

3.2 Separability

(73)

We have the following general abstract result:

Theorem 3.8 Let $(X, \mathcal{A}, \mu)$ be a σ -finite measure space which is countably generated (i.e., $\exists \mathcal{C} \subseteq \mathcal{P}(X)$ countable such that $\mathcal{A} = \sigma(\mathcal{C})$). Let $p \in [1, \infty)$. Then L^p is separable.

Pf. See Behrends, "Mas- und Integrationsstheorie" (1987) Thm IV.1.5 p. 160.

Remark 3.9 The theorem does not hold for $p = \infty$ (in general).

We turn to the concrete case of $L^p(\Omega, dx^d)$ with $\Omega \subseteq \mathbb{R}^d$ open (non-empty) and λ^d Lebesgue measure, for which we will prove the result in 3.8.

Notation: (i) For $f: \Omega \rightarrow \mathbb{K}$, the (topological) support of f is

$$\text{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}} \quad \left(\begin{array}{l} \text{closure w.r.t. rel. top.} \\ \text{on } \Omega \text{ induced by} \\ \text{Euclidean top.} \end{array} \right)$$

(ii) For $k \in \mathbb{N}$: Space of k -times cont. diff. fct's

$$C^k(\Omega) := \left\{ f \in C(\Omega) \mid \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f \in C(\Omega) \quad \forall \alpha_1, \dots, \alpha_d \in \mathbb{N}_0, \sum_{j=1}^d \alpha_j \leq k \right\}$$

Also, $C^0(\Omega) := C(\Omega)$ and

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega) \quad (\text{(\u0022infinitely\u0022) \u0022smooth functions\u0022})$$

(iii) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k -times cont. diff. fct's with compact support:

$$C_c^k(\Omega) := \left\{ f \in C^k(\Omega) \mid \text{supp}(f) \subseteq \Omega \text{ is compact} \right\}$$

(iv) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of k -times cont. diff. fct.'s vanishing at ∞ (!):

$$C_0^k(\Omega) := \left\{ f \in C^k(\Omega) \mid \forall \varepsilon > 0 \exists K_\varepsilon \subseteq \Omega \text{ compact s.t. } |f(x)| \leq \varepsilon \quad \forall x \in \Omega \setminus K_\varepsilon \right\}$$



Rest this section:

Consider $(X, \mathcal{A}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$, with $\mathcal{B}(\mathbb{R}^d)$

Borel- σ -algebra on $\mathbb{R}^d$ (trace- σ -algebra to $\mathbb{R}^d$ of Borel- σ -algeb. on $\mathbb{R}^d$) and λ^d Lebesgue-Borel measure on $\mathbb{R}^d$ (see Handout).

Main result is for $L^p(\mathbb{R}^d) := L^p(\mathbb{R}^d, \lambda^d)$:

Theorem 3.10 Let $\phi \neq \emptyset \subseteq \mathbb{R}^d$ be open and let $p \in [1, \infty)$.

Then $\overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_p} = L^p(\mathbb{R}^d)$

(i.e. $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ w.r.t. $\|\cdot\|_p$ -norm).

Remark 3.11 (See exercises)

(a) Theorem 3.10 does not hold for $p = \infty$. This follows from:

$$\overline{C_c(\mathbb{R}^d)}^{\|\cdot\|_\infty} = C_0(\mathbb{R}^d).$$

(b) $C_c(\mathbb{R}^d)$ is separable w.r.t. $\|\cdot\|_\infty$.

Corollary 3.12 Let $\phi \neq \emptyset \subseteq \mathbb{R}^d$ be open, and let $p \in [1, \infty)$.

Then $L^p(\mathbb{R}^d)$ is separable.

Pf. From Remark 3.11(b) & Thm. 3.10 (see exercise for details).

