## 3.2 Separability

We have the following general abstract result:

**Theorem 3.8** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space which is countably generated (i.e., $\exists\, \mathcal{C} \subseteq \mathcal{P}(X)$ countable such that $\mathcal{A} = \sigma(\mathcal{C})$). Let $p \in [1, \infty)$. Then $L^p$ is separable.

Pf: See Behrends, "Maß- und Integrationstheorie" (1987) Thm IV.1.5 p. 160.

**Remark 3.9** The theorem does not hold for $p = \infty$ (in general).

We turn to the concrete case of $L^p(\Omega, d\lambda^d)$ with $\Omega \subseteq \mathbb{R}^d$ open (non-empty) and $\lambda^d$ Lebesgue measure, for which we will prove the result in 3.8.

Notation: (i) For $f : \Omega \to \mathbb{K}$, the (topological) support of $f$ is

$$\operatorname{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$$

(closure wrt. rel. top. on $\Omega$ induced by Euclidean top.)

(ii) For $k \in \mathbb{N}$: Space of $k$-times cont. diff. fct's

$$C^k(\Omega) := \left\{ f \in C(\Omega) \;\middle|\; \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f \in C(\Omega) \ \forall \alpha_1, \ldots, \alpha_d \in \mathbb{N}_0,\ \sum_{j=1}^{d} \alpha_j \leq k \right\}$$

Also, $C^0(\Omega) := C(\Omega)$ and

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega) \quad \text{((infinitely) smooth functions)}$$

(iii) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of $k$-times cont. diff. fct's with compact support:

$$C_c^k(\Omega) := \{ f \in C^k(\Omega) \mid \operatorname{supp}(f) \subseteq \Omega \text{ is compact} \}$$

(iv) For $k \in \mathbb{N}_0 \cup \{\infty\}$: Space of $k$-times cont. diff. fct.'s vanishing at $\infty$ (!):

$$C_0^k(\Omega) := \left\{ f \in C^k(\Omega) \;\middle|\; \begin{array}{l} \forall \varepsilon > 0 \; \exists K_\varepsilon \subseteq \Omega \text{ compact s.t.} \\ |f(x)| \le \varepsilon \;\; \forall x \in \Omega \setminus K_\varepsilon \end{array} \right\}$$

Rest this section:

Consider $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{B}(\Omega), \lambda^d)$, with $\mathcal{B}(\Omega)$ Borel-$\sigma$-algebra on $\Omega$ (trace-$\sigma$-algebra to $\Omega$ of Borel-$\sigma$-algeb. on $\mathbb{R}^d$) and $\lambda^d$ Lebesgue-Borel measure on $\mathbb{R}^d$ (see Handout).

Main result is for $L^p(\Omega) := L^p(\Omega, \lambda^d)$:

**Theorem 3.10** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open and let $p \in [1, \infty)$.

Then
$$\overline{C_c^\infty(\Omega)}^{\|\cdot\|_p} = L^p(\Omega)$$

(i.e. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ wrt. $\|\cdot\|_p$-norm).

**Remark 3.11** (See exercises)

(a) Theorem 3.10 does not hold for $p = \infty$. This follows from:
$$\overline{C_c(\Omega)}^{\|\cdot\|_\infty} = C_0(\Omega).$$

(b) $C_c(\Omega)$ is separable wrt. $\|\cdot\|_\infty$.

**Corollary 3.12** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ be open, and let $p \in [1, \infty)$. Then $L^p(\Omega)$ is separable.

Pf: From Rmk. 3.11 (b) & Thm. 3.10 (see exercise for details). ∎

The proof of Thm. 3.10 requires some preparations:

**Proposition 3.13** (Young's Inequality (for convolutions))

Let $p, q \in [1, \infty]$ with $1 \le \frac{1}{p} + \frac{1}{q} \le 2$, and let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$.

