Example 2.3 (a) Consider

$$c_c := \{ x = (x_j)_{j \in \mathbb{N}} \mid x_j \in \mathbb{C} \ \forall j \in \mathbb{N}, \text{ and } x_j \neq 0 \text{ for only finitely many } j\text{'s} \}$$

(see also exercise; the index 'c' stands for "compact support"; also: c_{00})

Let $e_n := (\ldots, 0, 1, 0, \ldots)$ with a 1 at the n'th position.

Claim: $B := \{ e_n \mid n \in \mathbb{N} \}$ is a Hamel basis for c_c

(b) Even though ℓ^1 is separable, there exists no countable Hamel basis for ℓ^1 (see exercise).

Theorem 2.4 Every vector space $X \neq \{0\}$ has a Hamel basis

Pf.: Uses Zorn's Lemma; see later.

Corollary 2.5 X has infinite dimension iff for every $n \in \mathbb{N}$ there exists $M_n \subseteq X$ such that $|M_n| = n$ and M_n is linearly indep.

Pf: Existence of Hamel basis with $|B| = \infty$ ∎

Example 2.6 Infinite dimensional vector spaces:

c_c, ℓ^p, $C(X)$ (where $\emptyset \neq X \subseteq \mathbb{R}^d$ open)

2.2. Banach Spaces

Definition 2.7 Let X be a vectorspace. A map $X \to [0, \infty)$, $x \mapsto \|x\|$ is a norm :$\Leftrightarrow$

(1) $\|x\| > 0 \quad \forall\, 0 \neq x \in X$

(2) $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{K} \ \forall x \in X \qquad (\Rightarrow \|x\| = 0)$

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

$(X, \|\cdot\|)$ is called a normed space.

If only (2) and (3) hold, $\|\cdot\|$ is called a semi-norm

Remark 2.8 Let $\mathfrak{X}$ be a normed space. Then $d(x,y) := \|x-y\|$ is a metric on $\mathfrak{X}$. Thus all topological notions and results from the theory of metric spaces are available.

A base of the norm topology on $\mathfrak{X}$:

$$\{ B_{\frac{1}{k}}(x) \mid x \in \mathfrak{X}, k \in \mathbb{N} \} = \{ x + B_{\frac{1}{k}}(0) \mid x \in \mathfrak{X}, k \in \mathbb{N} \}$$

[Minkowski sum of sets A, B: $A + B := \{ a+b \mid a \in A, b \in B \}$
and $a + B := \{a\} + B$]

Warning: Not every metric (on vector spaces) comes from a norm.

Example 2.9 (a) ℓ^p is a normed space with $\|\cdot\| = \|\cdot\|_p$ $\forall p \in [1,\infty]$

(b) $C(\mathfrak{X}; \mathbb{K})$ (where $\mathfrak{X}$ is a compact Hausdorff space) is a normed space with

$$\|f\| := \|f\|_\infty := \sup_{x \in \mathfrak{X}} |f(x)|$$

(c) $\mathbb{R}^d, \mathbb{C}^d$ are normed spaces for $d \in \mathbb{N}$ (w.r.t. p-norm)

Lemma 2.10 Let $\mathfrak{X}$ be normed space. Then the following maps are continuous:

(a) Norm: $\mathfrak{X} \to [0,\infty)$
$x \mapsto \|x\|$

(b) Addition: $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$
$(x,y) \mapsto x+y$

(c) Scalar multiplication: $\mathbb{K} \times \mathfrak{X} \to \mathfrak{X}$
$(\lambda, x) \mapsto \lambda x$

Pf: (a) Let $(x_k)_{k\in\mathbb{N}} \subseteq \mathfrak{X}$ with $x_k \to x$, $k \to \infty$, i.e., $\|x_k - x\| \to 0$, $k \to \infty$ (see Cor. 1.8). Hence, the claim follow from reverse triangle inequality:

$$\big| \|x\| - \|y\| \big| \le \|x+y\|$$

(b), (c) see exercise $\square$

Definition 2.11 A complete normed space $(X, \|\cdot\|)$ is called a Banach space

Examples 2.12 (a) All spaces in Ex. 2.9 are Banach spaces

(b) Let $X := C([0,1])$ with L^1-norm $\|f\|_1 := \int_0^1 |f(t)|\,dt$.
Then $(X, \|\cdot\|_1)$ is not a Banach space (recall Ex. 1.11 (b1)).

