Chapter I: Topological and metric spaces

1.1. Limits and continuity

Definition 1.1 Let X be a topological space.

(a) X is called *separable* $:\Leftrightarrow$ $\exists A \subseteq X$ countable with $\overline{A} = X$

(b) X is called *1^{st} (first) countable* $:\Leftrightarrow$ Every $x \in X$ has a countable neighbourhood base.

(c) X is called *2^{nd} (second) countable* $:\Leftrightarrow$ there exists a countable (sub)base for the topology.

[Note: countable base $\Leftrightarrow$ countable subbase (see exercise)]

Theorem 1.2 Let X be a topological space.

Then: X is 2^{nd} countable $\Rightarrow$ X is 1^{st} countable and separable.

Pf (proof): Let $\mathcal{B}$ be a countable base for the topology $\mathcal{T}$ on X.

1^{st} countable: Let $x \in X$ and $\mathcal{N}_x := \{B \in \mathcal{B} \mid x \in B\}$. Then $\mathcal{N}_x$ is countable (clear (why?)) and a neighbourhood base at x:

Indeed, (i) every element of $\mathcal{N}_x$ is a neighbourhood of x, and (ii) let N be any neighbourhood of x. Then $\exists C \in \mathcal{T}$ with $x \in C \subseteq N$. By the definition of a base, $C = \bigcup_{\alpha \in I} B_\alpha$, where I is an index set and $B_\alpha \in \mathcal{B}$ $\forall \alpha \in I$. Hence, $\exists \alpha_x \in I : x \in B_{\alpha_x}$, i.e., $B_{\alpha_x} \in \mathcal{N}_x$ and $B_{\alpha_x} \subseteq N$.

Separable: $\forall \emptyset \neq B \in \mathcal{B}$ choose $x_B \in B$, and let $A := \{x_B \mid \emptyset \neq B \in \mathcal{B}\}$.

We claim A is countable (trivial (why?)) and $\overline{A} = X$.

For all $x \in X$ and neighbourhoods U of x there exists $C \in \mathcal{T}$ such that $x \in C \subseteq U$. But $C \neq \emptyset$ is a union of sets in $\mathcal{B}$, so $\exists x_B \in C$. Thus, $A \cap U \neq \emptyset$.

Since U was arbitrary, x is an adherent point of A ∎

Definition 1.3 Let X be a topological space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. We say that $(x_n)_n$ converges to $x \in X$ $:\Leftrightarrow$ for every neighbourhood U of x there exists $n_0 \in \mathbb{N}$ so for all $n \geq n_0$: $x_n \in U$. We write: $\lim_{n\to\infty} x_n = x$, or $x_n \xrightarrow{n\to\infty} x$ or $x_n \to x$, $n \to \infty$.

Remark 1.4

(a) Convergence is harder for finer topologies.

(b) X Hausdorff $\Rightarrow$ limits are unique (see exercise)

Definition 1.5 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top. spaces and let $f: X \to Y$.

(a) f is sequentially (seq.) continuous $:\Leftrightarrow$
$x_n \xrightarrow{n\to\infty} x$ (in X) implies ($\Rightarrow$) $f(x_n) \xrightarrow{n\to\infty} f(x)$ (in Y)

(b) f is continuous $:\Leftrightarrow$ for every $A \in \mathcal{T}_Y$: $f^{-1}(A) \in \mathcal{T}_X$.
($f^{-1}(A) := \{x \in X \mid \exists y \in A: f(x) = y\}$ is the inverse image or pre-image of A)

(c) f is open (open map) $:\Leftrightarrow \forall A \in \mathcal{T}_X$: $f(A) \in \mathcal{T}_Y$
($f(A) := \{y \in Y \mid \exists x \in A: y = f(x)\}$ is the image of A (under f))

(d) f is a homeomorphism $:\Leftrightarrow$ f is bijective, open, and continuous
(i.e., a bijection compatible with topological structure).

Theorem 1.6 Let X, Y be topological spaces, $f: X \to Y$ a map.
Then:

(a) f is continuous $\Rightarrow$ f is seq. continuous

(b) f is seq. continuous & X is 1^{st} countable
$\Rightarrow$ f is continuous

Pf: (a) Let $x_n \xrightarrow{n\to\infty} x$ in X. Let $V_0 \subseteq Y$ be a neighbourhood of $f(x)$. Then $\exists\, V \subseteq V_0$ open with $f(x) \in V$. Set $U := f^{-1}(V)$. U is open (because f is continuous and V is open), and $x \in U$, so U is a neighbourhood of x. Hence, we can apply the def. of convergence of $(x_n)_n$ to x:
$\exists\, n_0 \in \mathbb{N}: \forall n \geq n_0: x_n \in U$. But this means: $\forall n \geq n_0$:
$f(x_n) \in V \subseteq V_0$. So $f(x_n) \xrightarrow{n\to\infty} f(x)$ in Y.

(b) By contradiction: Suppose f not continuous, i.e., there exists open subset $V \subseteq Y$ such that (s.t.)
$U := f^{-1}(V)$ is not open, i.e.,

(*) $\exists\, x \in U: \forall$ neighbourhood N of x: $N \cap U^c \neq \emptyset$

Let $\{N_k\}_{k\in\mathbb{N}}$ be a countable neighbourhood base at x.
Consider $\tilde{N}_k := \bigcap_{j=1}^{k} N_j$ for $k \in \mathbb{N}$. Then $\{\tilde{N}_k\}_{k\in\mathbb{N}}$ is a countable neighbourhood base of x, with $\tilde{N}_{k+1} \subseteq \tilde{N}_k$ $\forall k$.
$\forall k \in \mathbb{N}: \tilde{N}_k$ is a neighbourhood of $x \overset{(*)}{\Longrightarrow} \exists\, x_k \in \tilde{N}_k \cap U^c$.
Thus (1) $\forall k \in \mathbb{N}\ \forall l \geq k: x_l \in \tilde{N}_k$, so: $x_l \xrightarrow{l\to\infty} x$

(2) $\forall k \in \mathbb{N}: f(x_k) \in f(U^c) \subseteq V^c$. But $f(x) \in V$, so $\{f(x_k)\}_k$ cannot converge to $f(x)$ ↯ (contradiction) ☒

1.2. Metric spaces

Lemma 1.7 Let X be a metric space. Then, the open sets of X are precisely the ones of the metric topology, i.e., the topology generated by the base $\{B_{1/n}(x)\}_{n\in\mathbb{N},\, x\in X}$.
Moreover, X is 1^{st} countable and Hausdorff with respect to (w.r.t.) the metric top.

Pf: (i) Open sets coincide: Let $A \subseteq X$.

Claim: A open in metric space $\iff$ A open in metric top. (see Handout)

Pf claim: "$\Rightarrow$" Let A be open according to Def. 8(iii) in Handout.

Then: $\forall x \in A \; \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x) \subseteq A$, so $A = \bigcup_{x \in A} B_{\varepsilon_x}(x)$.

Now, for all $x \in A$ choose $\mathbb{N} \ni n_x > \frac{1}{\varepsilon_x}$, then $A = \bigcup_{x \in A} B_{\frac{1}{n_x}}(x)$

so A is open in the metric topology.

"$\Leftarrow$": Exercise!

(ii) 1^{st} countable follows from:

Claim: $\forall x \in X : \{ B_{1/n}(x) \}_{n \in \mathbb{N}}$ is a neighbourhood base at x.

Pf claim: (a) $B_{1/n}(x)$ is a neighbourhood of x for every $n \in \mathbb{N}$

(b) Let N be a neighbourhood of $x \Rightarrow \exists G$ open s.t. $x \in G \subseteq N$.

By (i), use Def. 8(iii) in Handout for openness: $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq G$.

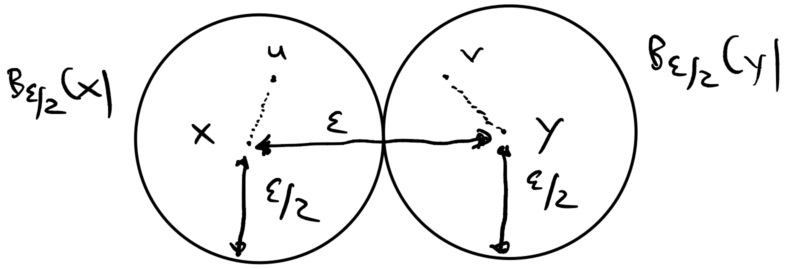
Let $\mathbb{N} \ni n > \frac{1}{\varepsilon} \Rightarrow B_{1/n}(x) \subseteq N$.

