4.2 Three consequences of Baire's theorem

Theorem 4.8 (Banach-Steinhaus, Uniform Boundedness Principle)

Let X be a Banach space, Y a normed (!) space, and $\mathcal{F} \subseteq BL(X, Y)$.

If
$$\sup_{T \in \mathcal{F}} \|Tx\| < \infty \quad \forall x \in X$$
then
$$\sup_{T \in \mathcal{F}} \|T\| < \infty .$$

Pf: For $n \in \mathbb{N}$, define

$$A_n := \{x \in X \mid \|Tx\| \leq n \; \forall T \in \mathcal{F}\} = \bigcap_{T \in \mathcal{F}} T^{-1}(\overline{B_n^Y(0)}) \quad \left(\text{closed, since } T \text{ cont.}\right)$$

Then, by hypothesis, $X = \bigcup_{n \in \mathbb{N}} A_n$. By Cor 1.49 (conseq. Baire) there exists $n_0 \in \mathbb{N}$ s.t. A_{n_0} is not no-where dense.

Since A_{n_0} is also closed: $\exists\, x_0 \in A_{n_0}$ & $r > 0$: $B_r(x_0) \subseteq A_{n_0}$.

Now let $0 \neq x \in X$ and $T \in \mathcal{F}$, then

$$\frac{r}{2\|x\|} \|Tx\| \leq \|T(\underbrace{\tfrac{r}{2\|x\|}x + x_0}_{\in B_r(x_0) \subseteq A_{n_0}})\| + \|Tx_0\| \leq n_0 + \|Tx_0\|,$$

hence,

$\|T\| \leq \frac{2}{r}(n_0 + \|Tx_0\|)$. Taking the supremum over all $T \in \mathcal{F}$ proves the claim. ∎

Theorem 4.9 (Open mapping theorem) Let X, Y be Banach spaces. Let $T \in BL(X, Y)$ be onto (surjective). Then T is open (an open map), i.e.:

$$A \subseteq X \text{ open} \Rightarrow T(A) \subseteq Y \text{ open}$$

Pf: The thm. follows from 3 claims:

Claim 1: $\exists r>0$: $T(B_r^X(0))$ has non-empty interior $\Rightarrow T$ is open (i.e. claim is "$\Rightarrow$" holds).

Pf: Assume $T(B_r^X(0))$ has non-empty interior.

Let $y := Tx$ for some $x \in B_r^X(0)$ be an interior point of $T(B_r^X(0))$, i.e., $\exists r_y$: $B_{r_y}^Y(y) \subset T(B_r^X(0))$.

Note: $B_r^X(0) \subseteq B_\delta^X(x)$ for some $\delta > 0$ (large enough, e.g. $3r$)

Hence, $B_{r_y}^Y(y) \subseteq T(B_\delta^X(x))$. (*)

By scaling, translation and linearity, we get $\forall r'>0$:

$$T(B_{r'}^X(x)) = T(B_{r'}^X(0)+x) = T\left(\frac{r'}{\delta} B_\delta^X(0) + x\right)$$

$$= \frac{r'}{\delta} T(B_\delta^X(0)) + Tx = \frac{r'}{\delta} T(B_\delta^X(x) - x) + Tx$$

$$\overset{(*)}{\supseteq} \frac{r'}{\delta}\Big(B_{r_y}^Y(y) - \underbrace{Tx}_{=y}\Big) + Tx = \frac{r'}{\delta} B_{r_y}^Y(0) + Tx$$

$$= B^Y_{\frac{r' r_y}{\delta}}(0) + Tx = B^Y_{\frac{r' r_y}{\delta}}(y). \qquad (**)$$

Now, let $A \subseteq X$ be open, let $y' \in T(A)$ be arbitrary, and choose $x' \in A$ s.t. $y' = Tx'$. Since A is open, $\exists r'>0$: $B_{r'}^X(x') \subseteq A$, hence

$$T(A) \supseteq T(B_{r'}^X(x')) = T(B_{r'}^X(x) + x' - x) \overset{(**)}{\supseteq} B^Y_{\frac{r' r_y}{\delta}}(y) + y' - y = B^Y_{\frac{r' r_y}{\delta}}(y').$$

i.e. $T(A)$ is open, so Claim 1 holds. ✓

From now on: center of <u>all</u> balls is 0 (unless otherwise noted) and we drop the center from the notation

Claim 2: $\exists \varepsilon > 0$:

$$B_\varepsilon^Y \subseteq \overline{T(B_1^X)} . \qquad (***)$$

Pf: $Y \overset{\text{onto}}{=} T(X) = T\left(\bigcup_{n\in\mathbb{N}} B_n^X\right) = \bigcup_{n\in\mathbb{N}} T(B_n^X)$.

