Chapter 2: Banach and Hilbert Spaces

2.1 Vector spaces

General assumption: $X \neq \{0\}$ is a $\mathbb{K}$-vectorspace, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1 Let $\emptyset \neq M \subseteq X$.

(i) M is *linearly independent* iff all non-empty finite (!) subsets $F \subseteq M$ are linearly independent, i.e., the following implication holds:

$$\sum_{f \in F} \underbrace{\alpha_f}_{\in \mathbb{K}} f = 0 \quad \Rightarrow \quad \alpha_f = 0 \quad \forall f \in F$$

(ii) M is *linearly dependent* iff M is not linearly independent.

(iii) $B \subseteq X$ is a *Hamel basis* (or *algebraic basis*) iff

(1) B is linearly independent

(2) Every $x \in X$ can be represented as a (finite!) linear combination of elements in B (B *spans* X).

(iv) X has *finite dimension* iff there exists a Hamel basis with $|B| < \infty$. Then $\dim X := |B|$ is called the *dimension of X*

(v) X has *infinite dimension* iff X does not have finite dimension

Remark 2.2 The dimension is well-defined: $|B|$ is the same for every Hamel basis in a given space (Pf: see linear algebra (LA)).

Example 2.3 (a) Consider

$$c_c := \{ x = (x_j)_{j\in\mathbb{N}} \mid x_j \in \mathbb{C}\ \forall j \in \mathbb{N}, \text{ and } x_j \neq 0 \text{ for only finitely many } j\text{'s} \}$$

(see also exercise; the index 'c' stands for "compact support"; also: c_{00})

Let $e_n := (\dots, 0, 1, 0, \dots)$ with a 1 at the n'th position.

Claim: $B := \{ e_n \mid n \in \mathbb{N} \}$ is a Hamel basis for c_c

(b) Even though ℓ^1 is separable, there exists no countable Hamel basis for ℓ^1 (see exercise).

Theorem 2.4 Every vector space $X \neq \{0\}$ has a Hamel basis

Pf.: Uses Zorn's Lemma; see later.

Corollary 2.5 X has infinite dimension iff for every $n \in \mathbb{N}$ there exists $M_n \subseteq X$ such that $|M_n| = n$ and M_n is linearly indep.

Pf: Existence of Hamel basis with $|B| = \infty$ ∎

Example 2.6 Infinite dimensional vector spaces:

c_c, ℓ^p, $C(X)$ (where $\emptyset \neq X \subseteq \mathbb{R}^d$ open)

2.2. Banach Spaces

Definition 2.7 Let X be a vectorspace. A map $X \to [0,\infty)$, $x \mapsto \|x\|$ is a norm :$\Leftrightarrow$

(1) $\|x\| > 0 \quad \forall\, 0 \neq x \in X$

(2) $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{K}\ \forall x \in X$ $\qquad (\Rightarrow \|x\| = 0)$

(3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

$(X, \|\cdot\|)$ is called a normed space.

If only (2) and (3) hold, $\|\cdot\|$ is called a semi-norm

Remark 2.8 Let X be a normed space. Then $d(x,y) := \|x-y\|$ is a metric on X. Thus all topological notions and results from the theory of metric spaces are available.

A base of the norm topology on X:

$$\{ B_{\frac{1}{k}}(x) \mid x \in X, k \in \mathbb{N} \} = \{ x + B_{\frac{1}{k}}(0) \mid x \in X, k \in \mathbb{N} \}$$

[Minkowski sum of sets A, B: $A + B := \{ a+b \mid a \in A, b \in B \}$

and $a + B := \{a\} + B$]

Warning: Not every metric (on vector spaces) comes from a norm.

Example 2.9 (a) ℓ^p is a normed space with $\|\cdot\| = \|\cdot\|_p$ $\forall p \in [1,\infty]$

(b) $C(X; \mathbb{K})$ (where X is a compact Hausdorff space) is a normed space with $\|f\| := \|f\|_\infty := \sup_{x \in X} |f(x)|$

(c) $\mathbb{R}^d, \mathbb{C}^d$ are normed spaces for $d \in \mathbb{N}$ (w.r.t. p-norm)

Lemma 2.10 Let X be normed space. Then the following maps are continuous:

(a) Norm: $X \to [0,\infty)$, $x \mapsto \|x\|$

(b) Addition: $X \times X \to X$, $(x,y) \mapsto x+y$

(c) Scalar multiplication: $\mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$

Pf: (a) Let $(x_k)_{k\in\mathbb{N}} \subseteq X$ with $x_k \to x$, $k \to \infty$, i.e., $\|x_k - x\| \to 0$, $k \to \infty$ (see Cor. 1.8). Hence, the claim follow from reverse triangle inequality:

$$\big| \|x\| - \|y\| \big| \leq \|x+y\|$$

(b), (c) see exercise $\square$

Definition 2.11 A complete normed space $(X, \|\cdot\|)$ is called a Banach space

Examples 2.12 (a) All spaces in Ex. 2.9 are Banach spaces

(b) Let $X := C([0,1])$ with L^1-norm $\|f\|_1 := \int_0^1 |f(t)|\,dt$.
Then $(X, \|\cdot\|_1)$ is not a Banach space (recall Ex. 1.11 (b1)).

Theorem 2.13 Every normed space X can be completed, so that X is isometric to a dense linear subspace W of a Banach space $\hat{X}$, which is unique up to isometric isomorphisms.

Pf: Analogous to the proof of Theorem 1.14. Note: The isometry is even a linear bijection (hence, an isomorphism) in this case (see Section 2.3 below).

Definition 2.14 Let X be a normed space. A sequence $(e_n)_n \subseteq X$ is called a (Schauder) basis in X :$\Leftrightarrow$ For all $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} x_n e_n \Big\| = 0$$

Notation: $x = \sum_{n\in\mathbb{N}} x_n e_n$, infinite linear combination, convergent series

Note: Linear independence not required for Schauder basis

Example 2.15 Let $p \in [1,\infty)$. Then $(e_n)_{n\in\mathbb{N}}$ with $e_n := (0,\dots,0,1,0,\dots)$ (the 1 in the n'th position) is a (Schauder) basis of ℓ^p:
For $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ we have

$$\Big\| x - \sum_{n=1}^{N} x_n e_n \Big\|^p = \sum_{n=N+1}^{\infty} |x_n|^p \xrightarrow{N\to\infty} 0.$$

Note: This construction fails for ℓ^∞ !

Lemma 2.16 Let X be a normed space. Then

X has a (Schauder) basis $\Rightarrow$ X is separable

Pf: Let $\mathbb{K}_0 := \mathbb{Q}$, if $\mathbb{K} = \mathbb{R}$, resp. $\mathbb{K}_0 := \mathbb{Q} + i\mathbb{Q}$ for $\mathbb{K} = \mathbb{C}$. Define

$$A_N := \left\{ \sum_{n=1}^{N} x_n e_n \mid x_n \in \mathbb{K}_0 \text{ for } n \in \{1, .., N\} \right\}$$

Then the union $A := \bigcup_{N \in \mathbb{N}} A_N$ is dense in X, and countable ☒

Remark 2.17 The implication "$\Leftarrow$" in Lemma 2.16 does not hold (Enflo, 1973)

Remark 2.18 All norms in finite-dimensional spaces are equivalent. That is, for norms $\|\cdot\|$ and $|||\cdot|||$ on $\mathbb{K}^n$ there exists constants $c, \tilde{c} > 0$ such that

$$c\|x\| \leq |||x||| \leq \tilde{c}\|x\| \quad \forall x \in \mathbb{K}^n.$$

(see exercise)

Theorem 2.19 Let X be a normed space, and $F \subseteq X$ a finite-dim. subspace. Then F is complete and closed.

Pf: Let $n := \dim F < \infty$. Fix a basis $\{e_1, .., e_n\}$ in F. For every $x \in F$ there exists unique $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{K}^n$ s.t. $x = \sum_{j=1}^{n} \alpha_j e_j$.

Let $$|||\alpha||| := \left\| \sum_{j=1}^{n} \alpha_j e_j \right\| \quad \forall \alpha = (\alpha_1, .., \alpha_n) \in \mathbb{K}^n.$$

Then the normed spaces $(F, \|\cdot\|)$ and $(\mathbb{K}^n, |||\cdot|||)$ are isometrically isomorphic via $x \mapsto \alpha$. Now, $\mathbb{K}^n$ is closed and complete w.r.t. the Euclidean norm, and all norms in $\mathbb{K}^n$ are equivalent (by 2.18). Hence, $(\mathbb{K}^n, |||\cdot|||)$ is closed and complete, and, because of the isometry, so is $(F, \|\cdot\|)$. ☒

As a preparation for Theorem 2.21 we prove the following Lemma:

Lemma 2.20 (Riesz' Lemma, 1918) Let X be a normed space, and $U \subsetneq X$ a closed (!) subspace. Then, for all $\lambda \in (0,1)$, there exists $x_\lambda \in X \setminus U$ such that

$$\|x_\lambda\| = 1 \quad \text{and} \quad \|x_\lambda - u\| \geq \lambda \quad \forall u \in U$$

Pf: Let $x \in X \setminus U$ (open!). Then $\exists \varepsilon_x > 0$: $B_{\varepsilon_x}(x) \subseteq X \setminus U$. Hence $d := \operatorname{dist}(x, U) = \inf_{u \in U} \|x - u\| \geq \varepsilon_x > 0$ for all $x \in X \setminus U$.