(iii) X is Hausdorff: Let $x, y \in X$ with $x \neq y$.

Then $\varepsilon := d(x,y) > 0$, and for all $u \in B_{\varepsilon/2}(x)$, $v \in B_{\frac{\varepsilon}{2}}(y)$, the triangle inequality yields:

$$\varepsilon = d(x,y) \leq d(x,u) + d(u,v) + d(v,y) < \frac{\varepsilon}{2} + d(u,v) + \frac{\varepsilon}{2}.$$

Hence $d(u,v) > 0 \Rightarrow u \neq v \Rightarrow B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(y) = \emptyset \Rightarrow X$ is Hausdorff



$\boxed{\text{Corollary 1.8}}$ All topological notions are available in a metric space X. In particular, for $(x_n)_n \subseteq X$, $x \in X$, $A \subseteq X$:

(a) $\lim_{n\to\infty} x_n = x$ (as in Def. 1.3 wrt. metric topology

$$\iff \lim_{n\to\infty} d(x_n, x) = 0 \quad (\text{in } \mathbb{R}, \text{ with } 1.1)$$

(b) $x \in \overline{A} \iff \exists (y_n)_n \subseteq A : \lim_{n\to\infty} y_n = x.$

(c) If Y is a topological space, and $f: X \to Y$, then

f is continuous $\iff$ f is sequentially continuous

Pf: (a), (b): simple exercises (do!) (c): See Theorem 1.6 ∎

$\boxed{\text{Theorem 1.9}}$ Let X be a metric space.

Then: X separable $\implies$ X 2^{nd} countable

(and, hence, $\iff$ by Theorem 1.2)

Pf: See exercises. Strategy: Let $A \subseteq X$ be countable and dense. Prove that for every open subset $G \subseteq X$ we have the representation

$$G = \bigcup_{x \in G} B_{\frac{1}{n(x)}}(a(x)) \quad \text{with } n(x) \in \mathbb{N} \text{ and } a(x) \in A \ \forall x \in G,$$

that is, $\{B_{\frac{1}{n}}(a)\}_{\substack{n\in\mathbb{N}\\ a\in A}}$ is a countable base ∎

$\boxed{\text{Example 1.10}}$ The above proof establishes the claim of the Handout (Ex. 2(2)): $\{B_{1/n}(q)\}_{\substack{n\in\mathbb{N}\\ q\in\mathbb{Q}^d}}$ is a base of the Euclidean topology on $\mathbb{R}^d$.

Example 1.11 Consider the space of all continuous (cont.) functions $f : [0,1] \to \mathbb{C}$:

$$C([0,1];\mathbb{C}) := C([0,1]) := \{ f : [0,1] \to \mathbb{C} \mid f \text{ is continuous} \}$$

with two different metrics:

(a) Supremum metric:

$$d_\infty(f,g) := \|f-g\|_\infty, \text{ where } \|f\|_\infty := \sup_{x\in[0,1]} |f(x)|.$$

The metric space $(C([0,1]), d_\infty)$ is complete and separable
(see Thms. 1.31 & 1.32 later)

(b) 1-metric:

$$d_1(f,g) := \|f-g\|_1, \text{ where } \|f\|_1 := \int_0^1 |f(x)|\, dx.$$

Then: (i) d_1 defines a metric on $C([0,1])$ (see Exercise)
(ii) $(C([0,1]), d_1)$ is separable because
$d_1(f,g) \le d_\infty(f,g)$ and $(C([0,1]), d_\infty)$ is separable
(iii) $(C([0,1]), d_1)$ is not complete (see Exercise)

Definition 1.12 Let (X, d_X), (Y, d_Y) be metric spaces, $T : X \to Y$ map

(i) T is called an isometry $:\Leftrightarrow$ $\forall x, x' \in X$:
$$d_X(x,x') = d_Y(T(x), T(y)).$$

(ii) X, Y are isometric $:\Leftrightarrow$ there exists a bijective
(correctly, (X, d_X) & (Y, d_Y)) isometry $T : X \to Y$

Remark 1.13 Let $T : X \to Y$ be an isometry. Then
(a) T is injective and continuous
(b) X and the range of T (or image; see 1.5(c))

$$\operatorname{ran} T := T(X) = \{ y \in Y \mid \exists x \in X \text{ with } y = T(x) \}$$ are isometric

$\boxed{\text{Theorem 1.14}}$ Let X be a metric space. Then there exists a complete metric space $\tilde{X}$, and an isometry $i : X \to \hat{X}$ such that $i(X)$ is dense in $\tilde{X}$. The space $\tilde{X}$ is unique up to isometric spaces.

Pf: Consists of 4 steps:

(a) Construct $\tilde{X}$

(b) Construct the isometry i, and $W := i(X)$ is dense in $\hat{X}$

(c) $\hat{X}$ is complete

(d) Uniqueness.

(a): Define an equivalence relation $\sim$ on the set of all Cauchy sequences $(x_n)_n \subseteq X$:

$$(x_n)_n \sim (y_n)_n :\Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$$

($\sim$ is clearly reflexive, symmetric, and transitive). Consider the equivalence class $[(x_n)_n] := \{(y_n)_n \subseteq X \text{ Cauchy} \mid (y_n)_n \sim (x_n)_n\}$ we obtain this way, and let

$$\tilde{X} := \{[(x_n)_n] \mid (x_n)_n \subseteq X \text{ Cauchy}\}.$$

Given $\tilde{x} \in \tilde{X}$, we write $(x_n)_n \in \tilde{x}$ for a representative $(x_n)_n$ of the equivalence class $\tilde{x}$. Define a metric on $\tilde{X}$:

$$\tilde{d}(\tilde{x}, \tilde{y}) := \lim_{n\to\infty} d(x_n, y_n) \quad , \quad \tilde{x}, \tilde{y} \in \tilde{X}, (x_n)_n \in \tilde{x}, (y_n)_n \in \tilde{y}$$

We have to check that:

(1) $\tilde{d}$ is well-defined:

(i) Existence of the limit:

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

$\Rightarrow \quad d(x_n,y_n) - d(x_m,y_m) \leq d(x_n,x_m) + d(y_m,y_n)$,

and the same inequality holds with m & n interchanged.

Thus we have:

$$(*) \qquad \Big| \underbrace{d(x_n,y_n)}_{=:\alpha_n} - \underbrace{d(x_m,y_m)}_{=:\alpha_m} \Big| \leq d(x_n,x_m) + d(y_m,y_n).$$

Since $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences, we have:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n,m \geq N$:

$$d(x_n,x_m) < \frac{\varepsilon}{2} \text{ and } d(y_n,y_m) < \frac{\varepsilon}{2}$$

So $|\alpha_n - \alpha_m| < \varepsilon$ for $n,m \geq N$ and so $(\alpha_n)_n \subseteq \mathbb{R}$ is a Cauchy sequence, thus convergent (because $\mathbb{R}$ is complete)

(ii) Independence of representatives: Let $(x_n)_n \sim (x_n')_n$ and $(y_n)_n \sim (y_n')_n$. We have to prove that

$$\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n',y_n')$$

Repeat the derivation of $(*)$ above, interchanging $x_m \leftrightarrow x_n'$ and $y_m \leftrightarrow y_n'$, and obtain

$$|d(x_n,y_n) - d(x_n',y_n')| \leq d(x_n,x_n') + d(y_n,y_n') \xrightarrow{n\to\infty} 0$$

by definition of $\sim$.

(2) $\tilde{d}$ fulfills the axioms of a metric:

(i) $\tilde{d} \geq 0$: $\quad \tilde{d}(\tilde{x},\tilde{y}) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n,y_n) = 0$ for some representatives

$\Leftrightarrow (x_n)_n \sim (y_n)_n \Leftrightarrow \tilde{x} = \tilde{y}$.

(ii) Symmetry

and (iii) Triangle inequality

are clear from the properties of d (do!).

