Because f is continuous & V_α open, $f^{-1}(V_\alpha)$ is open $\forall \alpha \in J$, (19)
hence (*) gives an open cover of X – which is compact.
Hence, $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N$ s.t. $X \subseteq \bigcup_{n=1}^{N} f^{-1}(V_\alpha)$, and therefore,

$$f(X) \subseteq \bigcup_{n=1}^{N} V_{\alpha_n}$$

(b), (c), (d): see Exercises.

1.5. Example: Spaces of continuous functions

General assumptions in this section:

(i) X is a compact Hausdorff space

(ii) $C(X)$ is equipped with the uniform (supremum) metric:

$$d_\infty(f,g) := \|f-g\|_\infty := \sup_{x \in X} |f(x)-g(x)|$$

(Note: sup = max is finite $\forall f,g \in C(X)$ by Thm. 1.30 (d))

$\boxed{\text{Theorem 1.31}}$ $C(X)$ is complete.

Pf: Follows from completeness of $C_b(X)$ (bounded cont. functions; see exercise), and that $C(X) = C_b(X)$, which follows from compactness of X, and Thm. 1.30 (d). ∎

$\boxed{\text{Theorem 1.32}}$ X is metrisable $\Leftrightarrow$ $C(X)$ is separable

(A topological space $(X, \mathcal{T})$ is metrisable :$\Leftrightarrow$ $\exists$ metric d on X that generates the topology $\mathcal{T}$)

Pf: For "$\Leftarrow$": See (f.ex) Bourbaki, "Elements of Mathematics, General Topology", Part 2, Sect. X 3.3, Thm. 1.1.

Here, we only prove:

"$\Rightarrow$": Fix any metric that is compatible with the topology.

For $m,n \in \mathbb{N}$, define

$$G_{m,n} := \left\{ f \in C(X) \;\middle|\; f(B_{\frac{1}{m}}(x)) \subseteq B_{\frac{1}{n}}(f(x)) \quad \forall x \in X \right\}$$

By compactness of X we get

(i) Any $f \in C(X)$ is even uniformly continuous, by Thm. 1.30 (c).

Choose $\varepsilon := \frac{1}{n}$ there, and let m large enough s.t. $\frac{1}{m} \le \delta$. Then $f \in G_{mn}$, so $C(\mathbb{X}) = \bigcup_{m \in \mathbb{N}} G_{mn} \quad \forall n \in \mathbb{N}$ (1)

(ii) For any $m \in \mathbb{N}$ we can find $k_m \in \mathbb{N}$ and $a_1, \dots, a_{k_m} \in \mathbb{X}$ such that $\mathbb{X}$ can be written as a union of open balls of radius $\frac{1}{m}$: $\mathbb{X} = \bigcup_{k=1}^{k_m} B_{\frac{1}{m}}(a_k)$ (2) ($\mathbb{X}$ is compact)

Now, $\mathbb{K}$ is separable, i.e. there exists countable set $\{x_\nu \in \mathbb{K} \mid \nu \in \mathbb{N}\}$ which is dense in $\mathbb{K}$.

For given $m \in \mathbb{N}$ and any $\varphi : \{1, \dots, k_m\} \to \mathbb{N}$, i.e. $\varphi = (\varphi(1), \dots, \varphi(k_m)) \in \mathbb{N}^{k_m}$, define

$$G^{\varphi}_{mn} := \left\{ g \in G_{mn} \;\middle|\; |g(a_k) - x_{\varphi(k)}| < \frac{1}{n} \;\; \forall k = 1, \dots, k_m \right\}$$

