2.5 Hilbert spaces

The main new feature is the "geometry" from the scalar product

Definition 2.40 Let X be a ($\mathbb{K}$-) vectorspace.

A map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}$ is a scalar product (or inner product) $:\Longleftrightarrow$

(i) $\langle x,x\rangle \geq 0 \quad \forall x \in X$, and $\langle x,x\rangle = 0 \Rightarrow x = 0$
(positive definit) (non-degenerate)

(ii) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle \quad \forall \alpha,\beta\in\mathbb{K}, \forall x,y,z\in X$
(linearity)

(iii) $\langle x,y\rangle = \overline{\langle y,x\rangle} \quad \forall x,y\in X$

$(X, \langle\cdot,\cdot\rangle)$ is an inner product space (or pre-Hilbert space)

Note: If $\mathbb{K} = \mathbb{R}$, (ii) & (iii) imply that $\langle\cdot,\cdot\rangle$ is bi-linear
If, however, $\mathbb{K} = \mathbb{C}$, (ii) & (iii) give that

$$\langle \alpha x, y\rangle = \bar{\alpha}\langle x,y\rangle \quad \forall \alpha\in\mathbb{C},\ x,y\in X \quad \left(\begin{array}{c}\text{conjugate}\\ \text{linear}\end{array}\right)$$

Then $\langle\cdot,\cdot\rangle$ is called sesqui-linear ("$1\frac{1}{2}$-linear")

Warning: In literatur also: $\langle\alpha x,y\rangle = \alpha\langle x,y\rangle$ & $\langle x,\alpha y\rangle = \bar{\alpha}\langle x,y\rangle$

Lemma 2.41 (Cauchy-Schwarz (Bunyakovsky) inequality)

Let X be an inner product space. Then

(C-S) $$|\langle x,y\rangle| \leq \langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2} \quad \forall x,y\in X$$

with equality iff x & y are linearly dependent

Proof: See Linear Algebra (and/or Ana 1-3). $\blacksquare$

Lemma 2.42 Let X be an inner product space.

Then X is a normed space with norm $\|x\| := \langle x, x\rangle^{1/2}$.

All notions & results from topological, metric, and normed spaces are available. In particular, the scalar product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is continuous.

Pf: (i) $\|\cdot\|$ fulfills all axioms of a norm, see LA (or do!)

(ii) Continuity of $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$:

Note that $X \times X$ is a normed space (f. ex. with norm $\|(x,y)\|_1 := \|x\| + \|y\|$), hence seq. cont. enough.

So let $x_n \xrightarrow{n\to\infty} x$, $y_n \xrightarrow{n\to\infty} y$ be two conv. seq.'s in X.

Then

$$
\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &\le |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\
&\overset{C\text{-}S}{\le} \|x_n\| \cdot \underbrace{\|y_n - y\|}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|y\|}_{<\infty} \cdot \underbrace{\|x_n - x\|}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to 0} 0
\end{aligned}
$$

since $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ (as $(x_n)_n$ converges & $\|\cdot\|$ cont.). $\blacksquare$

Definition 2.43 A complete inner product space is called a Hilbert space

Note: Hence, any Hilbert space is also a Banach space (for the other direction, see 2.46 below).

Theorem 2.44 Let X be an inner product space. Then there exists a Hilbert space $\tilde{X}$, a dense subspace $W \subseteq \tilde{X}$, and a unitary map $U : X \to W$.

(U is unitary :$\Leftrightarrow$ U is an isomorphism with
$$\langle x, y\rangle_X = \langle Ux, Uy\rangle_{\tilde{X}} \quad \forall x, y \in X)$$

Idea of pf: See the pf's of Thm's 2.13 & 1.14.
Additional aspect here: Define a scalar product on $\tilde{X} := \{$ equivalence classes $\tilde{x}$ of Cauchy sequences in $X\}$:

$$\langle \tilde{x}, \tilde{y}\rangle_{\tilde{X}} := \lim_{n\to\infty} \langle x_n, y_n\rangle_X$$

with $(x_n)_n, (y_n)_n \subseteq X$ being representatives of the equivalence classes $\tilde{x}$ and $\tilde{y}$, and $U : \begin{array}{l} X \to W \\ x \mapsto \tilde{x} \end{array}$
where $\tilde{x}$ is the equivalence class of the constant representative $(x, x, x, \dots)$ ∎

