

Chapter 3: L^p -spaces

(61)

3.1 Completeness and dual space

Notation: In this section: $(X, \mathcal{A}, \mu)$ a measure space, $K = \mathbb{R}$ or $\mathbb{C}$,
 $f: X \rightarrow K$ measurable (see also Handout!)

Definition 3.1 (L^p -spaces) Let $p \in (0, \infty)$.

(a)
$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p} \quad (\text{possibly } \infty)$$

$$\begin{aligned} \|f\|_\infty &:= \inf \{ \alpha > 0 \mid \mu(\{x \in X \mid |f(x)| > \alpha\}) = 0 \} \\ &= \inf_{\substack{V \in \mathcal{A} \\ \mu(V) = 0}} \sup_{x \in X \setminus V} |f(x)| =: \text{ess sup}_{x \in X} |f(x)| \quad (\mu\text{-}) \text{ essential supremum} \end{aligned}$$

(b) Vector space of p -integrable functions (wrt. μ):

$$\begin{aligned} \mathcal{L}^p &:= \mathcal{L}^p(\mu) := \mathcal{L}^p(X) := \mathcal{L}^p(X, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) := \\ &:= \{ f: X \rightarrow K \mid f \text{ measurable and } \|f\|_p < \infty \} = \{ f: X \rightarrow K \mid |f|^p \mu\text{-integrable} \} \end{aligned}$$

(c) Equivalence relation $\sim$ on $\mathcal{L}^p$:

$$f \sim g \iff f = g \text{ } \mu\text{-a.e.}$$

Vector space of equivalence classes of p -integrable fct.'s wrt μ :

$$\mathcal{L}^p := \mathcal{L}^p / \sim$$

Warning! Notation does not distinguish between equivalence classes of functions, and their representatives!

Lemma 3.2 (Hölder & Minkowski).

(a) $\forall r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$: $\forall f, g: X \rightarrow K$ meas.

$$\|fg\|_r \leq \|f\|_p \cdot \|g\|_q \quad (\text{generalised Hölder's ineq.})$$

(b) Let $(X_i, \mathcal{A}_i, \mu_i)$, $i=1,2$, be σ -finite measure spaces, and (62)
 $f: X_1 \times X_2 \rightarrow \mathbb{C}$ measurable. Then, for $p \in [1, \infty)$,

$$(*) \left\{ \int_{X_2} \left| \int_{X_1} f(x,y) d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \leq \int_{X_1} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

That is, (Minkowski's Integral Inequality)

$$\left\| \int_{X_1} f(x, \cdot) d\mu_1(x) \right\|_p \leq \int_{X_1} \|f(x, \cdot)\|_p d\mu_1(x)$$

(c) $\forall p \in [1, \infty]$:

$$(**) \|f+g\|_p \leq \|f\|_p + \|g\|_p \quad (\text{Minkowski's inequality})$$

Pf: (a) Ana III / Handout / see exercises.

(b) Note that

$$\left\{ \int_{X_2} \left| \int_{X_1} f(x,y) d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \leq \left\{ \int_{X_2} \left(\int_{X_1} |f(x,y)| d\mu_1(x) \right)^p d\mu_2(y) \right\}^{1/p}$$

hence, it suffices to consider $f \geq 0$.

Next, if $p=1$, (*) follows from Tonelli's Thm.

If $1 < p < \infty$, then

$$\begin{aligned} \int_{X_2} \left| \int_{X_1} f(x,y) d\mu_1(x) \right|^p d\mu_2(y) &= \int_{X_2} \left| \int_{X_1} f(x,y) d\mu_1(x) \right|^{p-1} \cdot \left| \int_{X_1} f(x,y) d\mu_1(x) \right| d\mu_2(y) \\ &\leq \int_{X_2} \left| \int_{X_1} f(t,y) d\mu_1(t) \right|^{p-1} \left[\int_{X_1} |f(x,y)| d\mu_1(x) \right] d\mu_2(y) \\ &= \int_{X_2} \left(\int_{X_1} \left| \int_{X_1} f(t,y) d\mu_1(t) \right|^{p-1} \cdot |f(x,y)| d\mu_1(x) \right) d\mu_2(y) \quad (\text{linearity } \int_{X_1} d\mu_1(x)) \end{aligned}$$

$$\stackrel{\text{Tonelli}}{=} \int_{X_1} \left(\int_{X_2} \left| \int_{X_1} f(t,y) d\mu_1(t) \right|^{p-1} \cdot |f(x,y)| d\mu_2(y) \right) d\mu_1(x) \quad (II)$$

Now, with $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $q = \frac{p}{p-1}$, Hölder gives

$$\begin{aligned} & \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t, y) d\mu_1(t) \right|^p \cdot |f(x, y)|^p d\mu_2(y) \\ & \leq \left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t, y) d\mu_1(t) \right|^{\frac{q(p-1)}{q}} d\mu_2(y) \right)^{1/q} \left(\int_{\mathbb{X}_2} |f(x, y)|^p d\mu_2(y) \right)^{1/p} \\ & = \left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t, y) d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{X}_2} |f(x, y)|^p d\mu_2(y) \right)^{1/p} \end{aligned} \quad (63)$$

Inserting this in (□) gives

$$\begin{aligned} & \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x, y) d\mu_1(x) \right|^p d\mu_2(y) \\ & \leq \int_{\mathbb{X}_1} \underbrace{\left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t, y) d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{indep. of } x} \cdot \left(\int_{\mathbb{X}_2} |f(x, y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x) \\ & = \underbrace{\left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t, y) d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{divide (!): } 1 - \frac{p-1}{p} = \frac{1}{p}} \cdot \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} |f(x, y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x) \end{aligned}$$

$$\Rightarrow \left\{ \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x, y) d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \leq \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} |f(x, y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