Since $0 < \lambda < 1$ there exists $u_\lambda \in U$ s.t. $d \leq \|x - u_\lambda\| \leq \frac{d}{\lambda}$.

Hence, $\gamma := \frac{1}{\|x - u_\lambda\|} \geq \frac{\lambda}{d}$. Define $x_\lambda := \gamma(x - u_\lambda) \in X \setminus U$ (!)

By definition of γ, we have $\|x_\lambda\| = \gamma \cdot \|x - u_\lambda\| = 1$, and

$$\|x_\lambda - u\| = \|\gamma(x - u_\lambda) - u\| = \|\gamma x - (u + \gamma u_\lambda)\| = \gamma \cdot \|x - \underbrace{(u_\lambda - \tfrac{u}{\gamma})}_{\in U}\|$$
$$\geq \gamma d \geq \lambda \quad \forall u \in U \quad \blacksquare$$

Warning 1.26 illustrates the following general result:

Theorem 2.21 Let X be a normed space. Then

$$\overline{B_1(0)} = \{x \in X \mid \|x\| \leq 1\} \text{ compact} \iff \dim X < \infty.$$

Pf: "$\Leftarrow$": Let $n := \dim X < \infty$. From (the proof of) Thm 2.19: X is isometric to $(\mathbb{K}^n, |||\cdot|||)$. The statement follows from Heine-Borel for $(\mathbb{K}^n, |\cdot|)$ and the equivalence of norms.

"$\Rightarrow$": Assume (for contradiction) that $\dim X = \infty$. Will prove: this implies $\overline{B_1(0)}$ is not seq. compact, by constructing a sequence $(x_n)_{n \in \mathbb{N}}$ in $\overline{B_1(0)}$ with no convergent subsequences:

(i) Let $x_1 \in X$ be arbitrary, s.t. $\|x_1\| = 1$. Let $U_1 := \operatorname{span}\{x_1\} \subsetneq X$ be the closed (!) subspace spanned by x_1. Riesz' Lemma, applied with $\lambda = \frac{1}{2}$, gives existence of $x_2 \in X \setminus U_1$ s.t. $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$. Let $U_2 := \operatorname{span}\{x_1, x_2\} \subsetneq X$.

(ii) Let $U_n := \overline{\mathrm{span}\{x_1, \dots, x_n\}} \subsetneq X$ be the closed subspace of the vectors constructed before. Again, by Riesz' Lemma, there exists $x_{n+1} \in X \setminus U_n$ s.t. $\|x_{n+1}\| = 1$ and $\mathrm{dist}(x_{n+1}, U_n) \geq \frac{1}{2}$.

By assumption, $\dim X = \infty$, hence this procedure does not stop.
We get a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \overline{B_1(0)}$ s.t. $\|x_n - x_m\| \geq \frac{1}{2}$ $\forall n \neq m$.
Clearly, $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence - contradicting the (seq.) compactness of $\overline{B_1(0)}$ $\lightning$ $\blacksquare$

2.3 Linear operators

Definition 2.22 Let X, Y be vector spaces (over the same field $\mathbb{K}$), $X_0 \subseteq X$ a (linear) subspace, and $T: X_0 \to Y$.

(i) T is a (linear) operator $:\Leftrightarrow$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \alpha, \beta \in \mathbb{K}, \ \forall x, y \in X_0$$

(ii) $\mathrm{dom}(T) := D(T) := X_0$ is the domain of T.

(iii) $\mathrm{ran}(T) := R(T) := T(X_0)$ is the range of T

(iv) $\ker(T) := N(T) := \{x \in X_0 \mid Tx = 0\}$ is the kernel (or nullspace) of T

(v) $U \subseteq \mathrm{dom}(T)$ a subspace,

$$T|_U : \begin{array}{l} U \to Y \\ x \mapsto Tx \end{array}$$ is the restriction of T to U

(vi) $W \supseteq \mathrm{dom}(T)$ a vector space, $\tilde{T}: W \to Y$ linear with $\tilde{T}|_{\mathrm{dom}(T)} = T$ is called a (!) linear extension of T to W.

Examples 2.23 Some linear operators:

(a) Identity operator on vector space X:

$$\mathbb{1} := \mathbb{1}_X : \begin{array}{l} X \to X \\ x \mapsto x \end{array}$$

(b) $\mathbb{X} = \mathbb{Y} = C([0,1])$

(i) Differentiation operator ($\mathbb{X}_0 = C^1([0,1])$):

$$\frac{d}{dx} : C^1([0,1]) \to C([0,1])$$
$$f \mapsto f'$$

(ii) Anti-derivative operator ($\mathbb{X}_0 = \mathbb{X}$):

$$T : C([0,1]) \to C([0,1])$$
$$f \mapsto Tf$$

with $\quad (Tf)(x) := \int_0^x f(t)\,dt \quad \forall x \in [0,1]$

(iii) Multiplication operator by argument: As above (in (ii)), with $(Tf)(x) := x f(x) \;\; \forall x \in [0,1]$

$\boxed{\text{Lemma 2.24}}$ Let $T : \mathbb{X} \supseteq \mathrm{dom}(T) \to \mathbb{Y}$ be a linear operator. Then

(a) $\mathrm{ran}(T)$ and $\ker(T)$ are vector subspaces (linear subspaces)

(b) $\dim \mathrm{ran}(T) \leq \dim \mathrm{dom}(T)$

(c) $\ker(T) = \{0\} \iff T$ injective

$\iff \exists$ inverse T^{-1} of T s.t. $T^{-1} : \mathrm{ran}(T) \to \mathrm{dom}(T)$,

$T^{-1}T = \mathbb{1}|_{\mathrm{dom}(T)}$, $\quad TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$

Pf: Copy from Linear Algebra ∎

$\boxed{\text{Remark 2.25}}$ (a) If T^{-1} exists, it is linear.

Pf: $\alpha x = T^{-1}T(\alpha x) = T^{-1}(\alpha T x) \;\; \forall x \in \mathrm{dom}(T), \; \forall \alpha \in \mathbb{K}$. Hence, $\alpha T^{-1} y = T^{-1}(\alpha y) \;\; \forall y \in \mathrm{ran}(T)$. Similarly for addition.

(b) Even if $T : \mathbb{X} \to \mathbb{X}$ with $\ker(T) = \{0\}$, then not necessarily $\mathrm{ran}(T) = \mathbb{X}$, nor $TT^{-1} = \mathbb{1}$ (unlike in the case $\dim \mathbb{X} < \infty$!) but only $TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$.

$\boxed{\text{Example 2.26}}$ (Illustrating Rmk 2.25)

Let $\mathbb{X} = \ell^\infty$.

Right-shift operator: $R: \ell^\infty \longrightarrow \ell^\infty$

$$(x_1, x_2, x_3, \dots) \longmapsto (0, x_1, x_2, \dots)$$

Clearly: $\ker(R) = \{0\}$, but $\operatorname{ran}(R) \subsetneq \ell^\infty$.

Left-shift operator: $L: \ell^\infty \longrightarrow \ell^\infty$

$$(x_1, x_2, x_3, \dots) \longmapsto (x_2, x_3, x_4, \dots)$$

Then $R^{-1} = L|_{\operatorname{ran}(R)}$ satisfies

$$R^{-1}R = \mathbb{1} \quad \text{and} \quad RR^{-1} = \mathbb{1}|_{\operatorname{ran}(R)}.$$

Definition 2.27 Let X, Y be normed spaces, $T: X \supseteq \operatorname{dom}(T) \to Y$.

T is bounded (operator) $:\Leftrightarrow$ its operator norm is finite, i.e.,

$$\|T\|_{\operatorname{dom}(T)\to Y} := \|T\| := \sup_{\substack{x \in \operatorname{dom}(T) \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in \operatorname{dom}(T) \\ \|x\| = 1}} \|Tx\| < \infty.$$

Examples 2.28 (Recall Ex. 2.23)

(a) $\mathbb{1}: (X, \|\cdot\|) \to (X, \|\cdot\|)$, $x \mapsto x$, is bounded with $\|\mathbb{1}\| = 1$.

(b) Let $(X, \|\cdot\|) = (C([0,1]), \|\cdot\|_\infty)$.

(i) Differentiation $\frac{d}{dx}: C^1([0,1]) \to C([0,1])$

$$f \mapsto f'$$

is unbounded (i.e., not bounded); see exercise.

