Chapter 3: L^p - spaces

3.1 Completeness and dual space

Notation: In this section: $(X, \mathcal{A}, \mu)$ a measure space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,
$f: X \to \mathbb{K}$ measurable (see also Handout!)

Definition 3.1 (L^p-spaces) Let $p \in (0, \infty)$.

(a)
$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} \quad \text{(possibly } \infty)$$

$$\|f\|_\infty := \inf \{ \alpha > 0 \mid \mu(\{x \in X \mid |f(x)| > \alpha\}) = 0 \}$$
$$= \inf_{\substack{N \in \mathcal{A} \\ \mu(N) = 0}} \sup_{x \in X \setminus N} |f(x)| =: \operatorname*{ess\,sup}_{x \in X} |f(x)| \quad (\mu\text{-}) \text{ essential supremum}$$

(b) Vectorspace of p-integrable functions (wrt. μ):

$$\mathcal{L}^p := \mathcal{L}^p(\mu) := \mathcal{L}^p(X) := \mathcal{L}^p(X, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) :=$$
$$:= \{ f: X \to \mathbb{K} \mid f \text{ measurable and } \|f\|_p < \infty \} = \{ f: X \to \mathbb{K} \mid |f|^p \ \mu\text{-integrable} \}$$

(c) Equivalence relation $\sim$ on $\mathcal{L}^p$:
$$f \sim g \ :\Leftrightarrow \ f = g \ \ \mu\text{-a.e.}$$
Vectorspace of equivalence classes of p-integrable fct.'s wrt μ:
$$L^p := \mathcal{L}^p / \sim .$$

Warning Notation does not distinguish between equivalence classes of functions, and their representatives!

Lemma 3.2 (Hölder & Minkowski).

(a) $\forall r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$: $\forall f, g: X \to \mathbb{K}$ meas.

$$\|fg\|_r \le \|f\|_p \cdot \|g\|_q \quad \text{(generalised Hölder's ineq.)}$$

(b) Let $(\mathbb{X}_i, \mathcal{A}_i, \mu_i)$, $i=1,2$, be σ-finite measure spaces, and $f: \mathbb{X}_1 \times \mathbb{X}_2 \to \mathbb{C}$ measurable. Then, for $p \in [1,\infty)$,

$$(*) \quad \left\{ \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

(Minkowski's Integral Inequality)

That is,

$$\left\| \int_{\mathbb{X}_1} f(x,\cdot)\, d\mu_1(x) \right\|_p \le \int_{\mathbb{X}_1} \|f(x,\cdot)\|_p \, d\mu_1(x)$$

(c) $\forall p \in [1,\infty]$:

$$(**) \qquad \|f+g\|_p \le \|f\|_p + \|g\|_p \quad \text{(Minkowski's inequality)}$$

Pf: (a) Ana III / Handout / see exercises.

(b) Note that

$$\left\{ \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \left\{ \int_{\mathbb{X}_2} \left(\int_{\mathbb{X}_1} |f(x,y)|\, d\mu_1(x) \right)^p d\mu_2(y) \right\}^{1/p},$$

hence, it suffices to consider $f \ge 0$.

Next, if $p=1$, $(*)$ follows from Tonelli's Thm.

If $1<p<\infty$, then

$$\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \overset{p=(p-1)+1}{=} \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^{p-1} \cdot \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right| d\mu_2(x)$$

$$\le \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \left[\int_{\mathbb{X}_1} |f(x,y)|\, d\mu_1(x) \right] d\mu_2(y)$$

$$= \int_{\mathbb{X}_2} \left(\int_{\mathbb{X}_1} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_1(x) \right) d\mu_2(y) \quad \left(\text{linearity } \int_{\mathbb{X}_1} d\mu_1(x)\right)$$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_2(y) \right) d\mu_1(x) \qquad (\mathrm{II})$$

Now, with $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $q = \frac{p}{p-1}$, Hölder gives

$$\int_{X_2} \left| \int_{X_1} f(t,y)\,d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_2(x)$$

$$\leq \left(\int_{X_2} \left| \int_{X_1} f(t,y)\,d\mu_1(t) \right|^{q(p-1)} d\mu_2(y) \right)^{1/q} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p}$$

$$= \left(\int_{X_2} \left| \int_{X_1} f(t,y)\,d\mu_1(t) \right|^{p} d\mu_2(y) \right)^{\frac{p-1}{p}} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p}$$

Inserting this in ($\square$) gives

$$\int_{X_2} \left| \int_{X_1} f(x,y)\,d\mu_1(x) \right|^p d\mu_2(y)$$

$$\leq \int_{X_1} \underbrace{\left(\int_{X_2} \left| \int_{X_1} f(t,y)\,d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{indep. of } x} \cdot \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

$$= \underbrace{\left(\int_{X_2} \left| \int_{X_1} f(\cancel{t}_x,y)\,d\mu_1(\cancel{t}_x) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{divide (!)} \,:\, 1-\frac{p-1}{p} = \frac{1}{p}} \cdot \int_{X_2} \left(\int_{X_1} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

$$\Rightarrow \left\{ \int_{X_2} \left| \int_{X_1} f(x,y)\,d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \leq \int_{X_1} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

To divide: we need integral to be finite; is the case if $\mu_i(X_i) < \infty$, $i=1,2$ and f bounded. For general case: Let $X_i^k \nearrow X_i$, $k \to \infty$, $\mu_i(X_i^k) < \infty$ (X_i σ-finite), consider the above for

$$\tilde{f}_k(x,y) := \mathbb{1}_{X_1^k}(x)\, \mathbb{1}_{X_2^k}(y)\, \mathbb{1}_{\{f \leq k\}} \cdot f(x,y) \text{ \& take limit } k \to \infty$$

(c) Take $X_1 = \{1,2\}$, $\mu_1 =$ counting measure, $\int_{X_1} \cdots d\mu_1(x) = \sum_{i=1}^{2}$,

then (*) becomes, with $f(x,y) = f_i(y)$,

$$(**) \quad \|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p, \quad f_i \in L^p(X_2, \mu_2) \quad \blacksquare$$

$\boxed{\text{Remark}}$ (a) Take $X_2 := \mathbb{N}$, $\mu_2 =$ counting measure, $\int_{X_2} d\mu_2(y) = \sum_{i=1}^{\infty}$,

then (**) becomes

$$(***) \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p, \quad x, y \in \underline{\ell^p}$$

(b) Take $X_2 := \{1, \dots, N\}$, $\mu_2 =$ counting measure, $\int_{X_2} d\mu_2 = \sum_{i=1}^{N}$

then (**) becomes

$$(****) \qquad \|x+y\|_p \le \|x\|_p + \|y\|_p, \quad x, y \in \mathbb{K}^N$$

(c) The proof above is "the same" as the standard proof of (**), (***), or (****) (check!).