Examples 2.45 (a) $\ell^2 = \ell^2(\mathbb{N})$ is a Hilbert space with scalar product $\langle x, y\rangle := \sum_{n\in\mathbb{N}} \overline{x_n} y_n$, where $x = (x_n)_n$, $y = (y_n)_n$.
Of course, $\ell^2(\{1, .., N\}) \equiv \mathbb{C}^N$ is also a Hilbert space.

(b) $C([0,1])$ is an inner product space with the scalar product
$$\langle f, g\rangle := \int_0^1 \overline{f(x)}\, g(x)\, dx$$
but not a Hilbert space (proof analogous to Ex. 1.11 (b)).
(Applying Thm. 2.44 to this example gives $L^2([0,1])$, see next chapter).

(c) ℓ^p for $p \neq 2$ is not an inner product space, because of the following theorem:

(Theorem 2.46) (Jordan-von Neumann) Let $(X, \|\cdot\|)$ be a normed space. Then X is an inner product space with $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ iff $\|\cdot\|$ satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x,y \in X.$$

Pf: "$\Rightarrow$": Elementary computation (do!)

"$\Leftarrow$": Define inner product (!) by polarisation:

$$\langle x,y\rangle := \begin{cases} \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) - \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) & \mathbb{K} = \mathbb{C} \end{cases}$$

We verify that this definition satisfies axioms of an inner product:

(i) Symmetry and definiteness are obvious (check!)

(ii) Bi-/sesqui-linearity follows from parallelogram identity (here only $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} \langle x,y\rangle + \langle x,z\rangle &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2) \\ &= \frac{1}{2}\left(\left\|x + \frac{y+z}{2}\right\|^2 - \left\|x - \frac{y+z}{2}\right\|^2\right) \\ &= 2\left\langle x, \frac{y+z}{2}\right\rangle \end{aligned} \qquad (1)$$

$z=0$ in (1) $\Rightarrow$ $\langle x,y\rangle = 2\langle x, \frac{y}{2}\rangle$ (2)

(1) & (2) $\Rightarrow$ $\langle x,y\rangle + \langle x,z\rangle = \langle x, y+z\rangle$ (3)

(2) with $2y$ instead of y: $\langle x, 2y\rangle = 2\langle x,y\rangle$ (4)

By induction from (3) & (4): $\langle x, ny\rangle = n\langle x,y\rangle \quad \forall n \in \mathbb{N}_0$ (5)

$z=-y$ in (3) $\Rightarrow$ $\langle x,y\rangle = -\langle x,-y\rangle \Rightarrow$ (5) holds $\forall n \in \mathbb{Z}$

Let $m \in \mathbb{Z}\setminus\{0\}$ and use (5) with $\frac{y}{m}$ instead of y (writing $\lambda := \frac{n}{m}$):

$$\langle x, \lambda y\rangle = n\left\langle x, \frac{y}{m}\right\rangle = \lambda m \left\langle x, \frac{y}{m}\right\rangle = \lambda\langle x,y\rangle.$$

This holds for all $\lambda \in \mathbb{Q}$, and thus for all $\lambda \in \mathbb{R}$ by continuity (2.10)

∎

Definition 2.47 Let X be an inner product space, and $A \subseteq X$ (not necessarily subspace)

(a) $x, y \in X$ are *orthogonal* $:\Leftrightarrow \langle x, y \rangle = 0$. In symbols: $x \perp y$

(b) *Orthogonal complement of A*: $A^\perp := \{x \in X \mid x \perp a \quad \forall a \in A\}$

(c) Let $J \neq \emptyset$ be an index set, and for all $\alpha \in J$, let $e_\alpha \in X$.
We call $\{e_\alpha\}_{\alpha \in J}$ an *orthonormal set* (or *family*) $:\Leftrightarrow$

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} \quad \forall \alpha, \beta \in J$$

(and *orthogonal set* if $\langle e_\alpha, e_\beta \rangle = 0 \quad \forall \alpha \neq \beta$)

(d) $\{e_\alpha\}_{\alpha \in J}$ is an *orthonormal basis* (ONB) or *complete orthonormal set* $:\Leftrightarrow$

(i) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal set.