(b): Define
$$i : X \to \tilde{X}$$
$$b \mapsto \tilde{b} := [(b, b, b, \dots)]$$

Set $W := i(X)$. The map i is an isometry since
$$\tilde{d}(i(a), i(b)) = \tilde{d}(\tilde{a}, \tilde{b}) = \lim_{n\to\infty} d(a,b) = d(a,b).$$

It remains to show that W is dense in $\tilde{X}$ with respect to $\tilde{d}$.
Let $\tilde{x} \in \tilde{X}$ and $\varepsilon > 0$. Pick any representative $(x_n)_n \in \tilde{x}$. As $(x_n)_n$ is a Cauchy sequence (in X) we have: $\exists N \in \mathbb{N}\ \forall n, m \geq N$: $d(x_n, x_m) < \frac{\varepsilon}{2}$. Let $\tilde{b} := [(x_N, x_N, x_N, \dots)] = i(x_N) \in W$.

Then $\tilde{d}(\tilde{b}, \tilde{x}) = \lim_{n\to\infty} d(x_N, x_n) \leq \frac{\varepsilon}{2} < \varepsilon$.

So W is dense in $\tilde{X}$.

(c): Let $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ be a Cauchy seq. in $\tilde{X}$; W is dense in $\tilde{X}$: $\forall k \in \mathbb{N}\ \exists \tilde{z}^{(k)} := [(z_k, z_k, \dots)] \in W$, $z_k \in X$, s.t.
$$\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)}) < \frac{1}{k}$$

For every $k, l \in \mathbb{N}$ we get (due to the isometry property of i):
$$(**)\ d(z_k, z_l) = \tilde{d}(\underbrace{i(z_k)}_{\tilde{z}^{(k)}}, \underbrace{i(z_l)}_{\tilde{z}^{(l)}}) \leq \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x}^{(k)})}_{< \frac{1}{k}} + \tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(l)}) + \underbrace{\tilde{d}(\tilde{x}^{(l)}, \tilde{z}^{(l)})}_{< \frac{1}{l}}$$

Let $\varepsilon > 0$. Since $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ is Cauchy (wrt. $\tilde{d}$), there exists $K \in \mathbb{N}$ such that $\tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(l)}) < \frac{\varepsilon}{3}$ for all $k, l \geq K$. Hence, $(**)$ implies that $d(z_k, z_l) < \varepsilon$ for all $k, l \geq \max\{\frac{3}{\varepsilon}, K\}$, that is, $(z_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy. Let $\tilde{x} := [(z_n)_n] \in \tilde{X}$.

We prove now that
$$\lim_{k\to\infty} \tilde{d}(\tilde{x}^{(k)}, \tilde{x}) = 0.$$

Indeed, let $\varepsilon > 0$. Then, as $(z_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. $d(z_k, z_n) < \frac{\varepsilon}{2}$ for all $k, n \geq N$. Thus, for all $k > \max\{\frac{2}{\varepsilon}, N\}$,
$$\tilde{d}(\tilde{x}^{(k)}, \tilde{x}) \leq \underbrace{\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)})}_{< \frac{1}{k}} + \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x})}_{= \lim_{n\to\infty} d(z_k, z_n)} < \varepsilon \qquad \square$$

(d) See exercise.

1.3. Example: sequence space ℓ^p

Definition 1.15 (ℓ^p-spaces)

Let, for $p \in [1,\infty)$ $(= [1,\infty[)$,

$$\ell^p := \ell^p(\mathbb{N}) := \{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_p := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p} < \infty \}$$

and ($p = \infty$)

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \}$$

($\|\cdot\|_p$ will be a norm for every $p \in [1,\infty]$, see later)

Lemma 1.16 For every $p \in [1,\infty]$, $d_p(x,y) := \|x-y\|_p$, $x,y \in \ell^p$, defines a metric d_p on ℓ^p

Pf: All properties clear (check!), except for triangle inequality, this follows from Lemma 1.17 (b) below (do!)

Lemma 1.17 (Hölder & Minkowski)

(a) Let $p,q \in [1,\infty]$ be (Hölder) conjugated exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ (convention: "$\frac{1}{\infty} = 0$").
Dual pairing and Hölder inequality: For all $x \in \ell^p$, $y \in \ell^q$:
$\langle x, y\rangle := \sum_{n\in\mathbb{N}} x_n y_n$ is well-defined, and

$$|\langle x,y\rangle| \le \sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$$

(b) Minkowski inequality:
For all $x,y \in \ell^p$: $\quad \|x+y\|_p \le \|x\|_p + \|y\|_p$

Pf: Both (a) & (b) follow from corresponding ineq.'s on $\mathbb{C}^N$, and then passing to the limit; for (a) f.ex.

$$\sum_{n=1}^N |x_n y_n| \le \Big(\sum_{n=1}^N |x_n|^p\Big)^{1/p} \Big(\sum_{n=1}^N |y_n|^q\Big)^{1/q}$$ (see ex. Forster, vol. 1) & carefully (!) take limit

Remark 1.18 (a) Note: alternative notation: ℓ_p

(b) Note: $\ell^p(\{1,\dots,n\}) := \{x=(x_1,\dots,x_n) \mid x_i \in \mathbb{C},\ \|x\|_p := (\sum_{i=1}^{n} |x_p|^p)^{1/p}\}$ ($n \in \mathbb{N}$) is nothing but $\mathbb{C}^n$ with $\|\cdot\|_p$-norm.

(c) ℓ^p-spaces can also be defined for $p \in (0,1)$ $(=]0,1[)$. But the Minkowski inequality does not remain true in this case. Instead, one has $\|x+y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$ $\forall p \in (0,1)$ (quasi-norm). Therefore we consider ℓ^p-spaces for $p \geq 1$ only.

Theorem 1.19 (a) $\forall p \in [1,\infty)$: ℓ^p is separable

(b) ℓ^∞ is not separable

Pf: (a) We use the separability of $\mathbb{C}$. For $n \in \mathbb{N}$, let

$$M_n := \{(x_1,\dots,x_n,0,\dots) \mid x_j \in \mathbb{Q} + i\mathbb{Q},\ j=1,\dots,n\}$$

So M_n is countable and $M := \bigcup_{n \in \mathbb{N}} M_n$ is also countable.

Claim: $\overline{M} = \ell^p$. Let $y \in \ell^p$ and $\varepsilon > 0$, then there exists $N \in \mathbb{N}$:

$$\sum_{j=N+1}^{\infty} |y_j|^p < \frac{\varepsilon^p}{2} \quad \left(\|y\|_p < \infty \Rightarrow \lim_{L\to\infty} \left(\sum_{j=L}^{\infty} |y_j|^p\right)^{1/p} = 0\right),$$

and since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$ there exists $x \in M_N$ such that

$$\sum_{j=1}^{N} |x_j - y_j|^p < \frac{\varepsilon^p}{2}.$$ This implies $(d_p(x,y))^p = \|x-y\|_p^p < \varepsilon^p$ ∎

(b) See exercise.

Theorem 1.20 ℓ^p is complete for every $p \in [1,\infty]$

(that is, $(\ell^p(\mathbb{N}), d_p)$ is a complete metric space)

Pf: (a) Case $p \in [1,\infty)$: Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq \ell^p$ be a Cauchy sequence w.r.t. $\|\cdot\|_p$, where $x^{(n)} := (x_1^{(n)}, x_2^{(n)}, \dots)$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$(*) \quad \sum_{j=J_1}^{J_2} |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j\in\mathbb{N}} |x_j^{(n)} - x_j^{(m)}|^p < \varepsilon^p \quad \forall n,m \geq N, \quad \forall J_1 \leq J_2 \in \mathbb{N}.$$

Step 1: Find a candidate for the limit using the completeness of $\mathbb{C}$. Let $J_1 = J_2 = J$, then $(*)$ implies $(x_J^{(n)})_n \subseteq \mathbb{C}$ is a Cauchy seq. $(\forall J \in \mathbb{N})$ and, since $\mathbb{C}$ is complete, there is some $x_J \in \mathbb{C}$ s.t.

