1.6 Baire's Theorem

- the basis of 3 (out of 4) fundamental theorems of FA.

Remark 1.42 Let X be a metric space, $A_1, A_2 \subseteq X$ both dense, and A_1 also open. Then $A_1 \cap A_2$ is also dense

Completeness gives more:

Theorem 1.43 (Baire) Let X be a complete metric space, and, for all $n \in \mathbb{N}$, let $A_n \subseteq X$ be open and dense.
Then $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.

Remark 1.44 (a) Completeness is essential: Consider $\mathbb{Q}$ (with metric from $\mathbb{R}$). Let $\{q_n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Define $A_n := \mathbb{Q} \setminus \{q_n\}$. This is open and dense in $\mathbb{Q}$ but $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ is not dense in $\mathbb{Q}$.

(b) Openness of $\bigcap_{n\in\mathbb{N}} A_n$ is false in general. Conside $A_n := \mathbb{R} \setminus \{q_n\}$.

(c) Baire's Theorem also holds if X is a locally compact Hausdorff space.

Pf (of 1.43): Define $D := \bigcap_{n\in\mathbb{N}} A_n$. Let $x_0 \in X$ be arbitrary and fix $\varepsilon > 0$.
To prove the denseness of D in X we have to prove that $D \cap B_\varepsilon(x_0) \neq \emptyset$.
We do this by constructing a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to some $x \in D \cap B_\varepsilon(x_0)$ (see drawing)
That A_1 is dense implies $A_1 \cap B_\varepsilon(x_0) \neq \emptyset$, so pick $x_1 \in A_1 \cap B_\varepsilon(x_0)$ and $\varepsilon_1 \in (0, \frac{\varepsilon}{2})$ s.t.

$$\overline{B_{\varepsilon_1}(x_1)} \subseteq A_1 \cap B_\varepsilon(x_0) \quad (\leftarrow \text{open})$$

Proceeding similarly with A_2 and $B_{\varepsilon_1}(x_1)$ we pick $x_2 \in A_2 \cap B_{\varepsilon_1}(x_1)$ and $\varepsilon_2 \in (0, \frac{\varepsilon_1}{2})$ s.t.

$$\overline{B_{\varepsilon_2}(x_2)} \subseteq A_2 \cap B_{\varepsilon_1}(x_1) \subseteq A_1 \cap A_2 \cap B_\varepsilon(x_0)$$

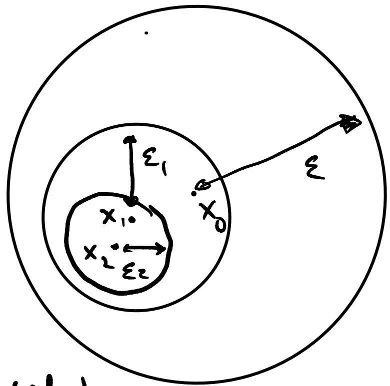
We proceed inductively in the same way, to get 2 sequences

(a) $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n < \frac{\varepsilon}{2^n}$ $\forall n \in \mathbb{N}$

(b) $(x_n)_n \subseteq X$ with, $\forall n \in \mathbb{N}$,

$$\overline{B_{\varepsilon_n}(x_n)} \subseteq A_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \subseteq A_n \cap \dots \cap A_1 \cap B_\varepsilon(x_0)$$

– hence, $\forall n \in \mathbb{N}\ \forall m \geq n$: $x_m \in B_{\varepsilon_n}(x_n)$. (*)



Now (*) & (a) implies $(x_n)_n$ is Cauchy,

and so: $\exists x \in X$: $x_n \to x$, $n \to \infty$ (since X complete)

But by (*) we have $x \in \overline{B_{\varepsilon_n}(x_n)}$ $\forall n \in \mathbb{N}$, and (b) gives

$$x \in \bigcap_{n\in\mathbb{N}} \overline{B_{\varepsilon_n}(x_n)} \subseteq D \cap B_\varepsilon(x_0), \text{ so } D \cap B_\varepsilon(x_0) \neq \emptyset \quad \blacksquare$$

Definition 1.45 Let X be a topological space and $A \subseteq X$.

(i) A is a G_δ(-set) $:\Leftrightarrow$ A is a countable intersection of open sets.

(ii) A is no-where dense $:\Leftrightarrow$ $\overline{A}$ has no interior points.

(iii) A is meagre (or of 1^{st} category) $:\Leftrightarrow$ A is a countable union of nowhere dense sets

(iv) A is non-meagre (or of 2^{nd} category) $:\Leftrightarrow$ A is not meagre

Example 1.46 $\mathbb{Q}$ is meagre in $\mathbb{R}$ $\left(\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}\right)$

Lemma 1.47 Let X be a topological space and $A \subseteq X$. Then

(a) A is nowhere dense $\Leftrightarrow$ $(\overline{A})^c$ dense

(b) A is meagre and $B \subseteq A$ $\Rightarrow$ B is meagre

(c) $A_n \subseteq X$ meagre $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcup_{n\in\mathbb{N}} A_n$ meagre.

Pf: (b) & (c) are clear by the very definitions (do!).

Statement (a) follows from equivalence

$$B \text{ has no interior points} \Leftrightarrow B^c \text{ dense}$$

This is equivalent to

$$\exists \text{ interior point of } B \Leftrightarrow B^c \text{ is } \underline{\text{not}} \text{ dense}$$

which is clearly true $\blacksquare$

We now prove 3 equivalent reformulations of Baire's Thm:

Lemma 1.48 Let X be a topological space. Then the following 4 statements are equivalent:

(i) $A_n \subseteq X$ open & dense $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ dense

(ii) $A_n \subseteq X$ is a dense G_δ $\forall n \in \mathbb{N}$ $\Rightarrow$ $\bigcap_{n\in\mathbb{N}} A_n$ is a dense G_δ

(iii) $A \subseteq X$ has an interior point $\Rightarrow$ A non-meagre

(iv) $A \subseteq X$ meagre $\Rightarrow$ A^c dense

If one (hence, all) of the above holds, X is called a Baire space.

Corollary 1.49 A complete metric space X is a Baire space.

In particular: If $X \neq \emptyset$, then X is non-meagre.

Pf (of 1.48): (i) $\Leftrightarrow$ (ii) by definition of G_δ

(i) $\Rightarrow$ (iii): Suppose A is meagre, that is, $A = \bigcup_{n\in\mathbb{N}} A_n$ with A_n nowhere dense $\forall n \in \mathbb{N}$. By 1.47(a) this is equivalent to $(\overline{A_n})^c$ dense & open $\forall n \in \mathbb{N}$, and (i) implies that the intersection $\bigcap_{n\in\mathbb{N}} (\overline{A_n})^c =: B$ is also dense. But then $B^c = \bigcup_{n\in\mathbb{N}} \overline{A_n} \supseteq A$ has no interior points, hence A has no interior points $\lightning$

(iii) $\Rightarrow$ (iv): See exercise

(iv) $\Rightarrow$ (i): See exercise