To divide: we need integral to be finite; is the case if $\mu_i(\mathbb{X}_i) < \infty$, $i=1,2$ and f bounded. For general case: Let $\mathbb{X}_i^k \nearrow \mathbb{X}_i$, $k \rightarrow \infty$, $\mu_i(\mathbb{X}_i^k) < \infty$ ($\mathbb{X}_i$ σ -finite), consider the above for

$$\tilde{f}_k(x, y) := \mathbb{1}_{\mathbb{X}_1^k}(x) \mathbb{1}_{\mathbb{X}_2^k}(y) \mathbb{1}_{\{f \leq k\}} \cdot f(x, y) \text{ \& take limit } k \rightarrow \infty$$

$$(c) \text{ Take } \mathbb{X}_1 = \{1, 2\}, \mu_1 = \text{counting measure}, \int_{\mathbb{X}_1} \dots d\mu_1(x) = \sum_{i=1}^2 \dots$$

then (*) becomes, with $f(x, y) = f_i(y)$,

$$(**) \quad \|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p, \quad f_i \in L^p(\mathbb{X}_2, \mu_2)$$

Remark (a) Take $\mathbb{X}_2 = \mathbb{N}$, $\mu_2 = \text{counting measure}$, $\int_{\mathbb{X}_2} d\mu_2(y) = \sum_{i=1}^{\infty} 1$,

then (**) becomes

$$(***) \quad \|x + y\|_p \leq \|x\|_p + \|y\|_p, \quad x, y \in \underline{\ell}^p$$

(b) Take $\mathbb{X}_2 := \{1, \dots, N\}$, $\mu_2 = \text{counting measure}$, $\int_{\mathbb{X}_2} d\mu_2 = \sum_{i=1}^N 1 = N$ (64)

then (**) becomes

$$(***) \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p, \quad x, y \in \mathbb{K}^N$$

(c) The proof above is "the same" as the standard proof of (**), (***), or (***) (check!).

Theorem 3.3 (Riesz-Fischer) Let $(\mathbb{X}, \mathcal{A}, \mu)$ be a measure space. Then $(L^p, \|\cdot\|_p)$ is a Banach space for all $p \in [1, \infty]$. Moreover, L^2 is a Hilbert space with scalar product

$$\langle f, g \rangle := \int_{\mathbb{X}} \overline{f(x)} g(x) d\mu(x) \quad \forall f, g \in L^2$$

Pf: Ana III.

(triangle inequality needs $p \geq 1$!)

Remark (a) Take $\mathbb{X} = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, $\mu = \text{counting measure}$, $\int_{\mathbb{X}} d\mu = \sum_{i=1}^{\infty} 1 = \infty$, then 3.3 says: $(\ell^p, \|\cdot\|_p)$ is Banach.

(b) Take $\mathbb{X} = \{1, \dots, N\}$, $\mathcal{A} = \mathcal{P}(\mathbb{X})$, $\mu = \text{counting measure}$, $\int_{\mathbb{X}} d\mu = \sum_{i=1}^N 1 = N$, then 3.3 says: $(\mathbb{K}^N, \|\cdot\|_p)$ is Banach.

Lemma 3.4 If $\mu(\mathbb{X}) < \infty$, $1 \leq r \leq p \leq \infty$, then $L^p (\subseteq L^r)$ is dense in L^r (wrt. $\|\cdot\|_r$), and

$$\|f\|_r \leq [\mu(\mathbb{X})]^{\frac{1}{r} - \frac{1}{p}} \|f\|_p \quad \forall f \in L^p.$$

Pf: Ana III.

The next theorem is about the geometry of L^p , but also serves as preparation to compute its dual:

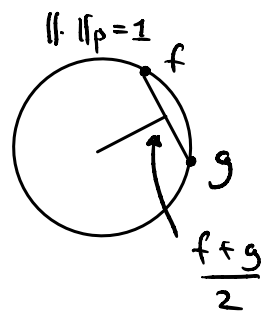
Theorem 3.5 Let $p \in (1, \infty)$. Then:

For all $\varepsilon > 0, r > 0$ there exists $\delta > 0$ s.t.:

For all $f, g \in L^p$:

$$\|f\|_p, \|g\|_p \leq r \text{ and } \|f - g\|_p \geq \varepsilon$$

$$\Rightarrow \left\| \frac{f+g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2} - \delta.$$



In particular, L^p is uniformly convex (case (see drawing) $r = \|f\|_p = \|g\|_p = 1$)

Pf (after Naoki Shioji)

Define the map $\alpha: \mathbb{K}^2 \rightarrow \mathbb{R}$
 $z = (z_1, z_2) \mapsto \frac{|z_1|^p + |z_2|^p}{2} - \left| \frac{z_1 + z_2}{2} \right|^p$

Claim: $\forall z \in \mathbb{K}^2: \alpha(z) \geq 0$, with " $=$ " iff $z_1 = z_2$ (*)

Pf Claim: $\left| \frac{z_1 + z_2}{2} \right|^p \leq \left(\frac{|z_1| + |z_2|}{2} \right)^p \leq \frac{|z_1|^p + |z_2|^p}{2}$
 Δ -ineq. $\mathbb{K}$ $(\cdot)^p$ strictly convex ($p > 1$)

Δ -ineq. is equality $\Leftrightarrow z_1 = 0$ or $z_2 = \lambda z_1$ for some $\lambda > 0$

Strict conv. ineq. is equality $\Leftrightarrow |z_1| = |z_2|$

$\Rightarrow$ (equality $\Leftrightarrow z_1 = z_2$) $\checkmark$ claim

Now, fix $\varepsilon, r > 0$, and choose $\gamma \in (0, \frac{\varepsilon^p}{2r^p})$ small enough
 (f.e. $< 2^{-p}$) such that

$$D := \{z \in \mathbb{K}^2 \mid |z_1|^p + |z_2|^p = 1 \text{ and } |z_1 - z_2|^p \geq \gamma\} \neq \emptyset$$

Then, by compactness (!) of D and continuity of α ,

$$\theta := \inf \{ \alpha(z) \mid z \in D \} = \min \{ \alpha(z) \mid z \in D \} > 0 \quad \text{by (*)}$$

Claim: $|z_1|^p + |z_2|^p \leq \frac{\alpha(z)}{\theta}$

$$\forall z \in F := \{ \tilde{z} \in \mathbb{K}^2 \mid |\tilde{z}_1 - \tilde{z}_2|^p \geq \gamma (|\tilde{z}_1|^p + |\tilde{z}_2|^p) \} \quad (**)$$

Pf claim: $\forall (\log, z \neq (0,0))$ (**) holds for $(0,0)$ (66)

Let $z \in F$, define $t := (|z_1|^p + |z_2|^p)^{1/p} > 0$ (since $z \neq 0$)

Note: (i) $|\frac{z_1}{t}|^p + |\frac{z_2}{t}|^p = 1$ (compute!)