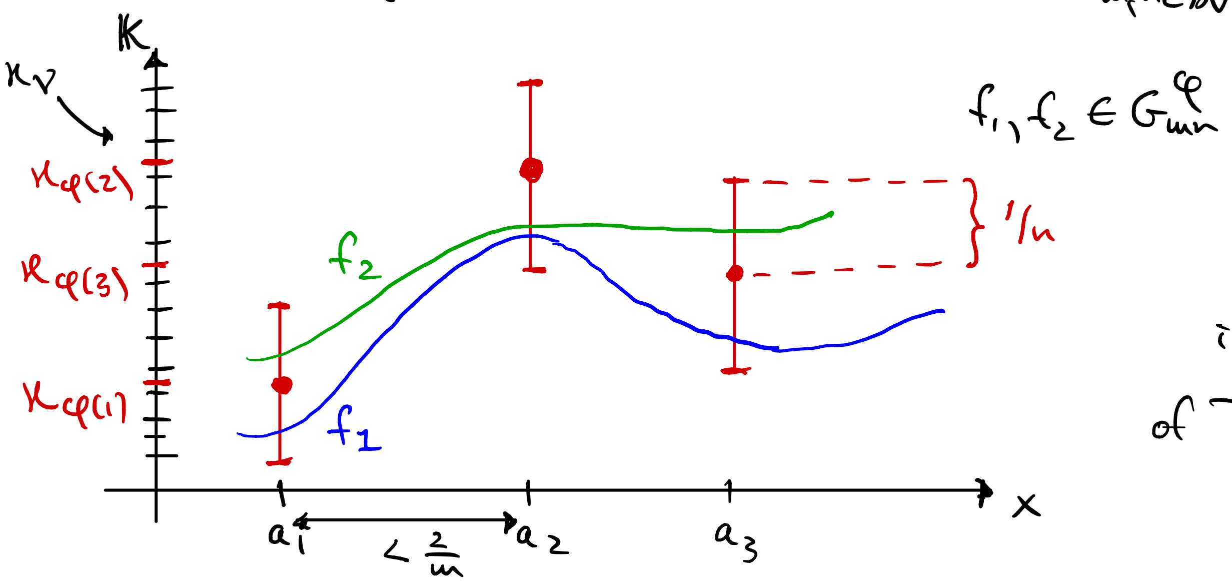
We only want to consider those φ with $G^{\varphi}_{mn} \neq \emptyset$, so let

$$\Phi_{mn} := \{ \varphi \in \mathbb{N}^{k_m} \mid G^{\varphi}_{mn} \neq \emptyset \}$$

Φ_{mn} is not empty (since $\varphi_\nu := (\nu, \dots, \nu) \in \Phi_{mn}$, because the constant function $f \equiv x_\nu \in G^{\varphi_\nu}_{mn}$).

For every $\varphi \in \Phi_{mn}$ pick some $g_\varphi \in G^{\varphi}_{mn}$. Now define

$$L_{mn} := \{ g_\varphi \mid \varphi \in \Phi_{mn} \} \quad \text{and} \quad L := \bigcup_{m,n \in \mathbb{N}} L_{mn}$$



Notions in proof of Thm. 1.32

Note: L_{mn} is countable because $\Phi_{mn} \subseteq \mathbb{N}^{K_m}$, hence L is countable. The theorem now follows from:

<u>Claim</u>: $\overline{L}^{d_\infty} = C(X)$

<u>Pf</u>: Fix $f \in C(X)$, $n \in \mathbb{N}$.

(1) $\Rightarrow \exists m \in \mathbb{N} : f \in G_{mn}$. Also, $\exists \varphi_f \in \Phi_{mn}$ s.t. $f \in G_{mn}^{\varphi_f}$: Simply choose $\varphi_f(k)$ such that $|f(a_k) - \kappa_{\varphi_f(k)}| < \frac{1}{n}$, which is possible by the denseness of $\{\kappa_r \mid r \in \mathbb{N}\}$ in $\mathbb{K}$.

(2) $\Rightarrow \forall x \in X \; \exists k_x \in \{1, \dots, K_m\} : x \in B_{\frac{1}{m}}(a_{k_x})$

Take $g_{\varphi_f} \in L_{mn} \subseteq L$ as approximant. Since $f, g_{\varphi_f} \in G_{mn}^{\varphi_f}$ we have, for all $x \in X$,

$$\begin{aligned}|f(x) - g_{\varphi_f}(x)| &\leq \underbrace{|f(x) - f(a_{k_x})|}_{< \frac{1}{n} \text{ by def. of } G_{mn}} + \underbrace{|f(a_{k_x}) - \kappa_{\varphi_f(k_x)}|}_{< \frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} \\ &+ \underbrace{|\kappa_{\varphi_f(k_x)} - g_{\varphi_f}(a_{k_x})|}_{< \frac{1}{n} \text{ by def. of } G_{mn}^{\varphi_f}} + \underbrace{|g_{\varphi_f}(a_{k_x}) - g_{\varphi_f}(x)|}_{< \frac{1}{n} \text{ by def. of } G_{mn}} \\ &< \frac{4}{n}.\end{aligned}$$

