2.4. Linear functionals and dual space

Definition 2.35 Let X be a normed space. A *linear functional* (on X) is a linear operator $\ell : X \supseteq \mathrm{dom}(\ell) \to \mathbb{K}$.

The *dual space* of X is $X^* := BL(X, \mathbb{K})$.

Notations for the norm on X^*: $\|\cdot\|_{X \to \mathbb{K}} =: \|\cdot\|_{X^*} =: \|\cdot\|_* =: \|\cdot\|$.

Note: X^* consists of the bounded (equivalently, continuous) linear functionals; X^* is therefore also called the topological dual - not to be confused with the algebraic dual of all linear functionals, $X' := \{f : X \to \mathbb{K} \mid f \text{ linear}\} = \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K})$, common in Linear Algebra!

Corollary 2.36 (to Thm. 2.34). $(X^*, \|\cdot\|_{X^*})$ is a Banach space (whether X is complete or not).

Examples 2.37 Let $X = C([a,b])$ (with $a < b \in \mathbb{R}$) equipped with $\|\cdot\|_\infty$

(a) For $f \in X$, let
$$\ell(f) := \int_a^b f(x)\, dx \in \mathbb{K}$$
This is clearly a linear functional $\ell : X \to \mathbb{K}$ with
$$|\ell(f)| \le \int_a^b \underbrace{|f(x)|}_{\le \|f\|_\infty} dx \le \|f\|_\infty (b-a) \quad (*)$$
so $\|\ell\|_{X^*} \le (b-a)$, and $f \equiv 1$ gives equality in $(*)$, so $\|\ell\|_{X^*} = b-a$.

(b) For $f \in X$ and $t_0 \in [a,b]$, let $\delta_{t_0}(f) := f(t_0) \in \mathbb{K}$.
This gives a linear functional $\delta_{t_0} : X \to \mathbb{K}$, called the *Dirac δ-functional (at t_0)*, with $|\delta_{t_0}(f)| = |f(t_0)| \le \|f\|_\infty$, and again equality for $f \equiv 1$. So $\|\delta_{t_0}\|_{X^*} = 1$.

NB! Boundedness (i.e., continuity) of δ_{t_0} depends on the norm on X. For ex., δ_{t_0} is not bounded, if X is equipped with the L^1-norm ($\|f\|_1 = \int_0^1 |f(x)|\, dx$).

Notation: Let X, Y be normed spaces. We write

$X \cong Y$ (or $X \simeq Y$) iff X is isometrically isomorphic to Y.

Theorem 2.38 Let $p \in [1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* \cong \ell^q$.

More precisely: The map $\ell^q \ni y = (y_n)_n \longmapsto f_y \in (\ell^p)^*$,

$$f_y(x) := \sum_{n \in \mathbb{N}} x_n y_n \quad , \quad x \in \ell^p$$

is a bijective isometry

Pf: (a) Case $1 < p < \infty$: Let $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$, $f \in (\ell^p)^*$.

We work with the canonical (Schauder) basis $(e_n)_{n \in \mathbb{N}}$ for ℓ^p (see 2.15)

Then x can be written as a $\|\cdot\|_p$-convergent series $x = \sum_{n \in \mathbb{N}} x_n e_n$

Since f is continuous (wrt. $\|\cdot\|_p$) and linear,

$$f(x) = f\Big(\lim_{N \to \infty} \sum_{n=1}^{N} x_n e_n\Big) = \sum_{n \in \mathbb{N}} f(x_n e_n) = \sum_{n \in \mathbb{N}} x_n f(e_n). \qquad (*)$$

For $N \in \mathbb{N}$ (fixed), define $\tilde{x} := (\tilde{x}_n)_{n \in \mathbb{N}} \in \ell^p$ by

$$\tilde{x}_n := \begin{cases} \dfrac{|f(e_n)|^q}{f(e_n)} & \text{if } n \le N \text{ and } f(e_n) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Apply $(*)$ to $\tilde{x}$:
$$f(\tilde{x}) = \sum_{n=1}^{N} |f(e_n)|^q \qquad (**)$$

so that $0 \le f(\tilde{x}) \le \|f\|_{(\ell^p)^*} \cdot \|\tilde{x}\|_p$, where (compute!)

$$\|\tilde{x}\|_p = \Big(\sum_{n=1}^{N} |f(e_n)|^{(q-1)p}\Big)^{1/p}.$$

Since $\frac{1}{p} = 1 - \frac{1}{q} \Leftrightarrow p = \frac{q}{q-1}$ (take note!), we have $p(q-1) = q$, so

$$(**) \Rightarrow \sum_{n=1}^{N} |f(e_n)|^q \le \|f\|_{(\ell^p)^*} \Big(\sum_{n=1}^{N} |f(e_n)|^q\Big)^{1/p}$$

$$\Rightarrow \Big(\sum_{n=1}^{N} |f(e_n)|^q\Big)^{1/q} \le \|f\|_{(\ell^p)^*} \quad \forall N \in \mathbb{N} \qquad (***)$$

Hence, the map
$$J: (\ell^p)^* \to \ell^q$$
$$f \mapsto (f(e_n))_{n\in\mathbb{N}}$$
is well-defined, and

(i) J is linear

(ii) $\|Jf\|_q = \|(f(e_n))_{n\in\mathbb{N}}\|_q \le \|f\|_{(\ell^p)^*}$ by (***)

(iii) J is onto/surjective: If $y = (y_n)_n \in \ell^q$, define
$$f_y : \begin{array}{l} \ell^p \to \mathbb{K} \\ x = (x_n)_n \mapsto f_y(x) := \sum_{n\in\mathbb{N}} x_n y_n \end{array}$$
Clearly, f_y is linear. It is well-defined: $\sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$ by Hölder's inequality, so
$$\|f_y\|_{(\ell^p)^*} \le \|y\|_q$$
and so $f_y \in (\ell^p)^*$. But $f_y(e_n) = y_n \ \forall n\in\mathbb{N}$, so $J(f_y) = y$.

(iv) (*) & Hölder imply:
$$\forall f\in(\ell^p)^* \ \forall x\in\ell^p: \ |f(x)| \le \|x\|_p \cdot \|\underbrace{(f(e_n))_{n\in\mathbb{N}}}_{Jf}\|_q$$
$$\Rightarrow \|f\|_{(\ell^p)^*} \le \|Jf\|_q$$

So $\|Jf\|_q = \|f\|_{(\ell^p)^*}$, and J is an isometric isomorphism.
(The map in the statement of the thm. is J^{-1}).

(b) The case $p=1$ is analogous, but instead of defining $\tilde{x}_n$, use
$$|f(e_n)| \le \|f\|_{(\ell^1)^*} \cdot \underbrace{\|e_n\|_1}_{=1}$$
This implies $\|(f(e_n))_n\|_\infty \le \|f\|_{(\ell^1)^*}$, which replaces (***), and the properties of J follow as above. ∎

$\boxed{\text{Remark 2.39}}$ (a) $(c_0)^* \cong \ell^1$ (see exercise)

(b) The map $\ell^1 \to (\ell^\infty)^*$, $y \mapsto f_y$, defined (as in Thm. 2.38) by
$$f_y(x) = \sum_{n\in\mathbb{N}} x_n y_n, \quad x\in\ell^\infty, \ y\in\ell^1$$
is well-defined (Hölder!), linear, isometric, but <u>not</u> onto!
In other words: $(\ell^\infty)^*$ is strictly "larger" than ℓ^1! (see later)