The proof of Thm. 3.10 requires some preparations:

Proposition 3.13 (Young's Inequality (for convolutions))

Let $p, q \in [1, \infty]$ with $1 \leq \frac{1}{p} + \frac{1}{q} \leq 2$, and let $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$

Then the convolution $f * g$ of f and g ,

$$\mathbb{R}^d \ni x \mapsto (f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y) dy \quad (*)$$

(where $dx := d\lambda^d(x)$, integration w.r.t. Lebesgue measure on $\mathbb{R}^d$)

exists for a.e. x , is commutative: $f * g = g * f$, and $f * g \in L^r(\mathbb{R}^d)$ with, for $\frac{1}{r} \stackrel{(**)}{=} -1 + \frac{1}{p} + \frac{1}{q}$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$),

$$(***) \quad \|f * g\|_r \leq \|f\|_p \|g\|_q \quad (\text{Young's Ineq.})$$

Corollary 3.14 Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ (I)

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then

$$I := I(f, g, h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) g(x-y) h(y) dx dy \right|$$

$$\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)| \cdot |g(x-y)| \cdot |h(y)| dx dy \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r \quad (\text{II})$$

Pf (of 3.14): Hölder & Young's ineq.: By 3.13, $f * g \in L^s$, $\frac{1}{s} = -1 + \frac{1}{p} + \frac{1}{q}$.

Hence, $\frac{1}{r} + \frac{1}{s} = 1$ by (I), and so (II) follows from Hölder's ineq.

Remark 3.14 (a) Note that, by scaling, the r in (**) is the only

one for which (***) could hold: For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$. Then

(change of variables), $\|f_\lambda\|_p = \lambda^{-1/p} \|f\|_p$ & $(f * g)_\lambda(x) = (\lambda f_\lambda * g_\lambda)(x)$.

Hence, if (***) holds for some $p, q, r \in [1, \infty]$, then also, $\forall \lambda > 0$

$$\|f * g\|_r = \lambda^{1/r} \|(f * g)_\lambda\|_r \leq \lambda^{1/r} \lambda \|f_\lambda\|_p \|g_\lambda\|_q = \lambda^{(1+\frac{1}{r} - \frac{1}{p} - \frac{1}{q})} \|f\|_p \|g\|_q$$

This is only possible if (**) holds (let $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ above).

(b) Often (in applications) one encounters

$$J_\alpha(f, h) := \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) |x-y|^{-\alpha} h(y) dx dy \right|, \text{ that is, (II) with } g(z) = |z|^{-\alpha}.$$

Now, $g \notin L^q(\mathbb{R}^d)$ if q . The replacement for (II) (or, for (***) in 3.13) is called the Hardy-Littlewood-Sobolev Ineq. (HLS):

$$J_\alpha(f, h) \leq C(p, \alpha, d) \|f\|_p \|h\|_r, \quad \frac{1}{p} + \frac{1}{r} + \frac{\alpha}{d} = 2 \text{ \& } 0 < \alpha < d$$

Its proof requires much more work.

Pf (of 3.13): Commutativity (if integral exists):

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy \stackrel{y := x-z}{=} \int_{\mathbb{R}^d} f(z) g(x-z) dz = (g * f)(x)$$

The cases $p=1 < q=r$ and $q=1 < p=r$ follow directly from (76) Rinkowski's Ineq. (3.2(b)) (a commutativity, for $q=1 < p=r$): ($r < \infty$!)

$$\left(\int_{\mathbb{R}^d} |f * g(x)|^r dx \right)^{1/r} = \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) g(x-y) dy \right|^r dx \right)^{1/r}$$

$$\stackrel{3.2(b)}{\leq} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y) g(x-y)|^r dx \right)^{1/r} dy = \int_{\mathbb{R}^d} |f(y)| \underbrace{\left(\int_{\mathbb{R}^d} |g(x-y)|^r dx \right)^{1/r}}_{= \|g\|_r \forall x} dy = \|f\|_1 \|g\|_r \quad \checkmark$$

Remains the most general case:

$$1 < p < r \text{ and } 1 < q < r, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \quad (**)$$

