Chapter 3: L^p-spaces

3.1 Completeness and dual space

Notation: In this section: $(X, \mathcal{A}, \mu)$ a measure space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,
$f : X \to \mathbb{K}$ measurable (see also Handout!)

Definition 3.1 (L^p-spaces) Let $p \in (0, \infty)$.

(a)
$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} \quad \text{(possibly } \infty\text{)}$$

$$\|f\|_\infty := \inf \{ \alpha > 0 \mid \mu(\{x \in X \mid |f(x)| > \alpha\}) = 0 \}$$
$$= \inf_{\substack{N \in \mathcal{A} \\ \mu(N) = 0}} \sup_{x \in X \setminus N} |f(x)| =: \operatorname*{ess\,sup}_{x \in X} |f(x)| \quad (\mu\text{-}) \text{ essential supremum}$$

(b) Vectorspace of p-integrable functions (wrt. μ):

$$\mathcal{L}^p := \mathcal{L}^p(\mu) := \mathcal{L}^p(X) := \mathcal{L}^p(X, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) :=$$
$$:= \{ f : X \to \mathbb{K} \mid f \text{ measurable and } \|f\|_p < \infty \} = \{ f : X \to \mathbb{K} \mid |f|^p \ \mu\text{-integrable} \}$$

(c) Equivalence relation $\sim$ on $\mathcal{L}^p$:
$$f \sim g \ :\Leftrightarrow \ f = g \ \mu\text{-a.e.}$$
Vectorspace of equivalence classes of p-integrable fct.'s wrt μ:
$$L^p := \mathcal{L}^p / \sim .$$

Warning Notation does not distinguish between equivalence classes of functions, and their representatives!

Lemma 3.2 (Hölder & Minkowski).

(a) $\forall r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$: $\forall f, g : X \to \mathbb{K}$ meas.

$$\|fg\|_r \le \|f\|_p \cdot \|g\|_q \quad \text{(generalised Hölder's ineq.)}$$

(b) Let $(\mathbb{X}_i, \mathcal{A}_i, \mu_i)$, $i=1,2$, be σ-finite measure spaces, and $f: \mathbb{X}_1 \times \mathbb{X}_2 \to \mathbb{C}$ measurable. Then, for $p \in [1,\infty)$,

$$(*)\quad \left\{ \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

That is, (Minkowski's Integral Inequality)

$$\left\| \int_{\mathbb{X}_1} f(x,\cdot)\, d\mu_1(x) \right\|_p \le \int_{\mathbb{X}_1} \| f(x,\cdot) \|_p \, d\mu_1(x)$$

(c) $\forall p \in [1,\infty]$:

$$(**)\qquad \|f+g\|_p \le \|f\|_p + \|g\|_p \quad \text{(Minkowski's inequality)}$$

Pf: (a) Ana III / Handout / see exercises.

(b) Note that

$$\left\{ \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \le \left\{ \int_{\mathbb{X}_2} \left(\int_{\mathbb{X}_1} |f(x,y)|\, d\mu_1(x) \right)^p d\mu_2(y) \right\}^{1/p},$$

hence, it suffices to consider $f \ge 0$.

Next, if $p=1$, $(*)$ follows from Tonelli's Thm.

If $1<p<\infty$, then

$$\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \overset{p=(p-1)+1}{=} \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right|^{p-1} \cdot \left| \int_{\mathbb{X}_1} f(x,y)\, d\mu_1(x) \right| d\mu_2(x)$$

$$\le \int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \left[\int_{\mathbb{X}_1} |f(x,y)|\, d\mu_1(x) \right] d\mu_2(y)$$

$$= \int_{\mathbb{X}_2} \left(\int_{\mathbb{X}_1} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_1(x) \right) d\mu_2(y) \quad \left(\text{linearity } \int_{\mathbb{X}_1} d\mu_1(x)\right)$$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{X}_1} \left(\int_{\mathbb{X}_2} \left| \int_{\mathbb{X}_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_2(y) \right) d\mu_1(x) \qquad (\square)$$