$$(**) \qquad \lim_{n\to\infty} x_J^{(n)} = x_J \qquad \forall J \in \mathbb{N}.$$

so our candidate is $x := (x_1, x_2, \dots)$

Step 2: Prove that the candidate belongs to the space: Use the Minkowski inequality in $\mathbb{C}^{J_2}$, $(**)$, and set $J_1 = 1$ in $(*)$:

$$\Big(\sum_{j=1}^{J_2} |x_j|^p\Big)^{1/p} \leq \lim_{m\to\infty} \Big(\sum_{j=1}^{J_2} |x_j^{(m)} - x_j^{(n)}|^p\Big)^{1/p} + \underbrace{\Big(\sum_{j=1}^{J_2} |x_j^{(n)}|^p\Big)^{1/p}}_{\leq \|x^{(n)}\|_p < \infty}$$

$$\leq \varepsilon + \|x^{(n)}\|_p \quad \forall J_2 \in \mathbb{N}, \forall n \geq N$$

so we get $\|x\|_p \leq \varepsilon + \|x^{(n)}\|_p$ for every $n \geq N$, and hence $x \in \ell^p$

Step 3: Prove that the sequence converges to the candidate, in the norm of the space: Set $J_1 = 1$ in $(*)$ and $m \to \infty$, using $(**)$. Then, for every $n \geq N$,

$$\sum_{j=1}^{J_2} |x_j^{(n)} - x_j|^p \leq \varepsilon^p \qquad \forall J_2 \in \mathbb{N},$$

so by sending $J_2 \to \infty$ in addition, we have

$$d_p(x^{(n)}, x) = \|x^{(n)} - x\|_p \leq \varepsilon, \quad \text{so } x^{(n)} \to x \text{ in } \ell^p.$$

(b) Case $p = \infty$: Replace "$\sum_{j=J_1}^{J_2}$" by "$\sup_{J_1 \leq j \leq J_2}$" and "$|\cdot|^p$" by "$|\cdot|$" ☒

1.4. Compactness

Definition 1.21 Let X be a topological space, and $A \subseteq X$

(a) A is *compact* $:\Leftrightarrow$ For every *open cover* $\bigcup_{\alpha\in I} B_\alpha \supseteq A$ with an index set $I \neq \emptyset$, and $B_\alpha \subseteq X$ open for all $\alpha \in I$, there exists a *finite open subcover*, i.e., $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in I$ such that $\bigcup_{n=1}^{N} B_{\alpha_n} \supseteq A$ (*Heine-Borel-property*).

(b) A is *sequentially compact* $:\Leftrightarrow$ Every sequence in A has a convergent subsequence with limit in A (*Bolzano-Weierstraß property*).

(c) A is *relatively (seq.) compact* $:\Leftrightarrow \bar{A}$ is (seq.) compact.

Remark 1.22 (a) Def. applies in particular to X. In this case we have "$=$" instead of "$\supseteq$".

(b) Some books (e.g. Bourbaki) use compactness only for Hausdorff spaces; they call our property *quasi-compact*.

Theorem 1.23 Let X be a topological space

(a) Assume X is 1^{st} countable. Then:
$$X \text{ compact} \Rightarrow X \text{ sequentially compact}$$

(b) Assume X is 2^{nd} countable. Then:
$$X \text{ compact} \Leftrightarrow X \text{ sequentially compact}.$$

In the proof we will use/need:

Theorem 1.24 (*Lindelöf's Lemma (!)*) Let X be a 2^{nd} countable top. space, and let $\bigcup_{\alpha\in I} A_\alpha = X$ be an open cover. Then there exists $(\alpha_k)_{k\in\mathbb{N}} \subseteq I$ such that $\bigcup_{k\in\mathbb{N}} A_{\alpha_k} = X$, i.e., there exists a *countable (!) open subcover*

Pf: Let $\{B_k\}_{k\in\mathbb{N}}$ be a countable base of the topology. Let

$K := \{k \in \mathbb{N} \mid \exists \alpha = \alpha_k \in I \text{ with } B_k \subseteq A_{\alpha_k}\}$. Then $\bigcup_{k\in K} B_k \subseteq \bigcup_{\alpha\in I} A_\alpha$ (1)

Now, every A_α is a union of certain B_k's, and the corresponding k's necessarily belong to K. Thus $\bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{k \in K} B_k$ (ΔΔ)

(Δ) & (ΔΔ) imply $X = \bigcup_{\alpha\in I} A_\alpha = \bigcup_{k\in K} B_k = \bigcup_{k\in K} A_{\alpha_k}$ ■

Pf. of Thm 1.23: (a) By contradiction: Assume X is compact, but that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ without convergent subsequence.

Claim: $\forall x \in X$ there exists a neighbourhood $U(x)$ such that $x_n \in U(x)$ for at most finitely many n.

Pf. of claim: Suppose claim false. Then $\exists x \in X$ and a countable neighbourhood base $\{V_k\}_{k\in\mathbb{N}}$ of x with $V_k \supseteq V_{k+1}$ $\forall k \in \mathbb{N}$ (holds wlog. (o.B.d.A), see pf. Thm. 1.6 (b)) such that

$$(*) \qquad \forall k \in \mathbb{N}: x_n \in V_k \text{ for infinitely many } n$$

So, for all $k \in \mathbb{N}$, define $n_k \in \mathbb{N}$ such that $x_{n_k} \in V_k$. Since (*) holds for infinitely many n, we can choose the n_k s.t. $n_k < n_{k+1}$ $\forall k \in \mathbb{N}$

But then $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and $\lim_{k\to\infty} x_{n_k} = x$ ↯

Wlog., the neighbourhoods $U(x)$ from the claim can be assumed to be open. (otherwise shrink $U(x)$ to the open set contained in it which itself contains x). Now, $X = \bigcup_{y\in X} U(y) = \bigcup_{j=1}^{n} U(y_j)$ for some $n \in \mathbb{N}$ and some $y_1, \dots, y_n \in X$ because X is compact. The claim implies that each $U(y_j)$ contains at most finitely many members of the sequence $(x_n)_{n\in\mathbb{N}}$, so X contains at most finitely many members of the sequence $(x_n)_{n\in\mathbb{N}}$ ↯

(b) "⟹" follows from Theorem 1.2 and (a). We prove "⟸" by contradiction:

Assume every sequence has a convergent subsequence, but there exists an open cover of X without finite subcover. As X is 2^{nd} countable, there exists a countable subcover $X = \bigcup_{j\in\mathbb{N}} C_j$ of this cover (by Thm. 1.24). For every $n \in \mathbb{N}$ pick a point

$$(**) \qquad x_n \in X \setminus \left(\bigcup_{j=1}^{n} C_j\right)$$

(possible $\forall n \in \mathbb{N}$, since there exists no finite subcover)

By hypothesis, $(x_n)_n \subseteq X$ has a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x \in X$. Then exists $N \in \mathbb{N}$ such that $x \in C_N^i$, so C_N^i is a neighbourhood of x. Now, $(x_{n_k})_{k\in\mathbb{N}}$ being convergent means $x_{n_k} \in C_N$ for finally all k (i.e. $\exists K : k \geq K \Rightarrow x_{n_k} \in C_N$). On the other hand, $n_k \geq N$ for finally all k, hence $x_{n_k} \notin C_N$ for finally all k by (**) $\lightning$ ∎

Theorem 1.25 Let X be a topological space and $A \subseteq X$. Then:

(a) X compact and A closed $\Rightarrow$ A compact

(b) X Hausdorff and A compact $\Rightarrow$ A closed.

Pf. (a) Let $\bigcup_{\alpha\in I} U_\alpha \supseteq A$ be an open cover. Since A is closed, A^c is open, and

$$X = A^c \cup \left(\bigcup_{\alpha\in I} U_\alpha\right)$$ is an open cover

Since X is compact, there exists $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in I$ such that

$$X = A^c \cup \left(\bigcup_{i=1}^n U_{\alpha_i}\right)$$

and so $\bigcup_{i=1}^n U_{\alpha_i} \supseteq A$ is a finite subcover.

(b) See Exercise.

Warning 1.26 Bounded and closed do NOT imply compact in general!