(ii) $|\frac{z_1}{t} - \frac{z_2}{t}|^p \geq \gamma$ (since $z \in F$)

so $\frac{z}{t} \in D$. Hence, by def. of Θ , we have $\Theta \leq \alpha(\frac{z}{t}) = \frac{\alpha(z)}{t^p}$ ✓ (claim)

Now, let $f, g \in L^p$ with $\|f\|_p, \|g\|_p \leq r$ and $\|f-g\|_p \geq \varepsilon$.

Define $M := \{x \in X \mid (f(x), g(x)) \in F\}$ ($X = M \cup M^c$)

and estimate:

$$\varepsilon^p \leq \int_X |f-g|^p d\mu = \int_{M^c} |f-g|^p d\mu + 2^p \int_M \left| \frac{f-g}{2} \right|^p d\mu =: I_1 + 2^p I_2 \quad (***)$$

By definition of M^c (& F), $M^c \subseteq X$

$$I_1 \leq \gamma \int_{M^c} (|f|^p + |g|^p) d\mu \leq \gamma (\|f\|_p^p + \|g\|_p^p) \leq 2\gamma r^p$$

Since $\alpha \geq 0$, we also get

$$I_2 \leq \frac{1}{2} \int_M (|f|^p + |g|^p) d\mu \stackrel{(**)}{\leq} \frac{1}{2\theta} \int_M \alpha(f(x), g(x)) d\mu(x)$$

$$\stackrel{M \subseteq X}{\leq} \frac{1}{2\theta} \int_X \left(\frac{|f|^p + |g|^p}{2} - \left| \frac{f+g}{2} \right|^p \right) d\mu.$$

Inserting the inequalities for I_1 & I_2 in (***) gives

$$\left\| \frac{f+g}{2} \right\|_p^p \leq \underbrace{\frac{\theta(2\gamma r^p - \varepsilon^p)}{2^{p-1}}}_{=: -\delta} + \frac{\|f\|_p^p + \|g\|_p^p}{2}$$

with $\delta > 0$ by the definition of γ $\square$

Theorem 3.6 (Riesz Representation Thm. for $(L^p)^*$) (67)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $p=1$, assume in addition that μ is σ -finite.

Then the map
$$J: L^q \rightarrow (L^p)^*, \quad \begin{aligned} f &\mapsto \ell_f \\ \ell_f: L^p &\rightarrow \mathbb{K} \\ g &\mapsto \int_X fg \, d\mu \end{aligned} \quad (*)$$

is an isometric isomorphism. Hence:

$$L^q \cong (L^p)^*, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \in [1, \infty)$$

Remark 3.7 (a) The map J is also well-defined and isometric for $p = \infty$, but fails to be surjective if $\text{supp}(\mu)$ has infinitely many elements (see exercise). I.e., $(L^\infty)^*$ is strictly larger than L^1 in this case (compare 2.38 and 2.39(b)).

(b) The case $p=1$ in 3.6 would be wrong without additional hypothesis on the measure space; σ -finiteness of μ is sufficient, but not necessary, because of:

Thm The map $J: L^\infty \rightarrow (L^1)^*$ is

(a) Injective $\Leftrightarrow$ isometric $\Leftrightarrow (X, \mathcal{A}, \mu)$ is semi-finite

(b) Bijective $\Leftrightarrow$ isometric $\Leftrightarrow (X, \mathcal{A}, \mu)$ is localisable & bijective

Note: $(X, \mathcal{A}, \mu)$ σ -finite $\Rightarrow$ localisable $\Rightarrow$ semi-finite

(For required def.'s & pf., see Salamon, "Measure and Integration", Eves, (2016) Sect. 4.5)

Pf (of 3.6): By linearity of the integral, the map J is linear.

Let $f \in L^q$. Then ℓ_f (see $(*)$) is linear (in g), $fg \in L^1 \, \forall g \in L^p$ by Hölder, and $|\ell_f(g)| \leq \|f\|_q \|g\|_p$. Hence, $\ell_f \in (L^p)^*$,

and $\|l_f\|_{(L^p)^*} \leq \|f\|_q$. $(**)$ (68)

Hence, J in $(*)$ is well-defined, linear, and bounded.

Claim: $\|l_f\|_{(L^p)^*} \stackrel{(***)}{=} \|f\|_q$ ($\Rightarrow J$ is isometric, hence inj.)

Pf claim: Wlog, $f \neq 0$ $(***)$ clear for $f=0$.

(i) Case $p \in (1, \infty)$: Let $g(x) := \begin{cases} \bar{f}(x) |f(x)|^{q-2} & , f(x) \neq 0 \\ 0 & , f(x) = 0 \end{cases}$

Then $|g|^p = |f|^{(q-1)p} = |f|^q \in L^1$ since $(q-1)p = q(1 - \frac{1}{q})p = q$

So $g \in L^p$, and $\|g\|_p = \|f\|_q^{q/p}$. Also $(\frac{1}{p} + \frac{1}{q} = 1)$

$$\frac{l_f(g)}{\|g\|_p} = \frac{\int |f|^q}{\|g\|_p} = \frac{\|f\|_q^q}{\|f\|_q^{q/p}} = \|f\|_q^{q(1-\frac{1}{p})} = \|f\|_q, \text{ hence}$$

$\|l_f\|_{(L^p)^*} \geq \|f\|_q$. This, and $(**)$, imply $(***)$ (the claim).

(ii) Case $p=1$: Since μ is σ -finite, we have:

$$\mathbb{X} = \bigcup_{n \in \mathbb{N}} \mathbb{X}_n, \quad \mathbb{X}_n \subseteq \mathbb{X}_{n+1}, \quad \mu(\mathbb{X}_n) < \infty \quad \forall n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and, for $f \in L^\infty$, define

$$M := \{x \in \mathbb{X} \mid |f(x)| \geq \|f\|_\infty - \varepsilon\}.$$

By definition of "ess sup", $\mu(M) > 0$. Then $M_n := M \cap \mathbb{X}_n$ satisfies $0 < \mu(M_n) < \infty$ for (all) n sufficiently large.