$\boxed{\text{Theorem 3.3}}$ (Riesz – Fischer) Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $(L^p, \|\cdot\|_p)$ is a Banach space for all $p \in [1, \infty]$. Moreover, L^2 is a Hilbert space with scalar product

$$\langle f, g \rangle := \int_X \overline{f(x)}\, g(x)\, d\mu(x) \quad \forall f, g \in L^2$$

Pf: Ana III. (triangle inequality needs $p \ge 1$!)

$\boxed{\text{Remark}}$ (a) Take $X = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, $\mu =$ counting measure, $\int_X d\mu = \sum_{i=1}^{\infty}$, then 3.3 says: $(\ell^p, \|\cdot\|_p)$ is Banach.

(b) Take $X = \{1, \dots, N\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu =$ counting measure, $\int_X d\mu = \sum_{i=1}^{N}$, then 3.3 says: $(\mathbb{K}^N, \|\cdot\|_p)$ is Banach.

$\boxed{\text{Lemma 3.4}}$ If $\mu(X) < \infty$, $1 \le r \le p \le \infty$, then L^p $(\subseteq L^r)$ is dense in L^r (wrt. $\|\cdot\|_r$), and

$$\|f\|_r \le [\mu(X)]^{\frac{1}{r} - \frac{1}{p}} \|f\|_p \qquad \forall f \in L^p.$$

Pf: Ana III.

The next theorem is about the geometry of L^p, but also serves as preparation to compute its dual:

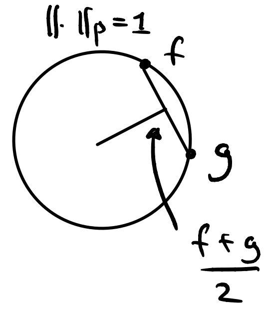
Theorem 3.5 Let $p \in (1,\infty)$. Then:

For all $\varepsilon > 0, r > 0$ there exists $\delta > 0$ s.t.:

For all $f, g \in L^p$:

$\|f\|_p, \|g\|_p \le r$ and $\|f-g\|_p \ge \varepsilon$

$$\Rightarrow \left\| \frac{f+g}{2} \right\|_p^p \le \frac{\|f\|_p^p + \|g\|_p^p}{2} - \delta.$$



In particular, L^p is uniformly convex (case (see drawing) $r = \|f\|_p = \|g\|_p = 1$)

Pf (after Naoki Shioji)

Define the map

$$\alpha : \mathbb{K}^2 \longrightarrow \mathbb{R}$$
$$z = (z_1, z_2) \longmapsto \frac{|z_1|^p + |z_2|^p}{2} - \left| \frac{z_1 + z_2}{2} \right|^p$$

Claim: $\forall z \in \mathbb{K}^2$: $\alpha(z) \ge 0$, with "=" iff $z_1 = z_2$ (*)

PfClaim: $\left|\frac{z_1+z_2}{2}\right|^p \underset{\Delta\text{-ineq. } \mathbb{K}}{\le} \left(\frac{|z_1|+|z_2|}{2}\right)^p \underset{(\cdot)^p \text{ strictly convex } (p>1)}{\le} \frac{|z_1|^p + |z_2|^p}{2}$

Δ-ineq. is equality $\Leftrightarrow$ $z_1 = 0$ or $z_2 = \lambda z_1$ for some $\lambda > 0$

Strict conv. ineq. is equality $\Leftrightarrow$ $|z_1| = |z_2|$

$\Rightarrow$ (equality $\Leftrightarrow$ $z_1 = z_2$) ✓ claim

Now, fix $\varepsilon, r > 0$, and choose $\eta \in (0, \frac{\varepsilon^p}{2r^p})$ small enough (f.ex. $< 2^{-p}$) such that

$$D := \{ z \in \mathbb{K}^2 \mid |z_1|^p + |z_2|^p = 1 \text{ and } |z_1 - z_2|^p \ge \eta \} \neq \emptyset$$

Then, by compactness (!) of D and continuity of α,

$$\theta := \inf\{\alpha(z) \mid z \in D\} = \min\{\alpha(z) \mid z \in D\} \underset{\text{by } (*)}{>} 0$$

Claim: $$|z_1|^p + |z_2|^p \le \frac{\alpha(z)}{\theta} \qquad (**)$$

$$\forall z \in F := \{ \tilde{z} \in \mathbb{K}^2 \mid |\tilde{z}_1 - \tilde{z}_2|^p \ge \eta (|\tilde{z}_1|^p + |\tilde{z}_2|^p) \}$$

Pf claim: W.l.o.g., $z \neq (0,0)$ ($(**)$ holds for $(0,0)$)

Let $z \in F$, define $t := \left(|z_1|^p + |z_2|^p\right)^{1/p} > 0$ (since $z \neq 0$)

Note: (i) $\left|\frac{z_1}{t}\right|^p + \left|\frac{z_2}{t}\right|^p = 1$ (compute!)

(ii) $\left|\frac{z_1}{t} - \frac{z_2}{t}\right|^p \geq \eta$ (since $z \in F$)

so $\frac{z}{t} \in D$. Hence, by def. of Θ, we have $\Theta \leq \alpha\left(\frac{z}{t}\right) = \frac{\alpha(z)}{t^p}$ ✓ (claim)

Now, let $f, g \in L^p$ with $\|f\|_p, \|g\|_p \leq r$ and $\|f-g\|_p \geq \varepsilon$.