Now, with $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $q = \frac{p}{p-1}$, Hölder gives

$$\int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^{p-1} \cdot |f(x,y)|\, d\mu_2(x)$$

$$\leq \left(\int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^{q(p-1)} d\mu_2(y) \right)^{1/q} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p}$$

$$= \left(\int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p}$$

Inserting this in ($\square$) gives

$$\int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y)$$

$$\leq \int_{X_1} \underbrace{\left(\int_{X_2} \left| \int_{X_1} f(t,y)\, d\mu_1(t) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{indep. of } x} \cdot \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

$$= \underbrace{\left(\int_{X_2} \left| \int_{X_1} f(\cancel{t},y)\, d\mu_1(\cancel{t}) \right|^p d\mu_2(y) \right)^{\frac{p-1}{p}}}_{\text{divide (!)} :\ 1 - \frac{p-1}{p} = \frac{1}{p}} \cdot \int_{X_2} \left(\int_{X_1} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

$$\Rightarrow \left\{ \int_{X_2} \left| \int_{X_1} f(x,y)\, d\mu_1(x) \right|^p d\mu_2(y) \right\}^{1/p} \leq \int_{X_1} \left(\int_{X_2} |f(x,y)|^p d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

To divide: we need integral to be finite; is the case if $\mu_i(X_i) < \infty$, $i = 1,2$ and f bounded. For general case: Let $X_i^k \nearrow X_i$, $k \to \infty$, $\mu_i(X_i^k) < \infty$ (X_i σ-finite), consider the above for

$$\tilde{f}_k(x,y) := \mathbb{1}_{X_1^k}(x)\, \mathbb{1}_{X_2^k}(y)\, \mathbb{1}_{\{f \leq k\}} \cdot f(x,y) \quad \& \text{ take limit } k \to \infty$$

(c) Take $X_1 = \{1,2\}$, μ_1 = counting measure, $\int_{X_1} \cdots d\mu_1(x) = \sum_{i=1}^{2}$,

then (*) becomes, with $f(x,y) = f_i(y)$,

(**) $\quad \|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p$, $f_i \in L^p(X_2, \mu_2)$ ∎

Remark (a) Take $X_2 := \mathbb{N}$, μ_2 = counting measure, $\int_{X_2} d\mu_2(y) = \sum_{i=1}^{\infty}$,

then (**) becomes

(***) $\quad \|x+y\|_p \leq \|x\|_p + \|y\|_p$, $x,y \in \ell^p$

(b) Take $X_2 := \{1,\dots,N\}$, $\mu_2 =$ counting measure, $\int_{X_2} d\mu_2 = \sum_{i=1}^{N}$

then (**) becomes

$$(****) \qquad \|x+y\|_p \le \|x\|_p + \|y\|_p \,, \quad x,y \in \mathbb{K}^N$$

(c) The proof above is "the same" as the standard proof of (**), (***), or (****) (check!).

Theorem 3.3 (Riesz–Fischer) Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $(L^p, \|\cdot\|_p)$ is a Banach space for all $p \in [1,\infty]$.
Moreover, L^2 is a Hilbert space with scalar product

$$\langle f, g\rangle := \int_X \overline{f(x)}\, g(x)\, d\mu(x) \qquad \forall f,g \in L^2$$

Pf: Ana III. (triangle inequality needs $p \ge 1$!)

Remark (a) Take $X = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, $\mu =$ counting measure,
$\int_X d\mu = \sum_{i=1}^{\infty}$, then 3.3 says: $(\ell^p, \|\cdot\|_p)$ is Banach.

(b) Take $X = \{1,\dots,N\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu =$ counting measure, $\int_X d\mu = \sum_{i=1}^{N}$,
then 3.3 says: $(\mathbb{K}^N, \|\cdot\|_p)$ is Banach.

Lemma 3.4 If $\mu(X) < \infty$, $1 \le r \le p \le \infty$, then L^p $(\subseteq L^r)$ is dense in L^r (wrt. $\|\cdot\|_r$), and

$$\|f\|_r \le [\mu(X)]^{\frac{1}{r}-\frac{1}{p}} \|f\|_p \qquad \forall f \in L^p.$$

Pf: Ana III.