Strategy: write $|f(y)g(x-y)|$ as product of 3 factors & use (gen.) Hölder:

$$|f(y)g(x-y)| = \left(|f(y)|^p |g(x-y)|^q \right)^{\frac{1}{s_1}} |g(x-y)|^{\frac{q}{s_2}} |f(y)|^{\frac{p}{s_3}} \quad (\nabla)$$

We must have (check!)

$$p \left(\frac{1}{s_1} + \frac{1}{s_3} \right) = 1, \quad q \left(\frac{1}{s_1} + \frac{1}{s_2} \right) = 1$$

As we want to estimate the L^r -norm of $f * g$, we want $s_1 = r$.

$$\text{Hence, } \begin{pmatrix} 1 & 0 & 0 \\ q & q & 0 \\ p & 0 & p \end{pmatrix} \begin{pmatrix} s_1^{-1} \\ s_2^{-1} \\ s_3^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \text{check!} \\ (\Rightarrow) \\ (pq \neq 0) \end{matrix} \begin{pmatrix} s_1^{-1} \\ s_2^{-1} \\ s_3^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ (1 - \frac{q}{r}) \frac{1}{q} \\ (1 - \frac{p}{r}) \frac{1}{p} \end{pmatrix} \quad (\nabla \nabla)$$

We see that $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} \stackrel{(A)}{=} 1$, and (∇) becomes

$$|f(y)g(x-y)| = \left(|f(y)|^p |g(x-y)|^q \right)^{\frac{1}{r}} |g(x-y)|^{1 - \frac{q}{r}} |f(y)|^{1 - \frac{p}{r}} \quad (\nabla \nabla \nabla)$$

and so, by gen. Hölder (Thm T26; 3 factors; use (A) & $(\nabla \nabla \nabla)$)

$$|f * g(x)| \leq \left(\int_{\mathbb{R}^d} |f(y)|^p |g(x-y)|^q dy \right)^{\frac{1}{r}} \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \quad (\nabla \nabla \nabla \nabla)$$

Hence (raise to power r & integrate w.r.t x),

$$\|f * g\|_r \leq \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|^p |g(x-y)|^q dy \right) dx \right)^{1/r}$$

$$\stackrel{\text{Tonelli}}{=} \|f\|_p^{1 - \frac{p}{r}} \|g\|_q^{1 - \frac{q}{r}} \left(\int_{\mathbb{R}^d} |f(y)|^p \cdot \underbrace{\left(\int_{\mathbb{R}^d} |g(x-y)|^q dx \right)}_{= \|g\|_q^q \forall x} dy \right)^{1/r}$$

$$= \|f\|_p \|g\|_q \quad \checkmark$$

If $r = \infty$, then $(**)$ follows from (standard) Hölder, as $\frac{1}{p} + \frac{1}{q} = 1$ □

Next, we will study mollification: L^p -functions are made smooth (C^∞) by convolution with a mollifier:

Let $L_c^p(\Omega) := \{ f \in L^p(\Omega) \mid \exists K \subseteq \Omega \text{ compact s.t. } f|_{\Omega \setminus K} = 0 \text{ a.e.} \}$

for the space of L^p -functions with compact support in Ω .

We identify $f \in L_c^p(\Omega)$ with its extension by 0 to all of $\mathbb{R}^d$.

(Lemma 3.15) Let $p \in [1, \infty)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open, $f \in L_c^p(\Omega)$.

Let $j \in C_c^\infty(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} j(x) dx = 1$, and for $\varepsilon > 0$

define the mollifier $j_\varepsilon: \mathbb{R}^d \rightarrow [0, \infty)$

$$x \mapsto \varepsilon^{-d} j\left(\frac{x}{\varepsilon}\right).$$

Then, for all $\varepsilon > 0$ sufficiently

small: $f_\varepsilon := j_\varepsilon * f \in C_c^\infty(\Omega)$ and $\|f_\varepsilon\|_p \leq \|f\|_p$.

(Remark) Note that j_ε is a pointwise defined function (and not just an equiv. class of integrable fct's, defined a.e.), the convolution $j_\varepsilon * f$ has the canonical representative

$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y) dy \quad (*)$$

which is pointwise well-defined $\forall x \in \mathbb{R}^d$, and independent of the choice of the representative for $f \in L^p$. It is the representative $(*)$ of the convolution $j_\varepsilon * f$ we denote by f_ε .