Define $M := \{x \in X \mid (f(x), g(x)) \in F\}$ $(X = M \cup M^c)$

and estimate:

$$\varepsilon^p \leq \int_X |f-g|^p d\mu = \int_{M^c} |f-g|^p d\mu + 2^p \int_M \left|\frac{f-g}{2}\right|^p d\mu =: I_1 + 2^p I_2 \quad (***)$$

By definition of M^c (& F) $M^c \subseteq X$

$$I_1 \leq \eta \int_{M^c} (|f|^p + |g|^p) d\mu \leq \eta \left(\|f\|_p^p + \|g\|_p^p\right) \leq 2\eta r^p$$

Since $\alpha \geq 0$, we also get

$$I_2 \leq \frac{1}{2} \int_M \left(|f|^p + |g|^p\right) d\mu \overset{(**)}{\leq} \frac{1}{2\Theta} \int_M \alpha(f(x), g(x))\, d\mu(x)$$

$$\overset{M \subseteq X}{\leq} \frac{1}{2\Theta} \int_X \left(\frac{|f|^p + |g|^p}{2} - \left|\frac{f+g}{2}\right|^p\right) d\mu.$$

Inserting the inequalities for I_1 & I_2 in $(***)$ gives

$$\left\|\frac{f+g}{2}\right\|_p^p \leq \underbrace{\frac{\Theta(2\eta r^p - \varepsilon^p)}{2^{p-1}}}_{=: -\delta} + \frac{\|f\|_p^p + \|g\|_p^p}{2}$$

with $\delta > 0$ by the definition of η ☒

$\boxed{\text{Theorem 3.6}}$ (Riesz' Representation Thm. for $(L^p)^*$)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$, assume in addition that μ is σ-finite. Then the map

$$J : L^q \to (L^p)^*, \quad f \mapsto \ell_f \quad , \quad \ell_f : L^p \to \mathbb{K}, \quad g \mapsto \int_X fg \, d\mu \qquad (*)$$

is an isometric isomorphism. Hence:

$$L^q \simeq (L^p)^*, \quad \frac{1}{p} + \frac{1}{q} = 1, \; p \in [1, \infty)$$

$\boxed{\text{Remark 3.7}}$ (a) The map J is also well-defined and isometric for $p = \infty$, but fails to be surjective if $\operatorname{supp}(\mu)$ has infinitely many elements (see exercise). I.e., $(L^\infty)^*$ is strictly larger than L^1 in this case (compare 2.38 and 2.39(b))

(b) The case $p = 1$ in 3.6 would be wrong without additional hypothesis on the measure space. σ-finiteness of μ is sufficient, but not necessary, because of:

$\boxed{\text{Thm}}$ The map $J : L^\infty \to (L^1)^*$ is

(a) Injective $\Leftrightarrow$ isometric $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is semi-finite

(b) Bijective $\Leftrightarrow$ isometric & bijective $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is localisable

Note: $(X, \mathcal{A}, \mu)$ σ-finite $\Rightarrow$ localisable $\Rightarrow$ semi-finite

(For required def.'s & pf., see Salamon, "Measure and Integration", EMS, (2016) Sect. 4.5)

Pf (of 3.6): By linearity of the integral, the map J is linear.

Let $f \in L^q$. Then ℓ_f (see $(*)$) is linear (in g), $fg \in L^1$ $\forall g \in L^p$ by Hölder, and $|\ell_f(g)| \leq \|f\|_q \|g\|_p$. Hence, $\ell_f \in (L^p)^*$, and $\quad \|\ell_f\|_{(L^p)^*} \le \|f\|_q . \quad (**)$

Hence, J in $(*)$ is well-defined, linear, and bounded.

<u>Claim:</u> $\quad \|\ell_f\|_{(L^p)^*} \overset{(***)}{=} \|f\|_q \qquad$ ($\Rightarrow J$ is isometric, hence inj.)

Pf <u>claim</u>: Wlog, $f \neq 0$ ((***) clear for $f=0$).

<u>(i) Case $p \in (1,\infty)$</u>: Let
$$g(x) := \begin{cases} \bar{f}(x)\,|f(x)|^{q-2} & , f(x) \neq 0 \\ 0 & , f(x) = 0 \end{cases}$$

Then
$$|g|^p = |f|^{(q-1)p} = |f|^q \in L^1 \quad \text{since } (q-1)p = q\left(1-\tfrac{1}{q}\right)p = q$$
$$\left(\tfrac{1}{p}+\tfrac{1}{q}=1\right).$$

So $g \in L^p$, and $\|g\|_p = \|f\|_q^{q/p}$. Also
$$\frac{\ell_f(g)}{\|g\|_p} = \frac{\int |f|^q}{\|g\|_p} = \frac{\|f\|_q^q}{\|f\|_q^{q/p}} = \|f\|_q^{q(1-\frac{1}{p})} = \|f\|_q \,, \text{ hence}$$

$\|\ell_f\|_{(L^p)^*} \ge \|f\|_q$. This, and $(**)$, imply $(***)$ (the claim).

<u>(ii) Case $p=1$</u>: Since μ is σ-finite, we have:
$$X = \bigcup_{n\in\mathbb{N}} X_n \,, \quad X_n \subseteq X_{n+1} \,, \quad \mu(X_n) < \infty \qquad \forall n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and, for $f \in L^\infty$, define
$$M := \{ x \in X \mid |f(x)| \ge \|f\|_\infty - \varepsilon \}.$$

By definition of "ess sup", $\mu(M) > 0$. Then $M_n := M \cap X_n$ satisfies $\quad 0 < \mu(M_n) < \infty \quad$ for (all) n sufficiently large.