(ii) "Completeness": If $\langle e_\alpha, x \rangle = 0$ for some $x \in X$ and all $\alpha \in J$, then $x = 0$ (I.e.: The zero vector is the only vector orthogonal to all e_α's).

Lemma 2.48 Let X be an inner product space.

(a) Let $A \subseteq X$ be a subset (not necessarily subspace). Then $A^\perp$ is a closed (linear) subspace in X.

(b) Every orthonormal set $\{e_\alpha\}_{\alpha \in J}$ is linearly independent.

Pf: (a) See exercises.

(b) Assume $\sum_{j=1}^{n} \lambda_j e_{\alpha_j} = 0$ for some $n \in \mathbb{N}$, some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, and some $\alpha_1, \dots, \alpha_n \in J$ with $\alpha_j \neq \alpha_k \ \forall j \neq k$. Then we have

$$0 = \langle e_{\alpha_k}, 0 \rangle = \langle e_{\alpha_k}, \sum_{j=1}^{n} \lambda_j e_{\alpha_j} \rangle = \sum_{j=1}^{n} \lambda_j \underbrace{\langle e_{\alpha_k}, e_{\alpha_j} \rangle}_{\delta_{kj}} = \lambda_k \quad \forall k = 1, \dots, n \qquad \blacksquare$$

Next we prove two lemmas/lemmata as preparation for Theorem 2.51:

Lemma 2.49 (Uncountable series) Let $\emptyset \neq J$ be an index set and $0 \leq c_\alpha < \infty$ $\forall \alpha \in J$. Then

$$\sum_{\alpha \in J} c_\alpha := \sup_{\substack{J_0 \subseteq J \\ |J_0| < \infty}} \sum_{\alpha \in J_0} c_\alpha < \infty$$

implies $c_\alpha = 0$ for all but countably many α's.

Note: The definition agrees with the usual one, if J is countable.

Proof: Define the sets $S_0 := \{\alpha \in J \mid c_\alpha \geq 1\}$

$$S_n := \{\alpha \in J \mid \tfrac{1}{n} > c_\alpha \geq \tfrac{1}{n+1}\}, \quad n \in \mathbb{N}$$

Claim: $|S_n| < \infty$ $\forall n \in \mathbb{N}_0$

True, since, if this was violated for some $n \in \mathbb{N}$, then

$\sum_{\alpha \in J} c_\alpha \geq \sum_{\alpha \in S_n} c_\alpha = \infty$. But then $\{\alpha \in J \mid c_\alpha > 0\} = \bigcup_{n \in \mathbb{N}_0} S_n$ is a countable union of finite sets, hence countable ☒

Lemma 2.50 (Series in Banach & Hilbert spaces)

(a) Let X be a Banach space and consider a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$.

Then
$$\sum_{n \in \mathbb{N}} \|x_n\| < \infty \implies \sum_{n \in \mathbb{N}} x_n := \lim_{N \to \infty} \left(\sum_{n=1}^{N} x_n\right) \text{ exists in } X.$$

(b) Let X be a Hilbert space, and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence of pairwise orthogonal vectors (i.e. an orthogonal family).

Then
$$\sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty \iff \sum_{n \in \mathbb{N}} x_n \text{ exists in } X$$

(Note: Recall that $\ell^1 \subsetneq \ell^2$)

Pf: Let $S_n := \sum_{j=1}^{n} x_j$ for $n \in \mathbb{N}$. Since X is complete, we have:

$$\sum_{n \in \mathbb{N}} x_n \text{ exists} \iff (S_n)_n \text{ is Cauchy (in } X).$$

(a) Let $m \geq n$. Then

$$\|S_m - S_n\| = \left\|\sum_{j=n+1}^{m} x_j\right\| \leq \sum_{j=n+1}^{m} \|x_j\| = \sigma_m - \sigma_n$$

with $\sigma_n := \sum_{j=1}^{n} \|x_j\|$. But by the assumption, $(\sigma_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges in $\mathbb{R}$, and hence is Cauchy.