Let $g(x) := \begin{cases} \frac{\bar{f}(x)}{|f(x)|} \frac{1}{\mu(M_n)} \mathbb{1}_{M_n}(x) & , f(x) \neq 0 \\ 0 & , f(x) = 0 \end{cases}$

Then $g \in L^1$ with $\|g\|_1 \leq 1$, and

$$l_f(g) = \int_{\mathbb{X}} fg \, d\mu = \frac{1}{\mu(M_n)} \int_{M_n} |f| \, d\mu \geq \|f\|_\infty - \varepsilon. \text{ As this holds } \forall \varepsilon > 0,$$

we get $\|l_f\|_{(L^1)^*} \geq \|f\|_\infty$. This, and $(**)$, imply the claim $(***)$.

It remains to prove that J is surjective.

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Let $\xi \in (L^p)^*$. Need to prove: $\exists f_\xi \in L^p$ s.t. $\xi = J_{f_\xi}$ (see *)

Case $p \in (1, \infty)$: (We follow again N. Shioji)

Define the non-linear functional $F: L^p \rightarrow \mathbb{R}$

$$g \mapsto F(g) := \frac{1}{p} \|g\|_p^p - R \xi(g)$$

Since $p > 1$ and

$$F(g) \geq \frac{1}{p} \|g\|_p^p - \|\xi\|_* \|g\|_p,$$

F is bounded from below:

$$m := \inf \{ F(g) \mid g \in L^p \} > -\infty$$

Claim: $\exists! f \in L^p: F(f) = m$ (i.e. exists unique minimiser of F)

[This is where we need/use uniform convexity!]

Pf claim: For $n \in \mathbb{N}$, let $C_n := \{ g \in L^p \mid F(g) \leq m + \frac{1}{n} \}$.

Then: (i) $C_n \neq \emptyset \forall n \in \mathbb{N}$ (by definition of the infimum)

(ii) C_n closed $\forall n \in \mathbb{N}$ (since F cont. and $(-\infty, m + \frac{1}{n}]$ closed)

(iii) $C_n \supseteq C_{n+1} \forall n \in \mathbb{N}$

(iv) $\text{diam}(C_n) \xrightarrow{n \rightarrow \infty} 0$ (proof below)

From (i)-(iv) & completeness of L^p follows (see exercise!) that

$\bigcap_{n \in \mathbb{N}} C_n = \{f\}$ for some (unique) $f \in L^p$.

This implies the claim, since $\bigcap_{n \in \mathbb{N}} C_n = \{ \varphi \in L^p \mid F(\varphi) = m \}$.

It remains to do the

Proof of (iv): Note that (a) $C_1 \neq \emptyset$, (b) $\forall g \in C_1: \frac{1}{p} \|g\|_p^p - \|\xi\|_* \|g\|_p \leq m+1$

and (c) $p > 1$ - so $r := \sup \{ \|g\|_p \mid g \in C_1 \} \in (0, \infty)$

Now let $\varepsilon > 0$, and let $g, h \in C_2$ (so $\|g\|_p, \|h\|_p \leq r$)

be arbitrary but fixed.

Assume (for contradiction!): $\|g-h\|_p \geq \varepsilon$ (□)

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Then, by Theorem 3.5, $\exists \delta > 0$ (indep. of g, h) s.t.

$$\left\| \frac{g+h}{2} \right\|_p^p \leq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta. \quad (\square\square)$$

On the other hand: $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \frac{p}{n} < \delta$.

Now restrict choice of g, h to: $g, h \in C_n, n \geq n_0$.

Then $\stackrel{\text{def of } m}{\leq}$

$$F\left(\frac{g+h}{2}\right) = \frac{1}{p} \left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2} \operatorname{Re} \int \zeta(g) - \frac{1}{2} \operatorname{Re} \int \zeta(h)$$

$$= \frac{1}{p} \left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2p} (\|g\|_p^p + \|h\|_p^p) + \underbrace{\frac{1}{2} (F(g) + F(h))}_{\leq m + \frac{1}{n} \text{ (by def of } C_n)}$$

Hence,

$$\left\| \frac{g+h}{2} \right\|_p^p \geq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \underbrace{\frac{p}{n}}_{< \delta} > \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta.$$

This contradicts (□□), hence (□) is wrong, so:

$\forall g, h \in C_n \forall n \geq n_0 : \|g-h\|_p < \varepsilon$. This proves (iv), hence the claim ✓

So, let $f \in L^p$ be the unique minimiser from the claim, i.e.

$F(f) = m$, and let $g \in L^p$ be arbitrary.

Define

$$\sigma_g: \mathbb{R} \rightarrow \mathbb{R}$$

$$t \mapsto \sigma_g(t) := F(f+tg) = \frac{1}{p} \int_{\mathbb{X}} |f+tg|^p d\mu - t \operatorname{Re} \int \zeta(g) - \operatorname{Re} \int \zeta(f)$$

Goal: Compute $\sigma_g'(0)$.

Lebesgue's Thm. on differentiability of parameter-dependent integrals (see Handout!) applies, since, for $I \subseteq \mathbb{R}$ open & bounded,

with $0 \in I$, we have:

$$(i) |f+tg|^p \in L^1 \quad \forall t \in I$$

(ii) $\varphi(t, \cdot) := \frac{\partial}{\partial t} |f + tg|^p \in L^1 \quad \forall t \in I$, since:

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$$\begin{aligned}\varphi(t, \cdot) &= \frac{\partial}{\partial t} [(f + tg)(\overline{f + tg})]^{p/2} \\ &= \frac{p}{2} |f + tg|^{p-2} [g(\overline{f + tg}) + (f + tg)\bar{g}] \\ &= p |f + tg|^{p-2} \operatorname{Re} [g(\overline{f + tg})]\end{aligned}$$

(in the case $p \in (1, 2)$ with the convention that

$$\varphi(t, x) := 0 \quad \text{if } |f(x) + tg(x)| = 0)$$

(iii) $\exists \Phi \in L^1 : |\varphi(t, \cdot)| \leq \Phi \quad \forall t \in I$.

