Chapter 4: The cornerstones of Functional Analysis

4.1 Hahn-Banach Theorem

Recall:

Definition Let M be a set, and $D \subseteq M \times M$. Let $\prec$ ("sub") be the associated binary relation: $x \prec y :\Leftrightarrow (x,y) \in D$.

(a) $\prec$ is a partial ordering (or, M is a partially ordered set)

$:\Leftrightarrow \forall x, y, z \in M$:

(i) $x \prec x$ (reflexive)

(ii) $(x \prec y \ \& \ y \prec x) \Rightarrow (x = y)$ (antisymmetry)

(iii) $(x \prec y \ \& \ y \prec z) \Rightarrow (x \prec z)$ (transitive)

(b) $x, y \in M$ are comparable $:\Leftrightarrow x \prec y$ or $y \prec x$

x, y are incomparable $:\Leftrightarrow x, y$ are not comparable.

(c) $\prec$ is a total ordering $:\Leftrightarrow$

(i) $\prec$ is a partial ordering

(ii) x, y are comparable for all $x, y \in M$

(d) Let $W \subseteq M$. Then $u \in M$ is an upper bound of/for W

$:\Leftrightarrow w \prec u \ \forall w \in W$.

(e) $m \in W$ is a maximal element of W $:\Leftrightarrow$ The following implication holds:

$$m \prec w \text{ for } w \in W \quad \Rightarrow \quad m = w.$$

(Note: A maximal element need not be an upper bound and vice versa - see examples below).

Examples (a) "$\leq$" is a total ordering on $\mathbb{R}$.

$W = [0,1)$ has no maximal element, but any $u \geq 1$ is an upper bound for W

(b) "$\subseteq$" is a partial ordering on $\mathcal{P}(X)$, but not a total ordering.

(c) $(x_1,x_2) \prec (y_1,y_2) :\Leftrightarrow x_j \le y_j\ ,\ j=1,2,$
is a partial ordering (but not total) on $M = \mathbb{R}^2$.
$W := \{(0,0),(1,0),(0,1)\}$ has 2 (!) maximal elements,
but none of them is an upper bound for W.

The following axiom is equivalent to the axiom of choice:

Axiom 4.1 (Zorn's Lemma) Let $M \neq \emptyset$ be a partially ordered set. Assume every totally ordered subset of M has an upper bound. Then M has a maximal element.

We now finish the

Proof of Thm. 2.53 (Remains: Every Hilbert space $X \neq \{0\}$ has an orthonormal basis)

Let $M := \{\mathcal{E} \subseteq X \mid \mathcal{E} \text{ orthonormal}\}$. Then:

(i) $M \neq \emptyset$

(ii) "$\subseteq$" is partial ordering on M

(iii) Let W be a totally ordered subset of M.

Then $\mathcal{U} := \bigcup_{\mathcal{E} \in W} \mathcal{E} \in M$:

If $x_1, x_2 \in \mathcal{U}$ then $\exists\, \mathcal{E}_j \in W :\ x_j \in \mathcal{E}_j\ ,\ j=1,2$.

As W is totally ordered, $\mathcal{E}_1 \subseteq \mathcal{E}_2$ (wlog.)

Hence, $x_1, x_2 \in \mathcal{E}_2 \Rightarrow x_1 \perp x_2 \Rightarrow \mathcal{U}$ orthonormal

That is, $\mathcal{U}$ is an upper bound for W.

By Zorn: $\exists$ maximal element $\mathcal{M} \in M$.

Claim: $\mathcal{M}$ is an orthonormal basis for X

(i) Orthonormal clear, since $\mathcal{M} \in \mathfrak{M}$

(ii) Assume $\mathcal{M}$ is not complete, i.e., not a basis.

Then (by Def. 2.47(d)), there is some $0 \neq x \in X$ such that $x \perp m$ $\forall m \in \mathcal{M}$. Hence,

$$\mathcal{M}' := \left\{ \frac{x}{\|x\|} \right\} \cup \mathcal{M} \in \mathfrak{M} \quad (\text{since } \mathcal{M}' \text{ is orthonormal})$$

So $\mathcal{M} \subsetneq \mathcal{M}'$ which contradicts $\mathcal{M}$ being maximal ∎

$\boxed{\text{Remark 4.2}}$ Similar arguments prove Thm. 2.4 (existence of Hamel basis; see exercise)