(ii) Anti-derivative $T: C([0,1]) \to C([0,1])$:

$$\|Tf\|_\infty = \sup_{x\in[0,1]} \left| \int_0^x f(t)\,dt \right| \le \|f\|_\infty \quad \forall f \in C([0,1])$$

Because of $\frac{\|Tf\|}{\|f\|} \le 1$ $(f \neq 0)$, we have $\|T\| \le 1$.

Also,
$$\|T1\|_\infty = \sup_{x\in[0,1]} \int_0^x dt = \sup_{x\in[0,1]} x = 1,$$

and, since $\|1\|_\infty = 1$, we obtain $\|T\| = 1$.

(iii) Multiplication operator $T: C([0,1]) \to C([0,1])$ by argument has $\|T\| = 1$, because

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |x f(x)| \le \|f\|_\infty, \text{ with equality for } f \equiv 1.$$

Theorem 2.29 Let X, Y be normed spaces, $T: X \supseteq \mathrm{dom}(T) \to Y$ a linear operator. Then the following statements are equivalent:

(i) T is continuous

(ii) T is continuous at some $x_0 \in \mathrm{dom}(T)$.

(iii) There exists $c \in (0,\infty)$ s.t.

$$\|Tx\| \le c\|x\| \quad \forall x \in \mathrm{dom}(T).$$

(iv) T is bounded

Pf: (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (iii): Step 1: Prove T is continuous at $x=0$.

For this, let $(x_n)_n \subseteq \mathrm{dom}(T)$ be a sequence s.t. $x_n \to 0$, $n \to \infty$.

Then (by 2.10(b)), $(x_n + x_0)_n \to x_0$, $n \to \infty$, and by linearity and continuity at x_0, we get

$$Tx_n + Tx_0 = T(x_n + x_0) \xrightarrow{n\to\infty} Tx_0, \text{ so } Tx_n \xrightarrow{n\to\infty} 0 \text{ (again, by 2.10(b))}.$$

Step 2: From Step 1 we get ('ε-δ criterion, with $\varepsilon = 1$')

$$\exists \delta > 0 : \forall x \in \mathrm{dom}(T) \text{ with } \|x\| < \delta : \quad \|Tx\| < 1 \qquad (*)$$

Hence, for all $0 \ne x \in \mathrm{dom}(T)$, by linearity:

$$\|Tx\| = \frac{2\|x\|}{\delta} \Big\| T\Big(\underbrace{\frac{\delta}{2\|x\|} x}_{\text{has norm } = \frac{\delta}{2} < \delta}\Big)\Big\| \overset{(*)}{<} \frac{2}{\delta}\|x\|.$$

(iii) $\Rightarrow$ (iv): We know that $\|Tx\| \le c\|x\|$, so

$$\sup_{\substack{x\in\mathrm{dom}(T)\\ x\ne 0}} \frac{\|Tx\|}{\|x\|} \le c < \infty.$$

(iv) $\Rightarrow$ (i): Let $(x_n)_n \subseteq \mathrm{dom}(T)$ with $x_n \xrightarrow{n\to\infty} x$, then

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \le \|T\| \cdot \|x_n - x\| \xrightarrow{n\to\infty} 0 \qquad \square$$

Lemma 2.30 Let T be linear and $\dim \operatorname{dom}(T) < \infty$. Then T is bounded.

Pf: T is continuous: Fix (finite!) basis of $\operatorname{dom}(T)$, expand $x \in \operatorname{dom}(T)$ w.r.t. this basis, and use linearity to deduce (sequential) continuity. The claim follows from Thm. 2.29 ∎

Theorem 2.31 (Bounded linear extension) Let X be a normed space, and Y a Banach (!) space. Let $T: X \to Y$ be a bounded linear operator. Let $\tilde{X}$ be the completion of X. Then there exists a bounded linear extension $\tilde{T}: \tilde{X} \to Y$ of T, which is unique if X is identified with a (dense) subspace of its completion $\tilde{X}$, i.e. $X = W$ in Theorem 2.13. Moreover, we have $\|\tilde{T}\| = \|T\|$.

Corollary 2.32 Let X be a normed space and Y a Banach space. Let $T: X \supseteq \operatorname{dom}(T) \to Y$ be a bounded linear operator with $\operatorname{dom}(T) \subseteq X$ a dense linear subspace. Then there exists a unique bounded linear extension $\hat{T}: X \to Y$ of T. Moreover, we have $\|\hat{T}\| = \|T\|$.

Pf (of 2.32): Because of denseness, the completion Z of $\operatorname{dom}(T)$ and $\tilde{X}$ of X are isometrically isomorphic (Check!). Wlog., we identify $Z = \tilde{X}$. Also identify X with a dense linear subspace of $\tilde{X}$, so that $\operatorname{dom}(T) \subseteq X \subseteq \tilde{X}$. The claim follows from Theorem 2.31 with $\hat{T} := \tilde{T}|_X$. ∎

Pf (of Thm. 2.31): We identify X with a dense subspace of $\tilde{X}$. Let $x \in \tilde{X}$. Then there exists a sequence $(x_n)_n \subseteq X$ with $x_n \xrightarrow{n\to\infty} x$ in $\tilde{X}$. Since $(x_n)_n$ is a Cauchy sequence, $(Tx_n)_n$ is a Cauchy sequence in Y because

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \le \|T\| \cdot \|x_n - x_m\|.$$

Using that Y is a Banach space, there exists $y \in T$ s.t. $Tx_n \xrightarrow{n\to\infty} y$ in Y. Define $\tilde{T}x := y$ (then $\tilde{T}|_X = T$: take constant sequences!). We have to check several things:

(i) Well-definedness, i.e., independence of approximating seq.: Let $x_m' \xrightarrow{m\to\infty} x$ in $\tilde{X}$. As above, $\exists\, y' \in Y$ s.t. $Tx_m' \xrightarrow{m\to\infty} y'$. Then we have

$$\|Tx_n - Tx_m'\| \le \|T\| \cdot \|x_n - x_m'\|.$$

In the limit $n, m \to \infty$ we get $\|y - y'\| = 0$, so $y = y'$.

(ii) Linearity: Let $x_n \xrightarrow{n\to\infty} x$, $x_n' \xrightarrow{n\to\infty} x'$. Then (by 2.10 (b) & (c)), $\alpha x_n + \alpha' x_n' \xrightarrow{n\to\infty} \alpha x + \alpha' x'$ $\forall \alpha, \alpha' \in \mathbb{K}$, and so (by def. of $\tilde{T}$)

$$\tilde{T}(\alpha x + \alpha' x') = \lim_{n\to\infty} T(\alpha x_n + \alpha' x_n') = \lim_{n\to\infty} \alpha T x_n + \lim_{n\to\infty} \alpha' T x_n'$$
$$= \alpha \tilde{T} x + \alpha \tilde{T} x'.$$

(iii) Norm: As $\tilde{T}$ is an extension, we have $\|\tilde{T}\| \ge \|T\|$, since

$$\|\tilde{T}\| = \sup_{\substack{x \in \tilde{X} \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} \overset{\tilde{X} \supseteq X}{\ge} \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|.$$

On the other hand, since $\|\cdot\|$ is continuous (Thm. 2.10 (a)),

$$\|\tilde{T}x\| = \lim_{\substack{n\to\infty \\ x_n \in X}} \|\tilde{T}x_n\| = \lim_{n\to\infty} \|Tx_n\| \le \lim_{n\to\infty} \|T\| \cdot \|x_n\|$$
$$= \|T\| \cdot \lim_{n\to\infty} \|x_n\| = \|T\| \cdot \|x\|.$$

So, $\dfrac{\|\tilde{T}x\|}{\|x\|} \le \|T\|$ $\forall x \in \tilde{X}$, hence $\|\tilde{T}\| \le \|T\|$, so $\|\tilde{T}\| = \|T\|$.

(iv) Uniqueness: The fact that we defined

$$\tilde{T}x := \lim_{m\to\infty} Tx_m$$ is necessary to ensure continuity at x. ∎

Definition 2.33 Let $\mathbb{X}, \mathbb{Y}$ be normed spaces. Define

$$BL(\mathbb{X},\mathbb{Y}) := \{ T : \mathbb{X} \to \mathbb{Y} \mid T \text{ is linear and bounded} \}$$

and set $BL(\mathbb{X}) := BL(\mathbb{X},\mathbb{X})$.

Warning: Notation varies - other choices: $B(\mathbb{X},\mathbb{Y})$, $L(\mathbb{X},\mathbb{Y})$, ...

Theorem 2.34 $(BL(\mathbb{X},\mathbb{Y}), \|\cdot\|_{\mathbb{X}\to\mathbb{Y}})$ is a normed space. If $\mathbb{Y}$ is complete, then so is $BL(\mathbb{X},\mathbb{Y})$.