Let
$$g(x) := \begin{cases} \dfrac{\bar{f}(x)}{|f(x)|} \dfrac{1}{\mu(M_n)} \mathbb{1}_{M_n}(x) & , f(x) \neq 0 \\ 0 & , f(x) = 0 \end{cases}$$

Then $g \in L^1$ with $\|g\|_1 \le 1$, and
$$\ell_f(g) = \int_X fg \, d\mu = \frac{1}{\mu(M_n)} \int_{M_n} |f| \, d\mu \ge \|f\|_\infty - \varepsilon.$$
As this holds $\forall \varepsilon > 0$, we get $\|\ell_f\|_{(L^1)^*} \ge \|f\|_\infty$. This, and $(**)$, imply the claim $(***)$.

It remains to prove that J is surjective.

Let $\xi \in (L^p)^*$. Need to prove: $\exists f_\xi \in L^q$ s.t. $\xi = \ell_{f_\xi}$ (see (*1))

Case $p \in (1,\infty)$: (We follow again N. Shioji)

Define the non-linear functional $F : L^p \to \mathbb{R}$

$$g \mapsto F(g) := \frac{1}{p}\|g\|_p^p - \operatorname{Re} \xi(g)$$

Since $p > 1$ and

$$F(g) \geq \frac{1}{p}\|g\|_p^p - \|\xi\|_* \|g\|_p ,$$

F is bounded from below:

$$m := \inf\{F(g) \mid g \in L^p\} > -\infty$$

Claim: $\exists!\, f \in L^p : F(f) = m$ (i.e. exists unique minimiser of F)

[This is where we need/use uniform convexity!]

Pf claim: For $n \in \mathbb{N}$, let $C_n := \{g \in L^p \mid F(g) \leq m + \frac{1}{n}\}$.

Then: (i) $C_n \neq \emptyset$ $\forall n \in \mathbb{N}$ (by definition of the infimum)

(ii) C_n closed $\forall n \in \mathbb{N}$ (since F cont. and $(-\infty, m + \frac{1}{n}]$ closed)

(iii) $C_n \supseteq C_{n+1}$ $\forall n \in \mathbb{N}$

(iv) $\operatorname{diam}(C_n) \xrightarrow{n\to\infty} 0$ (proof below)

From (i)-(iv) & completeness of L^p follows (see exercise!)

that $\bigcap_{n \in \mathbb{N}} C_n = \{f\}$ for some (unique) $f \in L^p$.

This implies the claim, since $\bigcap_{n\in\mathbb{N}} C_n = \{\varphi \in L^p \mid F(\varphi) = m\}$.

It remains to do the

Proof of (iv): Note that (a) $C_1 \neq \emptyset$, (b) $\forall g \in C_1 : \frac{1}{p}\|g\|_p^p - \|\xi\|_* \|g\|_p \leq m+1$

and (c) $p > 1$ – so $r := \sup\{\|g\|_p \mid g \in C_1\} \in (0,\infty)$

Now let $\varepsilon > 0$, and let $g, h \in C_2$ (so $\|g\|_p, \|h\|_p \leq r$)

be arbitrary but fixed.

Assume (for contradiction!): $\|g-h\|_p \geq \varepsilon$ $(\square)$

Then, by Theorem 3.5, $\exists \delta > 0$ (indep. of g, h) s.t.

$$\left\| \frac{g+h}{2} \right\|_p^p \leq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta. \qquad (\square\square)$$

On the other hand: $\exists n_0 \in \mathbb{N}: \forall n \geq n_0: \frac{p}{n} < \delta$.

Now restrict choice of g, h to: $g, h \in C_n$, $n \geq n_0$.

Then

$$m \overset{\text{def of } m}{\leq} F\left(\frac{g+h}{2}\right) = \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2}\mathrm{Re}\,\xi(g) - \frac{1}{2}\mathrm{Re}\,\xi(h)$$

$$= \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2p}\left(\|g\|_p^p + \|h\|_p^p\right) + \underbrace{\frac{1}{2}\left(F(g) + F(h)\right)}_{\leq m + \frac{1}{n} \text{ (by def of } C_n)}$$

Hence,

$$\left\| \frac{g+h}{2} \right\|_p^p \geq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \underbrace{\frac{p}{n}}_{<\delta} > \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta.$$

This contradicts $(\square\square)$, hence $(\square)$ is <u>wrong</u>, so:

$\forall g, h \in C_n$ $\forall n \geq n_0$: $\|g-h\|_p < \varepsilon$. This proves (iv), hence the claim ✓

So, let $f \in L^p$ be the unique minimiser from the claim, i.e. $F(f) = m$, and let $g \in L^p$ be arbitrary.

Define

$$\sigma_g : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sigma_g(t) := F(f+tg) = \frac{1}{p}\int_X |f+tg|^p \, d\mu - t\,\mathrm{Re}\,\xi(g) - \mathrm{Re}\,\xi(f) \end{array}$$

<u>Goal</u>: Compute $\sigma_g'(0)$.

Lebesgue's Thm. on differentiability of parameter-dependent integrals (see Handout!) applies, since, for $I \subseteq \mathbb{R}$ open & bounded, with $0 \in I$, we have:

(i) $|f+tg|^p \in L^1$ $\forall t \in I$

(ii) $\varphi(t,\cdot) := \frac{\partial}{\partial t}|f+tg|^p \in L^1 \quad \forall t \in I$, since:

$$\varphi(t,\cdot) = \frac{\partial}{\partial t}\left[(f+tg)\overline{(f+tg)}\right]^{p/2}$$
$$= \frac{p}{2}\left(|f+tg|\right)^{\frac{p}{2}-1}\left[g\overline{(f+tg)} + (f+tg)\bar{g}\right]$$
$$= p\,|f+tg|^{p-2}\,\mathrm{Re}\left[g\overline{(f+tg)}\right]$$

(in the case $p \in (1,2)$ with the convention that

$$\varphi(t,x) := 0 \text{ if } |f(x)+tg(x)| = 0)$$

(iii) $\exists\, \Phi \in L^1 : |\varphi(t,\cdot)| \leq \Phi \quad \forall t \in I$.