The next theorem is about the geometry of L^p, but also serves as preparation to compute its dual:

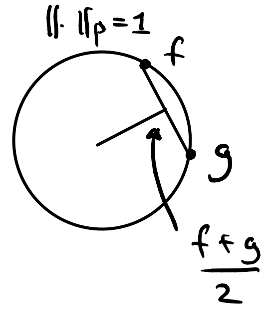
Theorem 3.5 Let $p \in (1,\infty)$. Then:

For all $\varepsilon > 0$, $r > 0$ there exists $\delta > 0$ s.t.:

For all $f, g \in L^p$:

$\|f\|_p, \|g\|_p \leq r$ and $\|f-g\|_p \geq \varepsilon$

$$\Rightarrow \left\| \frac{f+g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2} - \delta.$$



In particular, L^p is uniformly convex (case (see drawing) $r = \|f\|_p = \|g\|_p = 1$)

Pf (after Naoki Shioji)

Define the map

$$\alpha: \mathbb{K}^2 \to \mathbb{R}, \quad z = (z_1, z_2) \mapsto \frac{|z_1|^p + |z_2|^p}{2} - \left| \frac{z_1 + z_2}{2} \right|^p$$

Claim: $\forall z \in \mathbb{K}^2$: $\alpha(z) \geq 0$, with "=" iff $z_1 = z_2$ $(*)$

PfClaim: $\left| \frac{z_1+z_2}{2} \right|^p \underset{\uparrow}{\leq} \left(\frac{|z_1|+|z_2|}{2} \right)^p \underset{\uparrow}{\leq} \frac{|z_1|^p + |z_2|^p}{2}$

Δ-ineq. $\mathbb{K}$ $\quad$ $(\cdot)^p$ strictly convex $(p>1)$

Δ-ineq. is equality $\Leftrightarrow$ $z_1 = 0$ or $z_2 = \lambda z_1$ for some $\lambda > 0$

Strict conv. ineq. is equality $\Leftrightarrow$ $|z_1| = |z_2|$

$\Rightarrow$ (equality $\Leftrightarrow$ $z_1 = z_2$) $\checkmark$ claim

Now, fix $\varepsilon, r > 0$, and choose $\eta \in \left(0, \frac{\varepsilon^p}{2r^p}\right)$ small enough (f.ex. $< 2^{-p}$) such that

$$D := \{ z \in \mathbb{K}^2 \mid |z_1|^p + |z_2|^p = 1 \text{ and } |z_1 - z_2|^p \geq \eta \} \neq \emptyset$$

Then, by compactness (!) of D and continuity of α,

$$\theta := \inf \{ \alpha(z) \mid z \in D \} = \min \{ \alpha(z) \mid z \in D \} \underset{\uparrow \text{ by } (*)}{>} 0$$

Claim: $\quad |z_1|^p + |z_2|^p \leq \frac{\alpha(z)}{\theta}$ $\quad (**)$

$$\forall z \in F := \{ \tilde{z} \in \mathbb{K}^2 \mid |\tilde{z}_1 - \tilde{z}_2|^p \geq \eta (|\tilde{z}_1|^p + |\tilde{z}_2|^p) \}$$

Pf claim: W.l.o.g., $z \neq (0,0)$ ((∗∗) holds for $(0,0)$)

Let $z \in F$, define $t := \left(|z_1|^p + |z_2|^p\right)^{1/p} > 0$ (since $z \neq 0$)

Note: (i) $\left|\frac{z_1}{t}\right|^p + \left|\frac{z_2}{t}\right|^p = 1$ (compute!)

(ii) $\left|\frac{z_1}{t} - \frac{z_2}{t}\right|^p \geq \eta$ (since $z \in F$)

so $\frac{z}{t} \in D$. Hence, by def. of θ, we have $\theta \leq \alpha\left(\frac{z}{t}\right) = \frac{\alpha(z)}{t^p}$ ✓ (claim)

Now, let $f, g \in L^p$ with $\|f\|_p, \|g\|_p \leq r$ and $\|f-g\|_p \geq \varepsilon$.

