(ii) Let $U_n := \operatorname{span}\{x_1, \dots, x_n\} \subsetneq X$ be the closed subspace of the vectors constructed before. Again, by Riesz' Lemma, there exists $x_{n+1} \in X \setminus U_n$ s.t. $\|x_{n+1}\| = 1$ and $\operatorname{dist}(x_{n+1}, U_n) \geq \frac{1}{2}$.

By assumption, $\dim X = \infty$, hence this procedure does not stop. We get a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \overline{B_1(0)}$ s.t. $\|x_n - x_m\| \geq \frac{1}{2}$ $\forall n \neq m$. Clearly, $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence - contradicting the (seq.) compactness of $\overline{B_1(0)}$ ↯ ∎

2.3 Linear operators

Definition 2.22 Let X, Y be vector spaces (over the same field $\mathbb{K}$), $X_0 \subseteq X$ a (linear) subspace, and $T: X_0 \to Y$.

(i) T is a (linear) operator $:\Leftrightarrow$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \alpha, \beta \in \mathbb{K}, \ \forall x, y \in X_0$$

(ii) $\operatorname{dom}(T) := D(T) := X_0$ is the domain of T.

(iii) $\operatorname{ran}(T) := R(T) := T(X_0)$ is the range of T

(iv) $\ker(T) := N(T) := \{x \in X_0 \mid Tx = 0\}$ is the kernel (or nullspace) of T

(v) $U \subseteq \operatorname{dom}(T)$ a subspace,

$$T|_U : \begin{array}{l} U \to Y \\ x \mapsto Tx \end{array} \quad \text{is the restriction of } T \text{ to } U$$

(vi) $W \supseteq \operatorname{dom}(T)$ a vector space, $\tilde{T}: W \to Y$ linear with $\tilde{T}|_{\operatorname{dom}(T)} = T$ is called a (!) linear extension of T to W.

Examples 2.23 Some linear operators:

(a) Identity operator on vector space X:

$$\mathbb{1} := \mathbb{1}_X : \begin{array}{l} X \to X \\ x \mapsto x \end{array}$$

(b) $\mathbb{X} = Y = C([0,1])$

(i) Differentiation operator ($\mathbb{X}_0 = C^1([0,1])$):

$$\frac{d}{dx} : C^1([0,1]) \to C([0,1])$$
$$f \mapsto f'$$

(ii) Anti-derivative operator ($\mathbb{X}_0 = \mathbb{X}$):

$$T : C([0,1]) \to C([0,1])$$
$$f \mapsto Tf$$

with $\quad (Tf)(x) := \int_0^x f(t)\,dt \quad \forall x \in [0,1]$

(iii) Multiplication operator by argument: As above (in (ii)), with $(Tf)(x) := x f(x) \;\; \forall x \in [0,1]$

Lemma 2.24 Let $T : \mathbb{X} \supseteq \mathrm{dom}(T) \to Y$ be a linear operator. Then

(a) $\mathrm{ran}(T)$ and $\ker(T)$ are vector subspaces (linear subspaces)

(b) $\dim \mathrm{ran}(T) \leq \dim \mathrm{dom}(T)$

(c) $\ker(T) = \{0\} \iff T$ injective

$\iff \exists$ inverse T^{-1} of T s.t. $T^{-1} : \mathrm{ran}(T) \to \mathrm{dom}(T)$, $T^{-1}T = \mathbb{1}|_{\mathrm{dom}(T)}$, $TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$

Pf: Copy from Linear Algebra ∎

Remark 2.25 (a) If T^{-1} exists, it is linear.

Pf: $\alpha x = T^{-1}T(\alpha x) = T^{-1}(\alpha T x) \;\; \forall x \in \mathrm{dom}(T),\; \forall \alpha \in \mathbb{K}$. Hence, $\alpha T^{-1} y = T^{-1}(\alpha y) \;\; \forall y \in \mathrm{ran}(T)$. Similarly for addition.