Pf: (i) First of all, $BL(\mathbb{X},\mathbb{Y})$ is a $\mathbb{K}$-vectorspace with zero element $\mathbb{0} := 0 : \mathbb{X} \to \mathbb{Y}$, $x \longmapsto 0$

For $T_1, T_2 \in BL(\mathbb{X},\mathbb{Y})$ and $\alpha \in \mathbb{K}$, define $T_1 + T_2$ and αT_1 by

$$(T_1+T_2)x := T_1 x + T_2 x$$
$$(\alpha T_1)x := \alpha T_1 x \qquad \forall x \in \mathbb{X}.$$

Then clearly $T_1 + T_2, \alpha T_1 : \mathbb{X} \to \mathbb{Y}$ are linear.

(ii) $\|\cdot\|_{\mathbb{X}\to\mathbb{Y}}$ is a norm on $BL(\mathbb{X},\mathbb{Y})$:

(a) $\|T\|_{\mathbb{X}\to\mathbb{Y}} \geq 0$ ✓ and if $\|T\|_{\mathbb{X}\to\mathbb{Y}} = 0$, then $\|Tx\| = 0$ $\forall x \in \mathbb{X}$, so $Tx = 0$ $\forall x \in \mathbb{X}$, and so $T = 0$ $(= \mathbb{0})$.

(b) We have

$$\|\alpha T\|_{\mathbb{X}\to\mathbb{Y}} = \sup_{0 \neq x \in \mathbb{X}} \frac{\|(\alpha T)x\|}{\|x\|} = \sup_{0\neq x \in \mathbb{X}} \frac{\|\alpha T x\|}{\|x\|}$$
$$= \sup_{0 \neq x \in \mathbb{X}} |\alpha| \cdot \frac{\|Tx\|}{\|x\|} = |\alpha| \cdot \|T\|$$

Also, $\|(T_1+T_2)x\| = \|T_1 x + T_2 x\| \leq \|T_1 x\| + \|T_2 x\|$, so

$$\|T_1 + T_2\| \leq \sup_{0 \neq x \in \mathbb{X}} \left(\frac{\|T_1 x\|}{\|x\|} + \frac{\|T_2 x\|}{\|x\|} \right) \leq \|T_1\| + \|T_2\|.$$

Hence, if $T_1, T_2 \in BL(\mathbb{X},\mathbb{Y})$, then αT_1 and $T_1 + T_2$ are bounded, so $T_1 + T_2, \alpha T_1 \in BL(\mathbb{X},\mathbb{Y})$ (vectorspace!), and $\|\cdot\|_{\mathbb{X}\to\mathbb{Y}}$ is a norm.

(iii) Assume Y is complete. We prove completeness of $BL(X,Y)$. Let $(T_k)_{k\in\mathbb{N}} \subseteq BL(X,Y)$ be a Cauchy sequence (w.r.t. $\|\cdot\|_{X\to Y}$).

For every $x \in X$ and $k,l\in\mathbb{N}$, we have $\|T_k x - T_l x\| \leq \|T_k - T_l\|\cdot\|x\|$.

So,

$$\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall k,l\geq N\ \forall x\in X:\ \|T_k x - T_l x\| \leq \varepsilon\|x\| \quad (*)$$

This implies that $(T_k x)_{k\in\mathbb{N}} \subseteq Y$ is a Cauchy seq. (in Y) for every $x\in X$, and, since Y is complete, there is some limit

$$\lim_{k\to\infty} T_k x =: Tx \in Y.$$

This defines a map

$$T: X \to Y$$
$$x \longmapsto Tx := \lim_{k\to\infty} T_k x$$

(a) T is linear, since: Let $\alpha,\alpha'\in\mathbb{K}$, $x,x'\in X$, then

$$T(\alpha x + \alpha' x') = \lim_{k\to\infty} T_k(\alpha x+\alpha' x') = \lim_{k\to\infty}(\alpha T_k x + \alpha' T_k x')$$
$$= \alpha Tx + \alpha' Tx'$$

(b) T is bounded, and the norm limit of the T_k's:

Taking the limit $l\to\infty$ in $(*)$ above gives

$\|T_k x - Tx\| \leq \varepsilon\|x\|$ and so

$$\forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall k\geq N:\ \|T_k - T\| \leq \varepsilon \quad (**)$$

This gives two things:

(1) T is bounded: $T_k\in BL(X,Y)$, $T_k - T \overset{\text{by } (**)}{\in} BL(X,Y)$ and $BL(X,Y)$ is a vector space, so

$$T = T_k - (T_k - T) \in BL(X,Y).$$

(2) $T_k \xrightarrow{k\to\infty} T$ in $\|\cdot\|_{X\to Y}$ ☒

<u>Note</u>: Compare also with the proofs of Thm 1.31 (& exercise!) and Thm. 1.20 (for $p=\infty$).

2.4. Linear functionals and dual space

Definition 2.35 Let $\mathfrak{X}$ be a normed space. A *linear functional* (on $\mathfrak{X}$) is a linear operator $\ell : \mathfrak{X} \supseteq \mathrm{dom}(\ell) \to \mathbb{K}$.

The *dual space* of $\mathfrak{X}$ is $\mathfrak{X}^* := BL(\mathfrak{X}, \mathbb{K})$.

Notations for the norm on $\mathfrak{X}^*$: $\|\cdot\|_{\mathfrak{X}\to\mathbb{K}} =: \|\cdot\|_{\mathfrak{X}^*} =: \|\cdot\|_* =: \|\cdot\|$.

Note: $\mathfrak{X}^*$ consists of the bounded (equivalently, continuous) linear functionals; $\mathfrak{X}^*$ is therefore also called the topological dual – not to be confused with the algebraic dual of all linear functionals, $\mathfrak{X}' := \{f : \mathfrak{X} \to \mathbb{K} \mid f \text{ linear}\} = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{X}, \mathbb{K})$, common in Linear Algebra!

Corollary 2.36 (to Thm. 2.34). $(\mathfrak{X}^*, \|\cdot\|_{\mathfrak{X}^*})$ is a Banach space (whether $\mathfrak{X}$ is complete or not).

Examples 2.37 Let $\mathfrak{X} = C([a,b])$ (with $a < b \in \mathbb{R}$) equipped with $\|\cdot\|_\infty$.

(a) For $f \in \mathfrak{X}$, let
$$\ell(f) := \int_a^b f(x)\,dx \in \mathbb{K}$$
This is clearly a linear functional $\ell : \mathfrak{X} \to \mathbb{K}$ with
$$|\ell(f)| \le \int_a^b \underbrace{|f(x)|}_{\le \|f\|_\infty} dx \le \|f\|_\infty (b-a) \quad (*)$$
so $\|\ell\|_{\mathfrak{X}^*} \le (b-a)$, and $f \equiv 1$ gives equality in $(*)$, so $\|\ell\|_{\mathfrak{X}^*} = b-a$.

(b) For $f \in \mathfrak{X}$ and $t_0 \in [a,b]$, let $\delta_{t_0}(f) := f(t_0) \in \mathbb{K}$.
This gives a linear functional $\delta_{t_0} : \mathfrak{X} \to \mathbb{K}$, called the *Dirac δ-functional (at t_0)*, with $|\delta_{t_0}(f)| = |f(t_0)| \le \|f\|_\infty$, and again equality for $f \equiv 1$. So $\|\delta_{t_0}\|_{\mathfrak{X}^*} = 1$.

NB! Boundedness (i.e., continuity) of δ_{t_0} depends on the norm on $\mathfrak{X}$. For ex., δ_{t_0} is not bounded, if $\mathfrak{X}$ is equipped with the L^1-norm $\left(\|f\|_1 = \int_0^1 |f(x)|\,dx\right)$.

Notation: Let X, Y be normed spaces. We write

$X \cong Y$ (or $X \simeq Y$) iff X is isometrically isomorphic to Y.

Theorem 2.38 Let $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$.

More precisely: The map $\ell^q \ni y = (y_n)_n \mapsto f_y \in (\ell^p)^*$,

$$f_y(x) := \sum_{n \in \mathbb{N}} x_n y_n \quad , \quad x \in \ell^p$$

is a bijective isometry

Pf: (a) Case $1 < p < \infty$: Let $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$, $f \in (\ell^p)^*$.