Define $M := \{x \in X \mid (f(x), g(x)) \in F\}$ $(X = M \cup M^c)$

and estimate:

$$\varepsilon^p \leq \int_X |f-g|^p \, d\mu = \int_{M^c} |f-g|^p \, d\mu + 2^p \int_M \left|\frac{f-g}{2}\right|^p d\mu =: I_1 + 2^p I_2 \quad (∗∗∗)$$

By definition of M^c (& F)

$$I_1 \leq \eta \int_{M^c} (|f|^p + |g|^p) \, d\mu \overset{M^c \subseteq X}{\leq} \eta \left(\|f\|_p^p + \|g\|_p^p\right) \leq 2\eta r^p$$

Since $\alpha \geq 0$, we also get

$$I_2 \leq \frac{1}{2} \int_M \left(|f|^p + |g|^p\right) d\mu \overset{(∗∗)}{\leq} \frac{1}{2\theta} \int_M \alpha(f(x), g(x)) \, d\mu(x)$$

$$\overset{M \subseteq X}{\leq} \frac{1}{2\theta} \int_X \left(\frac{|f|^p + |g|^p}{2} - \left|\frac{f+g}{2}\right|^p\right) d\mu.$$

Inserting the inequalities for I_1 & I_2 in (∗∗∗) gives

$$\left\|\frac{f+g}{2}\right\|_p^p \leq \underbrace{\frac{\theta(2\eta r^p - \varepsilon^p)}{2^{p-1}}}_{=: -\delta} + \frac{\|f\|_p^p + \|g\|_p^p}{2}$$

with $\delta > 0$ by the definition of η ⊠

Theorem 3.6 (Riesz' Representation Thm. for $(L^p)^*$)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [1,\infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$, assume in addition that μ is σ-finite. Then the map

$$J : L^q \to (L^p)^*, \quad f \mapsto \ell_f \quad , \quad \ell_f : L^p \to \mathbb{K}, \quad g \mapsto \int_X fg \, d\mu \qquad (*)$$

is an isometric isomorphism. Hence:

$$L^q \simeq (L^p)^*, \quad \frac{1}{p} + \frac{1}{q} = 1, \; p \in [1,\infty)$$

Remark 3.7 (a) The map J is also well-defined and isometric for $p = \infty$, but fails to be surjective if $\mathrm{supp}(\mu)$ has infinitely many elements (see exercise). I.e., $(L^\infty)^*$ is strictly larger than L^1 in this case (compare 2.38 and 2.39(b))

(b) The case $p = 1$ in 3.6 would be wrong without additional hypothesis on the measure space. σ-finiteness of μ is sufficient, but not necessary, because of:

Thm The map $J : L^\infty \to (L^1)^*$ is

(a) Injective $\Leftrightarrow$ isometric $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is semi-finite

(b) Bijective $\Leftrightarrow$ isometric & bijective $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ is localisable

Note: $(X, \mathcal{A}, \mu)$ σ-finite $\Rightarrow$ localisable $\Rightarrow$ semi-finite

(For required def.'s & pf., see Salamon, "Measure and Integration", EMS, (2016) Sect. 4.5)

Pf (of 3.6): By linearity of the integral, the map J is linear.

Let $f \in L^q$. Then ℓ_f (see $(*)$) is linear (in g), $fg \in L^1 \; \forall g \in L^p$ by Hölder, and $|\ell_f(g)| \leq \|f\|_q \|g\|_p$. Hence, $\ell_f \in (L^p)^*$, and $$\|\ell_f\|_{(L^p)^*} \le \|f\|_q. \quad (**)$$

Hence, J in (*) is well-defined, linear, and bounded.

Claim: $\|\ell_f\|_{(L^p)^*} \overset{(***)}{=} \|f\|_q$ ($\Rightarrow J$ is isometric, hence inj.)

Pf claim: Wlog, $f \neq 0$ ((***) clear for $f=0$).