Since f and n were arbitrary, the claim follows $\boxtimes$

An alternative approach to separability:

$\boxed{\text{Definition 1.33}}$ (a) A $\mathbb{K}$-vectorspace A is a <u>$\mathbb{K}$-algebra</u> $:\Leftrightarrow$ there exists a multiplication $A \times A \to A$ which satisfies:

$$(a+b)c = ac + bc \qquad \forall a, b, c \in A$$

$$c(a+b) = ca + cb \qquad \forall a, b, c \in A$$

$$\lambda(ac) = (\lambda a)c = a(\lambda c) \qquad \forall a, c \in A, \lambda \in \mathbb{K}$$

(Example: $C(X)$ is a (commutative!) $\mathbb{K}$-algebra)

(b) A subspace $B \subseteq A$ is a subalgebra $:\Leftrightarrow$ B is closed under multiplication

(c) A subset $B \subseteq C(X)$ separates points in X $:\Leftrightarrow$ for all distinct points $x, y \in X$, $x \neq y$, there exists $f \in B$: $f(x) \neq f(y)$

Theorem 1.34 (Stone–Weierstraß) Let $B \subseteq C(X)$ be a subalgebra with the following properties:

(i) $1 \in B$ (where $1 : X \to \mathbb{K}$, $x \mapsto 1$ is the constant function) B is unital

(ii) B separates points.

(iii) If $\mathbb{K} = \mathbb{C}$: Assume further that B is closed under complex conjugation (i.e. $f \in B \Rightarrow \bar{f} \in B$)

Then: $\overline{B}^{d_\infty} = C(X)$ (that is, B is dense in $C(X)$)

Pf: Not here; see e.g. Reed–Simon, Vol. 1, App. to Sect. IV.3

Corollary 1.35 Let $X \subseteq \mathbb{R}^d$ be compact, $d \in \mathbb{N}$. Then the set of polynomials is dense (wrt. d_∞) in $C(X)$. Furthermore, $C(X)$ is separable.

Pf: Let B be the $\mathbb{K}$-subalgebra of the $\mathbb{K}$-algebra $C(X)$ generated by the monomials

$$(*) \quad M_{n,\alpha} : X \ni x = (x_1, \dots, x_d) \mapsto x_\alpha^n \in \mathbb{R}, \quad n \in \mathbb{N}_0, \alpha \in \{1, \dots, d\}$$

Then Stone–Weierstraß gives $\overline{B} = C(X)$ since, clearly,

(1) $1 \in B$ and, if $\mathbb{K} = \mathbb{C}$, then B is closed under complex conjugation, since $x \in X \subseteq \mathbb{R}^d$ and B is a $\mathbb{C}$-subspace.

(2) If $x, y \in X$, $x \neq y$, then $\exists \alpha \in \{1, \dots, d\} : x_\alpha \neq y_\alpha$, so $M_{1,\alpha}(x) \neq M_{1,\alpha}(y)$, and so B separates points.

We prove separability: Let $\mathbb{K}_0 := \begin{cases} \mathbb{Q}, & \mathbb{K} = \mathbb{R} \\ \mathbb{Q} + i\mathbb{Q}, & \mathbb{K} = \mathbb{C} \end{cases}$

and let B_0 be the $\mathbb{K}_0$-algebra generated by the monomials in (*) (Note: This is not a sub algebra of $C(\mathbb{X})$ - why?!)

(1) B_0 is countable (it consists of polynomials with rational coeff.)

(2) Since $\mathbb{X}$ is bounded:

$\forall \varepsilon > 0\ \forall n \in \mathbb{N}_0\ \forall \alpha \in \{1, .., d\}\ \forall c \in \mathbb{K}\ \exists q \in \mathbb{K}_0: d_\infty(c M_{n,\alpha}, q M_{n,\alpha}) < \varepsilon$

$\Rightarrow \quad \overline{B_0} = \overline{B} = C(\mathbb{X})$ ∎

Definition 1.36 (Let $\mathbb{X}$ be a metric space (not necessarily compact)) Let $\mathcal{F} \subseteq C(\mathbb{X})$.