Example: ℓ^p, $p \in [1,\infty]$, and (the closed unit ball)

$$\overline{B}_1(0) := \{x \in \ell^p \mid \|x\|_p \leq 1\} = \overline{\{x \in \ell^p \mid \|x\|_p < 1\}} = \overline{B_1(0)}$$

is bounded and closed but: consider $e^{(n)} := (\dots, 0, 1, 0, \dots) \in \overline{B}_1(0)$ (with 1 at the n-th position), $n \in \mathbb{N}$. Then

$$d_p(e^{(n)}, e^{(m)}) = \|e^{(n)} - e^{(m)}\|_p = \begin{cases} 2^{1/p}, & p < \infty \\ 1, & p = \infty \end{cases} \quad \forall n,m \in \mathbb{N},\ n \neq m,$$

so there exists no convergent subsequence, and $\overline{B}_1(0)$ is not seq. compact. Hence, by Thm. 1.23 (a), $\overline{B}_1(0)$ is not compact.

Theorem 1.27 Let X be a metric space. Then

$$\begin{array}{ccc} X \text{ is compact} & \overset{(a)}{\Longleftrightarrow} & X \text{ is sequentially compact} \\ & & \Downarrow (c) \\ X \text{ is } 2^{nd} \text{ countable} & \underset{(b)}{\Longleftrightarrow} & X \text{ is separable} \end{array}$$

(Recall that X is 1^{st} countable & Hausdorff (1.71))

Pf: (b) was proved in Thm. 1.9.

(a) "$\Rightarrow$": This is Thm. 1.23 (a) (since X is 1^{st} countable).

"$\Leftarrow$": Follows from pf. of (c) (see below), "$\Leftarrow$" of (b), and Thm. 1.23 (b).

(c): We prove that sequential compactness implies separability by constructing a countable set M with $\overline{M} = X$. Fix $n \in \mathbb{N}$ and use the following algorithm to define points $x_k^{(n)}$:

(i) Choose an arbitrary $x_1^{(n)} \in X$; set $k := 1$

(ii) WHILE $\quad R_k^{(n)} := X \setminus \left(\bigcup_{j=1}^{k} B_{\frac{1}{n}}(x_j^{(n)}) \right) \neq \emptyset \quad$ DO

$$\{ \text{pick } x_{k+1}^{(n)} \in R_k^{(n)} \text{ and increment } k \to k+1 \}$$

Claim: This algorithm stops after finitely many steps.

True, because for $k \neq l$, we have $d(x_k^{(n)}, x_l^{(n)}) \geq \frac{1}{n}$. Hence, if the algorithm did not stop after finitely many steps, we would have an infinite sequence $(x_k^{(n)})_{k \in \mathbb{N}} \subseteq X$ without a convergent subseq., in contradiction to X being sequentially compact. ✓

The claim now implies that: $\forall n \in \mathbb{N} \; \exists K_n \in \mathbb{N}$ such that

$$(*) \qquad X = \bigcup_{j=1}^{K_n} B_{\frac{1}{n}}(x_j^{(n)})$$

Set $M_n := \{ x_j^{(n)} \mid j = 1, \dots, K_n \}$ and $M := \bigcup_{n \in \mathbb{N}} M_n$. Then M is countable.

To prove denseness ($\overline{M} = X$), let $x \in X$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Then $(*) \Rightarrow \exists l \in \{1, \dots, K_n\}$ such that $x \in B_{\frac{1}{n}}(x_l^{(n)})$. Hence,

$$d(x, M) \leq d(x, x_l^{(n)}) < \frac{1}{n} < \varepsilon, \quad \text{so } \overline{M} = X \quad ■$$

Theorem 1.28 (Tychonoff's Theorem – or Tikhonov)

Let $J \neq \emptyset$ be an index set, and X_α a compact topological space for all $\alpha \in J$. Then

$$\bigtimes_{\alpha \in J} X_\alpha = \{ f : J \to \bigcup_{\alpha \in J} X_\alpha \mid f(\alpha) \in X_\alpha \}$$

is compact in the product topology.

Pf: See any textbook on topology (Kelley, Munkres, v. Querenburg f.ex.)

Definition 1.29 Let X, Y be topological spaces. Define

(i) $C(X,Y) := \{ f : X \to Y \mid f \text{ is continuous} \}$

In particular, for $Y = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, set $C(X) := C(X, \mathbb{K})$

(ii) $C_b(X) := \{ f \in C(X) \mid \|f\|_\infty < \infty \}$ (bounded continuous functions)

where $\|f\|_\infty := \sup_{x \in X} |f(x)| = \sup \{ |f(x)| \mid x \in X \}$

Theorem 1.30 Let X, Y be topological spaces, X compact, $f \in C(X, Y)$.

Then:

(a) $f(X)$ is compact

(b) Assume: Y is Hausdorff and f is a bijection.

Then f is a homeomorphism (i.e., f^{-1} is continuous)

(c) If X, Y are metric spaces, then f is uniformly continuous:

$$\forall \varepsilon > 0 \ \exists \delta = \delta_\varepsilon > 0 : \forall x \in X : f(B_\delta(x; d_X)) \subseteq B_\varepsilon(f(x); d_Y)$$

(equivalently: $\forall \varepsilon > 0 \ \exists \delta > 0 : d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$)

(d) Assume $Y = \mathbb{R}$ (i.e. $f : X \to \mathbb{R}$). Then f takes on its maximum and minimum: $\exists x_1, x_2 \in X$:

$$\forall x \in X : \quad f(x_1) \le f(x) \le f(x_2)$$

Pf: (a) Let $\bigcup_{\alpha \in J} V_\alpha \supseteq f(X)$ be an open cover. Then

$$(*) \qquad X \subseteq f^{-1}\Big(\bigcup_{\alpha \in J} V_\alpha\Big) = \bigcup_{\alpha \in J} f^{-1}(V_\alpha) \qquad (\text{"="} \text{ in fact})$$

Because f is continuous & V_α open, $f^{-1}(V_\alpha)$ is open $\forall \alpha \in J$, hence (*) gives an open cover of X – which is compact. Hence, $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N$ s.t. $X \subseteq \bigcup_{n=1}^{N} f^{-1}(V_{\alpha_n})$, and therefore,

$$f(X) \subseteq \bigcup_{n=1}^{N} V_{\alpha_n}$$

(b), (c), (d): see Exercises.

1.5. Example: Spaces of continuous functions

General assumptions in <u>this</u> section:

(i) X is a compact Hausdorff space

(ii) $C(X)$ is equipped with the uniform (supremum) metric:

$$d_\infty(f,g) := \|f-g\|_\infty := \sup_{x\in X} |f(x)-g(x)|$$

(Note: sup = max is finite $\forall f,g \in C(X)$ by Thm. 1.30 (d))

Theorem 1.31 $C(X)$ is complete.

Pf: Follows from completeness of $C_b(X)$ (bounded cont. functions; see exercise), and that $C(X) = C_b(X)$, which follows from compactness of X, and Thm. 1.30 (d). ∎

Theorem 1.32 X is metrisable $\iff$ $C(X)$ is separable

(A topological space $(X, \mathcal{T})$ is metrisable :$\iff$ $\exists$ metric d on X that generates the topology $\mathcal{T}$)

Pf: For "$\Leftarrow$": See (f.ex) Bourbaki, "Elements of Mathematics, General Topology", Part 2, Sect. X 3.3, Thm. 1.1.

Here, we only prove:

"$\Rightarrow$": Fix any metric that is compatible with the topology.

For $m, n \in \mathbb{N}$, define

$$G_{m,n} := \{ f \in C(X) \mid f(B_{\frac{1}{m}}(x)) \subseteq B_{\frac{1}{n}}(f(x)) \ \ \forall x \in X \}$$

By compactness of X we get

(i) Any $f \in C(X)$ is even uniformly continuous, by Thm. 1.30(c).

Choose $\varepsilon := \frac{1}{n}$ there, and let m large enough s.t. $\frac{1}{m} \le \delta$. Then $f \in G_{mn}$, so $C(X) = \bigcup_{m \in \mathbb{N}} G_{mn}$ $\forall n \in \mathbb{N}$ (1)

(ii) For any $m \in \mathbb{N}$ we can find $k_m \in \mathbb{N}$ and $a_1, \dots, a_{k_m} \in X$ such that X can be written as a union of open balls of radius $\frac{1}{m}$: $X = \bigcup_{k=1}^{k_m} B_{\frac{1}{m}}(a_k)$ (2) (X is compact)

Now, $\mathbb{K}$ is separable, i.e. there exists countable set $\{x_\nu \in \mathbb{K} \mid \nu \in \mathbb{N}\}$ which is dense in $\mathbb{K}$.