(b) Consider again the sequence of partial sums $(S_n)_n$. Let $m \geq n$, then

$$\|S_m - S_n\|^2 = \Big\|\sum_{j=n+1}^{m} x_j\Big\|^2 = \sum_{j,k=n+1}^{m} \langle x_j, x_k\rangle = \sum_{j=n+1}^{m} \|x_j\|^2 = \tau_m - \tau_n$$

with $\tau_n := \sum_{j=1}^{n} \|x_j\|^2$. Hence, $(S_n)_n$ is Cauchy (in $\mathfrak{X}$) iff. $(\tau_n)_{n\in\mathbb{N}}$ is Cauchy (in $\mathbb{R}$) ∎

Theorem 2.51 Let $\mathfrak{X}$ be an inner product space, $x \in \mathfrak{X}$, and $\{e_\alpha\}_{\alpha\in J}$ an orthonormal set. Then:

$$\|x\|^2 \geq \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Bessel's inequality)}.$$

If $\mathfrak{X}$ is even a Hilbert space, and if $\{e_\alpha\}_{\alpha\in J}$ is even an ONB, then

$$x = \sum_{\alpha\in J} \langle e_\alpha, x\rangle e_\alpha$$

where at most countably many terms are $\neq 0$, and

$$\|x\|^2 = \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 \qquad \text{(Parseval's equality)}.$$

Pf: Let $J_0 \subseteq J$ be any finite subset. Then we can write x as follows:

$$x = \underbrace{\sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha}_{=:u} + \underbrace{\Big(x - \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha\Big)}_{=:v}$$

Using that $\{e_\alpha\}_{\alpha\in J_0}$ is an orthonormal set, we get:

$$\langle u, u\rangle = \Big\langle \sum_{\alpha\in J_0} \langle e_\alpha, x\rangle e_\alpha, \sum_{\beta\in J_0} \langle e_\beta, x\rangle e_\beta \Big\rangle = \sum_{\alpha,\beta\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\beta, x\rangle \langle e_\alpha, e_\beta\rangle$$

$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha, x\rangle} \langle e_\alpha, x\rangle = \sum_{\alpha\in J_0} |\langle e_\alpha, x\rangle|^2,$$

$$\langle u, v\rangle = \langle u, x-u\rangle = \langle u, x\rangle - \langle u, u\rangle$$
$$= \sum_{\alpha\in J_0} \overline{\langle e_\alpha, x\rangle}\langle e_\alpha, x\rangle - \langle u, u\rangle = 0.$$

Hence, we can estimate $\|x\|^2$:

$$\|x\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + \underbrace{\langle v, v\rangle}_{\geq 0} \geq \sum_{\alpha\in J_0} |\langle e_\alpha, x\rangle|^2 \quad \forall J_0 \subseteq J,\ |J_0| < \infty$$

Taking sup over all finite subsets $J_0 \subseteq J$ gives Bessel's inequality.

Now, Lemma 2.49 $\Rightarrow \exists\, \mathcal{N} \in \mathbb{N}\cup\{\infty\}$ and $\alpha_n \in J$ $\forall n \in \{1, .., \mathcal{N}\}$ s.t.

$$(*) \quad \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2 = \sum_{n=1}^{\mathcal{N}} |\langle e_{\alpha_n}, x\rangle|^2 \leq \|x\|^2 < \infty \quad \text{(using Bessel's inep.)}$$

Then $(*)$ and Lemma 2.50(b) (which requires completeness!) imply that

$$x' := \sum_{n=1}^{\mathcal{N}} \langle e_{\alpha_n}, x\rangle e_{\alpha_n} \quad \text{exists in } X.$$

We need to prove that $x = x'$:

We have $x - x' \perp e_{\alpha_m}$ $\forall m \in \{1, .., \mathcal{N}\}$:

$$\langle e_{\alpha_m}, x-x'\rangle = \langle e_{\alpha_m}, x\rangle - \langle e_{\alpha_m}, x'\rangle = \langle e_{\alpha_m}, x\rangle - \sum_{n=1}^{\mathcal{N}} \langle e_{\alpha_n}, x\rangle \underbrace{\langle e_{\alpha_m}, e_{\alpha_n}\rangle}_{\delta_{mn}}$$
$$= 0 \quad \forall m \in \{1, .., \mathcal{N}\}$$

If $\alpha \notin \{\alpha_n \mid n \in \{1, .., \mathcal{N}\}\}$ then we have:

$$\langle e_\alpha, x-x'\rangle = \langle e_\alpha, x\rangle - \underbrace{\sum_{n=1}^{\mathcal{N}} \langle e_{\alpha_n}, x\rangle \underbrace{\langle e_\alpha, e_{\alpha_n}\rangle}_{=0}}_{=0} = \langle e_\alpha, x\rangle \overset{(*)}{=} 0$$

Taken together, we have $x - x' \perp e_\alpha$ $\forall \alpha \in J \overset{\text{ONB}}{\Rightarrow} x = x'$.

Moreover (if $\mathcal{N} < \infty$, no limits are needed):

$$\|x\|^2 = \left\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j} \right\|^2 \overset{\|\cdot\| \text{ cont.}}{=} \lim_{n\to\infty} \left\| \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j} \right\|^2$$
$$= \lim_{n\to\infty} \left\langle \sum_{j=1}^{n} \langle e_{\alpha_j}, x\rangle e_{\alpha_j}, \sum_{k=1}^{n} \langle e_{\alpha_k}, x\rangle e_{\alpha_k} \right\rangle$$
$$= \lim \sum_{j=1}^{n}\sum_{k=1}^{n} \overline{\langle e_{\alpha_j}, x\rangle}\langle e_{\alpha_k}, x\rangle \underbrace{\langle e_{\alpha_j}, e_{\alpha_k}\rangle}_{\delta_{jk}} = \sum_{j=1}^{\mathcal{N}} |\langle e_{\alpha_j}, x\rangle|^2$$
$$\overset{(*)}{=} \sum_{\alpha\in J} |\langle e_\alpha, x\rangle|^2. \quad \blacksquare$$

Lemma 2.50 (b) and Theorem 2.51 gives

Corollary 2.52 Let X be a Hilbert space and $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set in X

(a) If $(c_\alpha)_{\alpha \in J} \subseteq \mathbb{K}$ with $\sum_{\alpha \in J} |c_\alpha|^2 < \infty$, then $\sum_{\alpha \in J} c_\alpha e_\alpha$ is well-defined in X

(b) If J is countable, and $\{e_\alpha \mid \alpha \in J\}$ is even an orthonormal basis (ONB), then $\{e_\alpha\}_{\alpha \in J}$ is a Schauder basis.

Theorem 2.53 Every Hilbert space $X \neq \{0\}$ has an ONB. Moreover,

X separable $\Leftrightarrow$ there exists a countable ONB.

Pf: The existence of an ONB in the non-separable case will be proven later (when Zorn's Lemma is available).

"$\Leftarrow$": Suppose there exists a countable ONB. By Cor. 2.52(b), this is a Schauder basis. Thus, X is separable by Lemma 2.16. ✓

"$\Rightarrow$": Let $\{x_n \mid n \in \mathbb{N}\}$ be dense in X. We construct an ONB using the Gram-Schmidt procedure:

(i) If $x_1 \neq 0$, define $e_1 := \frac{x_1}{\|x_1\|}$ (otherwise, use x_{n_0} with $n_0 := \min\{k \in \mathbb{N} \mid x_k \neq 0\}$)

(ii) Throw away all x_n's that are linearly dependent of e_1.
Let $n_1 := \min\{k \in \mathbb{N} \mid x_k \notin \operatorname{span}\{e_1\}\} > n_0$ be the smallest index of the remaining elements. Set

$$\tilde{e}_2 := x_{n_1} - \langle e_1, x_{n_1} \rangle e_1 \quad \text{and} \quad e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|}.$$

(iii) Throw away all x_n's that are linearly dependent of $\{e_1, e_2\}$. Let $n_2 := \min\{k \in \mathbb{N} \mid x_k \notin \operatorname{span}\{e_1, e_2\}\} > n_1$ be the smallest index of the remaining elements. Set

$$\tilde{e}_3 := x_{n_2} - \langle e_1, x_{n_2} \rangle e_1 - \langle e_2, x_{n_2} \rangle e_2 \quad \text{and} \quad e_3 := \frac{\tilde{e}_3}{\|\tilde{e}_3\|}.$$

(iv) Continue this procedure.