Hence, since f minimises F (so δ_g has minimum at $t=0$),

$$0 = \delta_g'(0) = \int_{\mathbb{X}} |f|^{p-2} \operatorname{Re}(\bar{f}g) d\mu - \operatorname{Re} \zeta(g)$$

$$\Rightarrow \operatorname{Re} \zeta(g) = \operatorname{Re} \left(\int_{\mathbb{X}} f_{\zeta} g d\mu \right), \quad f_{\zeta} := |f|^{p-2} \bar{f} \in L^q \text{ (check!)}$$

If $\mathbb{K} = \mathbb{R}$, the claim follows

If $\mathbb{K} = \mathbb{C}$, consider also $\tau_g : \mathbb{R} \ni t \mapsto \tau_g(t) := F(f - itg)$

By similar arguments, we get

$$0 = \tau_g'(0) = \int_{\mathbb{X}} |f|^{p-2} \operatorname{Im}(\bar{f}g) d\mu - \operatorname{Im}(\zeta(g))$$

$$\Rightarrow \operatorname{Im} \zeta(g) = \operatorname{Im} \left(\int_{\mathbb{X}} f_{\zeta} g d\mu \right) \quad \checkmark \quad (p \in (1, \infty) \text{ done})$$

Case $p=1$: Idea: Step 1: Reduce to the above case $p \in (1, \infty)$, if μ is finite. Step 2: Generalise to σ -finite μ .

Step 1: Assume $\mu(\mathbb{X}) < \infty$. Let $r > 1$. Recall from Lemma 3.4:

$L^\infty \subseteq L^r \subseteq L^1$ (the inclusion being dense).

Then:

$$|\zeta(g)| \leq \|\zeta\|_{(L^1)^*} \|g\|_1 \leq \underbrace{\|\zeta\|_{(L^1)^*} [\mu(\mathbb{X})]^{1-\frac{1}{r}}}_{=: R_r} \|g\|_r \quad \forall g \in L^r$$

Hence, $\tilde{f}_r := \tilde{f}|_{L^r} \in (L^r)^*$. From the case $p \in (1, \infty)$ proved above (on r) (72)

we get: $\exists! f_r \in L^{\frac{r}{r-1}} \left(\frac{1}{r} + \frac{1}{\frac{r}{r-1}} = 1 \right) : \tilde{f}_r = \mathcal{I}_{f_r}$ (see (*)), and

$\|f_r\|_{\frac{r}{r-1}} = \|\tilde{f}_r\|_{(L^r)^*} \leq M_r$. Hence: $\forall r_1, r_2 \geq 1, \forall h \in L^\infty (\subseteq L^{r_i}, i=1,2)$:

$$\mathcal{I}_{f_{r_1}}(h) = \tilde{f}_{r_1}(h) = \tilde{f}(h) = \tilde{f}_{r_2}(h) = \mathcal{I}_{f_{r_2}}(h) \quad \text{i.e.} \quad \int_{\mathbb{X}} (f_{r_1} - f_{r_2}) h d\mu = 0$$

Choose $h := \frac{(f_{r_1} - f_{r_2})}{|f_{r_1} - f_{r_2}|} \in L^\infty$, then $f_{r_1} = f_{r_2} =: f$ is independent of

the chosen/used r . Hence, $f \in \bigcap_{q \in (1, \infty)} L^{\frac{q}{q-1}}$ and

$$\limsup_{q \rightarrow \infty} \|f\|_q \leq \limsup_{q \rightarrow \infty} \underbrace{\|f\|_q}_{\leq M_r} \leq \limsup_{q \rightarrow \infty} \left\{ \|\tilde{f}\|_{(L^1)^*} [\mu(\mathbb{X})]^{1/q} \right\} = \|\tilde{f}\|_{(L^1)^*} < \infty$$

$1 - \frac{1}{q} = \frac{1}{q}$

Then (see exercise!) $f \in L^\infty$, and so $\mathcal{I}_f \in (L^1)^*$ (see (*))

Since L^∞ dense in L^1 , we get: $\tilde{f}$ and $\mathcal{I}_f$ agree on a dense subspace, so (by Thm. 2.31) $\tilde{f} = \mathcal{I}_f$ on L^1 .

Step 2: Generalise to μ being σ -finite. See exercise

Remark (a) Compare with Thm 2.38 (for $\mathcal{I}_p$). Recall that

$$\mathcal{I}^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \quad \text{with } \mu \text{ counting measure.}$$

(b) Compare with Thm. 2.57 & 2.58 (for Hilb. space) Recall that

$$L^2 \text{ is a Hilbert space, with } \langle f, g \rangle := \int_{\mathbb{X}} \bar{f} g d\mu. \text{ Note}$$

the complex conjugate (also in 2.57 & 2.58) $\left(\frac{1}{2} + \frac{1}{2} = 1 \right)$.

(Recall also 2.54 & see next section)

(c) Another proof uses Radon-Nikodym Thm. to prove surjectivity of the map $\mathcal{I}$; see Werner, "Funktionalanalysis", 8. Aufl., Satz II.2.4 p. 65.

(d) A more abstract proof uses the

uniform convexity & Milman - Pettis Thm, to conclude

L^p , $p \in (1, \infty)$, is reflexive, to identify $(L^p)^*$; see 4.17 &

4.18 later.

3.2 Separability

(73)

We have the following general abstract result:

Theorem 3.8 Let $(X, \mathcal{A}, \mu)$ be a σ -finite measure space which is countably generated (i.e., $\exists \mathcal{C} \subseteq \mathcal{P}(X)$ countable such that $\mathcal{A} = \sigma(\mathcal{C})$). Let $p \in [1, \infty)$. Then L^p is separable.

Pf. See Behrends, "Mas- und Integrationstheorie" (1987) Thm IV.1.5 p. 160.

Remark 3.9 The theorem does not hold for $p = \infty$ (in general).

We turn to the concrete case of $L^p(\Omega, dx^d)$ with $\Omega \subseteq \mathbb{R}^d$ open (non-empty) and λ^d Lebesgue measure, for which we will prove the result in 3.8.