For given $m \in \mathbb{N}$ and any $\varphi : \{1, \dots, k_m\} \to \mathbb{N}$, i.e. $\varphi = (\varphi(1), \dots, \varphi(k_m)) \in \mathbb{N}^{k_m}$, define

$$G_{mn}^{\varphi} := \left\{ g \in G_{mn} \;\middle|\; |g(a_k) - x_{\varphi(k)}| < \frac{1}{n} \;\; \forall k = 1, \dots, k_m \right\}$$

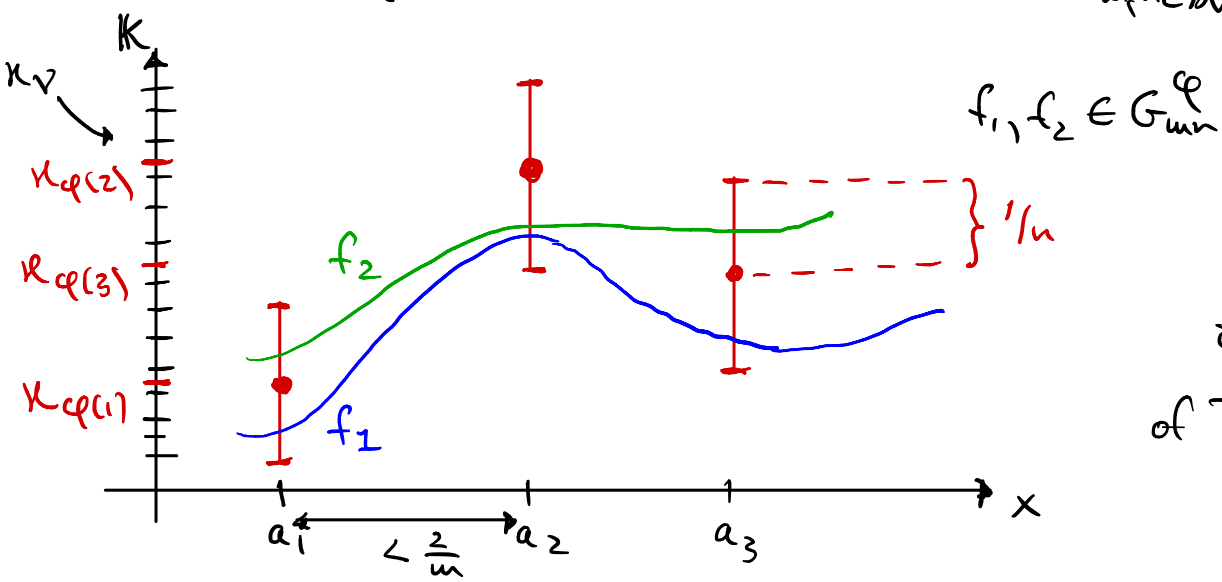
We only want to consider those φ with $G_{mn}^{\varphi} \neq \emptyset$, so let

$$\Phi_{mn} := \{ \varphi \in \mathbb{N}^{k_m} \mid G_{mn}^{\varphi} \neq \emptyset \}$$

Φ_{mn} is not empty (since $\varphi_\nu := (\nu, \dots, \nu) \in \Phi_{mn}$, because the constant function $f \equiv x_\nu \in G_{mn}^{\varphi_\nu}$).

For every $\varphi \in \Phi_{mn}$ pick some $g_\varphi \in G_{mn}^{\varphi}$. Now define

$$L_{mn} := \{ g_\varphi \mid \varphi \in \Phi_{mn} \} \quad \text{and} \quad L := \bigcup_{m,n \in \mathbb{N}} L_{mn}$$



Notions in proof of Thm. 1.32

Note: L_{mn} is countable because $\Phi_{mn} \subseteq \mathbb{N}^{K_m}$, hence L is countable. The theorem now follows from:

<u>Claim</u>: $\overline{L}^{d_\infty} = C(\mathbb{X})$

<u>Pf</u>: Fix $f \in C(\mathbb{X})$, $n \in \mathbb{N}$.

(1) $\Rightarrow \exists m \in \mathbb{N}: f \in G_{mn}$. Also, $\exists \varphi_f \in \Phi_{mn}$ s.t $f \in G_{mn}^{\varphi_f}$: Simply choose $\varphi_f(k)$ such that $|f(a_k) - \kappa_{\varphi_f(k)}| < \frac{1}{n}$, which is possible by the denseness of $\{\kappa_r \mid r \in \mathbb{N}\}$ in $\mathbb{K}$.

(2) $\Rightarrow \forall x \in \mathbb{X}\ \exists k_x \in \{1, \dots, K_m\}: x \in B_{\frac{1}{m}}(a_{k_x})$

Take $g_{\varphi_f} \in L_{mn} \subseteq L$ as approximant. Since $f, g_{\varphi_f} \in G_{mn}^{\varphi_f}$ we have, for all $x \in \mathbb{X}$,

$$\begin{aligned}|f(x) - g_{\varphi_f}(x)| &\leq \underbrace{|f(x) - f(a_{k_x})|}_{<\frac{1}{n} \text{ by def. of } G_{mn}} + \underbrace{|f(a_{k_x}) - \kappa_{\varphi_f(k_x)}|}_{<\frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} \\ &\quad + \underbrace{|\kappa_{\varphi_f(k_x)} - g_{\varphi_f}(a_{k_x})|}_{<\frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} + \underbrace{|g_{\varphi_f}(a_{k_x}) - g_{\varphi_f}(x)|}_{<\frac{1}{n} \text{ by def. of } G_{mn}} \\ &< \frac{4}{n}.\end{aligned}$$

Since f and n were arbitrary, the claim follows $\boxtimes$

An alternative approach to separability:

$\boxed{\text{Definition 1.33}}$ (a) A $\mathbb{K}$-vectorspace A is a <u>$\mathbb{K}$-algebra</u> $:\Leftrightarrow$ there exists a multiplication $A \times A \to A$ which satisfies:

$$(a+b)c = ac + bc \qquad \forall a,b,c \in A$$

$$c(a+b) = ca + cb \qquad \forall a,b,c \in A$$

$$\lambda(ac) = (\lambda a)c = a(\lambda c) \qquad \forall a,c \in A,\ \lambda \in \mathbb{K}$$

(Example: $C(\mathbb{X})$ is a (commutative!) $\mathbb{K}$-algebra)

(b) A subspace $B \subseteq A$ is a subalgebra $:\Leftrightarrow$ B is closed under multiplication

(c) A subset $B \subseteq C(X)$ separates points in X $:\Leftrightarrow$ for all distinct points $x, y \in X$, $x \neq y$, there exists $f \in B$: $f(x) \neq f(y)$

Theorem 1.34 (Stone–Weierstraß) Let $B \subseteq C(X)$ be a subalgebra with the following properties:

(i) $1 \in B$ (where $1 : X \to \mathbb{K}$, $x \mapsto 1$ is the constant function) B is unital

(ii) B separates points.

(iii) If $\mathbb{K} = \mathbb{C}$: Assume further that B is closed under complex conjugation (i.e. $f \in B \Rightarrow \bar{f} \in B$)

Then: $\overline{B}^{d_\infty} = C(X)$ (that is, B is dense in $C(X)$)

Pf: Not here; see e.g. Reed–Simon, Vol. 1, App. to Sect. IV.3

Corollary 1.35 Let $X \subseteq \mathbb{R}^d$ be compact, $d \in \mathbb{N}$. Then the set of polynomials is dense (wrt. d_∞) in $C(X)$. Furthermore, $C(X)$ is separable.

Pf: Let B be the $\mathbb{K}$-subalgebra of the $\mathbb{K}$-algebra $C(X)$ generated by the monomials

(*) $M_{n,\alpha} : X \ni x = (x_1, \dots, x_d) \mapsto x_\alpha^n \in \mathbb{R}$, $n \in \mathbb{N}_0$, $\alpha \in \{1, \dots, d\}$

Then Stone–Weierstraß gives $\overline{B} = C(X)$ since, clearly,

(1) $1 \in B$ and, if $\mathbb{K} = \mathbb{C}$, then B is closed under complex conjugation, since $x \in X \subseteq \mathbb{R}^d$ and B is a $\mathbb{C}$-subspace.