Theorem 2.13 Every normed space X can be completed, so that X is isometric to a dense linear subspace W of a Banach space $\widehat{X}$, which is unique up to isometric isomorphisms.

Pf: Analogous to the proof of Theorem 1.14. Note: The isometry is even a linear bijection (hence, an isomorphism) in this case (see Section 2.3 below).

Definition 2.14 Let X be a normed space. A sequence $(e_n)_n \subseteq X$ is called a (Schauder) basis in X :$\Leftrightarrow$ For all $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n e_n \right\| = 0$$

Notation: $x = \sum_{n\in\mathbb{N}} x_n e_n$, infinite linear combination, convergent series

Note: Linear independence not required for Schauder basis

Example 2.15 Let $p \in [1,\infty)$. Then $(e_n)_{n\in\mathbb{N}}$ with $e_n := (0,\dots,0,1,0,\dots)$ (the 1 in the n'th position) is a (Schauder) basis of ℓ^p:
For $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ we have

$$\left\| x - \sum_{n=1}^{N} x_n e_n \right\|^p = \sum_{n=N+1}^{\infty} |x_n|^p \xrightarrow{N\to\infty} 0.$$

Note: This construction fails for ℓ^∞ !

Lemma 2.16 Let X be a normed space. Then

X has a (Schauder) basis $\Rightarrow$ X is separable

Pf: Let $\mathbb{K}_0 := \mathbb{Q}$, if $\mathbb{K} = \mathbb{R}$, resp. $\mathbb{K}_0 := \mathbb{Q} + i\mathbb{Q}$ for $\mathbb{K} = \mathbb{C}$. Define

$$A_N := \left\{ \sum_{n=1}^{N} x_n e_n \;\middle|\; x_n \in \mathbb{K}_0 \text{ for } n \in \{1, .., N\} \right\}$$

Then the union $A := \bigcup_{N \in \mathbb{N}} A_N$ is dense in X, and countable ☒

Remark 2.17 The implication "$\Leftarrow$" in Lemma 2.16 does not hold (Enflo, 1973)

Remark 2.18 All norms in finite-dimensional spaces are equivalent. That is, for norms $\|\cdot\|$ and $|||\cdot|||$ on $\mathbb{K}^n$ there exists constants $c, \tilde{c} > 0$ such that $c\|x\| \leq |||x||| \leq \tilde{c}\|x\| \quad \forall x \in \mathbb{K}^n$.
(see exercise)

Theorem 2.19 Let X be a normed space, and $F \subseteq X$ a finite-dim. subspace. Then F is complete and closed.

Pf: Let $n := \dim F < \infty$. Fix a basis $\{e_1, .., e_n\}$ in F. For every $x \in F$ there exists unique $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{K}^n$ s.t. $x = \sum_{j=1}^{n} \alpha_j e_j$.
Let $|||\alpha||| := \|\sum_{j=1}^{n} \alpha_j e_j\| \quad \forall \alpha = (\alpha_1, .., \alpha_n) \in \mathbb{K}^n$.

Then the normed spaces $(F, \|\cdot\|)$ and $(\mathbb{K}^n, |||\cdot|||)$ are isometrically isomorphic via $x \mapsto \alpha$. Now, $\mathbb{K}^n$ is closed and complete w.r.t. the Euclidean norm, and all norms in $\mathbb{K}^n$ are equivalent (by 2.18). Hence, $(\mathbb{K}^n, |||\cdot|||)$ is closed and complete, and, because of the isometry, so is $(F, \|\cdot\|)$. ☒

As a preparation for Theorem 2.21 we prove the following Lemma:

Lemma 2.20 (Riesz' Lemma, 1918) Let X be a normed space, and $U \subsetneq X$ a closed (!) subspace. Then, for all $\lambda \in (0,1)$, there exists $x_\lambda \in X \setminus U$ such that

$$\|x_\lambda\| = 1 \quad \text{and} \quad \|x_\lambda - u\| \geq \lambda \quad \forall u \in U$$

Pf: Let $x \in X \setminus U$ (open!). Then $\exists \varepsilon_x > 0$: $B_{\varepsilon_x}(x) \subseteq X \setminus U$. Hence $d := \operatorname{dist}(x, U) = \inf_{u \in U} \|x - u\| \geq \varepsilon_x > 0$ for all $x \in X \setminus U$.