Then the convolution $f * g$ of $f$ and $g$,

$$\mathbb{R}^d \ni x \mapsto (f*g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) \, dy \qquad (*)$$

(where $dx := d\lambda^d(x)$, integration wrt. Lebesgue measure on $\mathbb{R}^d$)

exists for a.e. $x$, is commutative: $f*g = g*f$, and $f*g \in L^r(\mathbb{R}^d)$ with, for $\frac{1}{r} \overset{(**)}{:=} -1 + \frac{1}{p} + \frac{1}{q}$ (i.e. $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$),

$$(***) \quad \|f*g\|_r \le \|f\|_p \|g\|_q \qquad \text{(Young's Ineq.)}.$$

**Corollary 3.14** Let $p,q,r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$ $(\square)$

Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $h \in L^r(\mathbb{R}^d)$. Then

$$I := I(f,g,h) := \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\, g(x-y)\, h(y)\, dx\, dy \right|$$
$$\le \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)| \cdot |g(x-y)| \cdot |h(y)|\, dx\, dy \le \|f\|_p \cdot \|g\|_q \cdot \|h\|_r \quad (\square\square)$$

Pf (of 3.14): Hölder & Young's ineq.: By 3.13, $f*g \in L^s$, $\frac{1}{s} = -1 + \frac{1}{p} + \frac{1}{q}$.
Hence, $\frac{1}{r}+\frac{1}{s} = 1$ by $(\square)$, and so $(\square\square)$ follows from Hölder's ineq.

**Remark 3.14** (a) Note that, by scaling, the $r$ in $(**)$ is the only one for which $(***)$ could hold: For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$. Then (change of variables!), $\|f_\lambda\|_p = \lambda^{-1/p}\|f\|_p$ & $(f*g)_\lambda(x) = (\lambda f_\lambda * g_\lambda)(x)$.
Hence, if $(***)$ holds for some $p,q,r \in [1,\infty]$, then also, $\forall \lambda > 0$

$$\|f*g\|_r = \lambda^{1/r}\|(f*g)_\lambda\|_r \le \lambda^{1/r}\lambda \|f_\lambda\| \cdot \|g_\lambda\|_q = \lambda^{(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q})}\|f\|_p \|g\|_q$$

This is only possible if $(**)$ holds (let $\lambda \searrow 0$ and $\lambda \nearrow \infty$ above).

(b) Often (in applications) one encounters

$$J_\alpha(f,h) := \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\, |x-y|^{-\alpha}\, h(y)\, dx\, dy \right|,$$ that is, $(\square\square)$ with $g(z) = |z|^{-\alpha}$.

Now, $g \notin L^q(\mathbb{R}^d)$ $\forall q$. The replacement for $(\square\square)$ (or, for $(***)$ in 3.13) is called the Hardy-Littlewood-Sobolev Ineq. (HLS):

$$J_\alpha(f,h) \le C(p,\alpha,d)\|f\|_p \|h\|_r, \quad \frac{1}{p}+\frac{1}{r}+\frac{\alpha}{d} = 2 \ \& \ 0 < \alpha < d$$

Its proof requires much more work.

Pf (of 3.13): Commutativity (if integral exists):

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \overset{y := x-z}{=} \int_{\mathbb{R}^d} f(z)\, g(x-z)\, dz = (g*f)(x)$$

The cases $p=1<q=r$ and $q=1<p=r$ follow directly from Minkowski's Ineq. (3.2(b)) (& commutativity, for $q=1<p=r$): ($r<\infty$ !)

$$\left(\int_{\mathbb{R}^d}\left|(f*g)(x)\right|^r dx\right)^{1/r} = \left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(y)g(x-y)\,dy\right|^r dx\right)^{1/r}$$

$$\overset{3.2(b)}{\le} \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)g(x-y)|^r dx\right)^{1/r} dy = \int_{\mathbb{R}^d}|f(y)|\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^r dx\right)^{1/r}}_{=\|g\|_r\ \forall x} dy = \|f\|_1\|g\|_r \quad ✓$$

Remains the most general case:

$1<p<r$ and $1<q<r$, $\quad \frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. (**)

Strategy: write $|f(y)g(x-y)|$ as product of 3 factors & use (gen.) Hölder:

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{s_1}}|g(x-y)|^{\frac{q}{s_2}}|f(y)|^{\frac{p}{s_3}} \quad (\nabla)$$

We must have (check!)

$$p\left(\frac{1}{s_1}+\frac{1}{s_3}\right)=1, \quad q\left(\frac{1}{s_1}+\frac{1}{s_2}\right)=1$$

As we want to estimate the $L^r$-norm of $f*g$, we want $\underline{\underline{s_1=r}}$.