$\boxed{\text{Theorem 4.3}}$ (Hahn-Banach (H-B)) Let X be a vectorspace, let $p: X \to \mathbb{R}$ be convex, i.e. $\forall x, x' \in X$, $\forall \alpha \in [0,1]$:

$$p(\alpha x + (1-\alpha)x') \leq \alpha p(x) + (1-\alpha) p(x'). \quad (*)$$

Let Y be a subspace, and let $\lambda: Y \to \mathbb{K}$ be linear, with

$$\operatorname{Re} \lambda(y) \leq p(y) \quad \forall y \in Y \quad (\square)$$

Then there exists $\Lambda: X \to \mathbb{K}$ linear such that:

(i) $\Lambda|_Y = \lambda$ (i.e. Λ is extension of λ)

(ii) $\operatorname{Re} \Lambda(x) \leq p(x)$ $\forall x \in X$ (i.e. $(\square)$ is preserved)

If, in addition, p also satisfies

$$(**) \qquad p(\alpha x) \leq p(x) \quad \forall x \in X, \forall \alpha \in \mathbb{K} \text{ with } |\alpha| = 1$$

then we even have $|\Lambda(x)| \leq p(x)$ $\forall x \in X$.

$\boxed{\text{Remark 4.4}}$ (a) Condition $(**)$ is equivalent to:

$$p(\alpha x) = p(x) \quad \forall x \in \mathbb{K} \text{ with } |\alpha| = 1 \quad (\text{check!})$$

(b) If p is a (semi-) norm on X, then p satisfies $(*)$ & $(**)$.

<u>Pf (of 4.3):</u> 4 Steps : Steps 1 & 2 prove the main part for $\mathbb{R}$- vectorspaces, step 3 for $\mathbb{C}$-vector-spaces, and step 4 proves the addendum under add. condition (**).

Wlog: $Y \subsetneq X$ (otherwise trivial).

<u>Step 1:</u> Case $\mathbb{K} = \mathbb{R}$. Extend by <u>one</u> dimension – a preparation for Step 2.

By assumption: $\exists\, z \in X \setminus Y$ (so $z \neq 0$). Let $\tilde{Y} := \operatorname{span}(Y, \{z\})$.

For every $\tilde{y} \in \tilde{Y}$, $\exists!$ decomposition $\tilde{y} = y + \alpha z$, $y \in Y$, $\alpha \in \mathbb{R}$

Candidate for extension $\tilde{\lambda}$ of λ to $\tilde{Y}$:

$$\tilde{\lambda}(\tilde{y}) := \lambda(y) + \alpha \xi \qquad \text{for } \underline{\text{some}}\ \xi \in \mathbb{R}$$

(to be chosen)

Interpretation: $\xi = \tilde{\lambda}(z)$. Clearly, $\tilde{\lambda}$ is ($\mathbb{R}$-) linear on $\tilde{Y}$, and $\tilde{\lambda}|_Y = \lambda$. Will choose ξ s.t. $\tilde{\lambda} \le p$ on $\tilde{Y}$ (recall (ii)):

Let $\beta_1, \beta_2 > 0$, $y_1, y_2 \in Y$, then

$$\beta_1 \lambda(y_1) + \beta_2 \lambda(y_2) = (\beta_1 + \beta_2)\, \underbrace{\lambda\left(\frac{\beta_1}{\beta_1+\beta_2} y_1 + \frac{\beta_2}{\beta_1+\beta_2} y_2\right)}_{} \quad (\nabla)$$

$$\overset{\text{by } (\square)}{\le} p\left(\underbrace{\frac{\beta_1}{\beta_1+\beta_2} y_1 + \frac{\beta_2}{\beta_1+\beta_2} y_2}_{}\right)$$

$$= \frac{\beta_1}{\beta_1+\beta_2}(y - \beta_2 z) + \frac{\beta_2}{\beta_1+\beta_2}(y_2 + \beta_1 z)$$

Hence,

$$(\nabla) \overset{p \text{ convex}}{\le} \beta_1 p(y_1 - \beta_2 z) + \beta_2 p(y_2 + \beta_1 z).$$