Notation: (i) For $f: \Omega \rightarrow \mathbb{K}$, the (topological) support of f is

$$\text{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}} \quad \left(\begin{array}{l} \text{closure w.r.t. rel. top.} \\ \text{on } \Omega \text{ induced by} \\ \text{Euclidean top.} \end{array} \right)$$

(ii) For $k \in \mathbb{N}$: Space of k -times cont. diff. fct's

$$C^k(\Omega) := \left\{ f \in C(\Omega) \mid \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f \in C(\Omega) \quad \forall \alpha_1, \dots, \alpha_d \in \mathbb{N}_0, \sum_{j=1}^d \alpha_j \leq k \right\}$$

Also, $C^0(\Omega) := C(\Omega)$ and

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega) \quad (\text{(\u0022infinitely\u0022) \u0022smooth functions\u0022})$$

(iii) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k -times cont. diff. fct's with compact support:

$$C_c^k(\Omega) := \left\{ f \in C^k(\Omega) \mid \text{supp}(f) \subseteq \Omega \text{ is compact} \right\}$$

(iv) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k -times cont. diff. fct.'s vanishing at ∞ (!):

$$C_0^k(\Omega) := \left\{ f \in C^k(\Omega) \mid \forall \varepsilon > 0 \exists K_\varepsilon \subseteq \Omega \text{ compact s.t. } |f(x)| \leq \varepsilon \quad \forall x \in \Omega \setminus K_\varepsilon \right\}$$



Rest this section:

Consider $(X, \mathcal{A}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$, with $\mathcal{B}(\mathbb{R}^d)$

Borel- σ -algebra on $\mathbb{R}^d$ (trace- σ -algebra to $\mathbb{R}^d$ of Borel- σ -algebra on $\mathbb{R}^d$) and λ^d Lebesgue-Borel measure on $\mathbb{R}^d$ (see Handout).

Main result is for $L^p(\mathbb{R}^d) := L^p(\mathbb{R}^d, \lambda^d)$:

Theorem 3.10 Let $\phi \neq \emptyset \subseteq \mathbb{R}^d$ be open and let $p \in [1, \infty)$.

Then $\overline{C_c^\infty(\phi)}^{\|\cdot\|_p} = L^p(\phi)$

(i.e. $C_c^\infty(\phi)$ is dense in $L^p(\phi)$ w.r.t. $\|\cdot\|_p$ -norm).

Remark 3.11 (See exercises)

(a) Theorem 3.10 does not hold for $p = \infty$. This follows from:

$$\overline{C_c(\phi)}^{\|\cdot\|_\infty} = C_0(\phi).$$

(b) $C_c(\phi)$ is separable w.r.t. $\|\cdot\|_\infty$.

Corollary 3.12 Let $\phi \neq \emptyset \subseteq \mathbb{R}^d$ be open, and let $p \in [1, \infty)$.

Then $L^p(\phi)$ is separable.

Pf. From Remark 3.11(b) & Thm. 3.10 (see exercise for details).

The proof of Thm. 3.10 requires some preparations:

Proposition 3.13 (Young's Inequality (for convolutions))

Let $p, q \in [1, \infty]$ with $1 \leq \frac{1}{p} + \frac{1}{q} \leq 2$, and let $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$

Then the convolution $f * g$ of f and g ,

$$\mathbb{R}^d \ni x \mapsto (f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y) dy \quad (*)$$

(where $dx := d\lambda^d(x)$, integration w.r.t. Lebesgue measure on $\mathbb{R}^d$)

exists for a.e. x , is commutative: $f * g = g * f$, and $f * g \in L^r(\mathbb{R}^d)$ with, for $\frac{1}{r} \stackrel{(**)}{=} -1 + \frac{1}{p} + \frac{1}{q}$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$),

$$(***) \quad \|f * g\|_r \leq \|f\|_p \|g\|_q \quad (\text{Young's Ineq.})$$

Corollary 3.14 Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ (I)

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then

$$I := I(f, g, h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) g(x-y) h(y) dx dy \right|$$

$$\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)| \cdot |g(x-y)| \cdot |h(y)| dx dy \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r \quad (\text{II})$$

Pf (of 3.14): Hölder & Young's ineq.: By 3.13, $f * g \in L^s$, $\frac{1}{s} = -1 + \frac{1}{p} + \frac{1}{q}$.

Hence, $\frac{1}{r} + \frac{1}{s} = 1$ by (I), and so (II) follows from Hölder's ineq.

Remark 3.14 (a) Note that, by scaling, the r in (**) is the only

one for which (***) could hold: For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$. Then

(change of variables), $\|f_\lambda\|_p = \lambda^{-1/p} \|f\|_p$ & $(f * g)_\lambda(x) = (\lambda f_\lambda * g_\lambda)(x)$.

Hence, if (***), holds for some $p, q, r \in [1, \infty]$, then also, $\forall \lambda > 0$

$$\|f * g\|_r = \lambda^{1/r} \|(f * g)_\lambda\|_r \leq \lambda^{1/r} \lambda \|f_\lambda\|_p \|g_\lambda\|_q = \lambda^{(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q})} \|f\|_p \|g\|_q$$

This is only possible if (**) holds (let $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ above).

(b) Often (in applications) one encounters

$$J_\alpha(f, h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) |x-y|^{-\alpha} h(y) dx dy \right|, \text{ that is, (II) with } g(z) = |z|^{-\alpha}.$$

Now, $g \notin L^q(\mathbb{R}^d)$ if q . The replacement for (II) (or, for (***) in 3.13) is called the Hardy-Littlewood-Sobolev Ineq. (HLS):

$$J_\alpha(f, h) \leq C(p, \alpha, d) \|f\|_p \|h\|_r, \quad \frac{1}{p} + \frac{1}{r} + \frac{\alpha}{d} = 2 \text{ \& } 0 < \alpha < d$$

Its proof requires much more work.