(b) Even if $T : \mathbb{X} \to \mathbb{X}$ with $\ker(T) = \{0\}$, then not necessarily $\mathrm{ran}(T) = \mathbb{X}$, nor $TT^{-1} = \mathbb{1}$ (unlike in the case $\dim \mathbb{X} < \infty$!) but only $TT^{-1} = \mathbb{1}|_{\mathrm{ran}(T)}$.

Example 2.26 (Illustrating Rmk 2.25)

Let $\mathbb{X} = \ell^\infty$.

Right-shift operator: $R: \ell^\infty \longrightarrow \ell^\infty$
$$(x_1, x_2, x_3, \dots) \longmapsto (0, x_1, x_2, \dots)$$

Clearly: $\ker(R) = \{0\}$, but $\operatorname{ran}(R) \subsetneq \ell^\infty$.

Left-shift operator: $L: \ell^\infty \longrightarrow \ell^\infty$
$$(x_1, x_2, x_3, \dots) \longmapsto (x_2, x_3, x_4, \dots)$$

Then $R^{-1} = L|_{\operatorname{ran}(R)}$ satisfies
$$R^{-1}R = \mathbb{1} \quad \text{and} \quad RR^{-1} = \mathbb{1}|_{\operatorname{ran}(R)}.$$

Definition 2.27 Let X, Y be normed spaces, $T: X \supseteq \operatorname{dom}(T) \to Y$.

T is bounded (operator) $:\Leftrightarrow$ its operator norm is finite, i.e.,
$$\|T\|_{\operatorname{dom}(T)\to Y} := \|T\| := \sup_{\substack{x \in \operatorname{dom}(T)\\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in \operatorname{dom}(T)\\ \|x\| = 1}} \|Tx\| < \infty.$$

Examples 2.28 (Recall Ex. 2.23)

(a) $\mathbb{1}: (X, \|\cdot\|) \to (X, \|\cdot\|)$, $x \mapsto x$, is bounded with $\|\mathbb{1}\| = 1$.

(b) Let $(X, \|\cdot\|) = (C([0,1]), \|\cdot\|_\infty)$.

(i) Differentiation $\frac{d}{dx}: C^1([0,1]) \to C([0,1])$
$$f \mapsto f'$$
is unbounded (i.e., not bounded); see exercise.

(ii) Anti-derivative $T: C([0,1]) \to C([0,1])$:
$$\|Tf\|_\infty = \sup_{x \in [0,1]} \left| \int_0^x f(t)\,dt \right| \le \|f\|_\infty \quad \forall f \in C([0,1])$$

Because of $\frac{\|Tf\|}{\|f\|} \le 1$ $(f \neq 0)$, we have $\|T\| \le 1$.

Also,
$$\|T1\|_\infty = \sup_{x\in[0,1]} \int_0^x dt = \sup_{x \in [0,1]} x = 1,$$
and, since $\|1\|_\infty = 1$, we obtain $\|T\| = 1$.

(iii) Multiplication operator $T: C([0,1]) \to C([0,1])$ by argument has $\|T\| = 1$, because

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |x f(x)| \le \|f\|_\infty, \text{ with equality for } f \equiv 1.$$

Theorem 2.29 Let X, Y be normed spaces, $T: X \supseteq \mathrm{dom}(T) \to Y$ a linear operator. Then the following statements are equivalent:

(i) T is continuous

(ii) T is continuous at some $x_0 \in \mathrm{dom}(T)$.

(iii) There exists $c \in (0,\infty)$ s.t.

$$\|Tx\| \le c\|x\| \quad \forall x \in \mathrm{dom}(T).$$

(iv) T is bounded

Pf: (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (iii): Step 1: Prove T is continuous at $x = 0$.