Re-arranging, we get

$$\frac{1}{\beta_2}\left(\lambda(y_1) - p(y_1 - \beta_2 z)\right) \le \frac{1}{\beta_1}\left(p(y_2 + \beta_1 z) - \lambda(y_2)\right)$$

$$\forall \beta_1, \beta_2 > 0, \ \forall y_1, y_2 \in Y$$

Hence, there exists $a \in \mathbb{R}$:

$$(\nabla\nabla) \quad \sup_{\substack{\beta_2 > 0 \\ y_1 \in Y}} \left[\frac{1}{\beta_2} \left(\lambda(y_1) - p(y_1 - \beta_2 z) \right) \right] \le a \le \inf_{\substack{\beta_1 > 0 \\ y_2 \in Y}} \left[\frac{1}{\beta_1} \left(p(y_2 + \beta_1 z) - \lambda(y_2) \right) \right]$$

Set $\xi = \tilde{\lambda}(z) := a$. Then, $\forall \tilde{y} = y + \alpha z \in \tilde{Y}$ with $\alpha > 0$, the <u>right</u> inequality in $(\nabla\nabla)$ implies

$$\lambda(\tilde{z}) \le \frac{1}{\alpha}\left(p(\tilde{y}) - \lambda(y) \right), \text{ so } \tilde{\lambda}(\tilde{y}) \le p(\tilde{y}).$$

If $\tilde{y} = y - \alpha z$ with $\alpha > 0$, use the <u>left</u> ineq. in $(\nabla\nabla)$ instead.

Therefore, $\tilde{\lambda} \le p$ on $\tilde{Y}$.

<u>Step 2</u>: Case $\mathbb{K} = \mathbb{R}$. Idea: Use Zorn to construct the extension.

Let $M := \{ (\mathbb{R}\text{-})$ linear extensions e of λ with $e \le p$ on $\mathrm{dom}(e) \}$

We have

(i) $M \ne \emptyset$ since $\lambda \in M$

(ii) Define partial ordering $<$ on M:

$$e_1 < e_2 :\Longleftrightarrow \mathrm{dom}(e_1) \subseteq \mathrm{dom}(e_2) \wedge e_2|_{\mathrm{dom}(e_1)} = e_1.$$

(iii) Let $W \subseteq M$ be totally ordered.

Define $u : \bigcup_{e \in W} \mathrm{dom}(e) \to \mathbb{R}$, $x \longmapsto \tilde{e}(x)$ where $\tilde{e}$ is any element of W such that $x \in \mathrm{dom}(\tilde{e})$.

<u>u is well-defined</u>: (i.e. indep. of the chosen $\tilde{e}$ among allowed elements):

Let $x \in \mathrm{dom}(\tilde{e}_1) \cap \mathrm{dom}(\tilde{e}_2)$. Since W is totally ordered, we have (wlog.) $\tilde{e}_2$ is an extension of $\tilde{e}_1$. Hence, $\tilde{e}_2(x) = \tilde{e}_1(x)$.

<u>u is $(\mathbb{R}\text{-})$ linear</u>: Let $x_1, x_2 \in \bigcup_{e \in W} \mathrm{dom}(e)$, $\gamma \in \mathbb{R}$. Hence, $\exists \tilde{e}_j \in W : x_j \in \mathrm{dom}(\tilde{e}_j)$, $j = 1, 2$. Since W is totally ordered, (wlog.) $\tilde{e}_2$ is extension of $\tilde{e}_1$. Then $x_1, x_2, x_1 + \gamma x_2 \in \mathrm{dom}(\tilde{e}_2)$ & linearity of u follows from that of $\tilde{e}_2$.

Lastly: $u(x) = \tilde{e}(x) \le p(x)$ $\forall x \in \bigcup_{e \in W} \mathrm{dom}(e)$.

Hence: $u \in M$, and (check!) u is an upper bound for W.

By Zorn's Lemma: M has a maximal element Λ.

<u>Claim</u>: $\mathrm{dom}(\Lambda) = X$: Suppose $\mathrm{dom}(\Lambda) \subsetneq X$. Then there is some $0 \neq z \in X \setminus \mathrm{dom}(\Lambda)$. Let $\tilde{Y} := \mathrm{span}(\mathrm{dom}(\Lambda), \{z\})$.

By Step 1, Λ has some ($\mathbb{R}$-) linear extension $\tilde{\Lambda} \in M$ to $\tilde{Y}$, which contradicts M being maximal.

Hence, the main part of the thm. follows for $\mathbb{K} = \mathbb{R}$.

<u>Step 3</u>: Case $\mathbb{K} = \mathbb{C}$. Reduce to the real case.