Now, Y is complete, so we can apply Baire's Thm (in form of Cor. 1.49): There exists $n\in\mathbb{N}$: $T(B_n^X)$ is not no-where dense, i.e., $\exists y \in \overline{T(B_n^X)}$ and $\varepsilon > 0$: $B_\varepsilon^Y(y) \subseteq \overline{T(B_n^X)}$, or, equivalently, $B_\varepsilon^Y \subseteq \overline{T(B_n^X)} - y$.

We have:

(i) $\exists (x_k)_{k\in\mathbb{N}} \subseteq B_n^X$ s.t. $y = \lim_{k\to\infty} Tx_k$

(ii) $\forall k\in\mathbb{N}$: $\overline{T(B_n^X)} - Tx_k = \overline{T(B_n^X - x_k)} \subseteq \overline{T(B_{2n}^X)}$

hence $B_\varepsilon^Y \subseteq \overline{T(B_{2n}^X)}$, so $B^Y_{\frac{\varepsilon}{2n}} = \frac{1}{2n} B_\varepsilon^Y \subseteq \frac{1}{2n}\overline{T(B_{2n}^X)} = \overline{T(\frac{1}{2n}B_{2n}^X)}$
$= \overline{T(B_1^X)}$. ✓

Claim 3: $\overline{T(B_1^X)} \subseteq T(B_2^X)$

Pf: Let ε be as in Claim 2. Let $y \in \overline{T(B_1^X)}$.

Then there exists $x_1 \in B_1^X$ such that

$$y - Tx_1 \in B^Y_{\varepsilon/2} \overset{(***)}{\subseteq} \overline{T(B^X_{1/2})}.$$

Similarly, there exists $x_2 \in B^X_{1/2}$ such that

$$y - Tx_1 - Tx_2 = (y - Tx_1) - Tx_2 \in B^Y_{\varepsilon/4} \overset{(***)}{\subseteq} \overline{T(B^X_{1/4})}$$

Inductively we get: $\forall n\in\mathbb{N}\ \exists x_n \in B^X_{\frac{1}{2^{n-1}}}$ s.t.

$$y - \sum_{j=1}^n Tx_j \in B^Y_{\varepsilon/2^n} \qquad (****)$$

Since $\sum_{n\in\mathbb{N}} \underbrace{\|x_n\|}_{<2^{-(n-1)}} < 2 < \infty$, and X is a Banach space,

$\sum_{j\in\mathbb{N}} x_j =: x \in X$ exists by Lemma 2.50(a). As T is cont., we have $Tx = \sum_{j\in\mathbb{N}} Tx_j$. Using $(****)$ and the continuity of $\|\cdot\|$,

$$\|y - Tx\| = \lim_{n\to\infty} \|y - \sum_{j=1}^n Tx_j\| = 0, \text{ hence } y = Tx,\ \|x\| < 2$$

i.e. $y \in T(B_2^X)$ ∎

Corollary 4.10 (Inverse mapping theorem)
Let $\mathfrak{X}, Y$ be Banach spaces, and $T \in BL(\mathfrak{X}, Y)$ a bijection. Then $T^{-1} \in BL(Y, \mathfrak{X})$

Pf: Clearly, T^{-1} exists and is linear (recall 2.25).
By Thm. 4.9, T is open, that is, T^{-1} is continuous $\blacksquare$

Definition 4.11 Let $\mathfrak{X}, Y$ be normed spaces, and $T : \mathfrak{X} \supseteq \mathrm{dom}(T) \to Y$ a linear operator.

(a) Graph of T:

$$G(T) := \{(x, Tx) \subseteq \mathfrak{X} \times Y \mid x \in \mathrm{dom}(T)\}$$

We equip $\mathfrak{X} \times Y$ with the norm

$$\|(x,y)\|_{\mathfrak{X} \times Y} := \|x\|_{\mathfrak{X}} + \|y\|_{Y}$$

Then: $\mathfrak{X}, Y$ complete $\Rightarrow$ $X \times Y$ complete.

(b) T is closed (operator) $:\Leftrightarrow$ $G(T) \subseteq X \times Y$ is closed (in $X \times Y, \|\cdot\|_{X \times Y}$)

Remark 4.12
(a) T is closed if and only if the following implication holds:

$(x_n)_n \subseteq \mathrm{dom}(T)$ with: $x_n \xrightarrow{n\to\infty} x \in \mathfrak{X}$ & $Tx_n \xrightarrow{n\to\infty} y \in Y$

$\Longrightarrow$ $x \in \mathrm{dom}(T)$ and $y = Tx$

(b) Compare (a) with the definition of T (seq.) cont., where convergence of $(Tx_n)_n$ must be proved. Here, it is given / assumed!