We work with the canonical (Schauder) basis $(e_n)_{n \in \mathbb{N}}$ for ℓ^p (see 2.15)

Then x can be written as a $\|\cdot\|_p$-convergent series $x = \sum_{n \in \mathbb{N}} x_n e_n$

Since f is continuous (wrt. $\|\cdot\|_p$) and linear,

$$f(x) = f\Big(\lim_{N \to \infty} \sum_{n=1}^{N} x_n e_n\Big) = \sum_{n \in \mathbb{N}} f(x_n e_n) = \sum_{n \in \mathbb{N}} x_n f(e_n). \qquad (*)$$

For $N \in \mathbb{N}$ (fixed), define $\tilde{x} := (\tilde{x}_n)_{n \in \mathbb{N}} \in \ell^p$ by

$$\tilde{x}_n := \begin{cases} \dfrac{|f(e_n)|^q}{f(e_n)} & \text{if } n \le N \text{ and } f(e_n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Apply $(*)$ to $\tilde{x}$:
$$f(\tilde{x}) = \sum_{n=1}^{N} |f(e_n)|^q \qquad (**)$$

so that $0 \le f(\tilde{x}) \le \|f\|_{(\ell^p)^*} \cdot \|\tilde{x}\|_p$, where (compute!)

$$\|\tilde{x}\|_p = \Big(\sum_{n=1}^{N} |f(e_n)|^{(q-1)p}\Big)^{1/p}.$$

Since $\frac{1}{p} = 1 - \frac{1}{q} \Leftrightarrow p = \frac{q}{q-1}$ (take note!), we have $p(q-1) = q$, so

$$(**) \Rightarrow \sum_{n=1}^{N} |f(e_n)|^q \le \|f\|_{(\ell^p)^*} \Big(\sum_{n=1}^{N} |f(e_n)|^q\Big)^{1/p}$$

$$\Rightarrow \Big(\sum_{n=1}^{N} |f(e_n)|^q\Big)^{1/q} \le \|f\|_{(\ell^p)^*} \quad \forall N \in \mathbb{N} \qquad (***)$$

Hence, the map
$$J : (\ell^p)^* \to \ell^q$$
$$f \mapsto (f(e_n))_{n\in\mathbb{N}}$$
is well-defined, and

(i) J is linear

(ii) $\|Jf\|_q = \|(f(e_n))_{n\in\mathbb{N}}\|_q \le \|f\|_{(\ell^p)^*}$ by $(**)$

(iii) J is onto/surjectiv: If $y = (y_n)_n \in \ell^q$, define
$$f_y : \begin{array}{l} \ell^p \to \mathbb{K} \\ x = (x_n)_n \mapsto f_y(x) := \sum_{n\in\mathbb{N}} x_n y_n \end{array}$$
Clearly, f_y is linear. It is well-defined: $\sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$ by Hölder's inequality, so
$$\|f_y\|_{(\ell^p)^*} \le \|y\|_q$$
and so $f_y \in (\ell^p)^*$. But $f_y(e_n) = y_n \ \forall n\in\mathbb{N}$, so $J(f_y) = y$.

(iv) $(*)$ & Hölder imply:
$$\forall f \in (\ell^p)^* \ \forall x \in \ell^p : \ |f(x)| \le \|x\|_p \cdot \underbrace{\|(f(e_n))_{n\in\mathbb{N}}\|_q}_{Jf}$$
$$\Rightarrow \quad \|f\|_{(\ell^p)^*} \le \|Jf\|_q$$

So $\|Jf\|_q = \|f\|_{(\ell^p)^*}$, and J is an isometric isomorphism.
(The map in the statement of the thm. is J^{-1}).

(b) The case $p=1$ is analogous, but instead of defining $\tilde{x}_n$, use
$$|f(e_n)| \le \|f\|_{(\ell^1)^*} \cdot \underbrace{\|e_n\|_1}_{=1}$$
This implies $\|(f(e_n))_n\|_\infty \le \|f\|_{(\ell^1)^*}$, which replaces $(**)$, and the properties of J follow as above. ∎

$\boxed{\text{Remark 2.39}}$ (a) $(c_0)^* \cong \ell^1$ (see exercise)

(b) The map $\ell^1 \to (\ell^\infty)^*$, $y \mapsto f_y$, defined (as in Thm. 2.38) by
$$f_y(x) = \sum_{n\in\mathbb{N}} x_n y_n, \quad x \in \ell^\infty, \ y \in \ell^1$$
is well-defined (Hölder!), linear, isometric, but <u>not</u> onto!
In other words: $(\ell^\infty)^*$ is strictly "larger" than ℓ^1! (see later)

2.5 Hilbert spaces

The main new feature is the "geometry" from the scalar product

Definition 2.40 Let X be a ($\mathbb{K}$-) vectorspace.

A map $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{K}$ is a scalar product (or inner product) $:\Leftrightarrow$

(i) $\langle x, x \rangle \geq 0 \quad \forall x \in X$, and $\langle x, x\rangle = 0 \Rightarrow x = 0$
(positive definit) (non-degenerate)

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle \quad \forall \alpha, \beta \in \mathbb{K}, \forall x, y, z \in X$
(linearity)

(iii) $\langle x, y \rangle = \overline{\langle y, x\rangle} \quad \forall x, y \in X$

$(X, \langle \cdot , \cdot \rangle)$ is an inner product space (or pre-Hilbert space)

Note: If $\mathbb{K} = \mathbb{R}$, (ii) & (iii) imply that $\langle \cdot , \cdot \rangle$ is bi-linear
If, however, $\mathbb{K} = \mathbb{C}$, (ii) & (iii) give that

$$\langle \alpha x, y \rangle = \bar{\alpha} \langle x, y \rangle \quad \forall \alpha \in \mathbb{C},\ x, y \in X \quad \left(\text{conjugate linear}\right)$$

Then $\langle \cdot , \cdot \rangle$ is called sesqui-linear ("$1\frac{1}{2}$-linear")

Warning: In literatur also: $\langle \alpha x, y \rangle = \alpha \langle x, y\rangle$ & $\langle x, \alpha y\rangle = \bar{\alpha} \langle x, y \rangle$

Lemma 2.41 (Cauchy-Schwarz (Bunyakovsky) inequality)

Let X be an inner product space. Then

$$\text{(C-S)} \qquad |\langle x, y\rangle| \leq \langle x, x\rangle^{1/2} \langle y, y \rangle^{1/2} \quad \forall x, y \in X$$

with equality iff x & y are linearly dependent

Proof: See Linear Algebra (and/or Ana 1-3). $\blacksquare$

Lemma 2.42 Let X be an inner product space.

Then X is a normed space with norm $\|x\| := \langle x, x\rangle^{1/2}$.
All notions & results from topological, metric, and normed spaces are available. In particular, the scalar product
$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is continuous.

Pf: (i) $\|\cdot\|$ fulfills all axioms of a norm, see LA (or do!)
(ii) Continuity of $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$:
Note that $X \times X$ is a normed space (f. ex. with norm $\|(x,y)\|_1 := \|x\| + \|y\|$), hence seq. cont. enough.
So let $x_n \xrightarrow{n\to\infty} x$, $y_n \xrightarrow{n\to\infty} y$ be two conv. seq.'s in X.
Then

$$
\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &\le |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\
&\overset{C\text{-}S}{\le} \|x_n\| \cdot \underbrace{\|y_n - y\|}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|y\|}_{<\infty} \cdot \underbrace{\|x_n - x\|}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to 0} 0
\end{aligned}
$$

since $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ (as $(x_n)_n$ converges & $\|\cdot\|$ cont.). ∎

Definition 2.43 A complete inner product space is called a Hilbert space

Note: Hence, any Hilbert space is also a Banach space (for the other direction, see 2.46 below).

Theorem 2.44 Let X be an inner product space. Then there exists a Hilbert space $\tilde{X}$, a dense subspace $W \subseteq \tilde{X}$, and a <u>unitary</u> map $U: X \to W$.

(U is <u>unitary</u> :$\Leftrightarrow$ U is an isomorphism with

$$\langle x, y\rangle_X = \langle Ux, Uy\rangle_{\tilde{X}} \quad \forall x, y \in X)$$

<u>Idea of pf</u>: See the pf's of Thm's 2.13 & 1.14.

Additional aspect here: Define a scalar product on $\tilde{X} := \{$ equivalence classes $\tilde{x}$ of Cauchy sequences in $X\}$:

$$\langle \tilde{x}, \tilde{y}\rangle_{\tilde{X}} := \lim_{n\to\infty} \langle x_n, y_n\rangle_X$$

with $(x_n)_n, (y_n)_n \subseteq X$ being representatives of the equivalence classes $\tilde{x}$ and $\tilde{y}$, and $U: \begin{array}{l} X \to W \\ x \mapsto \tilde{x} \end{array}$

where $\tilde{x}$ is the equivalence class of the constant representative $(x, x, x, \ldots)$ $\boxtimes$

Examples 2.45 (a) $\ell^2 = \ell^2(\mathbb{N})$ is a Hilbert space with scalar product $\langle x, y\rangle := \sum_{n\in\mathbb{N}} \overline{x_n} y_n$, where $x = (x_n)_n$, $y = (y_n)_n$.

Of course, $\ell^2(\{1, \ldots, N\}) \equiv \mathbb{C}^N$ is also a Hilbert space.

(b) $C([0,1])$ is an inner product space with the scalar product

$$\langle f, g\rangle := \int_0^1 \overline{f(x)} g(x)\, dx$$

but <u>not</u> a Hilbert space (proof analogous to Ex. 1.11 (b)).

(Applying Thm. 2.44 to this example gives $L^2([0,1])$, see next chapter).

