Chapter 2: Banach and Hilbert Spaces

2.1 Vector spaces

General assumption: $X \neq \{0\}$ is a $\mathbb{K}$-vectorspace, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1 Let $\emptyset \neq M \subseteq X$.

(i) M is *linearly independent* iff all non-empty *finite* (!) subsets $F \subseteq M$ are linearly independent, i.e., the following implication holds:

$$\sum_{f \in F} \underbrace{\alpha_f}_{\in \mathbb{K}} f = 0 \Rightarrow \alpha_f = 0 \quad \forall f \in F$$

(ii) M is *linearly dependent* iff M is not linearly independent.

(iii) $B \subseteq X$ is a *Hamel basis* (or *algebraic basis*) iff

(1) B is linearly independent

(2) Every $x \in X$ can be represented as a (finite!) linear combination of elements in B (B *spans* X).

(iv) X has *finite dimension* iff there exists a Hamel basis with $|B| < \infty$. Then $\dim X := |B|$ is called the *dimension of* X

(v) X has *infinite dimension* iff X does not have finite dimension

Remark 2.2 The dimension is well-defined: $|B|$ is the same for every Hamel basis in a given space (Pf: see linear algebra (LA)).

Example 2.3 (a) Consider

$$c_c := \{ x = (x_j)_{j\in\mathbb{N}} \mid x_j \in \mathbb{C}\ \forall j \in \mathbb{N}, \text{ and } x_j \neq 0 \text{ for only finitely many } j\text{'s} \}$$

(see also exercise; the index 'c' stands for "compact support"; also: c_{00})

Let $e_n := (\ldots, 0, 1, 0, \ldots)$ with a 1 at the n'th position.

Claim: $B := \{ e_n \mid n \in \mathbb{N} \}$ is a Hamel basis for c_c

(b) Even though ℓ^1 is separable, there exists no countable Hamel basis for ℓ^1 (see exercise).

Theorem 2.4 Every vector space $X \neq \{0\}$ has a Hamel basis

Pf.: Uses Zorn's Lemma; see later.

Corollary 2.5 X has infinite dimension iff for every $n \in \mathbb{N}$ there exists $M_n \subseteq X$ such that $|M_n| = n$ and M_n is linearly indep.

Pf: Existence of Hamel basis with $|B| = \infty$ ∎

Example 2.6 Infinite dimensional vector spaces:

c_c, ℓ^p, $C(X)$ (where $\emptyset \neq X \subseteq \mathbb{R}^d$ open)

2.2. Banach Spaces

Definition 2.7 Let X be a vectorspace. A map $X \to [0,\infty)$, $x \mapsto \|x\|$

is a norm $:\Leftrightarrow$

(1) $\|x\| > 0 \quad \forall\, 0 \neq x \in X$

(2) $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{K}\ \forall x \in X$ $\qquad (\Rightarrow \|x\| = 0)$

(3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

$(X, \|\cdot\|)$ is called a normed space.

If only (2) and (3) hold, $\|\cdot\|$ is called a semi-norm