(2) If $x, y \in X$, $x \neq y$, then $\exists \alpha \in \{1, \dots, d\} : x_\alpha \neq y_\alpha$, so $M_{1,\alpha}(x) \neq M_{1,\alpha}(y)$, and so B separates points.

We prove separability: Let $\mathbb{K}_0 := \begin{cases} \mathbb{Q}, & \mathbb{K}=\mathbb{R} \\ \mathbb{Q}+i\mathbb{Q}, & \mathbb{K}=\mathbb{C} \end{cases}$ (23)

and let B_0 be the $\mathbb{K}_0$-algebra generated by the monomials in (*). (Note: This is not a sub algebra of $C(\mathbb{X})$ - why?!)

(1) B_0 is countable (it consists of polynomials with rational coeff.)

(2) Since $\mathbb{X}$ is bounded:

$$\forall \varepsilon>0\ \forall n\in\mathbb{N}_0\ \forall \alpha\in\{1,..,d\}\ \forall c\in\mathbb{K}\ \exists q\in\mathbb{K}_0 : d_\infty(cM_{n,\alpha}, qM_{n,\alpha})<\varepsilon$$

$$\Rightarrow \overline{B_0} = \overline{B} = C(\mathbb{X}) \quad \blacksquare$$

Definition 1.36 (Let $\mathbb{X}$ be a metric space (not necessarily compact)). Let $\mathcal{F}\subseteq C(\mathbb{X})$.

(i) $\mathcal{F}$ is equicontinuous $:\Leftrightarrow \forall\varepsilon>0\ \forall x\in\mathbb{X}\ \exists\delta>0 : \forall f\in\mathcal{F}$:

$$f(B_\delta(x))\subseteq B_\varepsilon(f(x))$$

(i.e. $d_{\mathbb{X}}(x,y)<\delta \Rightarrow |f(x)-f(y)|<\varepsilon \quad \forall f\in\mathcal{F}$; $\delta=\delta(\varepsilon,x)$)

(ii) $\mathcal{F}$ is uniformly equicontinuous $:\Leftrightarrow \forall\varepsilon>0\ \exists\delta>0 : \forall x\in\mathbb{X}\ \forall f\in\mathcal{F}$

$$f(B_\delta(x))\subseteq B_\varepsilon(f(x))$$

(i.e., $\delta=\delta(\varepsilon)$, but uniformly in $f\in\mathcal{F}$ & $x,y\in\mathbb{X}$)

Remark 1.37 (a) If $\mathbb{X}$ is even compact (by Thm. 1.30 (c)): equicontinuous $\Leftrightarrow$ uniformly equicontinuous.

(b) Examples:

(i) Any finite collection of continuous functions is equicontinuous

(ii) $\mathbb{X}=[0,1]$: $\{x\mapsto \cos(\frac{x}{n}) \mid n\in\mathbb{N}\}$ is uniformly equicontinuous

(iii) $\mathbb{X}=[0,1]$: $\{x\mapsto x^{1/n} \mid n\in\mathbb{N}\}$ is not equicontinuous

(iv) $\mathbb{X}=(0,\infty)$: $\{x\mapsto \arctan(nx) \mid n\in\mathbb{N}\}$ is equicontinuous but not uniformly equicont. However, each function is uniformly continuous.

Theorem 1.38 Let X be a metric space (not necessarily cpt.) and $(f_n)_{n\in\mathbb{N}} \subseteq C(X)$ an equicontinuous sequence. Assume there exists dense subset $D \subseteq X$ such that

$$\forall x \in D: \quad \lim_{n\to\infty} f_n(x) \quad \text{exists.}$$

Then $\lim_{n\to\infty} f_n(x) =: f(x)$ exists $\forall x \in X$, and the limit function $f \in C(X)$.

Pf: We first prove convergence for all $x \in X$:

Let $x \in X$ (fixed), $\varepsilon > 0$. By assumption there exists $\delta > 0$ s.t.

$$(*) \qquad \forall n \in \mathbb{N}\ \forall x' \in B_\delta(x): \quad |f_n(x) - f_n(x')| < \varepsilon.$$

Since $\overline{D} = X$, there exists $y \in B_\delta(x) \cap D$. At y, the seq. $(f_n(y))_{n\in\mathbb{N}}$ converges (by assumption), hence, it is Cauchy. So: $\exists N \in \mathbb{N}\ \forall n,m \geq N$:

$$(**) \qquad |f_n(y) - f_m(y)| < \varepsilon$$

and so, $\forall n,m \geq N$

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(y)|}_{<\varepsilon \text{ by } (*)} + \underbrace{|f_n(y) - f_m(y)|}_{<\varepsilon \text{ by } (**)} + \underbrace{|f_m(y) - f_m(x)|}_{<\varepsilon \text{ by } (*)}$$

$$< 3\varepsilon$$

So $\lim_{n\to\infty} f_n(x) =: f(x)$ exists for every $x \in X$ (since $(f_n(x))_{n\in\mathbb{N}}$ Cauchy & $\mathbb{K}$ complete)

We now prove continuity of f:

Note that $(*)$ holds for all $n \in \mathbb{N}$, and ε & δ there are indep. of n.

Hence, we can take the limit $n \to \infty$ in $(*)$ to get

$$\forall \varepsilon > 0\ \exists \delta > 0: \forall x' \in B_\delta(x): \quad |f(x) - f(x')| \leq \varepsilon \qquad \blacksquare$$

Corollary 1.39 Assume – in addition to the assumptions in Thm. 1.38 – that X is compact. Then in fact

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0$$

(i.e., the convergence is uniform on X)

Pf: Let $\varepsilon > 0$. As X is compact,

(i) $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous (see Rmk 1.37(a)), so:

$\exists \delta > 0$: $\quad \forall x \in X\ \forall n \in \mathbb{N}\ \forall x' \in X$ with $d(x,x') < \delta$:

$$|f_n(x) - f_n(x')| < \varepsilon \qquad (*),$$

(ii) there exists $K \in \mathbb{N}$ and $a_1, \dots, a_K \in X$ such that

$$X = \bigcup_{k=1}^{K} B_\delta(a_k). \qquad (**)$$

Using $(**)$, for every $x \in X$ we can choose $k_x \in \{1,\dots,K\}$ s.t. $x \in B_\delta(a_{k_x})$.
Then, $\forall n \in \mathbb{N}$,

$$|f_n(x) - f(x)| \le \underbrace{|f_n(x) - f_n(a_{k_x})|}_{=:T_1} + \underbrace{|f_n(a_{k_x}) - f(a_{k_x})|}_{=:T_2} + \underbrace{|f(a_{k_x}) - f(x)|}_{=:T_3}.$$

(1) $T_1 < \varepsilon$ for every $n \in \mathbb{N}$ by the definition of a_{k_x} and $(*)$)

(2) $\exists N \in \mathbb{N}: \forall n \ge N\ \forall x \in X$: $T_2 < \varepsilon$ by simultanous convergence of $(f_n(a_k))_n$ for $k = 1,\dots,K$ (finitely many!) by Thm. 1.38

(3) $T_3 \le \varepsilon$ by the definition of a_{k_x}, $(*)$, and the limit $n \to \infty$ in $(*)$)

Hence, we have

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall x \in X: \ |f_n(x) - f(x)| < 3\varepsilon \qquad \blacksquare$$

Theorem 1.40 (Arzelà–Ascoli) Let X be a compact metric space, and $(f_n)_{n\in\mathbb{N}} \subseteq C(X)$ an equicontinuous and pointwise bounded sequence, i.e. $\quad \forall x \in X: \ \sup_{n\in\mathbb{N}} |f_n(x)| < \infty$

Then there exists a uniformly convergent subsequence $(f_{n_j})_{j\in\mathbb{N}}$.

Equivalently: Every equicontinuous and pointwise bounded subset $\mathcal{F} \subseteq C(X)$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact).

Pf. The equivalence of the 2 statements follows from:

Z metric space, $A \subseteq Z$: A relatively compact $\iff$ every sequence in A has a convergent subsequence (with limit not necessarily in A but only in $\bar{A}$) (see exercise).