(c) ℓ^p for $p \neq 2$ is not an inner product space, because of the following theorem:

(Theorem 2.46) (Jordan-von Neumann) Let $(\mathfrak{X}, \|\cdot\|)$ be a normed space. Then $\mathfrak{X}$ is an inner product space with $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ iff $\|\cdot\|$ satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in \mathfrak{X}.$$

Pf: "$\Rightarrow$": Elementary computation (do!)

"$\Leftarrow$": Define inner product (!) by polarisation:

$$\langle x, y\rangle := \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) - \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right) & \mathbb{K} = \mathbb{C} \end{cases}$$

We verify that this definition satisfies axioms of an inner product:

(i) Symmetry and definiteness are obvious (check!)

(ii) Bi-/sesqui-linearity follows from parallelogram identity (here only $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} \langle x, y\rangle + \langle x, z\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2\right) \\ &= \frac{1}{2}\left(\left\|x + \frac{y+z}{2}\right\|^2 - \left\|x - \frac{y+z}{2}\right\|^2\right) \\ &= 2\left\langle x, \frac{y+z}{2}\right\rangle \end{aligned} \qquad (1)$$

$z = 0$ in (1) $\Rightarrow$ $\langle x, y\rangle = 2\langle x, \frac{y}{2}\rangle$ (2)

(1) & (2) $\Rightarrow$ $\langle x, y\rangle + \langle x, z\rangle = \langle x, y+z\rangle$ (3)

(2) with $2y$ instead of y: $\langle x, 2y\rangle = 2\langle x, y\rangle$ (4)

By induction from (3) & (4): $\langle x, ny\rangle = n\langle x, y\rangle \quad \forall n \in \mathbb{N}_0$ (5)

$z = -y$ in (3) $\Rightarrow$ $\langle x, y\rangle = -\langle x, -y\rangle \Rightarrow$ (5) holds $\forall n \in \mathbb{Z}$

Let $m \in \mathbb{Z}\setminus\{0\}$ and use (5) with $\frac{y}{m}$ instead of y (writing $\lambda := \frac{n}{m}$):

$$\langle x, \lambda y\rangle = n\langle x, \tfrac{y}{m}\rangle = \lambda m\langle x, \tfrac{y}{m}\rangle = \lambda\langle x, y\rangle.$$

This holds for all $\lambda \in \mathbb{Q}$, and thus for all $\lambda \in \mathbb{R}$ by continuity (2.10) $\blacksquare$

Definition 2.47 Let X be an inner product space, and $A \subseteq X$ (not necessarily subspace)

(a) $x, y \in X$ are orthogonal $:\Leftrightarrow \langle x, y\rangle = 0$. In symbols: $x \perp y$

(b) Orthogonal complement of A: $A^{\perp} := \{x \in X \mid x \perp a \quad \forall a \in A\}$

(c) Let $J \neq \emptyset$ be an index set, and for all $\alpha \in J$, let $e_\alpha \in X$.
We call $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set (or family) $:\Leftrightarrow$

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} \quad \forall \alpha, \beta \in J$$

(and orthogonal set if $\langle e_\alpha, e_\beta\rangle = 0 \quad \forall \alpha \neq \beta$)

(d) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal basis (ONB) or complete orthonormal set
$:\Leftrightarrow$ (i) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal set.
(ii) "Completeness": If $\langle e_\alpha, x\rangle = 0$ for some $x \in X$ and all $\alpha \in J$, then $x = 0$ (I.e.: The zero vector is the only vector orthogonal to all e_α's).

Lemma 2.48 Let X be an inner product space.

(a) Let $A \subseteq X$ be a subset (not necessarily subspace). Then $A^\perp$ is a closed (linear) subspace in X.

(b) Every orthonormal set $\{e_\alpha\}_{\alpha \in J}$ is linearly independent.

Pf: (a) See exercises.

(b) Assume $\sum_{j=1}^{n} \lambda_j e_{\alpha_j} = 0$ for some $n \in \mathbb{N}$, some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, and some $\alpha_1, \dots, \alpha_n \in J$ with $\alpha_j \neq \alpha_k \ \forall j \neq k$. Then we have

$$0 = \langle e_{\alpha_k}, 0\rangle = \langle e_{\alpha_k}, \sum_{j=1}^{n} \lambda_j e_{\alpha_j}\rangle = \sum_{j=1}^{n} \lambda_j \underbrace{\langle e_{\alpha_k}, e_{\alpha_j}\rangle}_{\delta_{kj}} = \lambda_k \quad \forall k = 1, \dots, n \ \blacksquare$$

Next we prove two lemmas/lemmata as preparation for Theorem 2.51:

Lemma 2.49 (Uncountable series) Let $\emptyset \neq J$ be an index set

and $0 \le c_\alpha < \infty$ $\forall \alpha \in J$. Then

$$\sum_{\alpha \in J} c_\alpha := \sup_{\substack{J_0 \subseteq J \\ |J_0| < \infty}} \sum_{\alpha \in J_0} c_\alpha < \infty$$

implies $c_\alpha = 0$ for all but countably many α's.

Note: The definition agrees with the usual one, if J is countable.

Proof: Define the sets $S_0 := \{\alpha \in J \mid c_\alpha \ge 1\}$

$$S_n := \{\alpha \in J \mid \tfrac{1}{n} > c_\alpha \ge \tfrac{1}{n+1}\}, \quad n \in \mathbb{N}$$

Claim: $|S_n| < \infty$ $\forall n \in \mathbb{N}_0$

True, since, if this was violated for some $n \in \mathbb{N}$, then

$\sum_{\alpha \in J} c_\alpha \ge \sum_{\alpha \in S_n} c_\alpha = \infty$. But then $\{\alpha \in J \mid c_\alpha > 0\} = \bigcup_{n \in \mathbb{N}_0} S_n$ is a countable union of finite sets, hence countable $\blacksquare$

Lemma 2.50 (Series in Banach & Hilbert spaces)

(a) Let X be a Banach space and consider a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$.

Then
$$\sum_{n\in\mathbb{N}} \|x_n\| < \infty \implies \sum_{n \in \mathbb{N}} x_n := \lim_{N\to\infty} \left(\sum_{n=1}^{N} x_n\right) \text{ exists in } X.$$

(b) Let X be a Hilbert space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence of pairwise orthogonal vectors (i.e. an orthogonal family).

Then
$$\sum_{n\in\mathbb{N}} \|x_n\|^2 < \infty \iff \sum_{n\in\mathbb{N}} x_n \text{ exists in } X$$

(Note: Recall that $\ell^1 \subsetneq \ell^2$)

Pf: Let $S_n := \sum_{j=1}^{n} x_j$ for $n \in \mathbb{N}$. Since X is complete, we have:

$\sum_{n\in\mathbb{N}} x_n$ exists $\iff$ $(S_n)_n$ is Cauchy (in X).

(a) Let $m \ge n$. Then

$$\|S_m - S_n\| = \left\|\sum_{j=n+1}^{m} x_j\right\| \le \sum_{j=n+1}^{m} \|x_j\| = \sigma_m - \sigma_n$$

with $\sigma_n := \sum_{j=1}^{n} \|x_j\|$. But by the assumption, $(\sigma_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges in $\mathbb{R}$, and hence is Cauchy.

(b) Consider again the sequence of partial sums $(S_n)_n$. Let $m \geq n$, then

$$\|S_m - S_n\|^2 = \Big\| \sum_{j=n+1}^{m} x_j \Big\|^2 = \sum_{j,k=n+1}^{m} \langle x_j, x_k \rangle = \sum_{j=n+1}^{m} \|x_j\|^2 = \tau_m - \tau_n$$

with $\tau_n := \sum_{j=1}^{n} \|x_j\|^2$. Hence, $(S_n)_n$ is Cauchy (in X) iff. $(\tau_n)_{n\in\mathbb{N}}$ is Cauchy (in $\mathbb{R}$) ∎

Theorem 2.51 Let X be an inner product space, $x \in X$, and $\{e_\alpha\}_{\alpha\in J}$ an orthonormal set. Then:

$$\|x\|^2 \geq \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Bessel's inequality)}.$$

If X is even a Hilbert space, and if $\{e_\alpha\}_{\alpha\in J}$ is even an ONB, then

$$x = \sum_{\alpha\in J} \langle e_\alpha, x\rangle e_\alpha$$

where at most countably many terms are $\neq 0$, and

$$\|x\|^2 = \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Parseval's equality)}.$$

Pf: Let $J_0 \subseteq J$ be any finite subset. Then we can write x as follows:

$$x = \underbrace{\sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha}_{=:u} + \underbrace{\Big(x - \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha\Big)}_{=:v}$$

Using that $\{e_\alpha\}_{\alpha\in J_0}$ is an orthonormal set, we get:

$$\langle u,u\rangle = \Big\langle \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha, \sum_{\beta\in J_0} \langle e_\beta, x\rangle e_\beta \Big\rangle = \sum_{\alpha,\beta\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\beta, x\rangle \langle e_\alpha, e_\beta\rangle$$

$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, x\rangle = \sum_{\alpha\in J_0} |\langle e_\alpha, x\rangle|^2,$$

$$\langle u,v\rangle = \langle u, x-u\rangle = \langle u,x\rangle - \langle u,u\rangle$$
$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha,x\rangle}\langle e_\alpha,x\rangle - \langle u,u\rangle = 0.$$

Hence, we can estimate $\|x\|^2$:

$$\|x\|^2 = \langle u+v,u+v\rangle = \langle u,u\rangle + \underbrace{\langle v,v\rangle}_{\geq 0} \geq \sum_{\alpha\in J_0} |\langle e_\alpha,x\rangle|^2 \quad \forall J_0\subseteq J,\ |J_0|<\infty$$

Taking sup over all finite subsets $J_0\subseteq J$ gives Bessel's inequality.