(i) Case $p \in (1,\infty)$: Let $$g(x) := \begin{cases} \bar{f}(x)|f(x)|^{q-2}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

Then $$|g|^p = |f|^{(q-1)p} = |f|^q \in L^1 \quad \text{since } (q-1)p = q(1-\tfrac{1}{q})p = q$$
$(\frac{1}{p}+\frac{1}{q}=1)$.

So $g \in L^p$, and $\|g\|_p = \|f\|_q^{q/p}$. Also
$$\frac{\ell_f(g)}{\|g\|_p} = \frac{\int |f|^q}{\|g\|_p} = \frac{\|f\|_q^q}{\|f\|_q^{q/p}} = \|f\|_q^{q(1-\frac{1}{p})} = \|f\|_q, \text{ hence}$$
$\|\ell_f\|_{(L^p)^*} \ge \|f\|_q$. This, and (**), imply (***) (the claim).

(ii) Case $p=1$: Since μ is σ-finite, we have:
$$X = \bigcup_{n \in \mathbb{N}} X_n, \quad X_n \subseteq X_{n+1}, \quad \mu(X_n) < \infty \quad \forall n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and, for $f \in L^\infty$, define
$$M := \{x \in X \mid |f(x)| \ge \|f\|_\infty - \varepsilon\}.$$

By definition of "ess sup", $\mu(M) > 0$. Then $M_n := M \cap X_n$ satisfies $0 < \mu(M_n) < \infty$ for (all) n sufficiently large.

Let $$g(x) := \begin{cases} \frac{\bar{f}(x)}{|f(x)|} \frac{1}{\mu(M_n)} \mathbb{1}_{M_n}(x), & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

Then $g \in L^1$ with $\|g\|_1 \le 1$, and
$$\ell_f(g) = \int_X fg \, d\mu = \frac{1}{\mu(M_n)} \int_{M_n} |f| \, d\mu \ge \|f\|_\infty - \varepsilon.$$ As this holds $\forall \varepsilon > 0$, we get $\|\ell_f\|_{(L^1)^*} \ge \|f\|_\infty$. This, and (**), imply the claim (***).

It remains to prove that J is surjective.

Let $\xi \in (L^p)^*$. Need to prove: $\exists f_\xi \in L^q$ s.t. $\xi = \ell_{f_\xi}$ (see (*1))

Case $p \in (1,\infty)$: (We follow again N. Shioji)

Define the non-linear functional $F : L^p \to \mathbb{R}$

$$g \mapsto F(g) := \frac{1}{p}\|g\|_p^p - \operatorname{Re}\xi(g)$$

Since $p > 1$ and

$$F(g) \geq \frac{1}{p}\|g\|_p^p - \|\xi\|_* \|g\|_p ,$$

F is bounded from below:

$$m := \inf\{F(g) \mid g \in L^p\} > -\infty$$

Claim: $\exists!\, f \in L^p : F(f) = m$ (i.e. exists unique minimiser of F)

[This is where we need/use uniform convexity!]

Pf claim: For $n \in \mathbb{N}$, let $C_n := \{g \in L^p \mid F(g) \leq m + \frac{1}{n}\}$.

Then: (i) $C_n \neq \emptyset$ $\forall n \in \mathbb{N}$ (by definition of the infimum)

(ii) C_n closed $\forall n \in \mathbb{N}$ (since F cont. and $(-\infty, m + \frac{1}{n}]$ closed)

(iii) $C_n \supseteq C_{n+1}$ $\forall n \in \mathbb{N}$

(iv) $\operatorname{diam}(C_n) \xrightarrow{n\to\infty} 0$ (proof below)

From (i)-(iv) & completeness of L^p follows (see exercise!)

that $\displaystyle\bigcap_{n\in\mathbb{N}} C_n = \{f\}$ for some (unique) $f \in L^p$.

This implies the claim, since $\displaystyle\bigcap_{n\in\mathbb{N}} C_n = \{\varphi \in L^p \mid F(\varphi) = m\}$.