Hence, since f minimises F (so σ_g has minimum at $t=0$),

$$0 = \sigma_g'(0) = \int_X |f|^{p-2}\,\mathrm{Re}(\bar{f}g)\,d\mu - \mathrm{Re}\,\xi(g)$$

$$\Rightarrow \quad \mathrm{Re}\,\xi(g) = \mathrm{Re}\Big(\int_X f_\xi g\,d\mu\Big), \quad f_\xi := |f|^{p-2}\bar{f} \in L^q \text{ (check!)}$$

If $\mathbb{K} = \mathbb{R}$, the claim follows

If $\mathbb{K} = \mathbb{C}$, consider also $\tau_g : \mathbb{R} \ni t \mapsto \tau_g(t) := F(f - itg)$

By similar arguments, we get

$$0 = \tau_g'(0) = \int_X |f|^{p-2}\,\mathrm{Im}(\bar{f}g)\,d\mu - \mathrm{Im}(\xi g)$$

$$\Rightarrow \quad \mathrm{Im}\,\xi(g) = \mathrm{Im}\Big(\int_X f_\xi g\,d\mu\Big) \quad ✓. \qquad (p \in (1,\infty) \text{ done})$$

<u>Case</u> $p \underline{=1}$: Idea: Step 1: Reduce to the above case $p \in (1,\infty)$, if μ is <u>finite</u>. Step 2: Generalise to σ-finite μ.

<u>Step 1</u>: Assume $\mu(X) < \infty$. Let $r > 1$. Recall from Lemma 3.4:

$L^\infty \subseteq L^r \subseteq L^1$ (the inclusion being dense).

Then:

$$|\xi(g)| \leq \|\xi\|_{(L^1)^*}\|g\|_1 \leq \underbrace{\|\xi\|_{(L^1)^*}\left[\mu(X)\right]^{1-\frac{1}{r}}}_{=:\,M_r}\|g\|_r \qquad \forall g \in L^r$$

Hence, $\xi_r := \xi|_{L^r} \in (L^r)^*$. From the case $p\in(1,\infty)$ proved above (on r)

we get: $\exists! f_r \in L^q$ $(\frac{1}{r}+\frac{1}{q}=1)$: $\xi_r = l_{f_r}$ (see (*1)), and

$\|f_r\|_q = \|\xi_r\|_{(L^r)^*} \le M_r$. Hence: $\forall r_1, r_2 \ge 1$, $\forall h \in L^\infty$ $(\subseteq L^{r_i}, i=1,2)$:

$$l_{f_{r_1}}(h) = \xi_{r_1}(h) = \xi(h) = \xi_{r_2}(h) = l_{f_{r_2}}(h) \quad \text{i.e.} \int_X (f_{r_1}-f_{r_2})\, h \, d\mu = 0$$

Choose $h := \frac{\overline{(f_{r_1}-f_{r_2})}}{|f_{r_1}-f_{r_2}|} \in L^\infty$, then $f_{r_1}=f_{r_2}=: f$ is <u>independent</u> of

the chosen/used r. Hence, $f \in \bigcap_{q\in(1,\infty)} L^q$ and

$$\limsup_{q\to\infty} \underbrace{\|f\|_q}_{\le M_r} \overset{1-\frac{1}{r}=\frac{1}{q}}{\le} \limsup_{q\to\infty} \left\{ \|\xi\|_{(L^1)^*} [\mu(X)]^{1/q} \right\} = \|\xi\|_{(L^1)^*} < \infty$$

Then (see exercise!) $f\in L^\infty$, and so $l_f \in (L^1)^*$ (see (*))

Since L^∞ dense in L^1, we get: ξ and l_f agree on a dense subspace, so (by Thm. 2.31) $\xi = l_f$ on L^1.

<u>Step 2</u>: Generalise to μ being σ-finite. See exercise ∎

$\boxed{\text{Remark}}$ (a) Compare with Thm 2.38 (for l_p). Recall that

$l^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with μ counting measure.

(b) Compare with Thm. 2.57 & 2.58 (for Hilb. space) Recall that L^2 is a Hilbert space, with $\langle f, g\rangle := \int_X \bar{f} g \, d\mu$. Note the complex conjugate (also in 2.57 & 2.58) $(\frac{1}{2}+\frac{1}{2}=1)$. (Recall also 2.54 & see next section)

(c) Another proof uses Radon-Nikodym Thm. to prove surjectivity of the map J; see Werner, "Funktionalanalysis", 8. Aufl., Satz II.2.4 p. 65.

(d) A more abstract proof uses the uniform convexity & Milman-Pettis's Thm, to conclude L^p, $p\in(1,\infty)$, is <u>reflexive</u>, to identify $(L^p)^*$; see 4.17 & 4.18 later.

3.2 Separability

We have the following general abstract result:

Theorem 3.8 Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space which is countably generated (i.e., $\exists\, \mathcal{C} \subseteq \mathcal{P}(X)$ countable such that $\mathcal{A} = \sigma(\mathcal{C})$). Let $p \in [1, \infty)$. Then L^p is separable.

Pf: See Behrends, "Maß- und Integrationstheorie" (1987) Thm IV.1.5 p. 160.

Remark 3.9 The theorem does not hold for $p = \infty$ (in general).

We turn to the concrete case of $L^p(\Omega, d\lambda^d)$ with $\Omega \subseteq \mathbb{R}^d$ open (non-empty) and λ^d Lebesgue measure, for which we will prove the result in 3.8.

Notation: (i) For $f: \Omega \to \mathbb{K}$, the (topological) support of f is

$$\operatorname{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$$

(closure w.r.t. rel. top. on Ω induced by Euclidean top.)

