $(e_n)_n$ is <u>not</u> weakly convergent:
Recall $(\ell^1)^* \cong \ell^\infty$, and choose $f \leftrightarrow (-1,1,-1,1,-1,\ldots) \in \ell^\infty$
then $f(e_n) = (-1)^n$ $\forall n \in \mathbb{N}$ which is not convergent as $n \to \infty$ (This is consistent with Rmk 4.27(e)).

(iii) If $1 < p \le \infty$, then $(e_n)_n$ is weakly convergent to 0:
In case $1 < p < \infty$, we have $(\ell^p)^* \cong \ell^q$ (Thm. 2.38) with $1 < q < \infty$. Hence, for $f \leftrightarrow y = (y_k)_k \in \ell^q$:
$f(e_n) = y_n \xrightarrow{n\to\infty} 0$ (since $\sum_{k\in\mathbb{N}} |y_k|^q < \infty$).
In the case $p = \infty$, use also exercises E6.1 & E6.3.

$\boxed{\text{Lemma 4.29}}$ Let X be a normed space, $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$, and $x_n \xrightarrow{w} x$. Then

(a) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$, (b) $\|x\| \le \liminf_{n\to\infty} \|x_n\|$

<u>Pf</u> (a): Let $x_n \xrightarrow{w} x$, so $f(x_n) \xrightarrow{n\to\infty} f(x)$ $\forall f \in X^*$, hence

$$\forall f \in X^* \text{ (fixed)}: \quad \sup_{n\in\mathbb{N}} \underbrace{|f(x_n)|}_{(Jx_n)(f)} < \infty,$$

with the canonical embedding $J : X \to X^{**}$. Since X^* is Banach, the Uniform Boundedness Principle (Thm. 4.9) gives

$$\sup_{n\in\mathbb{N}} \underbrace{\|Jx_n\|_{**}}_{=\|x_n\|} < \infty. \quad ✓$$

(b) By Cor. 4.6 (H-B!):

$\exists f_x \in X^* : \|f_x\|_* = 1$ and $f_x(x) = \|x\|$

Then

$$\|x\| = |f_x(x)| = \lim_{n\to\infty} |f_x(x_n)| = \liminf_{n\to\infty} |f_x(x_n)| \le \liminf_{n\to\infty} \underbrace{\|f_x\|_*}_{=1} \|x_n\|$$ ∎

Theorem 4.30 Let X be a normed space. Then $x_n \xrightarrow{w} x$ iff the following two statements hold:

(i) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$

(ii) $\exists F \subseteq X^*$ with $\mathrm{span}(F)$ dense (wrt. $\|\cdot\|_*$) in X^* s.t.

$$\forall f \in F: \qquad \lim_{n\to\infty} f(x_n) = f(x). \qquad (*)$$

Pf: "⇒": From Lem. 4.29 and by definition of weak convergence

"⇐": Use an $\frac{\varepsilon}{3}$-argument: Let $\varepsilon > 0$ and $g \in X^*$. Let

$K := \frac{1}{2}(\|x\| + \sup_{n\in\mathbb{N}} \|x_n\|) < \infty$. Since $\mathrm{span}(F)$ is dense in X^*, there exists $f \in \mathrm{span}(F)$: $\|f-g\|_* < \frac{\varepsilon}{3K}$. Also, for $f \in \mathrm{span}(F)$, $\exists N \in \mathbb{N}: \forall n \ge N: \quad |f(x_n) - f(x)| < \frac{\varepsilon}{3}$

(Note: $(*)$ holds also for $f \in \mathrm{span}(F)$ as finite lin. comb.)

Hence, $\forall n \ge N$:

$$|g(x) - g(x_n)| \le |g(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - g(x_n)|$$

$$\le \underbrace{\|g-f\|_*}_{< \frac{\varepsilon}{3K}} \underbrace{(\|x\| + \|x_n\|)}_{\le 2K} + \underbrace{|f(x) - f(x_n)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$ ∎

Theorem 4.31 (Eberlein–Šmulian) Let X be a Banach space and $A \subseteq X$. Then A weakly compact ⇔ A weakly sequentially compact.