Hence,
$$\begin{pmatrix}1&0&0\\ q&q&0\\ p&0&p\end{pmatrix}\begin{pmatrix}s_1^{-1}\\ s_2^{-1}\\ s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\ 1\\ 1\end{pmatrix} \quad \underset{(pq\neq 0)}{\overset{\text{check!}}{\Longleftrightarrow}} \quad \begin{pmatrix}s_1^{-1}\\ s_2^{-1}\\ s_3^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{r}\\ \left(1-\frac{q}{r}\right)\frac{1}{q}\\ \left(1-\frac{p}{r}\right)\frac{1}{p}\end{pmatrix} \quad (\nabla\nabla)$$

We see that $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\overset{(\Delta)}{=}1$, and $(\nabla)$ becomes

$$|f(y)g(x-y)| = \left(|f(y)|^p|g(x-y)|^q\right)^{\frac{1}{r}}|g(x-y)|^{1-\frac{q}{r}}|f(y)|^{1-\frac{p}{r}} \quad (\nabla\nabla\nabla)$$

and so, by gen. Hölder (Tut T26; 3 factors; use $(\Delta)$ & $(\nabla\nabla)$)

$$|(f*g)(x)| \le \left(\int_{\mathbb{R}^d}|f(y)|^p\cdot|g(x-y)|^q\,dy\right)^{\frac{1}{r}}\|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}} \quad (\nabla\nabla\nabla\nabla)$$

Hence (raise to power $r$ & integrate wrt $x$),

$$\|f*g\|_r \le \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(y)|^p|g(x-y)|^q\,dy\right)dx\right)^{1/r}$$

$$\overset{\text{Tonelli}}{=} \|f\|_p^{1-\frac{p}{r}}\|g\|_q^{1-\frac{q}{r}}\left(\int_{\mathbb{R}^d}|f(y)|^p\cdot\underbrace{\left(\int_{\mathbb{R}^d}|g(x-y)|^q dx\right)}_{=\|g\|_q^q\ \forall x}dy\right)^{1/r}$$

$$= \|f\|_p\|g\|_q \quad ✓$$

If $r=\infty$, then (***) follows from (standard) Hölder, as $\frac{1}{p}+\frac{1}{q}=1$. □

Next, we will study <u>mollification</u>: $L^p$-functions are made smooth ($C^\infty$) by convolution with a mollifier:

Let
$$L^p_c(\Omega) := \{ f \in L^p(\Omega) \mid \exists K \subseteq \Omega \text{ compact s.t. } f|_{\Omega\setminus K} = 0 \text{ a.e.} \}$$
for the space of $L^p$-functions with compact support in $\Omega$.

We identify $f \in L^p_c(\Omega)$ with its extension by $0$ to all of $\mathbb{R}^d$.

**Lemma 3.15** Let $p \in [1,\infty)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ open, $f \in L^p_c(\Omega)$.
Let $j \in C^\infty_c(\mathbb{R}^d)$, $j \geq 0$, $\int_{\mathbb{R}^d} j(x)\,dx = 1$, and for $\varepsilon > 0$ define the <u>mollifier</u>
$$j_\varepsilon : \mathbb{R}^d \to [0,\infty)$$
$$x \mapsto \varepsilon^{-d} j\left(\frac{x}{\varepsilon}\right).$$
Then, for all $\varepsilon > 0$ sufficiently small:
$$f_\varepsilon := j_\varepsilon * f \in C^\infty_c(\Omega) \text{ and } \|f_\varepsilon\|_p \leq \|f\|_p.$$

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**Remark** Note that $j_\varepsilon$ is a pointwise defined function (and not just an equiv. class of integrable fct's, defined a.e.), the convolution $j_\varepsilon * f$ has the canonical representative
$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} j_\varepsilon(x-y) f(y)\,dy \qquad (*)$$
which is pointwise well-defined $\forall x \in \mathbb{R}^d$, and <u>independent</u> of the <u>choice</u> of the representative for $f \in L^p$. It is the representative $(*)$ of the convolution $j_\varepsilon * f$ we denote by $f_\varepsilon$

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Pf: <u>Norm</u>: Use Thm 3.13 with $r = p$. Then $q = 1$ then, and so $f_\varepsilon \in L^p(\mathbb{R}^d)$ and $\|f_\varepsilon\|_p \leq \|f\|_p \|j_\varepsilon\|_1 = \|f\|_p$.