This terminates iff $\dim X < \infty$. It is clear, that $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal set. Also, $\operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ is dense in X because each x_n is a finite linear combination of e_n's. It remains to prove that this is a basis: Assume there is $y \in X$ such that $y \perp e_n$ $\forall n$. By denseness, $\exists (y_k)_k \subseteq \operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ such that $y_k \xrightarrow{k\to\infty} y$ in X. Thus $\langle y, y_k \rangle = 0$ $\forall k$ and

$$\|y\|^2 = \langle y, y \rangle = \lim_{k\to\infty} \underbrace{\langle y, y_k \rangle}_{=0} = 0 \qquad \blacksquare$$

Definition Let X, X' be inner product spaces.

X, X' are <u>unitarily equivalent</u> $:\Leftrightarrow$ $\exists$ isometric isomorphism $U : X \to X'$

<u>Note:</u> U isometric & linear $\Rightarrow$ U preserves scalar products:

$$\langle x, y \rangle_X = \langle Ux, Uy \rangle_{X'}, \quad \forall x, y \in X \quad \text{(follows from polarisation, see proof of Thm. 2.46)}$$

Theorem 2.54 Let X be a Hilbert space. If $n := \dim X < \infty$ then X is unitarily equivalent to $\mathbb{K}^n$ (with Euclidean scalar product). If $\dim X = \infty$ and X is <u>separable</u>, then X is unit. equiv. to ℓ^2

<u>Pf: 2nd case</u>: Fix a countable ONB $\{e_n\}_{n\in\mathbb{N}}$ (exists by Thm. 2.53 b/c X is separable) and consider the map

$$U : X \to \ell^2$$
$$x \mapsto (\langle e_n, x \rangle)_{n\in\mathbb{N}}$$

(i) U is well-defined by Bessel's inequality, and linear.

(ii) U is surjective by Cor. 2.52 (a)

(iii) U is isometric ($\Rightarrow$ injective) by Parseval's equality.

<u>1st case</u>: Analogous $\blacksquare$

Theorem 2.55 (<u>Projection theorem</u>) Let X be a Hilbert space and $A \subseteq X$ a <u>closed</u> subspace. Then: For all $x \in X$ there exists a <u>unique</u> $a \in A$ and a <u>unique</u> $w \in A^\perp$ such that

$$x = a + w$$

The proof relies on:

Lemma 2.56 Let X, A be as in Thm. 2.55.
For every $x \in X$ there exists a unique $a \in A$ such that
$\operatorname{dist}(x, A) = \|x - a\|$. That is, a is the "closest element" to x in A.

Pf (2.56): Existence: Let $d := d(x, A) = \inf_{y \in A} \|x - y\|$.
There exists a sequence $(y_n)_{n \in \mathbb{N}} \subseteq A$ s.t.

$$(*) \qquad d = \lim_{n \to \infty} \underbrace{\|x - y_n\|}_{\geq d} \qquad \text{(by def. of "inf"; } (y_n)_n \text{ is a minimizing sequence)}$$

Using the parallelogram identity
$\|u + v\|^2 = 2(\|u\|^2 + \|v\|^2) - \|u - v\|^2$ (Thm. 2.46) we get

$$\begin{aligned}
\|y_n - y_m\|^2 &= \|(y_n - x) + (x - y_m)\|^2 \\
&= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \|2x - y_m - y_n\|^2 \\
(**) \quad &= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \underbrace{4\,\|x - \underbrace{\tfrac{y_m + y_n}{2}}_{\in A \text{ (subspace)}}\|^2}_{\geq 4d^2} \\
&\leq 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4d^2 \\
&\longrightarrow 0, \quad m, n \to \infty \quad (\text{by } (*)).
\end{aligned}$$

Hence, $(y_n)_n$ is Cauchy, so (X complete) $\exists a \in X : y_n \to a, n \to \infty$.
But $(y_n)_n \subseteq A$ and A is closed, so $a \in A$, and $d = \|x - a\|$ by $(*)$ and continuity of the norm.