Now, Lemma 2.49 $\Rightarrow \exists\, N\in\mathbb{N}\cup\{\infty\}$ and $\alpha_n\in J$ $\forall n\in\{1,\dots,N\}$ s.t.

$$(*)\quad \sum_{\alpha\in J} |\langle e_\alpha,x\rangle|^2 = \sum_{n=1}^{N} |\langle e_{\alpha_n},x\rangle|^2 \leq \|x\|^2 < \infty \quad \text{(using Bessel's inequ.)}$$

Then $(*)$ and Lemma 2.50 (b) (which requires $\underline{\text{completeness}}$!) imply that

$$x' := \sum_{n=1}^{N} \langle e_{\alpha_n},x\rangle e_{\alpha_n} \text{ exists in } X.$$

We need to prove that $x = x'$:

We have $x-x' \perp e_{\alpha_m}$ $\forall m\in\{1,\dots,N\}$:

$$\langle e_{\alpha_m},x-x'\rangle = \langle e_{\alpha_m},x\rangle - \langle e_{\alpha_m},x'\rangle = \langle e_{\alpha_m},x\rangle - \sum_{n=1}^{N}\langle e_{\alpha_n},x\rangle \underbrace{\langle e_{\alpha_m},e_{\alpha_n}\rangle}_{\delta_{mn}}$$
$$= 0 \quad \forall m\in\{1,\dots,N\}$$

If $\alpha\notin\{\alpha_n \mid n\in\{1,\dots,N\}\}$ then we have:

$$\langle e_\alpha,x-x'\rangle = \langle e_\alpha,x\rangle - \underbrace{\sum_{n=1}^{N}\langle e_{\alpha_n},x\rangle \underbrace{\langle e_\alpha,e_{\alpha_n}\rangle}_{=0}}_{=0} = \langle e_\alpha,x\rangle \overset{(*)}{=} 0$$

Taken together, we have $x-x'\perp e_\alpha$ $\forall\alpha\in J \overset{\text{ONB}}{\Rightarrow} x = x'$.

Moreover (if $N<\infty$, no limits are needed):

$$\|x\|^2 = \left\|\lim_{n\to\infty}\sum_{j=1}^{n}\langle e_{\alpha_j},x\rangle e_{\alpha_j}\right\|^2 \overset{\|\cdot\|\text{ cont.}}{=} \lim_{n\to\infty}\left\|\sum_{j=1}^{n}\langle e_{\alpha_j},x\rangle e_{\alpha_j}\right\|^2$$
$$= \lim_{n\to\infty}\left\langle \sum_{j=1}^{n}\langle e_{\alpha_j},x\rangle e_{\alpha_j}, \sum_{k=1}^{n}\langle e_{\alpha_k},x\rangle e_{\alpha_k}\right\rangle$$
$$= \lim \sum_{j=1}^{n}\sum_{k=1}^{n} \overline{\langle e_{\alpha_j},x\rangle}\langle e_{\alpha_k},x\rangle \underbrace{\langle e_{\alpha_j},e_{\alpha_k}\rangle}_{\delta_{jk}} = \sum_{j=1}^{N}|\langle e_{\alpha_j},x\rangle|^2$$
$$\overset{(*)}{=} \sum_{\alpha\in J}|\langle e_\alpha,x\rangle|^2. \quad \blacksquare$$

Lemma 2.50 (b) and Theorem 2.51 gives

Corollary 2.52 Let X be a Hilbert space and $\{e_\alpha\}_{\alpha\in J}$ an orthonormal set in X

(a) If $(c_\alpha)_{\alpha\in J} \subseteq \mathbb{K}$ with $\sum_{\alpha\in J} |c_\alpha|^2 < \infty$, then $\sum_{\alpha\in J} c_\alpha e_\alpha$ is well-defined in X

(b) If J is countable, and $\{e_\alpha \mid \alpha\in J\}$ is even an orthonormal basis (ONB), then $\{e_\alpha\}_{\alpha\in J}$ is a Schauder basis.

Theorem 2.53 Every Hilbert space $X \neq \{0\}$ has an ONB. Moreover,

X separable $\iff$ there exists a countable ONB.

Pf: The existence of an ONB in the non-separable case will be proven later (when Zorn's Lemma is available).

"$\Leftarrow$": Suppose there exists a countable ONB. By Cor. 2.52(b), this is a Schauder basis. Thus, X is separable by Lemma 2.16. ✓

"$\Rightarrow$": Let $\{x_n \mid n\in\mathbb{N}\}$ be dense in X. We construct an ONB using the Gram-Schmidt procedure:

(i) If $x_1 \neq 0$, define $e_1 := \frac{x_1}{\|x_1\|}$ (otherwise, use x_{n_0} with $n_0 := \min\{k\in\mathbb{N} \mid x_k \neq 0\}$)

(ii) Throw away all x_n's that are linearly dependent of e_1.
Let $n_1 := \min\{k\in\mathbb{N} \mid x_k \notin \operatorname{span}\{e_1\}\} > n_0$ be the smallest index of the remaining elements. Set

$$\tilde{e}_2 := x_{n_1} - \langle e_1, x_{n_1}\rangle e_1 \quad \text{and} \quad e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|}.$$

(iii) Throw away all x_n's that are linearly dependent of $\{e_1, e_2\}$. Let $n_2 := \min\{k\in\mathbb{N} \mid x_k \notin \operatorname{span}\{e_1,e_2\}\} > n_1$ be the smallest index of the remaining elements. Set

$$\tilde{e}_3 := x_{n_2} - \langle e_1, x_{n_2}\rangle e_1 - \langle e_2, x_{n_2}\rangle e_2 \quad \text{and} \quad e_3 := \frac{\tilde{e}_3}{\|\tilde{e}_3\|}.$$

(iv) Continue this procedure.

This terminates iff $\dim X < \infty$. It is clear, that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set. Also, $\operatorname{span}\{e_n \mid n\in\mathbb{N}\}$ is dense in X because each x_n is a finite linear combination of e_n's. It remains to prove that this is a basis: Assume there is $y \in X$ such that $y \perp e_n$ $\forall n$. By denseness, $\exists (y_k)_k \subseteq \operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ such that $y_k \xrightarrow{k\to\infty} y$ in X. Thus $\langle y, y_k\rangle = 0$ $\forall k$ and

$$\|y\|^2 = \langle y, y\rangle = \lim_{k\to\infty} \underbrace{\langle y, y_k\rangle}_{=0} = 0 \qquad \blacksquare$$

Definition Let X, X' be inner product spaces.

X, X' are unitarily equivalent $:\Leftrightarrow$ $\exists$ isometric isomorphism $U: X \to X'$

Note: U isometric & linear $\Rightarrow$ U preserves scalar products:

$$\langle x, y\rangle_X = \langle Ux, Uy\rangle_{X'}, \quad \forall x, y \in X$$
(follows from polarisation, see proof of Thm. 2.46)

Theorem 2.54 Let X be a Hilbert space. If $n := \dim X < \infty$ then X is unitarily equivalent to $\mathbb{K}^n$ (with Euclidean scalar product). If $\dim X = \infty$ and X is separable, then X is unit. equiv. to ℓ^2

Pf: 2nd case: Fix a countable ONB $\{e_n\}_{n\in\mathbb{N}}$ (exists by Thm. 2.53 b/c X is separable) and consider the map

$$U: X \to \ell^2$$
$$x \mapsto (\langle e_n, x\rangle)_{n\in\mathbb{N}}$$

(i) U is well-defined by Bessel's inequality, and linear.

(ii) U is surjective by Cor. 2.52 (a)

(iii) U is isometric ($\Rightarrow$ injective) by Parseval's equality.

1st case: Analogous $\blacksquare$

Theorem 2.55 (Projection theorem) Let X be a Hilbert space and $A \subseteq X$ a closed subspace. Then: For all $x \in X$ there exists a unique $a \in A$ and a unique $w \in A^\perp$ such that

$$x = a + w$$

The proof relies on:

$\boxed{\text{Lemma 2.56}}$ Let X, A be as in Thm. 2.55.
For every $x \in X$ there exists a unique $a \in A$ such that
$$\operatorname{dist}(x, A) = \|x - a\|.$$ That is, a is the "closest element" to x in A.