Define $\ell(y) := \mathrm{Re}\,\lambda(y)$ $\forall y \in Y$. Then $\ell : X \to \mathbb{R}$ is $\mathbb{R}$-linear, and $\ell \le p$ on Y. So, Step 2 implies: There exists $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ with $L|_Y = \ell$ and $L \le p$ on X.

Note: $\lambda(y) = \ell(y) + i\,\ell(-iy)$ $\forall y \in Y$ since:

$$\ell(-iy) = \mathrm{Re}\,\lambda(-iy) \overset{\lambda\ \mathbb{C}\text{-lin}}{=} \mathrm{Re}\,[-i\,\lambda(y)] = \mathrm{Im}\,\lambda(y)$$

Define $\Lambda(x) := L(x) + i\,L(-ix)$, $x \in X$.

Then, by Step 2,

(i) Λ is $\mathbb{R}$-linear on X

(ii) $\Lambda|_Y = \lambda$

(iii) $\mathrm{Re}\,\Lambda \overset{L(-ix) \in \mathbb{R}}{=} L \le p$ on X

Also, Λ is <u>$\mathbb{C}$</u>-linear, since it is $\mathbb{R}$-linear and

$$\Lambda(ix) = L(ix) + i\,L(x) \overset{L\ \mathbb{R}\text{-lin}}{=} i\,(L(x) + i\,L(-ix)) = i\,\Lambda(x),$$

proving the main part for $\mathbb{K} = \mathbb{C}$.

<u>Step 4</u>: Addendum: We fix $\in X$ and use the polar repr.

$$\Lambda(x) = |\Lambda(x)|\,e^{i\theta(x)} \quad [\text{If } \mathbb{K} = \mathbb{R} \text{ then } e^{i\theta(x)} \in \{-1, 1\}]$$

Then $|\Lambda(x)| = e^{-i\theta(x)}\Lambda(x) \overset{\mathbb{K}\text{-lin}}{=} \Lambda(e^{-i\theta(x)}x) \overset{\text{LHS} \in \mathbb{R}}{=} \mathrm{Re}\,\Lambda(e^{-i\theta(x)}x)$

$$\overset{\text{main part}}{\le} p(e^{-i\theta(x)}x) \overset{(**)}{\le} p(x)$$

$\blacksquare$

Corollary 4.5 Let X be a normed space, $Y \subseteq X$ a subspace, and $\varphi \in Y^*$. Then there exists $f \in X^*$ with $f|_Y = \varphi$ and $\|f\|_{X^*} = \|\varphi\|_{Y^*}$.

Pf: Apply Thm. 4.3 with $p(x) := \|\varphi\|_{Y^*} \cdot \|x\|$ for $x \in X$ (fulfills assumptions - check!) to $\lambda = \varphi$. Then $\exists f : X \to \mathbb{K}$ linear with $|f(x)| \leq \|\varphi\|_{Y^*}\|x\|$, hence $\|f\|_{X^*} \leq \|\varphi\|_{Y^*}$.

But
$$\|f\|_{X^*} = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \geq \sup_{0 \neq x \in Y} \frac{|\varphi(x)|}{\|x\|} = \|\varphi\|_{Y^*} \qquad \blacksquare$$

Corollary 4.6 Let X be a normed space and let $0 \neq x_0 \in X$. Then there exists $f \in X^*$ with $f(x_0) = \|x_0\|$ and $\|f\|_{X^*} = 1$.

Pf: Let $Y = \operatorname{span}\{x_0\}$. If $y \in Y$ then $y = \alpha x_0$ for some unique $\alpha \in \mathbb{K}$. Define φ on Y by $\varphi(y) := \alpha \cdot \|x_0\|$. This implies $\varphi(x_0) = \|x_0\|$ and $\varphi \in Y^*$ with $\|\varphi\|_{Y^*} = 1$, since $|\varphi(y)| = \|y\|$. By Cor. 4.5, $\exists f : X \to \mathbb{K}$ linear, with $f|_Y = \varphi$ (in particular, $f(x_0) = \|x_0\|$) and $\|f\|_{X^*} = \|\varphi\|_{Y^*} = 1$ $\blacksquare$

Corollary 4.7 Let X be a normed space, $Z \subseteq X$ a closed subspace, and $x_0 \in X \setminus Z$ with $0 < \operatorname{dist}(x_0, Z) =: d$. Then there exists $f \in X^*$ with $f|_Z = 0$, $f(x_0) = d$ and $\|f\|_{X^*} = 1$.

Proof: See exercise.