(ii) For $k \in \mathbb{N}$: Space of k-times cont. diff. fct's

$$C^k(\Omega) := \Big\{ f \in C(\Omega) \;\Big|\; \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f \in C(\Omega) \;\; \forall \alpha_1, \dots, \alpha_d \in \mathbb{N}_0,\; \sum_{j=1}^{d} \alpha_j \leq k \Big\}$$

Also, $C^0(\Omega) := C(\Omega)$ and

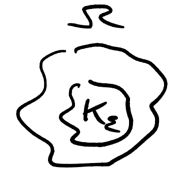
$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega) \quad \text{((infinitely) smooth functions)}$$

(iii) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k-times cont. diff. fct's with compact support:

$$C_c^k(\Omega) := \{ f \in C^k(\Omega) \mid \operatorname{supp}(f) \subseteq \Omega \text{ is compact} \}$$

(iv) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k-times cont. diff. fct.'s vanishing at ∞ (!):

$$C_0^k(\Omega) := \left\{ f \in C^k(\Omega) \;\middle|\; \begin{array}{l} \forall \varepsilon > 0 \ \exists K_\varepsilon \subseteq \Omega \text{ compact s.t.} \\ |f(x)| \le \varepsilon \ \ \forall x \in \Omega \setminus K_\varepsilon \end{array} \right\}$$



Rest this section:

Consider $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{B}(\Omega), \lambda^d)$, with $\mathcal{B}(\Omega)$ Borel-σ-algebra on Ω (trace-σ-algebra to Ω of Borel-σ-algeb. on $\mathbb{R}^d$) and λ^d Lebesgue-Borel measure on $\mathbb{R}^d$ (see Handout).

Main result is for $L^p(\Omega) := L^p(\Omega, \lambda^d)$:

Theorem 3.10 Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open and let $p \in [1, \infty)$.

Then $$\overline{C_c^\infty(\Omega)}^{\|\cdot\|_p} = L^p(\Omega)$$

(i.e. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ w.r.t. $\|\cdot\|_p$-norm).

Remark 3.11 (See exercises)

(a) Theorem 3.10 does not hold for $p = \infty$. This follows from:
$$\overline{C_c(\Omega)}^{\|\cdot\|_\infty} = C_0(\Omega).$$

(b) $C_c(\Omega)$ is separable w.r.t. $\|\cdot\|_\infty$.

Corollary 3.12 Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open, and let $p \in [1, \infty)$. Then $L^p(\Omega)$ is separable.

Pf: From Rmk. 3.11 (b) & Thm. 3.10 (see exercise for details). ∎

The proof of Thm. 3.10 requires some preparations:

Proposition 3.13 (Young's Inequality (for convolutions))

Let $p, q \in [1, \infty]$ with $1 \le \frac{1}{p} + \frac{1}{q} \le 2$, and let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$.

Then the convolution $f * g$ of f and g,

$$\mathbb{R}^d \ni x \mapsto (f*g)(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \qquad (*)$$

(where $dx := d\lambda^d(x)$, integration w.r.t. Lebesgue measure on $\mathbb{R}^d$)

exists for a.e. x, is commutative: $f*g = g*f$, and $f*g \in L^r(\mathbb{R}^d)$ with, for $\frac{1}{r} \overset{(**)}{:=} -1 + \frac{1}{p} + \frac{1}{q}$ (i.e. $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$),

$$(***) \qquad \|f*g\|_r \le \|f\|_p \|g\|_q \qquad \text{(Young's Ineq.)}.$$

Corollary 3.14 Let $p,q,r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$ ($\square$)

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then

$$I := I(f,g,h) := \left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\,g(x-y)\,h(y)\,dx\,dy\right|$$
$$\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |f(x)|\cdot|g(x-y)|\cdot|h(y)|\,dx\,dy \le \|f\|_p\cdot\|g\|_q\cdot\|h\|_r \quad (\square\square)$$

Pf (of 3.14): Hölder & Young's ineq.: By 3.13, $f*g \in L^s$, $\frac{1}{s} = -1+\frac{1}{p}+\frac{1}{q}$. Hence, $\frac{1}{r}+\frac{1}{s} = 1$ by ($\square$), and so ($\square\square$) follows from Hölder's ineq.

Remark 3.14~ (a) Note that, by scaling, the r in ($**$) is the only one for which ($***$) could hold: For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$. Then (change of variables!), $\|f_\lambda\|_p = \lambda^{-1/p}\|f\|_p$ & $(f*g)_\lambda(x) = (\lambda f_\lambda * g_\lambda)(x)$.

Hence, if ($***$) holds for some $p,q,r \in [1,\infty]$, then also, $\forall \lambda > 0$

$$\|f*g\|_r = \lambda^{1/r}\|(f*g)_\lambda\|_r \le \lambda^{1/r}\lambda\,\|f_\lambda\|\cdot\|g_\lambda\|_q = \lambda^{(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q})}\|f\|_p\|g\|_q$$

This is only possible if ($**$) holds (let $\lambda \searrow 0$ and $\lambda \nearrow \infty$ above).

(b) Often (in applications) one encounters

$$J_\alpha(f,h) := \left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\,|x-y|^{-\alpha}\,h(y)\,dx\,dy\right|, \text{ that is, } (\square\square) \text{ with } g(z) = |z|^{-\alpha}.$$

Now, $g \notin L^q(\mathbb{R}^d)$ $\forall q$. The replacement for ($\square\square$) (or, for ($***$) in 3.13) is called the Hardy-Littlewood-Sobolev Ineq. (HLS):

$$J_\alpha(f,h) \le C(p,\alpha,d)\,\|f\|_p\|h\|_r, \quad \frac{1}{p}+\frac{1}{r}+\frac{\alpha}{d} = 2 \;\&\; 0<\alpha<d$$

Its proof requires much more work.