Pf (of 3.13): Commutativity (if integral exists):

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy \stackrel{y := x-z}{=} \int_{\mathbb{R}^d} f(z) g(x-z) dz = (g * f)(x)$$

The cases $p=1 < q=r$ and $q=1 < p=r$ follow directly from (76) Rinkowski's Ineq. (3.2(b)) (a commutativity, for $q=1 < p=r$): ($r < \infty$!)

$$\left(\int_{\mathbb{R}^d} |f * g(x)|^r dx \right)^{1/r} = \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) g(x-y) dy \right|^r dx \right)^{1/r}$$

$$\stackrel{3.2(b)}{\leq} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y) g(x-y)|^r dx \right)^{1/r} dy = \int_{\mathbb{R}^d} |f(y)| \underbrace{\left(\int_{\mathbb{R}^d} |g(x-y)|^r dx \right)^{1/r}}_{= \|g\|_r \forall x} dy = \|f\|_1 \|g\|_r \quad \checkmark$$

Remains the most general case:

$$1 < p < r \text{ and } 1 < q < r, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \quad (**)$$

Strategy: write $|f(y)g(x-y)|$ as product of 3 factors & use (gen.) Hölder:

$$|f(y)g(x-y)| = \left(|f(y)|^p |g(x-y)|^q \right)^{\frac{1}{s_1}} |g(x-y)|^{\frac{q}{s_2}} |f(y)|^{\frac{p}{s_3}} \quad (\nabla)$$

We must have (check!)

$$p \left(\frac{1}{s_1} + \frac{1}{s_3} \right) = 1, \quad q \left(\frac{1}{s_1} + \frac{1}{s_2} \right) = 1$$

As we want to estimate the L^r -norm of $f * g$, we want $s_1 = r$.

$$\text{Hence, } \begin{pmatrix} 1 & 0 & 0 \\ q & q & 0 \\ p & 0 & p \end{pmatrix} \begin{pmatrix} s_1^{-1} \\ s_2^{-1} \\ s_3^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \text{check!} \\ (\Rightarrow) \\ (pq \neq 0) \end{matrix} \begin{pmatrix} s_1^{-1} \\ s_2^{-1} \\ s_3^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ (1 - \frac{q}{r}) \frac{1}{q} \\ (1 - \frac{p}{r}) \frac{1}{p} \end{pmatrix} \quad (\nabla \nabla)$$

We see that $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} \stackrel{(A)}{=} 1$, and (∇) becomes

$$|f(y)g(x-y)| = \left(|f(y)|^p |g(x-y)|^q \right)^{\frac{1}{r}} |g(x-y)|^{1 - \frac{q}{r}} |f(y)|^{1 - \frac{p}{r}} \quad (\nabla \nabla \nabla)$$

and so, by gen. Hölder (Thm T26; 3 factors; use (A) & $(\nabla \nabla \nabla)$)

$$|f * g(x)| \leq \left(\int_{\mathbb{R}^d} |f(y)|^p |g(x-y)|^q dy \right)^{\frac{1}{r}} \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \quad (\nabla \nabla \nabla \nabla)$$

Hence (raise to power r & integrate w.r.t x),

$$\|f * g\|_r \leq \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|^p |g(x-y)|^q dy \right) dx \right)^{1/r}$$

$$\stackrel{\text{Tonelli}}{=} \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \left(\int_{\mathbb{R}^d} |f(y)|^p \cdot \underbrace{\left(\int_{\mathbb{R}^d} |g(x-y)|^q dx \right)}_{= \|g\|_q^q \forall x} dy \right)^{1/r}$$

$$= \|f\|_p \|g\|_q \quad \checkmark$$

If $r = \infty$, then $(**)$ follows from (standard) Hölder, as $\frac{1}{p} + \frac{1}{q} = 1$ □

Next, we will study mollification: L^p -functions are made smooth (C^∞) by convolution with a mollifier:

Let $L_c^p(\Omega) := \{ f \in L^p(\Omega) \mid \exists K \subseteq \Omega \text{ compact s.t. } f|_{\Omega \setminus K} = 0 \text{ a.e.} \}$

for the space of L^p -functions with compact support in Ω .

We identify $f \in L_c^p(\Omega)$ with its extension by 0 to all of $\mathbb{R}^d$.

(Lemma 3.15) Let $p \in [1, \infty)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open, $f \in L_c^p(\Omega)$.

Let $j \in C_c^\infty(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} j(x) dx = 1$, and for $\varepsilon > 0$

define the mollifier $j_\varepsilon: \mathbb{R}^d \rightarrow [0, \infty)$

$$x \mapsto \varepsilon^{-d} j\left(\frac{x}{\varepsilon}\right).$$

Then, for all $\varepsilon > 0$ sufficiently

small: $f_\varepsilon := j_\varepsilon * f \in C_c^\infty(\Omega)$ and $\|f_\varepsilon\|_p \leq \|f\|_p$.

(Remark) Note that j_ε is a pointwise defined function (and not just an equiv. class of integrable fct's, defined a.e.), the convolution $j_\varepsilon * f$ has the canonical representative

$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y) dy \quad (*)$$

which is pointwise well-defined $\forall x \in \mathbb{R}^d$, and independent of the choice of the representative for $f \in L^p$. It is the representative (*) of the convolution $j_\varepsilon * f$ we denote by f_ε .

Pf: Norm: Use Thm 3.13 with $r=p$. Then $q=1$ then, and so $f_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j_\varepsilon\|_1 = \|f\|_p$.

Support: By assumption, $\exists K \subseteq \Omega$ compact with $f|_{\Omega \setminus K} = 0$.

Then (by T10(ii)) $\delta := \text{dist}(K, \partial\Omega) > 0$ (since K compact & $\partial\Omega$ closed)

By def. of the mollifier, $\exists \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:

$\text{supp } j_\varepsilon \subseteq B_{\delta/2}(0)$. From (*) we conclude that

$$\text{supp } f_\varepsilon \subseteq \bigcup_{y \in K} B_{\delta/2}(y) \subseteq \Omega.$$

In particular, $\text{supp } f_\varepsilon$ is bounded (!) and (since closed by def.), also compact.

Derivatives: Let $e \in \mathbb{R}^d$, $|e|=1$, be a unit vector, and $\lambda \in \mathbb{R} \setminus \{0\}$.