<u>Support</u>: By assumption, $\exists K \subseteq \Omega$ compact with $f|_{\Omega\setminus K} = 0$.
Then (by T10(ii)) $\delta := \operatorname{dist}(K, \partial\Omega) > 0$ (since $K$ compact & $\partial\Omega$ closed)
By def. of the mollifier, $\exists \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$:

supp $j_\varepsilon \subseteq B_{\delta/2}(0)$. From (*) we conclude that

$$\operatorname{supp} f_\varepsilon \subseteq \bigcup_{y\in K} B_{\delta/2}(y) \subseteq \Omega.$$

In particular, supp $f_\varepsilon$ is bounded (!) and (since closed by def.), also compact.

Derivatives: Let $e\in\mathbb{R}^d$, $|e|=1$, be a unit vector, and $\lambda\in\mathbb{R}\setminus\{0\}$.

Then, $\forall x\in\mathbb{R}^d$,

$$F_\lambda(x) := \frac{1}{\lambda}\big(f_\varepsilon(x+\lambda e) - f_\varepsilon(x)\big) = \frac{1}{\lambda}\int_{\mathbb{R}^d}\big(j_\varepsilon(x-y+\lambda e) - j_\varepsilon(x-y)\big) f(y)\,dy$$

$$\overset{HDI}{=} \frac{1}{\lambda}\int_{\mathbb{R}^d}\Big(\underbrace{\int_0^1 \frac{d}{dt}\big[j_\varepsilon(x-y+t\lambda e)\big]dt}_{\lambda e\cdot(\nabla j_\varepsilon)(x-y+t\lambda e)}\Big) f(y)\,dy$$

$$= \int_{\mathbb{R}^d}\underbrace{\Big(\int_0^1 (e\cdot\nabla j_\varepsilon)(x-y+t\lambda e)\,dt\Big) f(y)}_{=:D}dy$$

Since $|D| \le \| |\nabla j_\varepsilon| \|_\infty < \infty$ and $f\in L^1$ (because $f\in L^p$ & of compact support), using Dominated Convergence (twice, in $dt$ & $dy$ integral) gives that

$$e\cdot(\nabla f_\varepsilon)(x) = \lim_{\lambda\to 0} F_\lambda(x) = \int_{\mathbb{R}^d} e\cdot(\nabla j_\varepsilon)(x-y) f(y)\,dy = \big((e\cdot\nabla j_\varepsilon) * f\big)(x)$$

exists $\forall x\in\mathbb{R}^d$. The existence of higher derivatives follows by induction in the same way. ☒

**Lemma 3.16** Let $p\in[1,\infty)$, let $A\in\mathcal{B}(\mathbb{R}^d)$ be bounded.

Then

$$\lim_{\varepsilon\downarrow 0} \|\mathbb{1}_A - j_\varepsilon * \mathbb{1}_A\|_{L^p(\mathbb{R}^d)} = 0.$$

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Proof: Step 1: It suffices to prove the lemma for $p=1$, since:

Let $g_\varepsilon := \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \in L^\infty_c(\mathbb{R}^d)$; then $g_\varepsilon \in L^p(\mathbb{R}^d)$ $\forall p\in[1,\infty)$ and

$$\|g_\varepsilon\|_p^p \le \underbrace{(\|g_\varepsilon\|_\infty)}_{\le 2}{}^{p-1}\cdot\|g_\varepsilon\|_1.$$

Step 2: The case $p=1$. Since Lebesgue measure is (outer) regular (see Handout):

$$\forall \delta>0\ \exists B \supseteq A,\ B \text{ open and } \int_{B\setminus A} dx < \delta.$$

So, fix $\delta>0$, and let $B$ be as above (so, also bounded).