Pf. Not here; see Whitley, Math. Ann. 172, 116–118 (1967). ∎

Definition 4.32 Let X be a normed space. Then the weak* topology on X^* ("weak-star") is the locally convex topology on X^* induced by the family of seminorms $\{f \mapsto |f(x)|\}_{x\in X}$

Lemma 4.33 Let X be a normed space. Then the weak* topology is

(a) Hausdorff

(b) The coarsest vector-space topology on X^* s.t. $\forall x \in X$, the map
$$\begin{aligned} X^* &\to \mathbb{K} \\ f &\mapsto f(x) \end{aligned} \quad \text{is continuous}$$

(c) coarser than the weak topology on X^*, and the two coincide iff X is reflexive.

(d) such that $(f_n)_n \subseteq X^*$ converges to $f \in X^*$ in the weak*-top. iff
$$f_n(x) \xrightarrow{n\to\infty} f(x) \quad \forall x \in X$$
Notation: $f_n \xrightarrow{w^*} f$.

Pf: (a) $\{f \mapsto |f(x)|\}_{x\in X}$ is a separating family of seminorms: For $f \neq 0$ there exists $x \in X$ with $f(x) \neq 0$.

(b) Lemma 4.24(a) with $Z = \mathbb{K}$ and $S_\alpha = Jx$ ($J: X \to X^{**}$ canonical embedding)

(c) $J(X) \subseteq X^{**}$ with equality iff X is reflexive.

(d) Lemma 4.24 (b).

The next theorem is the analogue of Lem. 4.29 & Thm. 4.30. However: Note that here, X must be complete in order to apply the Uniform Boundedness Principle in (a)(i).

Theorem 4.34 Let X be a normed space, $f \in X^*$, $(f_n)_n \subseteq X^*$

(a) If $f_n \xrightarrow{w^*} f$, then

(i) If X is even a Banach space, then $\sup_{n\in\mathbb{N}} \|f_n\|_* < \infty$

(ii) $\|f\|_* \leq \liminf_{n\to\infty} \|f_n\|_*$

(b) If (i) $\sup_{n\in\mathbb{N}} \|f_n\|_* < \infty$

(ii) $\exists A \subseteq X$ with $\mathrm{span}(A)$ dense in X (w.r.t. $\|\cdot\|$) s.t.
$$\forall x \in A: \quad f_n(x) \xrightarrow{n\to\infty} f(x)$$
then $f_n \xrightarrow{w^*} f$. (If X Banach, the reverse holds by (a)(i))

Pf: See exercise ☒

$\boxed{\text{Theorem 4.35}}$ (Banach–Alaoglu) Let X be a normed space. Then the closed unit ball in X^*, $\overline{B}_1^* := \{ f \in X^* \mid \|f\|_* \leq 1 \}$

is compact in the weak* topology.

Pf: Equip the set of maps $\{f : X \to \mathbb{K}\} = \mathbb{K}^X$ with the product topology $\mathcal{T}_{prod} := \bigtimes_{x \in X} \mathcal{T}_{\mathbb{K}}$ of the Euclidean topology $\mathcal{T}_{\mathbb{K}}$ on $\mathbb{K}$. Then:

(i) $\mathcal{T}_{prod}$ is the coarsest topology on $\mathbb{K}^X$ such that the projection/evaluation map $\pi_x : \mathbb{K}^X \to \mathbb{K}$, $f \mapsto f(x)$ is continuous for all $x \in X$ (see T2.3, Tut Sheet 2)

(ii) $\mathcal{T}_{prod}$ is a vector-space topology on $\mathbb{K}^X$:

Addition: $A : \mathbb{K}^X \times \mathbb{K}^X \to \mathbb{K}^X$, $(f,g) \mapsto f+g$. It suffices to prove openness of $A^{-1}(U)$ in $\mathbb{K}^X \times \mathbb{K}^X$ for U in a base of $\mathcal{T}_{prod}$, that is, for $U = \bigtimes_{x \in U} U_x$ with $U_x \in \mathcal{T}_{\mathbb{K}}$, and $U_x = \mathbb{K}$ for all but at most finitely many $x \in X$. Addition in the field $\mathbb{K}$, $A_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(z, z') \mapsto z + z'$ is continuous.