Pf: Norm: Use Thm 3.13 with $r=p$. Then $q=1$ then, and so $f_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j_\varepsilon\|_1 = \|f\|_p$.

Support: By assumption, $\exists K \subseteq \Omega$ compact with $f|_{\Omega \setminus K} = 0$.

Then (by T10(ii)) $\delta := \text{dist}(K, \partial\Omega) > 0$ (since K compact & $\partial\Omega$ closed)

By def. of the mollifier, $\exists \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:

$\text{supp } j_\varepsilon \subseteq B_{\delta/2}(0)$. From (*) we conclude that

$$\text{supp } f_\varepsilon \subseteq \bigcup_{y \in K} B_{\delta/2}(y) \subseteq \Omega.$$

In particular, $\text{supp } f_\varepsilon$ is bounded (!) and (since closed by def.), also compact.

Derivatives: Let $e \in \mathbb{R}^d$, $|e|=1$, be a unit vector, and $\lambda \in \mathbb{R} \setminus \{0\}$.

Then, $\forall x \in \mathbb{R}^d$,

$$F_\lambda(x) := \frac{1}{\lambda} (f_\varepsilon(x + \lambda e) - f_\varepsilon(x)) = \frac{1}{\lambda} \int_{\mathbb{R}^d} (j_\varepsilon(x - \gamma + \lambda e) - j_\varepsilon(x - \gamma)) f(\gamma) d\gamma$$

$$\stackrel{\text{HDI}}{=} \frac{1}{\lambda} \int_{\mathbb{R}^d} \left(\int_0^1 \frac{d}{dt} [j_\varepsilon(x - \gamma + t\lambda e)] dt \right) f(\gamma) d\gamma$$

$$\lambda e \cdot (\nabla j_\varepsilon)(x - \gamma + t\lambda e)$$

$$= \int_{\mathbb{R}^d} \left(\int_0^1 (e \cdot \nabla j_\varepsilon)(x - \gamma + t\lambda e) dt \right) f(\gamma) d\gamma$$

$$=: D$$

Since $|D| \leq \|\nabla j_\varepsilon\|_\infty < \infty$ and $f \in L^1$ (because $f \in L^p$ & of compact support) using Dominated Convergence (twice, in dt & $d\gamma$ integral) gives that

$$e \cdot (\nabla f_\varepsilon)(x) = \lim_{\lambda \rightarrow 0} F_\lambda(x) = \int_{\mathbb{R}^d} e \cdot (\nabla j_\varepsilon)(x - \gamma) f(\gamma) d\gamma = ((e \cdot \nabla j_\varepsilon) * f)(x)$$

exists $\forall x \in \mathbb{R}^d$. The existence of higher derivatives follows by induction in the same way. $\square$

Lemma 3.16 Let $p \in [1, \infty)$, let $A \in \mathcal{B}(\mathbb{R}^d)$ be bounded.

Then

$$\lim_{\varepsilon \rightarrow 0} \| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_{L^p(\mathbb{R}^d)} = 0.$$

Proof: Step 1: It suffices to prove the lemma for $p=1$, since:

Let $g_\varepsilon := \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \in L_c^\infty(\mathbb{R}^d)$; then $g_\varepsilon \in L^p(\mathbb{R}^d) \forall p \in [1, \infty)$

$$\text{and } \|g_\varepsilon\|_p^p \leq \underbrace{(\|g_\varepsilon\|_\infty)^{p-1}}_{\leq 2} \|g_\varepsilon\|_1.$$

Step 2: The case $p=1$. Since Lebesgue measure is (outer) regular (see Handout):

(79)

$$\forall \delta > 0 \exists B \supseteq A, B \text{ open and } \int_{B \setminus A} dx < \delta.$$

So, fix $\delta > 0$, and let B be as above (so, also bounded).