Note that

$$\| \mathbb{1}_A - j_\varepsilon * \mathbb{1}_A \|_1 \le \| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 + \underbrace{\| \mathbb{1}_{B\setminus A} \|_1}_{<\delta} + \| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1$$

with $\| j_\varepsilon * \mathbb{1}_{B\setminus A} \|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_\varepsilon(x-y)\, \mathbb{1}_{B\setminus A}(y)\, dy\, dx$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \underbrace{\Big( \int_{\mathbb{R}^d} j_\varepsilon(x-y)\, dx \Big)}_{=1\ \forall y} \mathbb{1}_{B\setminus A}(y)\, dy < \delta.$$

Hence, it suffices to prove that $\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 \to 0$, $\varepsilon \downarrow 0$.

We have $0 \le j_\varepsilon * \mathbb{1}_B \le 1$, hence

$$\| \mathbb{1}_B - j_\varepsilon * \mathbb{1}_B \|_1 = \int_B \big(1 - (j_\varepsilon * \mathbb{1}_B)(x)\big)\, dx + \int_{\mathbb{R}^d \setminus B} (j_\varepsilon * \mathbb{1}_B)(x)\, dx$$

$$= 2|B| - 2 \int_B \int_{\mathbb{R}^d} j(z)\, \mathbb{1}_B(x - \varepsilon z)\, dz\, dx.$$

Now, since $B$ is open: $\forall x \in B\ \forall z \in \mathbb{R}^d: \mathbb{1}_B(x-\varepsilon z) \to 1$, $\varepsilon \downarrow 0$ (think!)

Also, $B$ is bounded and $j \in L^1(\mathbb{R}^d)$, so, using Lebesgue's Thm. on Dominated Convergence twice

$$\lim_{\varepsilon \downarrow 0} \int_B \Big( \int_{\mathbb{R}^d} j(z)\, \mathbb{1}_B(x-\varepsilon z)\, dz \Big) dx = \int_B \Big( \int_{\mathbb{R}^d} j(z)\, dz \Big) dx = |B| \quad \blacksquare$$

Proof of Thm. 3.6: (Here $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$).

Step 1: Let $\varepsilon>0$, $f \in L^p(\Omega)$. For $n \in \mathbb{N}$, define

$$K_n := \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge \tfrac{1}{n} \text{ and } |x| \le n \} \subseteq \Omega$$

Now, $\Omega$ is open, so $\Omega = \bigcup_{n\in\mathbb{N}} K_n$, and $K_n \subseteq K_{n+1}$ $\forall n \in \mathbb{N}$.

By dominated convergence, $\left( \lim_{n\to\infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_{\Omega}(x) \right)$ 

$$\lim_{n\to\infty} \|f - f\mathbb{1}_{K_n}\|_p = \lim_{n\to\infty} \|f(1-\mathbb{1}_{K_n})\|_p = 0.$$

Hence: $\exists n \in \mathbb{N}: \|f - f\mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$.

Step 2: Recall (Handout & exercise) that step functions are dense in $L^p$ : $\exists L \in \mathbb{N}, c_1, \dots, c_L \in \mathbb{K}$ and $A_1, \dots, A_L \in \mathcal{B}(\mathbb{R}^d)$ with $A_\ell \subseteq K_n$ (!) $\forall \ell = 1, \dots, L$ such that: For $g := \sum_{\ell=1}^{L} c_\ell \mathbb{1}_{A_\ell}$ we have

$$g \in L^p_c(\Omega) \quad \text{and} \quad \|g - f\mathbb{1}_{K_n}\|_p < \frac{\varepsilon}{3}$$

Step 3: Mollify step function: $\exists \delta > 0$ s.t.

(i) $j_\delta * g \in C_c^\infty(\Omega)$ (by Lemma 3.15)

(ii) $\|g - j_\delta * g\|_p \leq \sum_{\ell=1}^{L} |c_\ell| \cdot \|\mathbb{1}_{A_\ell} - j_\delta * \mathbb{1}_{A_\ell}\|_p < \frac{\varepsilon}{3}$ (by Lemma 3.16)

Hence,

$$\|f - \underbrace{j_\delta * g}_{\in C_c^\infty(\Omega)}\|_p < \varepsilon \qquad \boxtimes$$