We prove the "sequence version":

Since X is compact, X is also separable (by Thm. 1.27), so there exists a dense subset $\{a_\ell \in X \mid \ell \in \mathbb{N}\} \subseteq X$. Pointwise boundedness gives

$$\sup_{n\in\mathbb{N}} |f_n(a_\ell)| < \infty \qquad \forall \ell \in \mathbb{N}, \qquad (*)$$

so, by the Bolzano-Weierstraß Thm. (in $\mathbb{K}$), for every ℓ there exists a subsequence $(n_j^{(\ell)})_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{j\to\infty} f_{n_j^{(\ell)}}(a_\ell) \quad \text{exists.} \qquad (**)$$

But: We need simultaneous convergence on the set of points $\{a_\ell \in X \mid \ell \in \mathbb{N}\}$ for a single subsequence. Use diagonal sequence trick: W.l.o.g., assume $(n_j^{(\ell+1)})_{j\in\mathbb{N}} \subseteq (n_j^{(\ell)})_{j\in\mathbb{N}}$

[Indeed, $\sup_{j\in\mathbb{N}} |f_{n_j^{(1)}}(a_2)| < \infty$ by $(*)$. Thus, $(**)$ holds, with $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$. Now proceed by induction].

Define $v_j := n_j^{(j)}$ for $j \in \mathbb{N}$ ("diagonal sequence"). Then $(v_j)_{j\geq \ell} \subseteq (n_j^{(\ell)})_j$ and $\lim_{j\to\infty} f_{v_j}(a_\ell)$ exists for all $\ell \in \mathbb{N}$.

Since the set of the a_ℓ's is dense, Cor. 1.39 gives the claim ∎

Remark 1.41 Both assumptions – compactness of X and equicontinuity – are essential for Thm 1.40 to turn pointwise boundness of a sequence into (uniform) convergence of a subsequence.

1.6 Baire's Theorem

– the basis of 3 (out of 4) fundamental theorems of FA.

Remark 1.42 Let X be a metric space, $A_1, A_2 \subseteq X$ both dense, and A_1 also open. Then $A_1 \cap A_2$ is also dense.

Completeness gives more:

Theorem 1.43 (Baire) Let X be a complete metric space, and, for all $n \in \mathbb{N}$, let $A_n \subseteq X$ be open and dense. Then $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.

Remark 1.44 (a) Completeness is essential: Consider $\mathbb{Q}$ (with metric from $\mathbb{R}$). Let $\{q_n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Define $A_n := \mathbb{Q} \setminus \{q_n\}$. This is open and dense in $\mathbb{Q}$ but $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ is not dense in $\mathbb{Q}$.

(b) Openness of $\bigcap_{n\in\mathbb{N}} A_n$ is false in general. Consider $A_n := \mathbb{R} \setminus \{q_n\}$.

(c) Baire's Theorem also holds if X is a locally compact Hausdorff space.

Pf (of 1.43): Define $D := \bigcap_{n\in\mathbb{N}} A_n$. Let $x_0 \in X$ be arbitrary and fix $\varepsilon > 0$. To prove the denseness of D in X we have to prove that $D \cap B_\varepsilon(x_0) \neq \emptyset$. We do this by constructing a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to some $x \in D \cap B_\varepsilon(x_0)$ (see drawing).

That A_1 is dense implies $A_1 \cap B_\varepsilon(x_0) \neq \emptyset$, so pick $x_1 \in A_1 \cap B_\varepsilon(x_0)$ and $\varepsilon_1 \in (0, \frac{\varepsilon}{2})$ s.t.

$$\overline{B_{\varepsilon_1}(x_1)} \subseteq A_1 \cap B_\varepsilon(x_0) \quad (\leftarrow \text{open})$$

Proceeding similarly with A_2 and $B_{\varepsilon_1}(x_1)$ we pick $x_2 \in A_2 \cap B_{\varepsilon_1}(x_1)$ and $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2})$ s.t.

$$\overline{B_{\varepsilon_2}(x_2)} \subseteq A_2 \cap B_{\varepsilon_1}(x_1) \subseteq A_1 \cap A_2 \cap B_\varepsilon(x_0)$$

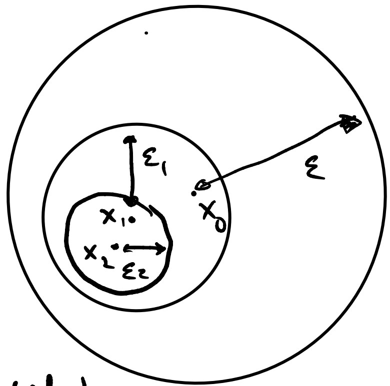
We proceed inductively in the same way, to get 2 sequences

(a) $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n < \frac{\varepsilon}{2^n}$ $\forall n\in\mathbb{N}$

(b) $(x_n)_n \subseteq X$ with, $\forall n \in \mathbb{N}$,

$$\overline{B_{\varepsilon_n}(x_n)} \subseteq A_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \subseteq A_n \cap \dots \cap A_1 \cap B_\varepsilon(x_0)$$

– hence, $\forall n \in \mathbb{N}\ \forall m \geq n$: $x_m \in B_{\varepsilon_n}(x_n)$. (*)



Now (*) & (a) implies $(x_n)_n$ is Cauchy,

and so: $\exists x \in X$: $x_n \to x$, $n \to \infty$ (since X complete)

But by (*) we have $x \in \overline{B_{\varepsilon_n}(x_n)}$ $\forall n \in \mathbb{N}$, and (b) gives

$$x \in \bigcap_{n\in\mathbb{N}} \overline{B_{\varepsilon_n}(x_n)} \subseteq D \cap B_\varepsilon(x_0), \text{ so } D \cap B_\varepsilon(x_0) \neq \emptyset \quad \blacksquare$$

Definition 1.45 Let X be a topological space and $A \subseteq X$.

(i) A is a G_δ (-set) :$\iff$ A is a countable intersection of open sets.

(ii) A is no-where dense :$\iff$ $\overline{A}$ has no interior points.

(iii) A is meagre (or of 1^{st} category): $\iff$ A is a countable union of nowhere dense sets

(iv) A is non-meagre (or of 2^{nd} category) :$\iff$ A is not meagre

Example 1.46 $\mathbb{Q}$ is meagre in $\mathbb{R}$ ($\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}$)

Lemma 1.47 Let X be a topological space and $A \subseteq X$. Then

(a) A is nowhere dense $\iff$ $(\overline{A})^c$ dense

(b) A is meagre and $B \subseteq A$ $\implies$ B is meagre

(c) $A_n \subseteq X$ meagre $\forall n \in \mathbb{N}$ $\implies$ $\bigcup_{n\in\mathbb{N}} A_n$ meagre.

Pf: (b) & (c) are clear by the very definitions (do!).

Statement (a) follows from equivalence

$$B \text{ has no interior points} \iff B^c \text{ dense}$$

This is equivalent to

$$\exists \text{ interior point of } B \iff B^c \text{ is not dense}$$

which is clearly true $\blacksquare$

We now prove 3 equivalent reformulations of Baire's Thm:

Lemma 1.48 Let X be a topological space. Then the following 4 statements are equivalent:

(i) $A_n \subseteq X$ open & dense $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ dense

(ii) $A_n \subseteq X$ is a dense G_δ $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ

(iii) $A \subseteq X$ has an interior point $\Rightarrow$ A non-meagre

(iv) $A \subseteq X$ meagre $\Rightarrow$ A^c dense

If one (hence, all) of the above holds, X is called a Baire space.

Corollary 1.49 A complete metric space X is a Baire space.

In particular: If $X \neq \emptyset$, then X is non-meagre.

Pf (of 1.48): (i) $\Leftrightarrow$ (ii) by definition of G_δ

(i) $\Rightarrow$ (iii): Suppose A is meagre, that is, $A = \bigcup_{n\in\mathbb{N}} A_n$ with A_n nowhere dense $\forall n \in \mathbb{N}$. By 1.47(a) this is equivalent to $(\overline{A_n})^c$ dense & open $\forall n \in \mathbb{N}$, and (i) implies that the intersection $\bigcap_{n\in\mathbb{N}} (\overline{A_n})^c =: B$ is also dense. But then $B^c = \bigcup_{n\in\mathbb{N}} \overline{A_n} \supseteq A$ has no interior points, hence A has no interior points $\lightning$

(iii) $\Rightarrow$ (iv): See exercise

(iv) $\Rightarrow$ (i): See exercise