(i) $\mathcal{F}$ is *equicontinuous* $:\Leftrightarrow \forall \varepsilon > 0\ \forall x \in \mathbb{X}\ \exists \delta > 0: \forall f \in \mathcal{F}:$

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

(i.e. $d_{\mathbb{X}}(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \underline{\forall f \in \mathcal{F}}$; $\delta = \delta(\varepsilon, \underset{*}{x})$)

(ii) $\mathcal{F}$ is *uniformly equicontinuous* $:\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0: \forall x \in \mathbb{X}\ \forall f \in \mathcal{F}$

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

(i.e., $\delta = \delta(\varepsilon)$, but uniformly in $f \in \mathcal{F}$ & $x, y \in \mathbb{X}$)

Remark 1.37 (a) If $\mathbb{X}$ is even compact (by Thm. 1.30 (c)): equicontinuous $\Leftrightarrow$ uniformly equicontinuous.

(b) Examples:

(i) Any <u>finite</u> collection of continuous functions is equicontinuous

(ii) $\mathbb{X} = [0,1]$: $\{x \mapsto \cos(\frac{x}{n}) \mid n \in \mathbb{N}\}$ is uniformly equicontinuous

(iii) $\mathbb{X} = [0,1]$: $\{x \mapsto x^{1/n} \mid n \in \mathbb{N}\}$ is <u>not</u> equicontinuous

(iv) $\mathbb{X} = (0, \infty)$: $\{x \mapsto \arctan(nx) \mid n \in \mathbb{N}\}$ is equicontinuous but not uniformly equicont. However, each function <u>is</u> uniformly continuous.

Theorem 1.38 Let X be a metric space (not necessarily cpt.) and $(f_n)_{n\in\mathbb{N}} \subseteq C(X)$ an equicontinuous sequence. Assume there exists dense subset $D \subseteq X$ such that

$$\forall x \in D: \quad \lim_{n\to\infty} f_n(x) \quad \text{exists.}$$

Then $\lim_{n\to\infty} f_n(x) =: f(x)$ exists $\forall x \in X$, and the limit function

$f \in C(X)$.

Pf: We first prove convergence for all $x \in X$:

Let $x \in X$ (fixed), $\varepsilon > 0$. By assumption there exists $\delta > 0$ s.t.

$$(*) \qquad \forall n \in \mathbb{N}\ \forall x' \in B_\delta(x): \quad |f_n(x) - f_n(x')| < \varepsilon.$$

Since $\overline{D} = X$, there exists $y \in B_\delta(x) \cap D$. At y, the seq. $(f_n(y))_{n\in\mathbb{N}}$ converges (by assumption), hence, it is Cauchy. So: $\exists N \in \mathbb{N}\ \forall n,m \geq N$:

$$(**) \qquad |f_n(y) - f_m(y)| < \varepsilon$$

and so, $\forall n,m \geq N$

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(y)|}_{<\varepsilon \text{ by } (*)} + \underbrace{|f_n(y) - f_m(y)|}_{<\varepsilon \text{ by } (**)} + \underbrace{|f_m(y) - f_m(x)|}_{<\varepsilon \text{ by } (*)}$$

$$< 3\varepsilon$$

So $\lim_{n\to\infty} f_n(x) =: f(x)$ exists for every $x \in X$ (since $(f_n(x))_{n\in\mathbb{N}}$ Cauchy & $\mathbb{K}$ complete)

We now prove continuity of f:

Note that $(*)$ holds for all $n \in \mathbb{N}$, and ε & δ there are indep. of n.

Hence, we can take the limit $n \to \infty$ in $(*)$ to get

$$\forall \varepsilon > 0\ \exists \delta > 0: \forall x' \in B_\delta(x): \quad |f(x) - f(x')| \leq \varepsilon \qquad \blacksquare$$

Corollary 1.39 Assume – in addition to the assumptions in Thm. 1.38 – that X is compact. Then in fact

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0$$

(i.e., the convergence is uniform on X)

<u>Pf</u>: Let $\varepsilon>0$. As X is compact,

(i) $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous (see Rmk 1.37(a)), so:

$\exists \delta>0$: $\quad \forall x\in X\ \forall n\in\mathbb{N}\ \forall x'\in X$ with $d(x,x')<\delta$:

$$|f_n(x)-f_n(x')|<\varepsilon \qquad (*),$$

(ii) there exists $K\in\mathbb{N}$ and $a_1,\dots,a_K\in X$ such that

$$X=\bigcup_{k=1}^{K} B_\delta(a_k). \qquad (**)$$

Using $(**)$, for every $x\in X$ we can choose $k_x\in\{1,\dots,K\}$ s.t. $x\in B_\delta(a_{k_x})$

Then, $\forall n\in\mathbb{N}$,

$$|f_n(x)-f(x)|\le \underbrace{|f_n(x)-f_n(a_{k_x})|}_{=:T_1}+\underbrace{|f_n(a_{k_x})-f(a_{k_x})|}_{=:T_2}+\underbrace{|f(a_{k_x})-f(x)|}_{=:T_3}.$$

(1) $T_1<\varepsilon$ for every $n\in\mathbb{N}$ by the definition of a_{k_x} and $(*)$)

(2) $\exists N\in\mathbb{N}: \forall n\ge N\ \forall x\in X$: $T_2<\varepsilon$ by sim<u>ul</u>tanous convergence of $(f_n(a_k))_n$ for $k=1,\dots,K$ (finitely many!) by Thm. 1.38

(3) $T_3\le\varepsilon$ by the definition of a_{k_x}, $(*)$, and the limit $n\to\infty$ in $(*)$)

Hence, we have

$$\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\ge N\ \forall x\in X: |f_n(x)-f(x)|<3\varepsilon \qquad \blacksquare$$

$\boxed{\text{Theorem 1.40}}$ (<u>Arzelà-Ascoli</u>) Let X be a compact metric space, and $(f_n)_{n\in\mathbb{N}}\subseteq C(X)$ an equicontinuous and pointwise bounded sequence, i.e. $\quad \forall x\in X: \sup_{n\in\mathbb{N}}|f_n(x)|<\infty$

Then there exists a uniformly convergent subsequence $(f_{n_j})_{j\in\mathbb{N}}$.

Equi<u>valently</u>: Every equicontinuous and pointwise bounded subset $\mathcal{F}\subseteq C(X)$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact).

Pf. The equivalence of the 2 statements follows from:

Z metric space, $A \subseteq Z$: A relatively compact $\iff$ every sequence in A has a convergent subsequence (with limit not necessarily in A but only in $\bar{A}$) (see exercise).

We prove the "sequence version":

Since X is compact, X is also separable (by Thm. 1.27), so there exists a dense subset $\{a_\ell \in X \mid \ell \in \mathbb{N}\} \subseteq X$. Pointwise boundedness gives

$$\sup_{n \in \mathbb{N}} |f_n(a_\ell)| < \infty \qquad \forall \ell \in \mathbb{N}, \qquad (*)$$

so, by the Bolzano-Weierstraß Thm. (in $\mathbb{K}$), for every ℓ there exists a subsequence $(n_j^{(\ell)})_{j \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$\lim_{j \to \infty} f_{n_j^{(\ell)}}(a_\ell) \quad \text{exists.} \qquad (**)$$

But: We need simultaneous convergence on the set of points $\{a_\ell \in X \mid \ell \in \mathbb{N}\}$ for a single subsequence. Use diagonal sequence trick: W.l.o.g., assume $(n_j^{(\ell+1)})_{j \in \mathbb{N}} \subseteq (n_j^{(\ell)})_{j \in \mathbb{N}}$

[Indeed, $\sup_{j \in \mathbb{N}} |f_{n_j^{(1)}}(a_2)| < \infty$ by $(*)$. Thus, $(**)$ holds, with $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$. Now proceed by induction].

Define $\nu_j := n_j^{(j)}$ for $j \in \mathbb{N}$ ("diagonal sequence"). Then $(\nu_j)_{j \geq \ell} \subseteq (n_j^{(\ell)})_j$ and $\lim_{j \to \infty} f_{\nu_j}(a_\ell)$ exists for all $\ell \in \mathbb{N}$.

Since the set of the a_ℓ's is dense, Cor. 1.39 gives the claim ∎

Remark 1.41 Both assumptions – compactness of X and equicontinuity – are essential for Thm 1.40 to turn pointwise boundness of a sequence into (uniform) convergence of a subsequence.