Theorem 4.13 (Closed graph theorem)

Let $\mathfrak{X}, Y$ be Banach spaces and $T: \mathfrak{X} \supseteq \operatorname{dom} T \to Y$ a closed linear operator. Then

$$\operatorname{dom}(T) \text{ closed} \iff T \text{ bounded.}$$

Pf: "$\Leftarrow$": Let $(x_n)_n \subseteq \operatorname{dom} T$ with $x_n \xrightarrow{n\to\infty} x \in \mathfrak{X}$. Then $(x_n)_n$ is Cauchy in $\mathfrak{X}$. By hypothesis, T is bounded, so $(Tx_n)_n$ is Cauchy in Y. But Y is complete, so $\exists y \in Y: Tx_n \xrightarrow{n\to\infty} y$ and, since T is closed, it follows that $x = \lim_{n\to\infty} x_n \in \operatorname{dom}(T)$ (and $Tx = y$).

"$\Leftarrow$": Define the projection $P_1: \begin{array}{l} G(T) \to \operatorname{dom}(T) \\ (x, Tx) \mapsto x \end{array}$

(i) $G(T)$ is closed in $X \times Y$, hence $G(T)$ is a Banach space.

(ii) $\operatorname{dom}(T)$ is closed in $\mathfrak{X}$ (by hypothesis), hence $\operatorname{dom}(T)$ is a Banach space.

(iii) P_1 is a bijection

(iv) P_1 is bounded: Let $z = (x, Tx) \in G(T)$, then

$$\|P_1 z\|_{\mathfrak{X}} = \|x\|_{\mathfrak{X}} \le \|x\|_{\mathfrak{X}} + \|Tx\|_Y = \|z\|_{X\times Y}.$$

So $\|P_1\| \le 1$

By the inverse mapping theorem $P_1^{-1}: \begin{array}{l} \operatorname{dom}(T) \to G(T) \\ x \mapsto (x, Tx) \end{array}$ is also bounded, hence:

$\exists c < \infty: \|P_1^{-1} x\|_{X\times Y} \le c\|x\|_X$. Since $\|P_1^{-1}x\|_{X\times Y} = \|x\|_X + \|Tx\|_Y$ this implies

$$\|Tx\|_Y \le (c+1)\|x\|_{\mathfrak{X}}$$

So T is bounded $\blacksquare$

$\boxed{\text{Example 4.14}}$ Let $X = Y = C_0(\mathbb{R})$ with supremum norm (Banach!)

Consider $T := \frac{d}{dx} : \operatorname{dom}(T) \to C_0(\mathbb{R})$
$f \mapsto f'$

with $\operatorname{dom}(T) = \{ f \in C_0(\mathbb{R}) \mid f \in C^1(\mathbb{R}) \text{ and } f' \in C_0(\mathbb{R}) \} \subseteq C_0(\mathbb{R})$.

Claim: T is closed (a closed operator).

Pf: Let $(f_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}(T)$ be a sequence such that:

(i) $\exists g \in C_0(\mathbb{R})$: $\|f_n - g\|_\infty \xrightarrow{n\to\infty} 0$

(ii) $\exists h \in C_0(\mathbb{R})$: $\|f_n' - h\|_\infty \xrightarrow{n\to\infty} 0$

i.e., $(f_n, f_n')_n \in G(T)$, $(f_n, f_n')_n \xrightarrow{n\to\infty} (g, h)$ in $X \times Y$; we need to prove that $(g,h) \in G(T)$, equivalently, $g \in \operatorname{dom}(T)$ and $h = g'$.

By uniform convergence, we can exchange limits, so, $\forall x \in \mathbb{R}$,

$$\int_0^x h(t)\,dt = \int_0^x \lim_{n\to\infty} f_n'(t)\,dt = \lim_{n\to\infty} \underbrace{\int_0^x f_n'(t)\,dt}_{f_n(x) - f_n(0)} = g(x) - g(0)$$

(since $f_n \to g$ pointwise, by (i)).

Hence, $g(x) = g(0) + \int_0^x h(t)\,dt \quad \forall x \in \mathbb{R}$

So, by the Fundamental Theorem of Calculus (HDI): $g \in C^1(\mathbb{R})$ with $g' = h \in C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$, we have $g \in \operatorname{dom}(T)$, and so $(g, h) = (g, g') \in G(T)$. So T is closed.

Also: Since T is unbounded (cf. 2.28) $\overset{\text{Thm. 4.13}}{\Longrightarrow}$ $\operatorname{dom}(T)$ is not a closed subspace of $C_0(\mathbb{R})$ wrt. $\|\cdot\|_\infty$.

4.3 (Bi-) Dual spaces and weak topologies

$\boxed{\text{Theorem 4.15}}$ Let X be a normed space. Then, for every $x \in X$,

$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*}$$

Pf: See exercise.