It remains to do the

Proof of (iv): Note that (a) $C_1 \neq \emptyset$, (b) $\forall g \in C_1$: $\frac{1}{p}\|g\|_p^p - \|\xi\|_* \|g\|_p \leq m+1$

and (c) $p > 1$ – so $r := \sup\{\|g\|_p \mid g \in C_1\} \in (0,\infty)$

Now let $\varepsilon > 0$, and let $g, h \in C_1$ (so $\|g\|_p, \|h\|_p \leq r$)

be arbitrary but fixed.

Assume (for contradiction!): $\|g-h\|_p \geq \varepsilon$ $(\square)$

Then, by Theorem 3.5, $\exists \delta > 0$ (indep. of g, h) s.t.

$$\left\| \frac{g+h}{2} \right\|_p^p \leq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta. \qquad (\square\square)$$

On the other hand: $\exists n_0 \in \mathbb{N}: \forall n \geq n_0: \frac{p}{n} < \delta$.

Now restrict choice of g, h to: $g, h \in C_n$, $n \geq n_0$.

Then

$$m \overset{\text{def of } m}{\leq} F\left(\frac{g+h}{2}\right) = \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2}\operatorname{Re}\xi(g) - \frac{1}{2}\operatorname{Re}\xi(h)$$

$$= \frac{1}{p}\left\| \frac{g+h}{2} \right\|_p^p - \frac{1}{2p}\left(\|g\|_p^p + \|h\|_p^p\right) + \underbrace{\frac{1}{2}\left(F(g)+F(h)\right)}_{\leq m + \frac{1}{n} \text{ (by def of } C_n)}$$

Hence,

$$\left\| \frac{g+h}{2} \right\|_p^p \geq \frac{\|g\|_p^p + \|h\|_p^p}{2} - \underbrace{\frac{p}{n}}_{< \delta} > \frac{\|g\|_p^p + \|h\|_p^p}{2} - \delta.$$

This contradicts $(\square\square)$, hence $(\square)$ is wrong, so:

$\forall g, h \in C_n$ $\forall n \geq n_0$: $\|g-h\|_p < \varepsilon$. This proves (iv), hence the claim ✓

So, let $f \in L^p$ be the unique minimiser from the claim, i.e. $F(f) = m$, and let $g \in L^p$ be arbitrary.

Define

$$\sigma_g: \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sigma_g(t) := F(f+tg) = \frac{1}{p}\int_X |f+tg|^p \, d\mu - t\operatorname{Re}\xi(g) - \operatorname{Re}\xi(f) \end{array}$$

Goal: Compute $\sigma_g'(0)$.

Lebesgue's Thm. on differentiability of parameter-dependent integrals (see Handout!) applies, since, for $I \subseteq \mathbb{R}$ open & bounded, with $0 \in I$, we have:

(i) $|f+tg|^p \in L^1$ $\forall t \in I$

(ii) $\varphi(t,\cdot) := \frac{\partial}{\partial t} |f+tg|^p \in L^1 \quad \forall t \in I$, since:

$$\varphi(t,\cdot) = \frac{\partial}{\partial t} \left[(f+tg)\overline{(f+tg)} \right]^{p/2}$$
$$= \frac{p}{2} \left(|f+tg| \right)^{\frac{p}{2}-1} \left[g\overline{(f+tg)} + (f+tg)\bar{g} \right]$$
$$= p\, |f+tg|^{p-2} \, \mathrm{Re}\left[g\overline{(f+tg)} \right]$$

(in the case $p \in (1,2)$ with the convention that
$\varphi(t,x) := 0$ if $|f(x) + tg(x)| = 0$)

(iii) $\exists\, \Phi \in L^1 : |\varphi(t,\cdot)| \le \Phi \quad \forall t \in I$.