For this, let $(x_n)_n \subseteq \mathrm{dom}(T)$ be a sequence s.t. $x_n \to 0$, $n \to \infty$. Then (by 2.10(b)), $(x_n + x_0)_n \to x_0$, $n\to\infty$, and by linearity and continuity at x_0, we get

$$Tx_n + Tx_0 = T(x_n + x_0) \xrightarrow{n\to\infty} Tx_0, \text{ so } Tx_n \xrightarrow{n\to\infty} 0 \text{ (again, by 2.10(b))}.$$

Step 2: From Step 1 we get (ε-δ criterion, with $\varepsilon = 1$)

$$\exists \delta > 0 : \forall x \in \mathrm{dom}(T) \text{ with } \|x\| < \delta : \quad \|Tx\| < 1 \qquad (*)$$

Hence, for all $0 \neq x \in \mathrm{dom}(T)$, by linearity:

$$\|Tx\| = \frac{2\|x\|}{\delta} \Big\| T\Big(\underbrace{\frac{\delta}{2\|x\|} x}_{\text{has norm } = \frac{\delta}{2} < \delta}\Big)\Big\| \overset{(*)}{<} \frac{2}{\delta}\|x\|.$$

(iii) $\Rightarrow$ (iv): We know that $\|Tx\| \le c\|x\|$, so

$$\sup_{\substack{x\in \mathrm{dom}(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \le c < \infty.$$

(iv) $\Rightarrow$ (i): Let $(x_n)_n \subseteq \mathrm{dom}(T)$ with $x_n \xrightarrow{n\to\infty} x$, then

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \le \|T\|\cdot\|x_n - x\| \xrightarrow{n\to\infty} 0 \quad \blacksquare$$

Lemma 2.30 Let T be linear and $\dim \operatorname{dom}(T) < \infty$. Then T is bounded.

Pf: T is continuous: Fix (finite!) basis of $\operatorname{dom}(T)$, expand $x \in \operatorname{dom}(T)$ w.r.t. this basis, and use linearity to deduce (sequential) continuity. The claim follows from Thm. 2.29 ∎

Theorem 2.31 (Bounded linear extension) Let X be a normed space, and Y a Banach (!) space. Let $T: X \to Y$ be a bounded linear operator. Let $\tilde{X}$ be the completion of X. Then there exists a bounded linear extension $\tilde{T}: \tilde{X} \to Y$ of T, which is unique if X is identified with a (dense) subspace of its completion $\tilde{X}$, i.e. $X = W$ in Theorem 2.13. Moreover, we have $\|\tilde{T}\| = \|T\|$.

Corollary 2.32 Let X be a normed space and Y a Banach space. Let $T: X \supseteq \operatorname{dom}(T) \to Y$ be a bounded linear operator with $\operatorname{dom}(T) \subseteq X$ a dense linear subspace. Then there exists a unique bounded linear extension $\hat{T}: X \to Y$ of T. Moreover, we have $\|\hat{T}\| = \|T\|$.

Pf (of 2.32): Because of denseness, the completion Z of $\operatorname{dom}(T)$ and $\tilde{X}$ of X are isometrically isomorphic (Check!). Wlog, we identify $Z = \tilde{X}$. Also identify X with a dense linear subspace of $\tilde{X}$, so that $\operatorname{dom}(T) \subseteq X \subseteq \tilde{X}$. The claim follows from Theorem 2.31 with $\hat{T} := \tilde{T}|_X$. ∎

Pf (of Thm. 2.31): We identify X with a dense subspace of $\tilde{X}$. Let $x \in \tilde{X}$. Then there exists a sequence $(x_n)_n \subseteq X$ with $x_n \xrightarrow{n \to \infty} x$ in $\tilde{X}$. Since $(x_n)_n$ is a Cauchy sequence, $(Tx_n)_n$ is a Cauchy sequence in Y because

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \cdot \|x_n - x_m\|.$$

Using that Y is a Banach space, there exists $y \in T$ s.t. $Tx_n \xrightarrow{n\to\infty} y$ in Y. Define $\tilde{T}x := y$ (then $\tilde{T}|_X = T$: take constant sequences!). We have to check several things:

(i) <u>Well-definedness</u>, i.e., independence of approximating seq.:
Let $x_m' \xrightarrow{m\to\infty} x$ in $\tilde{X}$. As above, $\exists y' \in Y$ s.t. $Tx_m' \xrightarrow{m\to\infty} y'$.
Then we have

$$\|Tx_n - Tx_m'\| \leq \|T\| \cdot \|x_n - x_m'\|.$$

In the limit $n, m \to \infty$ we get $\|y - y'\| = 0$, so $y = y'$.

(ii) <u>Linearity</u>: Let $x_n \xrightarrow{n\to\infty} x$, $x_n' \xrightarrow{n\to\infty} x'$. Then (by 2.10 (b) & (c)), $\alpha x_n + \alpha' x_n' \xrightarrow{n\to\infty} \alpha x + \alpha' x'$ $\forall \alpha, \alpha' \in \mathbb{K}$, and so (by def. of $\tilde{T}$)

$$\tilde{T}(\alpha x + \alpha' x') = \lim_{n\to\infty} T(\alpha x_n + \alpha' x_n') = \lim_{n\to\infty} \alpha T x_n + \lim_{n\to\infty} \alpha' T x_n'$$
$$= \alpha \tilde{T} x + \alpha \tilde{T} x'.$$

(iii) <u>Norm</u>: As $\tilde{T}$ is an extension, we have $\|\tilde{T}\| \geq \|T\|$, since

$$\|\tilde{T}\| = \sup_{\substack{x \in \tilde{X} \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} \overset{\tilde{X} \supseteq X}{\geq} \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|.$$

On the other hand, since $\|\cdot\|$ is continuous (Thm. 2.10 (a)),

$$\|\tilde{T}x\| = \lim_{\substack{n\to\infty \\ x_n \in X}} \|\tilde{T}x_n\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\| \cdot \|x_n\|$$
$$= \|T\| \cdot \lim_{n\to\infty} \|x_n\| = \|T\| \cdot \|x\|.$$

So, $\dfrac{\|\tilde{T}x\|}{\|x\|} \leq \|T\|$ $\forall x \in \tilde{X}$, hence $\|\tilde{T}\| \leq \|T\|$, so $\|\tilde{T}\| = \|T\|$.

(iv) <u>Uniqueness</u>: The fact that we defined

$\tilde{T}x := \lim_{m\to\infty} Tx_m$ is necessary to ensure continuity at x. ∎

Definition 2.33 Let $\mathbb{X}, Y$ be normed spaces. Define

$$BL(\mathbb{X},Y) := \{T:\mathbb{X}\to Y \mid T \text{ is linear and bounded}\}$$

and set $BL(\mathbb{X}) := BL(\mathbb{X},\mathbb{X})$.

Warning: Notation varies - other choices: $B(\mathbb{X},Y)$, $L(\mathbb{X},Y)$, ...

Theorem 2.34 $(BL(\mathbb{X},Y), \|\cdot\|_{\mathbb{X}\to Y})$ is a normed space.
If $\underline{\underline{Y}}$ is complete, then so is $BL(\mathbb{X},Y)$.

Pf: (i) First of all, $BL(\mathbb{X},Y)$ is a $\mathbb{K}$-vectorspace with zero element $\mathbb{0} := 0 : \mathbb{X} \to Y$, $x \longmapsto 0$

For $T_1, T_2 \in BL(\mathbb{X},Y)$ and $\alpha \in \mathbb{K}$, define T_1+T_2 and αT_1 by

$$(T_1+T_2)x := T_1x + T_2x \qquad \forall x \in \mathbb{X}.$$
$$(\alpha T_1)x := \alpha T_1 x$$

Then clearly $T_1+T_2, \alpha T_1 : \mathbb{X}\to Y$ are linear.