Pf (of 3.13): Commutativity (if integral exists):

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy \overset{y:=x-z}{=} \int_{\mathbb{R}^d} f(z)\,g(x-z)\,dz = (g*f)(x)$$

The cases $p=1<q=r$ and $q=1<p=r$ follow directly from Minkowski's Ineq. (3.2(b)) (& commutativity, for $q=1<p=r$): ($r<\infty$!)

$$\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(f*g)(x)\right|^r dx\right)^{1/r} = \left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(y)g(x-y)\,dy\right|^r dx\right)^{1/r}$$

$$\overset{3.2(b)}{\leq} \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)g(x-y)|^r dx\right)^{1/r} dy = \int_{\mathbb{R}^d}|f(y)|\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^r dx\right)^{1/r}}_{=\|g\|_r\ \forall x} dy = \|f\|_1\|g\|_r \quad ✓$$

Remains the most general case:

$1<p<r$ and $1<q<r$, $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. (**)

Strategy: write $|f(y)g(x-y)|$ as product of 3 factors & use (gen.) Hölder:

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{s_1}}|g(x-y)|^{\frac{q}{s_2}}|f(y)|^{\frac{p}{s_3}} \quad (\nabla)$$

We must have (check!)

$$p\left(\frac{1}{s_1}+\frac{1}{s_3}\right)=1, \quad q\left(\frac{1}{s_1}+\frac{1}{s_2}\right)=1$$

As we want to estimate the L^r-norm of $f*g$, we want $\underline{\underline{s_1=r}}$.

Hence,
$$\begin{pmatrix}1&0&0\\q&q&0\\p&0&p\end{pmatrix}\begin{pmatrix}s_1^{-1}\\s_2^{-1}\\s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\1\\1\end{pmatrix} \underset{(pq\neq0)}{\overset{\text{check!}}{\iff}} \begin{pmatrix}s_1^{-1}\\s_2^{-1}\\s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\\left(1-\frac{q}{r}\right)\frac{1}{q}\\\left(1-\frac{p}{r}\right)\frac{1}{p}\end{pmatrix} \quad (\nabla\nabla)$$

We see that $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\overset{(\Delta)}{=}1$, and (∇) becomes

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{r}}|g(x-y)|^{1-\frac{q}{r}}|f(y)|^{1-\frac{p}{r}} \quad (\nabla\nabla\nabla)$$

and so, by gen. Hölder (Tut T26; 3 factors; use (Δ) & $(\nabla\nabla)$)

$$|(f*g)(x)| \leq \left(\int_{\mathbb{R}^d}|f(y)|^p\cdot|g(x-y)|^q dy\right)^{\frac{1}{r}}\|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}} \quad (\nabla\nabla\nabla\nabla)$$

Hence (raise to power r & integrate wrt x),

$$\|f*g\|_r \leq \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)|^p|g(x-y)|^q dy\right)dx\right)^{1/r}$$

$$\overset{\text{Tonelli}}{=} \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}|f(y)|^p\cdot\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^q dx\right)}_{=\|g\|_q^q\ \forall x}dy\right)^{1/r}$$

$$= \|f\|_p\|g\|_q \quad ✓$$

If $r=\infty$, then (***) follows from (standard) Hölder, as $\frac{1}{p}+\frac{1}{q}=1$

$\blacksquare$

Next, we will study <u>mollification</u>: L^p-functions are made smooth (C^∞) by convolution with a mollifier:

Let
$$L^p_c(\Omega) := \{ f \in L^p(\Omega) \mid \exists K \subseteq \Omega \text{ compact s.t. } f|_{\Omega\setminus K} = 0 \text{ a.e.} \}$$
for the space of L^p-functions with compact support in Ω.

We identify $f \in L^p_c(\Omega)$ with its extension by 0 to all of $\mathbb{R}^d$.

Lemma 3.15 Let $p \in [1,\infty)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open, $f \in L^p_c(\Omega)$.

Let $j \in C^\infty_c(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} j(x)\,dx = 1$, and for $\varepsilon > 0$ define the mollifier
$$j_\varepsilon : \mathbb{R}^d \to [0,\infty), \quad x \mapsto \varepsilon^{-d} j\left(\frac{x}{\varepsilon}\right).$$

Then, for all $\varepsilon > 0$ sufficiently small:
$$f_\varepsilon := j_\varepsilon * f \in C^\infty_c(\Omega) \text{ and } \|f_\varepsilon\|_p \leq \|f\|_p.$$

Remark Note that j_ε is a pointwise defined function (and not just an equiv. class of integrable fct's, defined a.e.), the convolution $j_\varepsilon * f$ has the canonical representative
$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y)\,dy \qquad (*)$$
which is pointwise well-defined $\forall x \in \mathbb{R}^d$, and independent of the choice of the representative for $f \in L^p$. It is the representative $(*)$ of the convolution $j_\varepsilon * f$ we denote by f_ε

Pf: Norm: Use Thm 3.13 with $r = p$. Then $q = 1$ then, and so $f_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j_\varepsilon\|_1 = \|f\|_p$.

Support: By assumption, $\exists K \subseteq \Omega$ compact with $f|_{\Omega\setminus K} = 0$.

Then (by T10 (ii)) $\delta := \operatorname{dist}(K, \partial\Omega) > 0$ (since K compact & $\partial\Omega$ closed)

By def. of the mollifier, $\exists \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:

$\operatorname{supp} j_\varepsilon \subseteq B_{\delta/2}(0)$. From (*) we conclude that

$$\operatorname{supp} f_\varepsilon \subseteq \bigcup_{y \in K} B_{\delta/2}(y) \subseteq \Omega.$$

In particular, $\operatorname{supp} f_\varepsilon$ is bounded (!) and (since closed by def.), also compact.

<u>Derivatives:</u> Let $e \in \mathbb{R}^d$, $|e|=1$, be a unit vector, and $\lambda \in \mathbb{R}\setminus\{0\}$.