Uniqueness: Assume there exists $(y_n')_n \subseteq A$ with

$$\lim_{n \to \infty} \|y_n' - x\| = d \quad \text{and} \quad a' = \lim_{n \to \infty} y_n'$$

Replacing y_m by y_m' in $(**)$ gives

$$\begin{aligned}
\|a - a'\|^2 &= \lim_{n \to \infty} \lim_{m \to \infty} \|y_n - y_m'\|^2 \\
&\leq \lim_{n \to \infty} \lim_{m \to \infty} 2(\|y_n - x\|^2 + \|y_m' - x\|^2) - 4d^2 = 0
\end{aligned}$$

$$\Rightarrow a = a' \qquad \blacksquare$$

Remark: Note that the proof works if A is closed and convex (not necessarily a subspace)

Pf (of 2-55): Existence: For given $x \in \mathfrak{X}$, let $a \in A$ be defined by Lemma 2-56, and set $w := x - a$. Need to prove: $w \in A^\perp$

Let $\xi \in \mathbb{K}$ and $y \in A$. Then

$$\|w\|^2 \le \|x - \underbrace{\underbrace{(a + \xi y)}_{\in A}}_{w - \xi y}\|^2 = \|w\|^2 + \|\xi y\|^2 - 2\,\mathrm{Re}\langle w, \xi y\rangle.$$

So for $0 \neq y \in A$ we have $|\xi|^2 - \dfrac{2\,\mathrm{Re}\langle w, \xi y\rangle}{\|y\|^2} \ge 0$

(i) For $\xi = t \in \mathbb{R}$ this gives

$$t^2 - \frac{2\,\mathrm{Re}\langle w, y\rangle}{\|y\|^2}\, t \ge 0 \quad \forall t \in \mathbb{R}$$

This implies $\mathrm{Re}\langle w, y\rangle = 0$. If $\mathbb{K} = \mathbb{R}$ we are done, since $y \in A$ was arbitrary.

For $\mathbb{K} = \mathbb{C}$ we also consider the following case:

(ii) For $\xi = it$, $t \in \mathbb{R}$:

$$t^2 + \frac{2\,\mathrm{Im}\langle w, y\rangle}{\|y\|^2}\, t \ge 0 \quad \forall t \in \mathbb{R}$$

So $\mathrm{Im}\langle w, y\rangle = 0$ for every $y \in A$. Together with (i) this gives $\langle w, y\rangle = 0$ and so $w \in A^\perp$.

Uniqueness: Assume there are $a, a' \in A$ and $w, w' \in A^\perp$ with $a + w = x = a' + w'$. Then $\underbrace{a - a'}_{\in A} = \underbrace{w' - w}_{\in A^\perp}$. Then $a = a'$ and $w = w'$ as $A \cap A^\perp = \{0\}$. $\blacksquare$

Theorem 2-57 (Riesz' Representation Thm.) Let $\mathfrak{X}$ be a Hilbert space, and let $\ell \in \mathfrak{X}^*$. Then there exists a unique $y_\ell \in \mathfrak{X}$ such that

$$\ell(x) = \langle y_\ell, x\rangle \quad \forall x \in \mathfrak{X} \qquad (*)$$

and

$$\|\ell\|_* = \|y_\ell\|.$$

Pf: If $\ker \ell = X \Rightarrow \ell = 0$ and thm. true with $y_\ell = 0$.

Assume therefor $\ker \ell \subsetneq X$. Note that $\ker \ell$ is closed.