Then, $\forall x \in \mathbb{R}^d$,

$$F_\lambda(x) := \frac{1}{\lambda} (f_\varepsilon(x + \lambda e) - f_\varepsilon(x)) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (j_\varepsilon(x - \gamma + \lambda e) - j_\varepsilon(x - \gamma)) f(\gamma) d\gamma$$

$$\stackrel{\text{HDI}}{=} \frac{1}{\lambda} \int_{\mathbb{R}^d} \left(\int_0^1 \frac{d}{dt} [j_\varepsilon(x - \gamma + t\lambda e)] dt \right) f(\gamma) d\gamma$$

$$\lambda e \cdot (\nabla j_\varepsilon)(x - \gamma + t\lambda e)$$

$$= \int_{\mathbb{R}^d} \left(\int_0^1 (e \cdot \nabla j_\varepsilon)(x - \gamma + t\lambda e) dt \right) f(\gamma) d\gamma$$

$$=: D$$

Since $|D| \leq \|\nabla j_\varepsilon\|_\infty < \infty$ and $f \in L^1$ (because $f \in L^p$ & of compact support) using Dominated Convergence (twice, in dt & $d\gamma$ integral) gives that

$$e \cdot (\nabla f_\varepsilon)(x) = \lim_{\lambda \rightarrow 0} F_\lambda(x) = \int_{\mathbb{R}^d} e \cdot (\nabla j_\varepsilon)(x - \gamma) f(\gamma) d\gamma = ((e \cdot \nabla j_\varepsilon) * f)(x)$$

exists $\forall x \in \mathbb{R}^d$. The existence of higher derivatives follows by induction in the same way. $\square$

Lemma 3.16 Let $p \in [1, \infty)$, let $A \in \mathcal{B}(\mathbb{R}^d)$ be bounded.

Then

$$\lim_{\varepsilon \rightarrow 0} \| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_{L^p(\mathbb{R}^d)} = 0.$$

Proof: Step 1: It suffices to prove the lemma for $p=1$, since:

Let $g_\varepsilon := \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \in L_c^\infty(\mathbb{R}^d)$; then $g_\varepsilon \in L^p(\mathbb{R}^d) \forall p \in [1, \infty)$

$$\text{and } \|g_\varepsilon\|_p^p \leq \underbrace{(\|g_\varepsilon\|_\infty)^{p-1}}_{\leq 2} \|g_\varepsilon\|_1.$$

Step 2: The case $p=1$. Since Lebesgue measure is (outer) regular (see Handout):

(79)

$$\forall \delta > 0 \exists B \supseteq A, B \text{ open and } \int_{B \setminus A} dx < \delta.$$

So, fix $\delta > 0$, and let B be as above (so, also bounded).

Note that

$$\| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_1 \leq \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 + \underbrace{\| \mathbb{1}_{B \setminus A} \|_1}_{< \delta} + \| j_\varepsilon * \mathbb{1}_{B \setminus A} \|_1$$

$$\text{with } \| j_\varepsilon * \mathbb{1}_{B \setminus A} \|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_\varepsilon(x-y) \mathbb{1}_{B \setminus A}(y) dy dx$$

$$\stackrel{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y) dx \right)}_{=1 \ \forall y} \mathbb{1}_{B \setminus A}(y) dy < \delta.$$

Hence, it suffices to prove that $\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 \rightarrow 0, \varepsilon \downarrow 0$.

We have $0 \leq j_\varepsilon * \mathbb{1}_B \leq 1$, hence

$$\begin{aligned} \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 &= \int_B (1 - (j_\varepsilon * \mathbb{1}_B)(x)) dx + \int_{\mathbb{R}^d \setminus B} (j_\varepsilon * \mathbb{1}_B)(x) dx \\ &= 2|B| - 2 \int_B \int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x-\varepsilon z) dz dx. \end{aligned}$$

Now, since B is open: $\forall x \in B \ \forall z \in \mathbb{R}^d: \mathbb{1}_B(x-\varepsilon z) \rightarrow 1, \varepsilon \downarrow 0$ (think!)

Also, B is bounded and $j \in L^1(\mathbb{R}^d)$, so, using

Lebesgue's Thm. on Dominated Convergence twice

$$\lim_{\varepsilon \downarrow 0} \int_B \left(\int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x-\varepsilon z) dz \right) dx = \int_B \left(\int_{\mathbb{R}^d} j(z) dz \right) dx = |B| \quad \blacksquare$$

Proof of Thm. 3.6: (Here $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$).

Step 1: Let $\varepsilon > 0, f \in L^p(\mathbb{R})$. For $n \in \mathbb{N}$, define

$$K_n := \left\{ x \in \mathbb{R} \mid \text{dist}(x, \partial \mathbb{R}) \geq \frac{1}{n} \text{ and } |x| \leq n \right\} \subseteq \mathbb{R}$$

Now, $\mathbb{R}$ is open, so $\mathbb{R} = \bigcup_{n \in \mathbb{N}} K_n$, and $K_n \subseteq K_{n+1} \ \forall n \in \mathbb{N}$.

By dominated convergence, $\left(\lim_{n \rightarrow \infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_{\mathcal{R}}(x) \right)$ (80)

$$\lim_{n \rightarrow \infty} \|f - f \mathbb{1}_{K_n}\|_p = \lim_{n \rightarrow \infty} \|f(1 - \mathbb{1}_{K_n})\|_p = 0.$$

Hence: $\exists n \in \mathbb{N}$: $\|f - f \mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$.

Step 2: Recall (handout & exercise) that step functions are dense in L^p : $\exists L \in \mathbb{N}, c_1, \dots, c_L \in \mathbb{K}$ and $A_1, \dots, A_L \in \mathcal{B}(\mathbb{R}^d)$ with $A_\ell \subseteq K_n (!) \forall \ell = 1, \dots, L$ such that: For $g := \sum_{\ell=1}^L c_\ell \mathbb{1}_{A_\ell}$ we have $g \in L_c^p(\mathcal{R})$ and $\|g - f \mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$

Step 3: Mollify step function: $\exists \delta > 0$ s.t.

(i) $j_\delta * g \in C_c^\infty(\mathcal{R})$ (by Lemma 3.15)

(ii) $\|g - j_\delta * g\|_p \leq \sum_{\ell=1}^L |c_\ell| \cdot \|\mathbb{1}_{A_\ell} - j_\delta * \mathbb{1}_{A_\ell}\|_p < \frac{\varepsilon}{3}$
(by Lemma 3.16)

Hence,

$$\|f - \underbrace{j_\delta * g}_{\in C_c^\infty(\mathcal{R})}\|_p < \varepsilon \quad \square$$