Pf (2.56): Existence: Let $d := d(x, A) = \inf_{y \in A} \|x - y\|$.
There exists a sequence $(y_n)_{n \in \mathbb{N}} \subseteq A$ s.t.

$$(*) \qquad d = \lim_{n \to \infty} \underbrace{\|x - y_n\|}_{\geq d}$$ (by def. of "inf"; $(y_n)_n$ is a minimizing sequence)

Using the parallelogram identity
$\|u + v\|^2 = 2(\|u\|^2 + \|v\|^2) - \|u - v\|^2$ (Thm. 2.46) we get

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|(y_n - x) + (x - y_m)\|^2 \\
&= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \|2x - y_m - y_n\|^2 \\
(**) \qquad &= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \underbrace{4\,\|x - \underbrace{\tfrac{y_m + y_n}{2}}_{\in A \text{ (subspace)}}\|^2}_{\geq 4d^2} \\
&\leq 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4d^2 \\
&\longrightarrow 0, \quad m, n \to \infty \quad (\text{by } (*)).
\end{aligned}$$

Hence, $(y_n)_n$ is Cauchy, so (X complete) $\exists a \in X : y_n \to a, n \to \infty$.
But $(y_n)_n \subseteq A$ and A is closed, so $a \in A$, and $d = \|x - a\|$ by $(*)$ and continuity of the norm.

Uniqueness: Assume there exists $(y_n')_n \subseteq A$ with
$$\lim_{n \to \infty} \|y_n' - x\| = d \quad \text{and} \quad a' = \lim_{n \to \infty} y_n'$$
Replacing y_m by y_m' in $(**)$ gives
$$\begin{aligned}
\|a - a'\|^2 &= \lim_{n \to \infty} \lim_{m \to \infty} \|y_n - y_m'\|^2 \\
&\leq \lim_{n \to \infty} \lim_{m \to \infty} 2(\|y_n - x\|^2 + \|y_m' - x\|^2) - 4d^2 = 0
\end{aligned}$$
$$\Rightarrow a = a' \quad \blacksquare$$

Remark: Note that the proof works if A is closed and convex (not necessarily a subspace)

Pf (of 2.55): Existence: For given $x \in X$, let $a \in A$ be defined by Lemma 2.56, and set $w := x - a$. Need to prove: $w \in A^\perp$

Let $\xi \in \mathbb{K}$ and $y \in A$. Then

$$\|w\|^2 \leq \|\underbrace{x - \underbrace{(a+\xi y)}_{\in A}}_{w - \xi y}\|^2 = \|w\|^2 + \|\xi y\|^2 - 2\,\mathrm{Re}\langle w, \xi y\rangle.$$

So for $0 \neq y \in A$ we have $|\xi|^2 - \dfrac{2\,\mathrm{Re}\langle w, \xi y\rangle}{\|y\|^2} \geq 0$

(i) For $\xi = t \in \mathbb{R}$ this gives

$$t^2 - \frac{2\,\mathrm{Re}\langle w, y\rangle}{\|y\|^2}\, t \geq 0 \qquad \forall t \in \mathbb{R}$$

This implies $\mathrm{Re}\langle w, y\rangle = 0$. If $\mathbb{K} = \mathbb{R}$ we are done, since $y \in A$ was arbitrary.

For $\mathbb{K} = \mathbb{C}$ we also consider the following case:

(ii) For $\xi = it$, $t \in \mathbb{R}$:

$$t^2 + \frac{2\,\mathrm{Im}\langle w, y\rangle}{\|y\|^2}\, t \geq 0 \qquad \forall t \in \mathbb{R}$$

So $\mathrm{Im}\langle w, y\rangle = 0$ for every $y \in A$. Together with (i) this gives $\langle w, y\rangle = 0$ and so $w \in A^\perp$.

Uniqueness: Assume there are $a, a' \in A$ and $w, w' \in A^\perp$ with $a + w = x = a' + w'$. Then $\underbrace{a - a'}_{\in A} = \underbrace{w' - w}_{\in A^\perp}$. Then $a = a'$ and $w = w'$ as $A \cap A^\perp = \{0\}$. $\blacksquare$

Theorem 2.57 (Riesz' Representation Thm.) Let X be a Hilbert space, and let $\ell \in X^*$. Then there exists a unique $y_\ell \in X$ such that

$$\ell(x) = \langle y_\ell, x\rangle \qquad \forall x \in X \qquad (*)$$

and

$$\|\ell\|_* = \|y_\ell\|.$$

Pf: If $\ker \ell = X \Rightarrow \ell = 0$ and thm. true with $y_\ell = 0$. (59)

Assume therefor $\ker \ell \subsetneq X$. Note that $\ker \ell$ is closed.

Existence of y_ℓ: From Thm. 2.55 we know $(\ker \ell)^\perp \supsetneq \{0\}$ and this allows to choose $0 \neq x_0 \in (\ker \ell)^\perp$. Define

$$y_\ell := \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0 \in X$$

(i) So (*) holds for every $x \in \ker \ell$ ($x_0 \perp \ker \ell$)

(ii) Let $x = \alpha x_0$, $\alpha \in \mathbb{K}$. Then $\ell(x) = \alpha \ell(x_0)$ and

$$\langle y_\ell, x \rangle = \left\langle \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0, \alpha x_0 \right\rangle = \frac{\ell(x_0)}{\|x_0\|^2} \alpha \langle x_0, x_0 \rangle = \alpha \ell(x_0),$$

so ℓ and $\langle y_\ell, \cdot \rangle$ agree on $\operatorname{span}\{\ker \ell, x_0\}$.

(iii) But $\operatorname{span}\{\ker \ell, x_0\} = X$ because, for all $x \in X$,

$$x = \underbrace{\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0 \right)}_{\in \ker \ell} + \underbrace{\frac{\ell(x)}{\ell(x_0)} \cdot x_0}_{\in \operatorname{span}\{x_0\}}$$

since
$$\ell\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0 \right) = \ell(x) - \frac{\ell(x)}{\ell(x_0)} \ell(x_0) = 0.$$

So $\ell = \langle y_\ell, \cdot \rangle$ on X by (i), (ii), and the linearity of ℓ and $\langle y_\ell, \cdot \rangle$.

Uniquess of y_ℓ: Assume there exists $y_\ell' \in X$ with $\ell = \langle y_\ell', \cdot \rangle$. Then, for every $x \in X$,

$$0 = \underbrace{\ell(x)}_{\langle y_\ell, x \rangle} - \underbrace{\ell(x)}_{\langle y_\ell', x \rangle} = \langle y_\ell - y_\ell', x \rangle$$

Choose $x = y_\ell - y_\ell'$, so that $0 = \| y_\ell - y_\ell' \|^2$, so $y_\ell = y_\ell'$

Norm: We have

$$\|\ell\|_0 = \sup_{0 \neq x \in X} \underbrace{\frac{|\ell(x)|}{\|x\|}}_{\frac{|\langle y_\ell, x \rangle|}{\|x\|}} \overset{C-S}{\leq} \|y_\ell\|$$

Note that $y_\ell \neq 0$ since $\ker \ell \subsetneq X$. Hence, we may choose $x = y_\ell$ in sup above:

$$\|\ell\|_* \geq \frac{|\ell(y_\ell)|}{\|y_\ell\|} = \frac{|\langle y_\ell, y_\ell \rangle|}{\|y_\ell\|} = \|y_\ell\|$$

hence, $\|\ell\|_* = \|y_\ell\|$ ☒

Corollary 2.58 Let X be a Hilbert space. Then the map

$$J: X^* \to X$$
$$\ell \mapsto y_\ell$$

defined by Thm. 2.57 is a semi-linear, isometric bijection (linear for $\mathbb{K} = \mathbb{R}$, conjugate linear for $\mathbb{K} = \mathbb{C}$).

Pf: Let $\alpha, \beta \in \mathbb{K}$, and $\ell_1, \ell_2 \in X^*$ with $\ell_j = \langle y_j, \cdot \rangle$, i.e. $J(\ell_j) = y_j$ for $j = 1, 2$. Then

$$\alpha \ell_1 + \beta \ell_2 = \alpha \langle y_1, \cdot \rangle + \beta \langle y_2, \cdot \rangle = \langle \bar{\alpha} y_1 + \bar{\beta} y_2, \cdot \rangle$$

So $J(\alpha \ell_1 + \beta \ell_2) = \bar{\alpha} J(\ell_1) + \bar{\beta} J(\ell_2)$ (by uniqueness in 2.57)

Hence, J is semi-linear. That J is an isometry was proved in 2.57 ($\Rightarrow$ J injective). Also, J is onto / surjective: For $y \in X$, $\ell_y := \langle y, \cdot \rangle \in X^*$ (use C-S, or Lem. 2.42, to prove continuity) and $J(\ell_y) = y$ (by uniqueness in 2.57) ☒