Existence of y_ℓ: From Thm. 2.55 we know $(\ker \ell)^\perp \supsetneq \{0\}$ and this allows to choose $0 \neq x_0 \in (\ker \ell)^\perp$. Define

$$y_\ell := \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0 \in X$$

(i) So (*) holds for every $x \in \ker \ell$ $(x_0 \perp \ker \ell)$

(ii) Let $x = \alpha x_0$, $\alpha \in \mathbb{K}$. Then $\ell(x) = \alpha \ell(x_0)$ and

$$\langle y_\ell, x\rangle = \left\langle \frac{\overline{\ell(x_0)}}{\|x_0\|^2} \cdot x_0, \alpha x_0 \right\rangle = \frac{\ell(x_0)}{\|x_0\|^2} \alpha \langle x_0, x_0\rangle = \alpha \ell(x_0),$$

so ℓ and $\langle y_\ell, \cdot\rangle$ agree on $\operatorname{span}\{\ker \ell, x_0\}$.

(iii) But $\operatorname{span}\{\ker \ell, x_0\} = X$ because, for all $x \in X$,

$$x = \underbrace{\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0\right)}_{\in \ker \ell} + \underbrace{\frac{\ell(x)}{\ell(x_0)} \cdot x_0}_{\in \operatorname{span}\{x_0\}}$$

since

$$\ell\left(x - \frac{\ell(x)}{\ell(x_0)} \cdot x_0\right) = \ell(x) - \frac{\ell(x)}{\ell(x_0)} \ell(x_0) = 0.$$

So $\ell = \langle y_\ell, \cdot\rangle$ on X by (i), (ii), and the linearity of ℓ and $\langle y_\ell, \cdot\rangle$.

Uniqueness of y_ℓ: Assume there exists $y_\ell' \in X$ with $\ell = \langle y_\ell', \cdot\rangle$. Then, for every $x \in X$,

$$0 = \underbrace{\ell(x)}_{\langle y_\ell, x\rangle} - \underbrace{\ell(x)}_{\langle y_\ell', x\rangle} = \langle y_\ell - y_\ell', x\rangle$$

Choose $x = y_\ell - y_\ell'$, so that $0 = \|y_\ell - y_\ell'\|^2$, so $y_\ell = y_\ell'$

Norm: We have

$$\|\ell\|_0 = \sup_{0 \neq x \in X} \underbrace{\frac{|\ell(x)|}{\|x\|}}_{\frac{|\langle y_\ell, x\rangle|}{\|x\|}} \overset{C-S}{\leq} \|y_\ell\|$$

Note that $y_\ell \neq 0$ since $\ker \ell \subsetneq X$. Hence, we may choose $x = y_\ell$ in sup above:

$$\|\ell\|_* \geq \frac{|\ell(y_\ell)|}{\|y_\ell\|} = \frac{|\langle y_\ell, y_\ell \rangle|}{\|y_\ell\|} = \|y_\ell\|$$

hence, $\|\ell\|_* = \|y_\ell\|$ ∎

Corollary 2.58 Let X be a Hilbert space. Then the map

$$J: X^* \to X$$
$$\ell \mapsto y_\ell$$

defined by Thm. 2.57 is a semi-linear, isometric bijection (linear for $\mathbb{K} = \mathbb{R}$, conjugate linear for $\mathbb{K} = \mathbb{C}$).

Pf: Let $\alpha, \beta \in \mathbb{K}$, and $\ell_1, \ell_2 \in X^*$ with $\ell_j = \langle y_j, \cdot \rangle$, i.e. $J(\ell_j) = y_j$ for $j = 1, 2$. Then

$$\alpha \ell_1 + \beta \ell_2 = \alpha \langle y_1, \cdot \rangle + \beta \langle y_2, \cdot \rangle = \langle \bar{\alpha} y_1 + \bar{\beta} y_2, \cdot \rangle$$

So $J(\alpha \ell_1 + \beta \ell_2) = \bar{\alpha} J(\ell_1) + \bar{\beta} J(\ell_2)$ (by uniqueness in 2.57)

Hence, J is semi-linear. That J is an isometry was proved in 2.57 ($\Rightarrow$ J injective). Also, J is onto / surjective:

For $y \in X$, $\ell_y := \langle y, \cdot \rangle \in X^*$ (use C-S, or Lem. 2.42, to prove continuity) and $J(\ell_y) = y$ (by uniqueness in 2.57) ∎