Hence, since f minimises F (so σ_g has minimum at $t=0$),

$$0 = \sigma_g'(0) = \int_X |f|^{p-2} \mathrm{Re}(\bar{f} g)\, d\mu - \mathrm{Re}\, \xi(g)$$

$$\Rightarrow \quad \mathrm{Re}\, \xi(g) = \mathrm{Re}\Big(\int_X f_\xi\, g\, d\mu \Big), \quad f_\xi := |f|^{p-2} \bar{f} \in L^q \text{ (check!)}$$

If $\mathbb{K} = \mathbb{R}$, the claim follows

If $\mathbb{K} = \mathbb{C}$, consider also $\tau_g : \mathbb{R} \ni t \mapsto \tau_g(t) := F(f - itg)$

By similar arguments, we get

$$0 = \tau_g'(0) = \int_X |f|^{p-2} \mathrm{Im}(\bar{f} g)\, d\mu - \mathrm{Im}(\xi g)$$

$$\Rightarrow \quad \mathrm{Im}\, \xi(g) = \mathrm{Im}\Big(\int_X f_\xi\, g\, d\mu \Big) \quad \checkmark. \quad (p \in (1,\infty) \text{ done})$$

Case $p=1$: Idea: Step 1: Reduce to the above case $p \in (1,\infty)$, if μ is finite. Step 2: Generalise to σ-finite μ.

Step 1: Assume $\mu(X) < \infty$. Let $r > 1$. Recall from Lemma 3.4:

$L^\infty \subseteq L^r \subseteq L^1$ (the inclusion being dense).

Then:

$$|\xi(g)| \le \|\xi\|_{(L^1)^*} \|g\|_1 \le \underbrace{\|\xi\|_{(L^1)^*} [\mu(X)]^{1-\frac{1}{r}}}_{=: \kappa_r} \|g\|_r \qquad \forall g \in L^r$$

Hence, $\xi_r := \xi|_{L^r} \in (L^r)^*$. From the case $p \in (1,\infty)$ proved above (on r) we get: $\exists! f_r \in L^q$ $(\frac{1}{r}+\frac{1}{q}=1)$: $\xi_r = \ell_{f_r}$ (see (*1)), and

$\|f_r\|_q = \|\xi_r\|_{(L^r)^*} \leq M_r$. Hence: $\forall r_1, r_2 > 1$, $\forall h \in L^\infty$ $(\subseteq L^{r_i}, i=1,2)$:

$$\ell_{f_{r_1}}(h) = \xi_{r_1}(h) = \xi(h) = \xi_{r_2}(h) = \ell_{f_{r_2}}(h) \quad \text{i.e.} \int_X (f_{r_1} - f_{r_2}) h \, d\mu = 0$$

Choose $h := \frac{\overline{(f_{r_1} - f_{r_2})}}{|f_{r_1} - f_{r_2}|} \in L^\infty$, then $f_{r_1} = f_{r_2} =: f$ is independent of the chosen/used r. Hence, $f \in \bigcap_{q \in (1,\infty)} L^q$ and

$$\limsup_{q\to\infty} \underbrace{\|f\|_q}_{\leq M_r} \leq \limsup_{q\to\infty} \left\{ \|\xi\|_{(L^1)^*} [\mu(X)]^{1/q} \right\} = \|\xi\|_{(L^1)^*} < \infty$$

($1-\frac{1}{r}=\frac{1}{q}$)

Then (see exercise!) $f \in L^\infty$, and so $\ell_f \in (L^1)^*$ (see (*))

Since L^∞ dense in L^1, we get: ξ and ℓ_f agree on a dense subspace, so (by Thm. 2.31) $\xi = \ell_f$ on L^1.

Step 2: Generalise to μ being σ-finite. See exercise ∎

Remark (a) Compare with Thm 2.38 (for ℓ_p). Recall that

$\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with μ counting measure.

(b) Compare with Thm. 2.57 & 2.58 (for Hilb. space) Recall that L^2 is a Hilbert space, with $\langle f, g \rangle := \int_X \bar{f} g \, d\mu$. Note the complex conjugate (also in 2.57 & 2.58) $(\frac{1}{2} + \frac{1}{2} = 1)$. (Recall also 2.54 & see next section)

(c) Another proof uses Radon-Nikodym Thm. to prove surjectivity of the map J; see Werner, "Funktionalanalysis", 8. Aufl., Satz II.2.4 p. 65.

(d) A more abstract proof uses the uniform convexity & Milman-Pettis's Thm, to conclude L^p, $p \in (1,\infty)$, is reflexive, to identify $(L^p)^*$; see 4.17 & 4.18 later.