(ii) $\|\cdot\|_{\mathbb{X}\to Y}$ is a norm on $BL(\mathbb{X},Y)$:

(a) $\|T\|_{\mathbb{X}\to Y} \geq 0$ ✓ and if $\|T\|_{\mathbb{X}\to Y} = 0$, then $\|Tx\| = 0$ $\forall x\in\mathbb{X}$, so $Tx = 0$ $\forall x \in \mathbb{X}$, and so $T = 0$ $(=\mathbb{0})$.

(b) We have

$$\|\alpha T\|_{\mathbb{X}\to Y} = \sup_{0\neq x\in\mathbb{X}} \frac{\|(\alpha T)x\|}{\|x\|} = \sup_{0\neq x\in\mathbb{X}} \frac{\|\alpha Tx\|}{\|x\|}$$
$$= \sup_{0\neq x\in\mathbb{X}} |\alpha|\cdot\frac{\|Tx\|}{\|x\|} = |\alpha|\cdot\|T\|$$

Also, $\|(T_1+T_2)x\| = \|T_1x + T_2x\| \leq \|T_1x\| + \|T_2x\|$, so

$$\|T_1+T_2\| \leq \sup_{0\neq x\in\mathbb{X}} \left(\frac{\|T_1x\|}{\|x\|} + \frac{\|T_2x\|}{\|x\|}\right) \leq \|T_1\| + \|T_2\|.$$

Hence, if $T_1, T_2 \in BL(\mathbb{X},Y)$, then αT_1 and T_1+T_2 are bounded, so $T_1+T_2, \alpha T_1 \in BL(\mathbb{X},Y)$ (vectorspace!), and $\|\cdot\|_{\mathbb{X}\to Y}$ is a norm.

(iii) Assume Y is complete. We prove completeness of $BL(X,Y)$. Let $(T_k)_{k\in\mathbb{N}} \subseteq BL(X,Y)$ be a Cauchy sequence (wrt. $\|\cdot\|_{X\to Y}$).

For every $x\in X$ and $k,l\in\mathbb{N}$, we have $\|T_k x - T_l x\| \leq \|T_k - T_l\|\cdot\|x\|$.

So,

$$\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall k,l\geq N\ \forall x\in X:\ \|T_k x - T_l x\| \leq \varepsilon\|x\| \quad (*)$$

This implies that $(T_k x)_{k\in\mathbb{N}} \subseteq Y$ is a Cauchy seq. (in Y) for every $x\in X$, and, since Y is complete, there is some limit

$$\lim_{k\to\infty} T_k x =: Tx \in Y.$$

This defines a map

$$T: X\to Y$$
$$x \longmapsto Tx := \lim_{k\to\infty} T_k x$$

(a) T is linear, since: Let $\alpha,\alpha'\in\mathbb{K}$, $x,x'\in X$, then

$$T(\alpha x+\alpha' x') = \lim_{k\to\infty} T_k(\alpha x+\alpha' x') = \lim_{k\to\infty}(\alpha T_k x + \alpha' T_k x')$$
$$= \alpha Tx + \alpha' Tx'$$

(b) T is bounded, and the norm limit of the T_k's:

Taking the limit $l\to\infty$ in $(*)$ above gives

$\|T_k x - Tx\| \leq \varepsilon\|x\|$ and so

$$\forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall k\geq N:\ \|T_k - T\|\leq\varepsilon \quad (**)$$

This gives two things:

(1) T is bounded: $T_k\in BL(X,Y)$, $T_k - T \overset{\text{by } (**)}{\in} BL(X,Y)$ and $BL(X,Y)$ is a vector space, so

$$T = T_k - (T_k - T) \in BL(X,Y).$$

(2) $T_k \xrightarrow{k\to\infty} T$ in $\|\cdot\|_{X\to Y}$ ☒

<u>Note</u>: Compare also with the proofs of Thm 1.31 (& exercise!) and Thm. 1.20 (for $p=\infty$).