Since $0 < \lambda < 1$ there exists $u_\lambda \in U$ s.t. $d \leq \|x - u_\lambda\| \leq \frac{d}{\lambda}$.

Hence, $\gamma := \frac{1}{\|x - u_\lambda\|} \geq \frac{\lambda}{d}$. Define $x_\lambda := \gamma(x - u_\lambda) \in X \setminus U$ (!)

By definition of γ, we have $\|x_\lambda\| = \gamma \cdot \|x - u_\lambda\| = 1$, and

$$\|x_\lambda - u\| = \|\gamma(x - u_\lambda) - u\| = \|\gamma x - (u + \gamma u_\lambda)\| = \gamma \cdot \|x - \underbrace{(u_\lambda - \tfrac{u}{\gamma})}_{\in U}\|$$

$$\geq \gamma d \geq \lambda \quad \forall u \in U \quad \blacksquare$$

Warning 1.26 illustrates the following general result:

Theorem 2.21 Let X be a normed space. Then

$$\overline{B_1(0)} = \{x \in X \mid \|x\| \leq 1\} \text{ compact } \iff \dim X < \infty.$$

Pf: "$\Leftarrow$": Let $n := \dim X < \infty$. From (the proof of) Thm 2.19: X is isometric to $(\mathbb{K}^n, |||\cdot|||)$. The statement follows from Heine-Borel for $(\mathbb{K}^n, |\cdot|)$ and the equivalence of norms.

"$\Rightarrow$": Assume (for contradiction) that $\dim X = \infty$. Will prove: this implies $\overline{B_1(0)}$ is not seq. compact, by constructing a sequence $(x_n)_{n \in \mathbb{N}}$ in $\overline{B_1(0)}$ with no convergent subsequences:

(i) Let $x_1 \in X$ be arbitrary, s.t. $\|x_1\| = 1$. Let $U_1 := \operatorname{span}\{x_1\} \subsetneq X$ be the closed (!) subspace spanned by x_1. Riesz' Lemma, applied with $\lambda = \frac{1}{2}$, gives existence of $x_2 \in X \setminus U_1$ s.t. $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$. Let $U_2 := \operatorname{span}\{x_1, x_2\} \subsetneq X$.

(iii) Let $U_n := \operatorname{span}\{x_1, \dots, x_n\} \subsetneq X$ be the closed subspace of the vectors constructed before. Again, by Riesz' Lemma there exists $x_{n+1} \in X \setminus U_n$ s.t. $\|x_{n+1}\| = 1$ and $\operatorname{dist}(x_{n+1}, U_n) \geq \frac{1}{2}$.

By assumption, $\dim X = \infty$, hence this procedure does not stop.
We get a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \overline{B_1(0)}$ s.t. $\|x_n - x_m\| \geq \frac{1}{2}$ $\forall n \neq m$.
Clearly, $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence - contradicting the (seq.) compactness of $\overline{B_1(0)}$ ↯ ∎

2.3 Linear operators

Definition 2.22 Let X, Y be vector spaces (over the same field $\mathbb{K}$), $X_0 \subseteq X$ a (linear) subspace, and $T : X_0 \to Y$.

(i) T is a (*linear*) *operator* :⟺
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \alpha, \beta \in \mathbb{K},\ \forall x, y \in X_0$$

(ii) $\operatorname{dom}(T) := D(T) := X_0$ is the *domain of T*.

(iii) $\operatorname{ran}(T) := R(T) := T(X_0)$ is the *range of T*

(iv) $\ker(T) := N(T) := \{x \in X_0 \mid Tx = 0\}$ is the *kernel* (or *nullspace*) of T

(v) $U \subseteq \operatorname{dom}(T)$ a subspace,
$$T|_U : \begin{array}{l} U \to Y \\ x \mapsto Tx \end{array} \quad \text{is the } \textit{restriction of } T \textit{ to } U$$

(vi) $W \supseteq \operatorname{dom}(T)$ a vector space, $\tilde{T} : W \to Y$ linear with $\tilde{T}|_{\operatorname{dom}(T)} = T$ is called a (!) linear extension of T to W.

Examples 2.23 Some linear operators:

(a) Identity operator on vector space X:
$$\mathbb{1} := \mathbb{1}_X : \begin{array}{l} X \to X \\ x \mapsto x \end{array}$$