Then, $\forall x \in \mathbb{R}^d$,

$$F_\lambda(x) := \frac{1}{\lambda}\left(f_\varepsilon(x+\lambda e) - f_\varepsilon(x)\right) = \frac{1}{\lambda}\int_{\mathbb{R}^d}\left(j_\varepsilon(x-y+\lambda e) - j_\varepsilon(x-y)\right) f(y)\,dy$$

$$\overset{HDI}{=} \frac{1}{\lambda}\int_{\mathbb{R}^d}\Big(\underbrace{\int_0^1 \frac{d}{dt}\left[j_\varepsilon(x-y+t\lambda e)\right]dt}_{\lambda\, e\cdot(\nabla j_\varepsilon)(x-y+t\lambda e)}\Big) f(y)\,dy$$

$$= \int_{\mathbb{R}^d}\underbrace{\Big(\int_0^1 (e\cdot\nabla j_\varepsilon)(x-y+t\lambda e)\,dt\Big)}_{=:D} f(y)\,dy$$

Since $|D| \le \| |\nabla j_\varepsilon| \|_\infty < \infty$ and $f \in L^1$ (because $f \in L^p$ & of compact support), using Dominated Convergence (twice, in dt & dy integral) gives that

$$e\cdot(\nabla f_\varepsilon)(x) = \lim_{\lambda\to 0} F_\lambda(x) = \int_{\mathbb{R}^d} e\cdot(\nabla j_\varepsilon)(x-y) f(y)\,dy = \left((e\cdot\nabla j_\varepsilon) * f\right)(x)$$

<u>exists</u> $\forall x \in \mathbb{R}^d$. The existence of higher derivatives follows by induction in the same way. ☒

$\boxed{\text{Lemma 3.16}}$ Let $p \in [1,\infty)$, let $A \in \mathcal{B}(\mathbb{R}^d)$ be bounded.

Then
$$\lim_{\varepsilon \searrow 0} \| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_{L^p(\mathbb{R}^d)} = 0.$$

<u>Proof:</u> <u>Step 1</u>: It suffices to prove the lemma for $p=1$, since:

Let $g_\varepsilon := \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \in L^\infty_c(\mathbb{R}^d)$; then $g_\varepsilon \in L^p(\mathbb{R}^d)$ $\forall p \in [1,\infty)$ and

$$\|g_\varepsilon\|_p^p \le \underbrace{\left(\|g_\varepsilon\|_\infty\right)^{p-1}}_{\le 2}\cdot \|g_\varepsilon\|_1.$$

Step 2: The case $p=1$. Since Lebesgue measure is (outer) regular (see Handout):

$$\forall \delta>0\ \exists B \supseteq A,\ B \text{ open and } \int_{B\setminus A} dx < \delta.$$

So, fix $\delta>0$, and let B be as above (so, also bounded).

Note that

$$\| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_1 \le \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 + \underbrace{\| \mathbb{1}_{B\setminus A} \|_1}_{<\delta} + \| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1$$

with $\| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_\varepsilon(x-y)\, \mathbb{1}_{B\setminus A}(y)\, dy\, dx$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y)\, dx \right)}_{=1\ \forall y} \mathbb{1}_{B\setminus A}(y)\, dy < \delta.$$

Hence, it suffices to prove that $\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 \to 0,\ \varepsilon \downarrow 0$.

We have $0 \le j_\varepsilon * \mathbb{1}_B \le 1$, hence

$$\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 = \int_B \left(1 - (j_\varepsilon * \mathbb{1}_B)(x)\right) dx - \int_{\mathbb{R}^d \setminus B} (j_\varepsilon * \mathbb{1}_B)(x)\, dx$$

$$= 2|B| - 2 \int_B \int_{\mathbb{R}^d} j(z)\, \mathbb{1}_B(x-\varepsilon z)\, dz\, dx.$$

Now, since B is <u>open</u>: $\forall x \in B\ \forall z \in \mathbb{R}^d: \mathbb{1}_B(x-\varepsilon z) \to 1,\ \varepsilon \downarrow 0$ (think!)

Also, B is bounded and $j \in L^1(\mathbb{R}^d)$, so, using Lebesgue's Thm. on Dominated Convergence twice

$$\lim_{\varepsilon \downarrow 0} \int_B \left(\int_{\mathbb{R}^d} j(z)\, \mathbb{1}_B(x-\varepsilon z)\, dz \right) dx = \int_B \left(\int_{\mathbb{R}^d} j(z)\, dz \right) dx = |B| \quad \blacksquare$$

<u>Proof of Thm. 3.6:</u> (Here $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$).

<u>Step 1:</u> Let $\varepsilon>0$, $f \in L^p(\Omega)$. For $n \in \mathbb{N}$, define

$$K_n := \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge \tfrac{1}{n} \text{ and } |x| \le n \} \subseteq \Omega$$

Now, Ω is open, so $\Omega = \bigcup_{n\in\mathbb{N}} K_n$, and $K_n \subseteq K_{n+1}$ $\forall n \in \mathbb{N}$.

By dominated convergence, $\left(\lim_{n\to\infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_{\Omega}(x) \right)$

$$\lim_{n\to\infty} \| f - f\mathbb{1}_{K_n} \|_p = \lim_{n\to\infty} \| f(1-\mathbb{1}_{K_n}) \|_p = 0.$$

Hence: $\exists n \in \mathbb{N}: \ \| f - f\mathbb{1}_{K_n} \|_p < \frac{\varepsilon}{3}$.

Step 2: Recall (Handout & exercise) that step functions are dense in L^p : $\exists L \in \mathbb{N}, c_1, .., c_L \in \mathbb{K}$ and $A_1, .., A_L \in \mathcal{B}(\mathbb{R}^d)$ with $A_\ell \subseteq K_n$ (!) $\forall \ell = 1, .., L$ such that : For $g := \sum_{\ell=1}^{L} c_\ell \mathbb{1}_{A_\ell}$ we have

$$g \in L^p_c(\Omega) \quad \text{and } \| g - f\mathbb{1}_{K_n} \|_p < \frac{\varepsilon}{3}$$

Step 3: Mollify step function : $\exists \delta > 0$ s.t.

(i) $j_\delta * g \in C_c^\infty(\Omega)$ (by Lemma 3.15)

(ii) $\| g - j_\delta * g \|_p \leq \sum_{\ell=1}^{L} |c_\ell| \cdot \| \mathbb{1}_{A_\ell} - j_\delta * \mathbb{1}_{A_\ell} \|_p < \frac{\varepsilon}{3}$ (by Lemma 3.16)

Hence,

$$\| f - \underbrace{j_\delta * g}_{\in C_c^\infty(\Omega)} \|_p < \varepsilon \qquad \boxtimes$$