Note that

$$\| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_1 \leq \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 + \underbrace{\| \mathbb{1}_{B \setminus A} \|_1}_{< \delta} + \| j_\varepsilon * \mathbb{1}_{B \setminus A} \|_1$$

$$\text{with } \| j_\varepsilon * \mathbb{1}_{B \setminus A} \|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_\varepsilon(x-y) \mathbb{1}_{B \setminus A}(y) dy dx$$

$$\stackrel{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y) dx \right)}_{=1 \ \forall y} \mathbb{1}_{B \setminus A}(y) dy < \delta.$$

Hence, it suffices to prove that $\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 \rightarrow 0, \varepsilon \downarrow 0$.

We have $0 \leq j_\varepsilon * \mathbb{1}_B \leq 1$, hence

$$\begin{aligned} \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 &= \int_B (1 - (j_\varepsilon * \mathbb{1}_B)(x)) dx + \int_{\mathbb{R}^d \setminus B} (j_\varepsilon * \mathbb{1}_B)(x) dx \\ &= 2|B| - 2 \int_B \int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x-\varepsilon z) dz dx. \end{aligned}$$

Now, since B is open: $\forall x \in B \ \forall z \in \mathbb{R}^d: \mathbb{1}_B(x-\varepsilon z) \rightarrow 1, \varepsilon \downarrow 0$ (think!)

Also, B is bounded and $j \in L^1(\mathbb{R}^d)$, so, using

Lebesgue's Thm. on Dominated Convergence twice

$$\lim_{\varepsilon \downarrow 0} \int_B \left(\int_{\mathbb{R}^d} j(z) \mathbb{1}_B(x-\varepsilon z) dz \right) dx = \int_B \left(\int_{\mathbb{R}^d} j(z) dz \right) dx = |B| \quad \blacksquare$$

Proof of Thm. 3.6: (Here $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$).

Step 1: Let $\varepsilon > 0, f \in L^p(\mathbb{R})$. For $n \in \mathbb{N}$, define

$$K_n := \left\{ x \in \mathbb{R} \mid \text{dist}(x, \partial \mathbb{R}) \geq \frac{1}{n} \text{ and } |x| \leq n \right\} \subseteq \mathbb{R}$$

Now, $\mathbb{R}$ is open, so $\mathbb{R} = \bigcup_{n \in \mathbb{N}} K_n$, and $K_n \subseteq K_{n+1} \ \forall n \in \mathbb{N}$.

By dominated convergence, $\left(\lim_{n \rightarrow \infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_{\mathcal{R}}(x) \right)$ (80)

$$\lim_{n \rightarrow \infty} \|f - f \mathbb{1}_{K_n}\|_p = \lim_{n \rightarrow \infty} \|f(1 - \mathbb{1}_{K_n})\|_p = 0.$$

Hence: $\exists n \in \mathbb{N}$: $\|f - f \mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$.

Step 2: Recall (handout & exercise) that step functions are dense in L^p : $\exists L \in \mathbb{N}, c_1, \dots, c_L \in \mathbb{K}$ and $A_1, \dots, A_L \in \mathcal{B}(\mathbb{R}^d)$ with $A_\ell \subseteq K_n (!) \forall \ell = 1, \dots, L$ such that: For $g := \sum_{\ell=1}^L c_\ell \mathbb{1}_{A_\ell}$ we have $g \in L_c^p(\mathcal{R})$ and $\|g - f \mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$

Step 3: Mollify step function: $\exists \delta > 0$ s.t.

(i) $j_\delta * g \in C_c^\infty(\mathcal{R})$ (by Lemma 3.15)

(ii) $\|g - j_\delta * g\|_p \leq \sum_{\ell=1}^L |c_\ell| \cdot \|\mathbb{1}_{A_\ell} - j_\delta * \mathbb{1}_{A_\ell}\|_p < \frac{\varepsilon}{3}$
(by Lemma 3.16)

Hence,

$$\|f - \underbrace{j_\delta * g}_{\in C_c^\infty(\mathcal{R})}\|_p < \varepsilon \quad \square$$