Hence, $A^{-1}(U) = \bigtimes_{x \in X} A_{\mathbb{K}}^{-1}(U_x)$ with

$$A_{\mathbb{K}}^{-1}(U_x) \subseteq \mathcal{T}_{\mathbb{K}} \times \mathcal{T}_{\mathbb{K}} \quad \forall x \in X$$

and

$$A_{\mathbb{K}}^{-1}(U_x) = \mathbb{K} \times \mathbb{K} \text{ except for at most finitely many } x \in X.$$

Hence, $A^{-1}(U) \in \mathcal{T}_{prod} \times \mathcal{T}_{prod}$, as finite intersection of a union of product sets $V_1 \times V_2$ with each $V_j \in \mathcal{T}_{prod}$, $j = 1, 2$, being a product set of factors all of which are equal to $\mathbb{K}$ except for one. ✓ The case of scalar mult. is analogous & simpler.

(iii) Given a linear subspace $S \subseteq \mathbb{K}^X$, the subspace topology $\mathcal{T}_{prod}|_S := \{ U \cap S \mid U \in \mathcal{T}_{prod} \}$ is a vector space topology on S wrt. which the restricted eval. maps $\pi_x|_S$ are continuous $\forall x \in X$

Now, choose $S = X^*$, and let $\mathcal{T}_{w*}$ denote the weak* top. on X^*. By 4.33(b), $\mathcal{T}_{w*} \subseteq \mathcal{T}_{prod}|_{X^*}$. Therefore, the map

$$\sigma : (X^*, \mathcal{T}_{prod}|_{X^*}) \longrightarrow (X^*, \mathcal{T}_{w*})$$
$$f \longmapsto f$$

is continuous, and the theorem follows if we prove: $\bar{B}_1^*$ is $\mathcal{T}_{prod}$-compact: We claim: Then it is $\mathcal{T}_{prod}|_{X^*}$-compact (and hence, by continuity of σ, $\mathcal{T}_{w*}$-compact).

Pf Claim: Consider a $\mathcal{T}_{prod}|_{X^*}$-open cover $\bigcup_\alpha V_\alpha$ of $\bar{B}_1^*$ and use that, for $V_\alpha \in \mathcal{T}_{prod}|_{X^*}$, there exists $U_\alpha \in \mathcal{T}_{prod}$ with $V_\alpha = U_\alpha \cap X^*$. The $\mathcal{T}_{prod}$-open cover $\bigcup_\alpha U_\alpha$ has a finite subcover (by assumption), and intersection with X^* gives finite subcover of $\bigcup_\alpha V_\alpha$. ✓ (claim).

To prove $\mathcal{T}_{prod}$-compactness: Define $A := \bigtimes_{x \in X} A_x$, where $A_x := \{ z \in \mathbb{K} \mid |z| \le \|x\| \}$ is compact in $\mathbb{K}$. By Tychonoff (Thm. 7.28), A is $\mathcal{T}_{prod}$-compact. Note that: $f \in A \Leftrightarrow f(x) \in A_x \ \forall x \in X$

$\Leftrightarrow \sup_{0 \ne x \in X} \frac{|f(x)|}{\|x\|} \le 1$. For $x, y \in X$, $\alpha, \beta \in \mathbb{K}$, let

$$L_{x,y,\alpha,\beta} := \{ f \in \mathbb{K}^X \mid \underbrace{f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)}_{= (\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y)(f)} = 0 \}$$

$$= (\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y)^{-1}(\{0\}).$$

But $\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y$ is $\mathcal{T}_{prod}$-$\mathcal{T}_{\mathbb{K}}$-continuous, and $\{0\} \subseteq \mathbb{K}$ is $\mathcal{T}_{\mathbb{K}}$-closed, hence $L_{x,y,\alpha,\beta}$ is $\mathcal{T}_{prod}$-closed in $\mathbb{K}^X$, and so also

$$L := \bigcap_{\substack{x,y \in X \\ \alpha, \beta \in \mathbb{K}}} L_{x,y,\alpha,\beta} = \{ f : X \to \mathbb{K} \text{ is linear} \}$$

is closed.

Finally, $\bar{B}_1^* = A \cap L$ is $\mathcal{T}_{prod}$-compact (as closed subset of the compact A) ∎

Theorem 4.36 (Helly; version 2 of Banach-Alaoglu)

Let X be a separable normed space. Then $\overline{B}_1^*$ is weak* sequentially compact.

Pf: Let $\{x_k \mid k \in \mathbb{N}\} \subseteq X$ be (countable) dense in X. We need to prove: Any sequence $(f_n)_n \subseteq \overline{B}_1^*$ has a weak*-convergent subsequence: Fix $k \in \mathbb{N}$ arbitrary. Consider the sequence $(f_n(x_k))_{n\in\mathbb{N}} \subseteq \mathbb{K}$. This sequence is bounded, since

$$|f_n(x_k)| \le \underbrace{\|f_n\|_*}_{\le 1} \|x_k\| \le \|x_k\| \quad (\& \ k \text{ fix})$$

Hence, by Balzano-Weierstraß, for all k fixed, $(f_n(x_k))_n$ has a convergent subsequence (in $\mathbb{K}$).

Claim: There exist a common subsequence $(n_j)_j \subseteq \mathbb{N}$:

$$\forall k \in \mathbb{N}: \quad (f_{n_j}(x_k))_{j\in\mathbb{N}} \text{ is convergent.}$$

Pf: Use Cantor's diagonal sequence trick:

There exists $(n_j^{(1)})_j \subseteq \mathbb{N}$ such that $(f_{n_j^{(1)}}(x_1))_j$ converges.

Then there exists $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$ s.t. $(f_{n_j^{(2)}}(x_2))_j$ converges.

Continuing this procedure, there exists $(n_j^{(\ell+1)})_j \subseteq (n_j^{(\ell)})_j$ such that $(f_{n_j^{(\ell+1)}}(x_{\ell+1}))_{j\in\mathbb{N}}$ converges. The claim then holds with $n_j := n_j^{(j)}$.

Now, define $g(x) := \lim_{j\to\infty} f_{n_j}(x) \quad \forall x \in \operatorname{span}\{x_k \mid k \in \mathbb{N}\} =: \operatorname{dom}(g)$.

(i) $\operatorname{dom}(g)$ is a dense subspace of X with respect to $\|\cdot\|$

(ii) $g: \operatorname{dom}(g) \to \mathbb{K}$ is linear

(iii) g is bounded: $|g(x)| = \lim_{j\to\infty} \underbrace{|f_{n_j}(x)|}_{\le \underbrace{\|f_{n_j}\|_*}_{\le 1} \|x\|} \le \|x\|$

Since $\mathbb{K}$ is complete, we can apply the Bounded Linear Ext. Thm. 2.31:

There exists $\tilde{g}: X \to \mathbb{K}$ such that $\tilde{g}|_{\operatorname{dom}(g)} = g$ & $\|\tilde{g}\|_* = \underbrace{\|g\|_*}_{\le 1}$

Hence, $\tilde{g} \in \overline{B}_1^*$. Then, by Thm. 4.34(b), $f_{n_j} \xrightarrow{w^*} \tilde{g}$ as $j \to \infty$ ■

Theorem 4.37 (Kakutani, version 3 of Banach-Alaoglu)

Let X be a Banach space. Then

$$X \text{ reflexive} \iff \overline{B}_1 := \{x \in X \mid \|x\| \leq 1\} \text{ is weakly compact}$$

Pf: "$\Rightarrow$": By Thm 4.19, $X^* =: Y$ is reflexive. Applying 4.33(c) to Y, we get: The weak topology on $Y^* = (X^*)^* = X$ coincides with the weak* topology on $Y^* = X$, hence the claim follows from Thm. 4.35 applied to Y. ■

"$\Leftarrow$": See f.ex. Werner, "Funktionalanalysis", 8. Aufl, Satz VIII.3.18. ■

Example 4.38 Compactness of $\overline{B}_1$ in different spaces.

Here, $p, q \in (1, \infty)$ are Hölder conjugate

	weak	weak*	seq. weak	seq. weak*
ℓ^1 ($\cong c_0^*$)	no	yes	no	yes
ℓ^p ($\cong (\ell^q)^*$)	yes	yes	yes	yes
ℓ^∞ ($\cong (\ell^1)^*$)	no	yes	no	yes
(*) L^1 (not a dual!)	no	—	no	—
L^p ($\cong (L^q)^*$)	yes	yes	yes	yes
L^∞ ($\cong (L^1)^*$)	no	yes	no	yes
Thm. used	4.37	4.35	4.31	4.36

(*) Cor. to Krein-Milman Theorem, see f.ex. Werner, Sect. VIII.4.